A NOTE ON LARGE SUMS OF FOURIER COEFFICIENTS OF HOLOMORPHIC CUSP FORMS

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Contents

1. Introduction

The study of mean values of multiplicative functions plays a central role in analytic number theory. A longstanding problem in the area – extensively studied over the past century and still attracting significant attention – is the asymptotic behavior of character sums. This article surveys the background on the subject and presents the author's recent results from [4] and [5]. In these works, we studied sums of the Fourier coefficients of holomorphic cusp forms, or more generally, sums of divisor-bounded multiplicative functions, providing a natural setting in which to investigate analogous questions more broadly. The main results provide new estimates for these quantities, obtained by adapting key ingredients from Pretentious Number Theory developed by Granville and Soundararajan. Our main results are stated as Theorems 3.1, 3.2, 3.3, and Corollary 3.4.

The organization of this article is as follows. In Section 2, we present an overview of the historical background and some key results for character sums, which motivate our later generalizations. Section 3 is devoted to the histrical context in the setting of the sums of Fourier coefficients and to the statement of our main results. Finally, Section 4

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describes the methods underlying our approach and outlines the ideas of the proofs of the main theorems.

Conventions and Notation. In this work, we adopt the following conventions and notation. Given two functions $f(x)$ and $g(x)$ we write $f(x) \ll g(x)$, $g(x) \gg f(x)$ or $f(x) = O(g(x))$ to mean there exists some positive constant M such that $|f(x)| \leq M|g(x)|$ for x large enough. The notation $f(x) \asymp g(x)$ is used when both estimates $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold simultaneously. We write $f(x) = o(g(x))$ when $g(x) \neq 0$ for sufficiently large x and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. The letter p will be exclusively used to represent a prime number.

2. Character Sums

Let χ be a primitive Dirichlet character modulo q and let $S(x,\chi) = \sum_{n\leq x} \chi(n)$ be the associated character sum. The sum $S(x,q)$ is trivially bounded as

$$|S(x,\chi)| \leq \min(x,q)$$

due to the orthogonality. A much deeper result was established independently by Pólya and Vinogradov [2, pages 135–137], which asserts that

$$\max_x |S(x,\chi)| \ll \sqrt{q}\log q.$$

Moreover, by assuming the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions, it was further improved to

$$\max_x |S(x,\chi)| \ll \sqrt{q}\log\log q,$$

by Montgomery and Vaughan [17]. Consequently, we have that

$$S(x,\chi) = o(x) \tag{2.1}$$

when $x \geq q^{\frac{1}{2}+\epsilon}$. We note that it is conjectured that (2.1) should hold in the range of $x \gg_\epsilon q^\epsilon$ for any $\epsilon > 0$. In this direction, Burgess made further improvements, proving that (2.1) holds in the range of

- $x > q^{\frac{1}{4}+o(1)}$ for any quadratic character χ when q is prime (1957),
- $x > q^{\frac{1}{4}+o(1)}$ for any character χ when q is cube-free (1962),
- $x > q^{\frac{3}{8}+o(1)}$ otherwise (1963).

While the unconditional results have seen little progress since then, some conditional results have been established. For example, Granville and Soundararajan [8] proved that, for a primitive character $\chi \mod q$, (2.1) holds in the range $\log x/\log\log q \to \infty$ assuming the GRH for $L(s,\chi)$, and also showed unconditionally that this range in x is best possible. In more recent work [10], they improved it by establishing that this asymptotic holds under the weaker assumption that "100%" of the zeros of $L(s,f)$ up to height $\frac{1}{4}$ lie on the critical line. More precisely, they proved the following:

Theorem 2.1. [10, Corollary 1.2] *Let χ be a primitive character modulo q. Let ϵ and T be real numbers with $1 \le T \le (\log q)^{1/200}$ and $\epsilon \ge (\log q)^{-1/3}$. Suppose that for every real ϕ with $|\phi| \le T$ the region $\{s : \Re(s) \ge 3/4, |\Im(s) - \phi| \le 1/4\}$ contains no more than $\epsilon^2(\log q)/1440$ zeros of $L(s,\chi)$. Then for all $x \ge q^\epsilon$, we have*

$$\left|\sum_{n\le x}\chi(n)\right| \ll \frac{x}{T}.$$

We were drawn to this work because it establishes strong connections between character sums and zeros of the associated L-functions through general results from multiplicative number theory pertaining to mean values of 1-bounded multiplicative functions in the pretentious framework. The paper also contains several additional interesting observations on large character sums. For example, they show that if χ_1 and χ_2 have large character sums then so does their product $\chi_1\chi_2$. In fact, they proved the following.

Theorem 2.2. [10, Corollary 1.7] *Suppose χ_1 and χ_2 are Dirichlet characters modulo q such that for some x_1, x_2 and some $\eta > 0$ we have (for $j = 1, 2$)*

$$\left|\sum_{n\le x_j}\chi_j(n)\right| \ge \eta x_j.$$

Then, with $\xi = c\eta^6$ for a suitable absolute constant $c > 0$, there exists $x \ge (\min(x_1,x_2))^\xi$ with

$$\left|\sum_{n\le x}(\chi_1\chi_2)(n)\right| \ge \xi x.$$

This result follows directly from [10, Theorem 6.2] which asserts that if the partial sums of two completely multiplicative 1-bounded functions are large, then the partial sum of their product is large as well.

It was our aim to generalize Theorems 2.1 and 2.2 to the setting of the sums of Fourier coefficients of a holomorphic cusp form in [4] and [5], respectively. We discuss these results in the following section.

3. Sums of Fourier Coefficients and Main Results

Let $f \in S_k(N)$ be a primitive cusp form of weight k and level N, given by the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz),$$

where $e(z) = e^{2\pi i z}$.

The Fourier coefficienst $\lambda_f(n)$ of a cusp form f satisifes $\lambda_f(n) \ll n^{\frac{1}{2}}$ for all $n \geq 1$, which is a consequence of standard arithmetic properties. A deep result of Deligne [3] provides a substantially sharper estimate, which states

$$|\lambda_f(n)| \leq d(n) \tag{3.1}$$

for any $n \geq 1$, where $d(n)$ is the number of positive divisors of n.

As in the case of the Dirichlet characters, a classical approach to understand the asymptotic behavior of the Fourier cofficients $\lambda_f(n)$ is to study them on average. In 1927, Hecke proved in [12] that

$$S(x, f) := \sum_{n \leq x} \lambda_f(n) \ll_f x^{\frac{1}{2}}$$

for x large enough. Later, Walfisz [18] improved this bound, using the Deligne's bound (3.1),

$$S(x, f) \ll_f x^{\frac{1}{3}+\epsilon} \tag{3.2}$$

for any $\epsilon > 0$. Hafner and Ivić [11] improved upon this result by removing the factor x^ϵ from (3.2). We note that the implicit constants in these estimates depend on the modular form f. In many applications, one seeks asymptotic estimates for $S(x, f)$ that are uniform in the level aspect or the weight aspect of the underlying modular form. If f is a primitive cusp form of weight k and a fixed level N, one could use Perron's formula and the convexity bound for $L(s, f)$ to prove that

$$S(x, f) \ll (xk)^{\frac{1}{2}} \log(xk), \tag{3.3}$$

as $x, k \to \infty$. This implies that

$$S(x, f) = o(x \log x) \tag{3.4}$$

in the range $x > k^{1+\epsilon}$. In fact, using subconvexity bounds for $L(s, f)$ (see, for example, [16, Theorem 1.1]), one sees that (3.4) is valid in the wider range $x > k^{1-\delta}$ for some $\delta > 0$. For a primitive cusp form f in $S_k(1)$, Lamzouri [14, Corollary 1.2] proved that (3.4) holds in the range $\log x / \log \log k \to \infty$ assuming the GRH for $L(s, f)$. He also proved unconditionally that this range in x is best possible [14, Corollary 1.4]. This work is a

GL(2) analogue of the work of Granville and Soundararajan [8] on large character sums. Building on this, we aimed to refine the result by implementing a weaker assumption than GRH. We achieved this goal by adapting the methods used to establish Theorem 2.1. More precisely, we proved the following.

Theorem 3.1. [4, Corollary 1.2] *Let $f \in S_k(N)$ be a primitive cusp form. Let ϵ and T be real numbers with $\epsilon \geq (\log k)^{-1/8}$ and $1 \leq T \leq (\log k)^{1/200}$. Suppose that for every real ϕ with $|\phi| \leq T$ the region*

$$\left\{ s \,:\, \Re(s) \geq \frac{3}{4},\, |\Im(s) - \phi| \leq \frac{1}{4} \right\} \tag{3.5}$$

contains no more than $\epsilon^2 \log k/5000$ zeroes of $L(s,f)$. Then for all $x \geq k^\epsilon$, we have

$$\left| \sum_{n \leq x} \lambda_f(n) \right| \ll \frac{x \log x}{T}. \tag{3.6}$$

It is known that, thanks to the work of Iwaniec and Kowalski, $L(s,f)$ has $O(\log k)$ zeros in the critical strip up to height $(\log k)^{1/200}$. This implies that, if the inequality (3.6) fails for $x = k^\epsilon$, then a positive propotion ($\gg \epsilon^2$) of these zeros lie off the critical line.

We derived this result from the theorem below.

Theorem 3.2. [4, Theorem 1.1] *Let $f \in S_k(N)$ be a primitive cusp form, and let $\exp(\sqrt{\log k}) \leq x \leq \sqrt{k}$ be such that $|S(x,f)| = \frac{x \log x}{Q}$ where $1 \leq Q \leq (\log x)^{1/100}$. Then there exists an absolute constant $C > 0$ such that for some real number ϕ with $|\phi| \leq CQ$ and any parameter $100CQ^3 \leq M \leq 40 \log x$, the region*

$$\left\{ s \,:\, |s - (1 + i\phi)| < \frac{M \log(k(\log k)^{1/\gamma})}{(\log x)^2} \right\}, \tag{3.7}$$

with $\gamma = M/100 \log x$, contains at least $M/625$ zeroes of $L(s,f)$.

Indeed, by choosing $M = \epsilon^2 \log k/8$ in Theorem 3.2, we observe that, for any $k^\epsilon \leq x \leq k$, the region given in (3.7) is contained in the region given in (3.5) since

$$\frac{L \log(k(\log k)^{1/\gamma})}{(\log x)^2} \leq \frac{2L \log k}{(\log x)^2} = \frac{2}{8} \left(\frac{\epsilon \log k}{\log x} \right)^2 \leq \frac{1}{4}.$$

This proves Theorem 3.1. We note that this result is recently improved by Kerr, Klurman, and Thorner in [13]. Further developments in this direction are made in, for example, Li and Luo [15], and Bassett and Hamieh [1], where the authors considered the large sums of coefficients of symmetric power L-functions.

In subsequent paper [5], we also studied the size of sums of products of Fourier coefficients, motivated by Theorem 2.2. In this direction, we succeeded in establishing a more general result involving sums of products of arbitrary divisor-bounded multiplicative functions. The precise statement of the theorem is as follows.

Theorem 3.3. [5, Theorem 1.1] *Let κ be a positive constant. Let f_1 and f_2 be multiplicative functions with $|f_j(n)| \leq \tau(n)^\kappa$ for all $n \in \mathbb{N}$. Suppose that, for $x_1, x_2 \gg 1$, there exists $\eta > \max_j\{(\log x_j)^{-1/100}\}$ such that*

$$\left|\sum_{n\leq x_j} f_j(n)\right| \geq \eta x_j (\log x_j)^{2^\kappa - 1}. \tag{3.8}$$

Then, with $\xi = C\eta^{1+2^{\kappa+3}}$ for some absolute constant $C > 0$, there exists $x \geq \min\{x_1, x_2\}^{\xi^2}$ such that

$$\left|\sum_{n\leq x} f_1(n) f_2(n)\right| \geq \xi x (\log x)^{2^{2\kappa}-1}.$$

As an immediate consequence of this theorem by taking $\kappa = 1$, we have the following result.

Corollary 3.4. [5, Corollary 1.2] *Let $g \in S_k(r)$ and $h \in S_\ell(q)$ be primitive cusp forms whose Fourier coefficients are denoted by $\lambda_g(n)$ and $\lambda_h(n)$, respectively. Suppose that*

$$\left|\sum_{n\leq x_1} \lambda_g(n)\right| \geq \eta x_1 \log x_1 \quad \textit{and} \quad \left|\sum_{n\leq x_2} \lambda_h(n)\right| \geq \eta x_2 \log x_2,$$

for some $x_1, x_2 \gg 1$ and $\eta > \max_j\{(\log x_j)^{-1/100}\}$. Then there exists $x \geq (\min(x_1, x_2))^{\xi^2}$ such that

$$\left|\sum_{n\leq x_1} \lambda_g(n)\lambda_h(n)\right| \geq \xi x (\log x)^3,$$

where $\xi = C\eta^{17}$ for some absolute positive constant C.

In the following section, we briefly discuss the proofs of Theorems 3.2 and 3.3.

4. Key Ideas of the Proofs

As previously mentioned, our approach to proving these theorems is to adapt key ingredients from Pretentious Number Theory, a framework developed by Andrew Granville and Kannan Soundararajan. For an introduction to this framework, see, for example, their work [6, 7, 9]. The departure of the topic is to define the following distance function: Let χ be a completely multiplicative function such that $|\chi(n)| \leq 1$ for all n. Then, the distance function $\mathbb{D}(\chi, n^{it}; x)^2$ defined as

$$\mathbb{D}(\chi, n^{it}; x)^2 = \sum_{p \leq x} \frac{1 - \Re(\chi(p)p^{-it})}{p}$$

is related to the L-value associated with χ in the following way,

$$L\left(1 + \frac{1}{\log x} + it, \chi\right) \asymp (\log x) \exp\left(-\mathbb{D}(\chi(n), n^{it}; x)^2\right).$$

We note that the distance function can be defined for any such two functions χ and ψ, but in practice, we only need $\psi(n) = n^{it}$ for these projects.

In order to adapt this to our setting, define, for a multiplicative function $f(n)$ such that $|f(n)| \leq d(n)^\kappa$ for any positive κ,

$$\mathbb{D}(f, n^{it}; x)^2 = \sum_{p \leq x} \frac{2^\kappa - \Re(f(p)p^{-it})}{p}.$$

Then, we observe the relation

$$L\left(1 + \frac{1}{\log x} + it, f\right) \asymp (\log x)^{2^\kappa} \exp\left(-\mathbb{D}(f(n), n^{it}; x)^2\right).$$

It is known that the standard distance function on 1-bounded multiplicative functions satisfies a triangle inequality. Although such a result does not hold in our setting, we established a triangle-type inequality,

$$\mathbb{D}(f_1, n^{it_1}; x_1) + \mathbb{D}(f_2, n^{it_2}; x_2) \geq \frac{1}{\sqrt{2^\kappa}} \mathbb{D}(f_1 f_2, n^{i(t_1+t_2)}; \min\{x_1, x_2\}). \tag{4.1}$$

We also record the following crucial propositions, which follow from applications of Halász Theorem and Lipschitz Theorem. Let $\phi_f(x, T_0)$ be a real number in the range $|t| \leq T_0$ for which the function

$$t \mapsto \left| L\left(1 + \frac{1}{\log x} + it, f\right) \right|$$

attains its maximum, and let $N_f(x, T_0)$ be a real number satisfying

$$\left| L\left(1 + \frac{1}{\log x} + i\phi_f(x, T_0), f\right) \right| = e^{-N_f(x,T_0)} (\log x)^{2^\kappa}. \tag{4.2}$$

Then we have the following.

Proposition 4.1. *Suppose f is a multiplicative function such that $|f(n)| \le \tau(n)^\kappa$, and let $\phi = \phi_f(x, (\log x)^{2^\kappa})$ be as given in* (4.2)*. Then*

$$\frac{1}{x}\sum_{n\le x} f(n) = \frac{x^{i\phi}}{1+i\phi}\cdot\frac{1}{x}\sum_{n\le x} f(n)n^{-i\phi} + O(E_\kappa(x)),$$

where

$$E_\kappa(x) = \begin{cases} (\log x)^{-1+4/\pi}(\log\log x)^{5-8/\pi} & \text{if } \kappa = 1 \\ (\log x)^{2^\kappa-2}(\log\log x)^3 & \text{if } \kappa > 1. \end{cases} \tag{4.3}$$

Furthermore, we obtain the following estimate.

Proposition 4.2. *Let f be a multiplicative function such that $|f(n)| \le \tau(n)^\kappa$. For sufficiently large x, let $N = N_f(x, (\log x)^{2^\kappa})$ and $\phi = \phi_f(x, (\log x)^{2^\kappa})$ be as given in* (4.2)*. Then, we have*

$$\frac{1}{x}\sum_{n\le x} f(n) \ll (\log x)^{2^\kappa-1}\left(\frac{(N+1)e^{-N}}{1+|\phi|} + \frac{E_\kappa(x)}{(\log x)^{2^\kappa-1}}\right),$$

where $E_\kappa(x)$ is given in (4.3)*.*

We utilize these propositions in the proof of Theorems 3.2 and 3.3, which are given in Sections 4.1 and 4.2, respectively.

4.1. **Proof of Theorem 3.2.** The proof proceeds by connecting large values of the partial sums $S(x,f)$ with the distribution of zeros of $L(s,f)$. We begin by relating the original partial sum $S(x,f) = \sum_{n\le x}\lambda_f(n)$ to the twisted sum $S(x,f_\phi) = \sum_{n\le x}\lambda_f(n)n^{-i\phi}$, by applying Proposition 4.1. This reduction allows us to work with a twisted multiplicative function that more closely captures the oscillatory behavior of $\lambda_f(n)$.

The second step is to relate the size of $S(x,f_\phi)$ to values of the associated L-function near the line $\Re(s) = 1$ via a Plancherel formula. More precisely, we establish the following result.

Lemma 4.3. *Let ϕ be a real number, T a positive real number, and γ a real number such that $0 \le \gamma \le \frac{1}{2}$. Set $S(x,f_\phi) := \sum_{n\le x}\lambda_f(n)n^{-i\phi}$. Then*

$$\begin{aligned}\sqrt{2\pi T}\int_{-\infty}^{\infty}\frac{S(e^y,f_\phi)}{e^y}\exp\left(\gamma y - \frac{T}{2}y^2\right)dy \\ = \int_{-\infty}^{\infty}\frac{L(1-\gamma+i\phi+i\xi,f)}{1-\gamma+i\xi}\exp\left(-\frac{\xi^2}{2T}\right)d\xi.\end{aligned}$$

The significance of this lemma is that a large value of the twisted partial sum $S(x,f_\phi)$ forces the L-function

$$L(1-\gamma+i(\phi+\xi),f)$$

to be large. In particular, one obtains a lower bound of the form

$$|L(1-\gamma+i(\phi+\xi),f)| \gg \frac{\log x}{Q}\exp\left(\frac{\gamma\log x}{2}\right),$$

where the parameters γ, ξ, and Q are the quantities arising in the analysis of [4].

The final step is to derive an upper bound for the same L-value. Using the analytic properties of $L(s,f)$, one obtains an estimate of the form

$$|L(1-\gamma+i(\phi+\xi),f)| \ll \frac{1}{\gamma^2}\exp\left(\sum_{\rho}\frac{2\gamma^2}{|1+\gamma+i(\phi+\xi)-\rho|^2}\right),$$

where the sum runs over the nontrivial zeros ρ of $L(s,f)$. Comparing this upper bound with the lower bound obtained in the previous step yields

$$\sum_{\rho}\frac{\gamma}{|1+\gamma+i(\phi+\xi)-\rho|^2} \geq \frac{\log x}{4}.$$

Thus, the existence of a large partial sum forces a substantial contribution from the zeros of the L-function near the point $1+\gamma+i(\phi+\xi)$. To convert this information into a statement about the location of zeros, one separates the contributions of zeros lying inside and outside the disk

$$|1+i\phi-\rho| < Y,$$

for a suitable choice of Y. The contribution of the zeros inside this disk is bounded above by

$$\frac{1}{\gamma}\#\{\rho : |1+i\phi-\rho| < Y\},$$

while the contribution from the remaining zeros can be shown to be sufficiently small. Combining these estimates with the preceding lower bound, one obtains

$$\#\{\rho : |1+i\phi-\rho| < Y\} \gg \gamma\log x.$$

Thus, a large value of $S(x,f)$ implies that the disk

$$|1+i\phi-\rho| < Y$$

contains many zeros of $L(s,f)$. Setting $\gamma \asymp M/\log x$ gives precisely the statement of Theorem 3.2.

4.2. **Proof of Theorem 3.3.** Proving Theorem 3.3 requires several auxiliary results. We begin with the following lemma, which is a consequence of Proposition 4.2.

Lemma 4.4. *Let f be a multiplicative function such that $|f(n)| \le \tau(n)^\kappa$ for all $n \in \mathbb{N}$. Let $N = N_f(x, (\log x)^{2^\kappa})$ and $\phi = \phi_f(x, (\log x)^{2^\kappa})$ as in* (4.2)*. Suppose that there exists $x \gg 1$ and $\eta > (\log x)^{-1/100}$ such that*

$$\left| \sum_{n \le x} f(n) \right| \ge \eta x (\log x)^{2^\kappa - 1}. \tag{4.4}$$

Then,

$$|\phi| \ll \frac{1}{\eta} \quad \text{and} \quad N \le 2 \log \left(\frac{1}{\eta} \right) + O(1).$$

Another crucial ingredient is a lower bound for the partial sums of multiplicative functions in terms of the distance function. To proceed, let $1 \le T_0 \ll (\log x)^{2^\kappa}$, and let $M_f(x, T_0)$ be a real number satisfying

$$\max_{|t| \le T_0} \left| \frac{L\left(1 + \frac{1}{\log x} + it, f\right)}{1 + \frac{1}{\log x} + it} \right| = e^{-M_f(x, T_0)} (\log x)^{2^\kappa}. \tag{4.5}$$

Roughly speaking, the following proposition shows that if the distance $\mathbb{D}(f, n^{i\psi}; x)$ is small, then the partial sums of f must be large over a suitable range.

Proposition 4.5. *Let f be a multiplicative function satisfying $|f(n)| \le \tau(n)^\kappa$ for all $n \in \mathbb{N}$. Let $\psi = \psi_f(x, (\log x)^{2^\kappa})$ be a number in the range $|t| \le (\log x)^{2^\kappa}$ where the maximum in* (4.5) *is attained. Let $x \gg 1$ and set $\lambda = \mathbb{D}(f, n^{i\psi}; x)^2 + \log(1 + |\psi|) + c$ where c is a suitably large constant. Then, there exists $y \in [x^\gamma, x]$, with $\gamma = 1/(\lambda e^\lambda)$, such that*

$$\left| \sum_{n \le y} f(n) \right| \gg \frac{\exp(-\mathbb{D}(f, n^{i\psi}; x)^2)}{1 + |\psi|} y (\log y)^{2^\kappa - 1}. \tag{4.6}$$

Now, the proof of Theorem 3.3 combines the lower bound for partial sums discussed in Proposition 4.5 with the triangle-type inequality (4.1) for the distance function. Applying the lower bound to the multiplicative function $f_1 f_2$, one obtains an estimate of the form

$$\left| \sum_{n \le y} f_1(n) f_2(n) \right| \gg \frac{\exp\big(-\mathbb{D}(f_1 f_2, n^{i\psi}; \min_j \{x_j\})^2\big)}{1 + |\psi|} y (\log y)^{2^\kappa - 1}, \tag{4.7}$$

for a suitable value of ψ.

To connect this estimate with the distances associated with f_1 and f_2, we invoke the triangle-type inequality (4.1), which shows that the distance between $f_1 f_2$ and the twist $n^{i(\phi_1+\phi_2)}$ is controlled by the corresponding distances for f_1 and f_2. More precisely, it yields

$$\mathbb{D}(f_1 f_2, n^{i(\phi_1+\phi_2)}; \min_j\{x_j\})^2 \le 2^{\kappa+2} \max_j \Big\{ \mathbb{D}(f_j, n^{i\phi_j}; x_j)^2 \Big\}.$$

Since the distance function is related to $N = N_f(x, (\log x)^{2^\kappa})$, defined in (4.2), in the following way:

$$e^{-N} = \frac{1}{(\log x)^{2^\kappa}} \left| L(1 + \frac{1}{\log x} + i\phi, f) \right| \asymp \exp(-\mathbb{D}^2(f, n^{i\phi}; x)), \tag{4.8}$$

it follows that

$$\exp\left(-\mathbb{D}^2(f_1 f_2, n^{i(\phi_1+\phi_2)}; x)\right) \ge \min_j \left\{ e^{-2^{\kappa+2} N_j} \right\}.$$

Furthermore, applying Lemma 4.4 yields

$$\exp\left(-\mathbb{D}^2(f_1 f_2, n^{i(\phi_1+\phi_2)}; x)\right) \ge \left(\eta^2\right)^{2^{\kappa+2}} = \eta^{2^{\kappa+3}}. \tag{4.9}$$

Substituting this estimate into Proposition 4.5, one obtains

$$\left| \sum_{n \le y} f_1(n) f_2(n) \right| \gg \eta^{\,1+2^{\kappa+3}}\, y (\log y)^{2^\kappa - 1}.$$

Thus, if both f_1 and f_2 exhibit large partial sums, then their product $f_1 f_2$ also possesses a large partial sum on a suitable range.

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