On congruences arising from p-adic limits of Eisenstein series

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Abstract

By our work with Kikuta and Gunji/Kkuta, the p-adic limit of a degree n Siegel Eisenstein series $E^n_{k_m}$ of weight $k_m = k_0 + (p-1)p^m$ equals an explicitly given Eisenstein series of weight k_0 for $\Gamma_0(p)$. We discuss congruences which arise by combining this result with pullback formulas of Garrett type, possibly involving differential operators. In the simplest case, one gets congruences between (finite averages of) critical values of standard L-functions of level one for weight k_m and for those of level $\Gamma_0(p)$ for weight k_0. We describe several other types of such congruences as well, without specifying the constants involved.

1 Short summary about p-adic limits of Siegel-Eisenstein series

We use standard notations concerning Siegel modular forms, in particular the symplectic group $Sp(n,\mathbb{R})$ acts on the Siegel upper half space $\mathbb{H}_n$ in the usual way. We denote by S^n_k the space of cusp forms of degree n and weight k for the full modular group. To define p-adic modular forms, we view $q_{ij} := e^{2\pi i z_{ij}}$ as a formal variable ($Z = (z_{ij}) \in \mathbb{H}_n$). A p-adic modular form is then defined as a formal power series

$$f = \sum_{T \in \Lambda_n} a_f(T) q^T$$

with coefficients $a_f(T) \in \mathbb{Q}_p$ and Λ_n denotes the space of half integral symmetrc matrices of size n; in the sum, only positive semidefinite matrices should occur; we request that there exists a sequence F_l of level one Siegel modular forms with rational Fourier coefficients such that $F_l \longrightarrow f$ in the sense that all the sequences $a_{F_l}(T)$ of Fourier coefficients converge p-adically to $a_f(T)$ (uniformly in T).

The main example for us (in fact the only example, because everything will be deduced from this) is the Siegel Eisenstein series, defined by

$$E_k^n(Z) := \sum_{C,D} \det(CZ+D)^{-k} \qquad (Z \in \mathbb{H}_n).$$

Here k is a positive even integer with $k > n+1$ and the summation goes over all "second rows" of elements in $Sp(n,\mathbb{Z})$ modulo action of $GL(n,\mathbb{Z})$ from the left. The Fourier expansion of such Eisenstein series, denoted by $\sum_T a_k^n(T)q^T$ is well known, the coefficients are rational with nice congruence properties.
Now we fix k_0 and an odd prime p and consider weights

$$k_m := k_0 + (p-1)p^{m-1} \qquad (m >> 0)$$

The sequence $\left(E_{k_m}^n\right)_m$ of such Eisenstein series was studied by many people: Serre, Nagaoka, Katsurada ... We used our theory of mod p^m singular modular forms to prove
Theorem *Under the conditions p odd and $k_0 \geq \frac{n+1}{2}$ the sequence $E_{k_m}^n$ converges p-adically to a p-adic modular form $g_{k_0}^n$, which turns out to be a classical modular form of weight k_0 for the congruence subgroup $\Gamma_0^n(p)$.*

One can identify the modular form $g_{k_0}^n$ in several ways, indeed it can be characterized in three different ways:

- as an explicit linear combination of genus theta series for quadratic forms of dimension $2k_0$
- as an element of the space of Eisenstein series which is an eigenform of the $U(p)$-operator with eigenvalue 1
- as an Eisenstein series $\mathbb{E}_{k_0,p}^n$ for $\Gamma_0(p)$

It is the last characterization, which we will use in the sequel, it holds under the condition $k_0 > \frac{n+1}{2}$, the other characterizations were obtained under

stronger conditions.
To define $\mathbb{E}$, we need a general Eisenstein series for the cusp $\omega^n := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Sp(n,\mathbb{Z})$, defined for $Re(s) >> 0$ by

$$\mathfrak{E}^n_{k_0}(Z,s) := \sum_\gamma det(CZ+D)^{-k} \det(Im(\gamma < Z >))^s,$$

where γ runs over all elements of the coset $\omega \cdot \Gamma_0(p)$ modulo the appropriate paprabolic. This series has analytic continuation to all of $\mathbb{C}$ and we put

$$\mathbb{E}^n_{k_0,p} := \mathfrak{E}^n_{k_0}(Z, \frac{n+1}{2} - k)$$

This defines a holomorphic modular form for $\Gamma_0(p)$ of weight k_0.
Note that $\mathbb{E}$ has a quite simple Fourier expansion (the p-factor in the singular series is just substituted by 1).

2 Garrett's pullback formula for level one and level p

We decompose $Z \in \mathbb{H}_{2n}$ as $Z = \begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathbb{H}_n$. Before restricting a degree $2n$ Eisenstein series to the subspace $z = 0$, we apply holomorphic differential operators $\mathcal{D}_{k,\nu}$, which change weights from k to $k+\nu$.
Garrett's pullback formula gives us

$$\mathcal{D}_{k,\nu} E^{2n}_k(\tau,\tau') = \delta_{\nu,0} \cdot \mathcal{E}(\tau,\tau') + \sum_i c(k,\nu) \frac{L(f_i, k-n)}{< f_i, f_i >} f_i(\tau) \cdot f_i(\tau')$$

Here (only for $\nu = 0$) $\mathcal{E}$ is an element in $(S^n_k)^\perp \otimes (S^n_k)^\perp$, f_i runs over an orthogonal basis of $S^n_{k+\nu}$ consisting of Hecke eigenforms, $<,>$ denotes the Petersson inner product and $L(f,s)$ is the standard L-function attached to f. The constant $c(k,\nu)$ is known explictly.
The pullback formula for the Eisenstein series $\mathbb{E}^{2n}_{k_0,p}$ is

$$\mathcal{D}_{k_0,\nu}(\mathbb{E}^{2n}_{k_0,p}(\tau,\tau') = \delta_{\nu,0} \cdot (*) + \sum_j d(k_0,\nu) \frac{L^{(p)}(g_j, 1-k_0+n)}{< g_j, g_j >} g_j(\tau) g_j \mid \omega^n(\tau')$$

where the g_j run over a Hecke eigen basis of S^n_k and $(*) = (*)(\tau,\tau')$ is some modular form in τ and τ' orthogonal to cusp forms.

3 Congruences and p-adic limits derived from the above

The pullback formulas when combined with the Theorem above allow us to deduce many seemingly new statements giving congruences, which are formulated as statements concerning p-adic limits, basically under the assumption $k_0 \geq \frac{2n+1}{2}$; note that we have to use p-adic properties for Siegel-Eisenstein series of degree $2n$.

3.1 Looking at Fourier coefficients

We fix k_0 and choose positive definite elements S and T in Λ_n and look (for $\nu > 0$) at the average of L-values

$$R(k_m, S, T) := c(k_m, \nu) \sum_i a(T, f_i) a(S, T_i) \frac{L(f_i, k_m - 1)}{< f_i, f_i >}$$

As a sequence concerning m, this converges p-adically to a similar expression for weight k_0, in particular there are congruences among the $R(k_m, S, T)$ when m varies.

3.2 Interpretation via Poincare series

To get a formulation similar to the one concerning the sequence $E^n_{k_m}$ for other modular forms, we may use *Poincare series of exponential type*, defined for positive definite $T \in \Lambda_n$ by

$$P^n_k(T, Z) := \sum_M det(CZ + D)^{-k} e^{2\pi i tr(M<Z>}$$

where M runs over $Sp(n, \mathbb{Z})$ modulo translations. The Fourier coefficients are not of arithmetic nature (involving Bessel functions and Kloosterman sums...); the situation becomes much better, if we apply the **symmetric square operator** $\mathbb{T}(\nu)$ which is the linear substitute for the L-value $L(f, k - n)$, i.e. it is a sum (infinte) of Hecke operators, such that

$$f \mid \mathbb{T}(\nu) = L(f, k - n) \cdot f$$

for all Hecke eigenforms f in $S^n_{k+\nu}$.
We get

Theorem: *For positive $T \in \Lambda_n$ and $\nu > 0$ the sequence*

$$\mathbb{T}(\nu) P^n_{k_m}(T, Z)$$

converges p-adically to a cusp form of weight $k_0 + \nu$ for $\Gamma_0(p)$.
Supplement: The p-adic limit itself can explicitly be described by the symmetric square of a Poincare series.
Comment: The Theorem looks quite similar to the statement about p-adic convergence for Siegel Eisenstein series.

3.3 Klingen type Eisenstein series

We only describe the simplest case here: We start from an embedding $\mathbb{H}_1 \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{n+1}$ and consider the restiction of a degree $n+1$ Siegel Eisenstein series

$$E_k^{n+1}(\begin{pmatrix} \tau & 0 \\ 0 & Z \end{pmatrix}) = E_k^1(\tau) \cdot E_k^n(Z) + c(n,k) \sum_i L(f_i, k-1) \frac{f_i(\tau) \cdot E_k^{n,1}(f_i)(Z)}{< f_i, f_i >}$$

where f_i runs over a normalized basis of cuspidal Hecke eigenforms and $E^{n,1}(f)$ denotes the degree n Klingen-Eisenstein series attached to f.
We take the first Fourier coefficient in the above (for τ) and we may ignore the Siegel-Eisenstein part (whose covergence properties for $k = k_m$ are known anyway) and we obtain that the sequence

$$\left(\sum_{f_i} \frac{L(f_i, k_m - 1}{< f_i, f_i >} \cdot E^{n,1}(f_i)(Z) \right)_{m \geq 1}$$

converges p-adically aginst a Klingen-Eisenstein series for weight $k = k_0$ for $\Gamma_0(p)$ In the above, the summation over f_i is over a Hecke basis of cusp forms of weight k_m.

3.4 Jacobi-Eisenstein series

Jacobi-Eisenstein series appear in the Fourier-Jacobi-expansion of Siegel Eisenstein series, see [1]. One may then deduce congruence properties and one may also consider p-adic limits of Jacobi-Eisenstein series.

3.5 Triple products

The ingredients for the integral representation of the triple product L-function are quite somilar to those for the standard L-function (using an embedding ${\mathbb{H}_1}^3 \hookrightarrow \mathbb{H}_3$ and the restriction of a degree 3 Siegel Eisenstein series. We may then obtain similar congruence properties involving critical values of the triple L-function.

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