Langlands correspondence for covering groups of algebraic tori

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Abstract. A global $\mathcal{S}$-group is expected, and known in some cases, to specify the automorphic representations in each global packet. For a covering group of an algebraic torus, it is not obvious from the definition whether the analogue of a global $\mathcal{S}$-group has finite order. In this note, we report our recent result that verifies this finiteness for a Brylinski-Deligne covering group of a torus.

Contents

1. Introduction

This note provides a naive background of our recent result [7] on the Langlands program for covering groups of tori. As in the case of reductive algebraic groups, their covering groups have also attracted arithmetic interest. For instance, Weil studied the Segal-Shale-Weil representations of the non-trivial double cover

$$\mathrm{Mp}_{2r} \to \mathrm{Sp}_{2r},$$

to prove an identity now known as the Siegel-Weil formula. Another notable example is an Eisenstein series on a double cover of GL_r, which yields a Rankin-Selberg integral representing the symmetric square L-function of a cuspidal representation of GL_r [2, 4, 9] . These examples motivate us to pursue the Langlands program for covering groups.

In this note, we focus on a covering group of a torus $T(\mathbb{A})$, which is regarded as one of the most basic cases. Here, we assume the torus T

Date: 2026.6.30, updated.

to be defined over a number field F, and write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of F. To formulate the Langlands correspondence, let

$$\widetilde{T(\mathbb{A}_F)} \to T(\mathbb{A}_F)$$

be a functorial n-fold cover defined by Brylinski and Deligne [1, §10]. Then we may define a global packet Π in the Langlands correspondence for $\widetilde{T(\mathbb{A}_F)}$.

An irreducible admissible genuine representation π of the covering group $\widetilde{T(\mathbb{A}_F)}$ is known to be factorized into a restricted tensor product $\bigotimes'_v \pi_v$, as usual [11, 4.1]. With this, a global packet Π of representations of $\widetilde{T(\mathbb{A}_F)}$ is also factorized into the restricted tensor product of a family $(\Pi_v)_v$ of local packets. At a place v of F, the packet Π_v is a finite set consisting of irreducible representations the group $\widetilde{T(F_v)}$.

In the local Langlands correspondence [10] at a place v, we have a finite abelian group $\mathcal{S}_v$ whose dual $\mathrm{Hom}(\mathcal{S}_v, \mathbb{C}^\times)$ corresponds bijectively to the packet Π_v. The dual $\mathrm{Hom}(\prod_v \mathcal{S}_v, \mathbb{C}^\times)$ of their product also corresponds bijectively to the global packet Π, but the automorphic representations in Π form a smaller subset Π_{aut}. Then, we have a subgroup $\mathcal{S}_{\widetilde{T(\mathbb{A})}} \subset \prod_v \mathcal{S}_v$ such that the dual

$$\mathrm{Hom}((\prod_v \mathcal{S}_v)/\mathcal{S}_{\widetilde{T(\mathbb{A})}}, \mathbb{C}^\times)$$

of the quotient corresponds bijectively to the set Π_{aut}. Thus, the group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ describes the difference between the product $\prod_v \mathcal{S}_v$ of the local $\mathcal{S}$-groups and the subset $\Pi_{\mathrm{aut}} \subset \Pi$ consisting of automorphic representations, i.e., plays a role analogous to a usual global $\mathcal{S}$-group for a linear group. Our main theorem claims the finiteness of the order of $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$, which is not obvious from our formulation.

Main Theorem (N. [7]; Theorem 4.7 in this note)**.** *In the global Langlands correspondence for the covering group $\widetilde{T(\mathbb{A})}$, the group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ has finite order.*

In the next section, we review the definition and the classification theorem for Brylinski-Deligne covering groups of an algebraic torus T. Before global discussion, we also review the formulation of the local $\mathcal{S}$-group for an n-fold Brylinski-Deligne covering group in Section 3. Then we formulate the global correspondence for the cover $\widetilde{T(\mathbb{A})}$ in the last section.

Notation. Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ respectively denote the ring of rational integers, the field of rational numbers, and the field of complex numbers. The symbol $\otimes$ always denotes the tensor product over $\mathbb{Z}$. For a group G, let $\mathrm{Z}(G) \subset G$ denote the center. For an irreducible representation π, let ω_π denote the central character.

Throughout this paper, F denotes a field, and $F^\times$ the multiplicative group consisting of non-zero elements. For a Galois module A, let $\mathrm{H}^i(F, A)$ denote the i-th Galois cohomology group. If F is a local or global field, then we always assume that F contains a primitive n-th root of unity, and let $\mu_n \subset F^\times$ denote the subgroup consisting of n-th roots of unity. In Section 4, we assume F to be a number field. Then, at each place v of F, let F_v denote the completion, and $\mathcal{O}_v \subset F_v$ for its valuation ring. Let $\mathbb{A}$ denote the adele ring of F.

2. Brylinski-Deligne covering group

In the literature on the Langlands program for general covering groups [3, 12], one often focuses on certain covering groups defined by Brylinski and Deligne [1]. More recently, Zhao refined them to define "étale metaplectic covers," but we do not discuss it. Here, we review the definition and the classification theorem for Brylinski-Deligne covering groups of an algebraic torus T defined over a field F.

A Brylinski-Deligne covering group is, roughly, a central extension by Milnor's group K_2 in algebraic K-theory, or its variation. The groups K_2 are generalized by Quillen [8] to be defined over schemes, and form a presheaf on the big Zariski site of a base field F. Let $\mathbb{K}_2$ be the sheaf associated with the presheaf K_2.

Definition 2.1 (Brylinski-Deligne [1]). Let T be an algebraic torus over a field F. Then, a *Brylinski-Deligne* $\mathbb{K}_2$*-extension* of T is a central extension

$$1 \to \mathbb{K}_2 \to \widetilde{T} \to T \to 1$$

as a sheaf of groups on the big Zariski site of $\operatorname{Spec} F$.

Here, we allow this extension to be non-abelian. Various covering groups induced from Definition 2.1 in the following way are called *Brylinski-Deligne covering groups*.

Example 2.2 (Brylinski-Deligne covering groups). (1) Let F be any field. Then, the sections

$$1 \to \mathrm{K}_2(F) \to \widetilde{T}(F) \to T \to 1$$

of a Brylinski-Deligne $\mathbb{K}_2$-extension $\widetilde{T} \to T$ over $\operatorname{Spec} F$ form a central extension as an abstract group.

(2) Let F be a local field containing a full set μ_n of n-th roots of unity. Then the push-out

$$1 \to \mu_n \to \widetilde{T(F)} \to T(F) \to 1$$

of the extension 1 via the Hilbert symbol $\mathrm{K}_2(F) \to \mu_n$ is called an n-fold *Brylinski-Deligne covering group* of $T(F)$. Here, if F is the field $\mathbb{C}$ of complex numbers, then we regard the Hilbert symbol as the trivial map.

This covering group is indeed a topological one [1, Construction 10.3].

(3) Let F be a number field containing a full set μ_n of n-th roots of unity, and $\mathbb{A} = \mathbb{A}_F$ the adele ring. At each place v of F, 2 gives a local covering group $\mu_n \to \widetilde{T(F_v)} \to T(F_v)$.

At almost every place v of F, Brylinski and Deligne found a split monomorphism $T(\mathcal{O}_v) \to \widetilde{T(F_v)}$ in this cover. Thus, we may construct the restricted direct product $\prod'_v \widetilde{T(F_v)}$ with respect to the family $\{T(\mathcal{O}_v)\}_v$ of compact groups. This restricted direct product is a central extension

$$1 \to \bigoplus_v \mu_n \to \prod_v{}' \widetilde{T(F_v)} \to T(\mathbb{A}) \to 1,$$

and gives a covering group

$$1 \to \mu_n \to \widetilde{T(\mathbb{A})} \to T(\mathbb{A}) \to 1$$

by taking the push-out via the product map $\bigoplus_v \mu_n \to \mu_n$. This is called an n-fold *Brylinski-Deligne covering group of the adele group* $T(\mathbb{A})$.

A Brylinski-Deligne covering group $\widetilde{T(\mathbb{A})} \to T(\mathbb{A})$ is known to split canonically over the diagonal embedding of $T(F)$. By this splitting, we regard $T(F)$ as a subgroup of the cover $\widetilde{T(\mathbb{A})}$.

Brylinski and Deligne generally defined such covering groups for any algebraic group; a metaplectic group $\mathrm{Mp}_{2r} \to \mathrm{Sp}_{2r}$ is one of the well-known examples.

For any torus T over a field F, we have a finite Galois extension L/F such that T splits over L. Then the following classification theorem gives a bilinear form $Y \times Y \to \mathbb{Z}$ on the cocharacter lattice, which is an essential ingredient to define a packet of representations of a Brylinski-Deligne covering group.

Theorem 2.3 (Classification Theorem of $\mathbb{K}_2$-extensions [1])**.** *The following two Picard categories are equivalent:*

(1) *the category of Brylinski-Deligne $\mathbb{K}_2$-extensions of the torus T;*

(2) *the category of the following data $(Q, \mathcal{E})$:*
- *$Q\colon Y \to \mathbb{Z}$ is a Galois-invariant quadratic form on the cocharacter lattice, which gives a bilinear form $B\colon Y \times Y \to \mathbb{Z}$ by*

$$(y, y') \mapsto Q(y + y') - Q(y) - Q(y').$$

- *$1 \to L^\times \to \mathcal{E} \xrightarrow{\overline{(\,)}} Y \to 1$ is a Galois-equivariant central extension such that for any $\epsilon, \zeta \in \mathcal{E}$,*

$$\epsilon\zeta\epsilon^{-1}\zeta^{-1} = (-1)^{B(\overline{\epsilon}, \overline{\zeta})}$$

in $L^\times$. Here, the symbols $\overline{\epsilon}$ and $\overline{\zeta}$ respectively denote the images of ϵ and ζ under the surjection $\mathcal{E} \to Y$.

- *For any data $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$,*

$$\mathrm{Hom}((Q, \mathcal{E}), (Q', \mathcal{E}')) = \begin{cases} \emptyset & \textit{if } Q \neq Q' \\ \{\textit{morphisms } \mathcal{E} \to \mathcal{E}' \textit{ of extensions}\} & \textit{if } Q = Q'. \end{cases}$$

Here, morphisms $\mathcal{E} \to \mathcal{E}'$ are also required to be Galois-equivariant.

More generally, Brylinski and Deligne proved this type of classification theorem for $\mathbb{K}_2$-extensions of reductive algebraic groups [1].

3. Local $\mathcal{S}$-groups

Before formulating the global Langlands correspondence for a covering group of a torus, we introduce the local $\mathcal{S}$-groups due to Weissman [10]. In this section, F denotes a local field, which is assumed to have characteristic 0, for simplicity. Let

$$\mu_n \to \widetilde{T(F)} \to T(F)$$

be an n-fold covering group defined by Brylinski and Deligne (see Example 2.2). Here, the base field F is assumed to contain a primitive n-th root of unity, which generates the cyclic group μ_n.

To establish a correspondence of irreducible representations π of $\widetilde{T(F)}$ in the style of Langlands, it suffices to parametrize the central characters

$$\omega_\pi \colon \mathrm{Z}(\widetilde{T(F)}) \to \mathbb{C}^\times,$$

by an analogue of the Stone-von Neumann theorem [11, Theorem 2.7]. Here, $\mathrm{Z}(\widetilde{T(F)})$ denotes the center of the group $\widetilde{T(F)}$.

To estimate the center $\mathrm{Z}(\widetilde{T(F)})$, Weissman chose the following isogeny $T^\# \xrightarrow{\iota} T$ of tori, which approximates the image $\mathrm{Z}^\dagger(F) \subset T(F)$ of the center $\mathrm{Z}(\widetilde{T(F)})$.

Definition 3.1 (key isogeny)**.** Let $Y = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T)$ denote the cocharacter lattice of the torus T. Let $\widetilde{T(F)}$ be an n-fold Brylinski-Deligne cover, $Q \colon Y \to \mathbb{Z}$ the associated Galois-invariant quadratic form (see Theorem 2.3), and $B \colon Y \times Y \to \mathbb{Z}$ the symmetric bilinear form defined by

$$(y, y') \mapsto Q(y + y') - Q(y) - Q(y').$$

(1) The sublattice $Y^\#$ of Y is defined as

$$Y^\# = \{y \in Y \mid B(y, Y) \subset n\mathbb{Z}\}.$$

Then, the map B induces a non-degenerate bilinear form

$$Y/Y^\# \times Y/Y^\# \to \mathbb{Z}/n,$$

which characterize this lattice $Y^\#$.

(2) We define $T^\# \xrightarrow{\iota} T$ as the isogeny of tori whose cocharacter lattices are $Y^\# \subset Y$.

More accurately, the quadratic form $Q\colon Y \to \mathbb{Z}$ is associated with a central extension of a sheaf defining the cover $\widetilde{T(F)}$. We should also note the inclusion $nY \subset Y^\#$.

For an integer $k > 0$ and an n-torsion Galois module M, let $M(k)$ denote the k-th Tate twist of M. For instance, $M(1) = M \otimes \mu_n$, and $M(2) = M \otimes \mu_n \otimes \mu_n$. Though $M(k) \cong M$ as Galois modules by our assumption, we adopt this symbol to indicate canonical identification.

To see that the isogenous torus $T^\#$ defined in 3.1 approximates the image $\mathrm{Z}^\dagger(F)$ of the center of $\widetilde{T(F)}$, we give the following exact sequences.

Lemma 3.2 (kernel of the isogeny)**.** *Let the symbol* (1) *denote the first Tate twist. Then, the kernel of the isogeny $T^\# \xrightarrow{\iota} T$ of tori is isomorphic to $(Y/Y^\#)(1)$. Thus, we have an exact sequence*

$$1 \to (Y/Y^\#)(1) \to T^\# \xrightarrow{\iota} T \to 1.$$

Proof. Let X denote the character lattice of the torus T, and $X^\#$ that of $T^\#$. Applying the functor $\mathrm{Hom}(\quad, \mathbb{G}_{\mathrm{m}})$ to the short exact sequence $0 \to X \to X^\# \to X^\#/X \to 0$, we have an exact sequence

$$1 \to \mathrm{Hom}(X^\#/X, \mathbb{G}_{\mathrm{m}}) \to T^\# \to T \to 1,$$

by the divisibility of the multiplicative group of an algebraically closed field.

The isomorphism $\mathrm{Hom}(X^\#/X, \mathbb{G}_{\mathrm{m}}) \cong (Y/Y^\#)(1)$ is given as follows. By the inclusion $nX^\# \subset X$, the pairing $X \otimes \mathbb{Q} \times Y \otimes \mathbb{Q} \to \mathbb{Q}$ gives a bilinear form $X^\# \times Y \to \frac{1}{n}\mathbb{Z}$, and hence a non-degenerate bilinear form

$$X^\#/X \times Y/Y^\# \to \left(\frac{1}{n}\mathbb{Z}\right)/\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}.$$

Therefore,

$$\begin{aligned}(Y/Y^\#)(1) = \mathrm{Hom}(X^\#/X, \mathbb{Z}/n) \otimes \mu_n &= \mathrm{Hom}(X^\#/X, \mu_n) \\ &= \mathrm{Hom}(X^\#/X, \mathbb{G}_{\mathrm{m}}).\end{aligned}$$

□

We previously gave another proof of Lemma 3.2, which was irrelevant to duality. Let $\Gamma = \Gamma_F$ denote the absolute Galois group of the field F. Then the lemma gives a long exact sequence

$$1 \to (Y/Y^\#)(1)^\Gamma \to T^\#(F) \xrightarrow{\iota} T(F) \xrightarrow{\partial} \mathrm{H}^1(F, (Y/Y^\#)(1)) \to \cdots.$$

Note that the bilinear form $(Y/Y^\#)(1) \times (Y/Y^\#)(1) \xrightarrow{B} (\mathbb{Z}/n)(2)$ induces the map

$$\mathrm{H}^1(F, (Y/Y^\#)(1)) \times \mathrm{H}^1(F, (Y/Y^\#)(1)) \to \mathrm{H}^2(F, (\mathbb{Z}/n)(2)) = \mu_n$$

by the cup product.

Fact 3.3 (description of the commutator map [1, Proposition 9.9]). *Let $\widetilde{T(F)} \to T(F)$ be an n-fold Brylinski-Deligne covering group. Then, the commutator map $\widetilde{T(F)} \times \widetilde{T(F)} \to \widetilde{T(F)}$ factors through the composite*

$$T(F)\times T(F) \xrightarrow{\partial\times\partial} \mathrm{H}^1(F,(Y/Y^{\#})(1))\times \mathrm{H}^1(F,(Y/Y^{\#})(1)) \to \mathrm{H}^2(F,(\mathbb{Z}/n)(2)) = \mu_n.$$

That is, for any $s, t \in T(F)$ and their lifts $\tilde{s}, \tilde{t} \in \widetilde{T(F)}$, we have

$$\mathrm{comm}(s,t) := \tilde{s}\tilde{t}\tilde{s}^{-1}\tilde{t}^{-1} = B(\partial s \cup \partial t).$$

Thus, $\mathrm{Z}^{\dagger}(F) = \left\{ s \in T(F) \mid \forall t \in T(F),\ B(\partial s \cup \partial t) = 0 \ \ \textit{in}\ \ \mathrm{H}^2(\Gamma,(\mathbb{Z}/n)(2)) \right\}$.

In the original exposition by Brylinski and Deligne, they choose another sublattice $nY \subset Y$, and claim that the commutator map on $\widetilde{T(F)}$ factors through the composite

$$T(F)\times T(F) \xrightarrow{\partial\times\partial} \mathrm{H}^1(F,(Y/nY)(1))\times \mathrm{H}^1(F,(Y/nY)(1)) \to \mathrm{H}^2(F,(\mathbb{Z}/n)(2)) = \mu_n,$$

where the left map is derived from the short exact sequence

$$1 \to (Y/nY)(1) \to T \xrightarrow{n} T \to 1,$$

and the right map is the cup product for the bilinear form

$$(Y/nY)(1) \times (Y/nY)(1) \xrightarrow{B} (\mathbb{Z}/n)(2).$$

Then the inclusion $nY \subset Y^{\#}$ induces the map $Y/nY \to Y/Y^{\#}$ and the commutative diagram

$$\begin{array}{ccccc} T(F)\times T(F) & \to & \mathrm{H}^1(F,(Y/nY)(1))\times \mathrm{H}^1(F,(Y/nY)(1)) & \to & \mathrm{H}^2(F,(\mathbb{Z}/n)(2)) \\ \| & & \downarrow & & \| \\ T(F)\times T(F) & \to & \mathrm{H}^1(F,(Y/Y^{\#})(1))\times \mathrm{H}^1(F,(Y/Y^{\#})(1)) & \to & \mathrm{H}^2(F,(\mathbb{Z}/n)(2)), \end{array}$$

which verifies our modification.

Fact 3.3 allows us to relate the group $\mathrm{Z}^{\dagger}(F)$ with the isogeny $T^{\#} \xrightarrow{\iota} T$ of tori as follows.

Proposition 3.4 (Weissman [11, Theorem 1.3]). *Let $\widetilde{T(F)} \to T(F)$ be an n-fold Brylinski-Deligne covering group. Then we have inclusions*

$$\iota(T^{\#}(F)) \subset \mathrm{Z}^{\dagger}(F) \subset T(F),$$

where the index $\left[T(F) : \iota(T^{\#}(F))\right]$ is finite.

Proof. By Fact 3.3, the group $\mathrm{Z}^{\dagger}(F)$ contains $\operatorname{Ker}\partial = \iota(T^{\#}(F))$. Since $(Y/Y^{\#})(1)$ is a finite Galois module, the cohomology group $\mathrm{H}^1(F,(Y/Y^{\#})(1))$ also has finite order. In particular, the order of

$$\partial(T(F)) \cong T(F)/\operatorname{Ker}\partial = T(F)/\iota(T^{\#}(F))$$

is finite. □

We note a few on the two inclusions $\iota(T^{\#}(F)) \subset \mathrm{Z}^{\dagger}(F) \subset T(F)$. First:

Definition 3.5 (local $\mathcal{S}$-group [10])**.** Let $\widetilde{T(F)} \to T(F)$ be an n-fold Brylinski-Deligne covering group. Then, we define a finite abelian group $\mathcal{S}_{\widetilde{T(F)}}$ as the quotient

$$\mathcal{S}_{\widetilde{T(F)}} = \mathrm{Z}^{\dagger}(F)/\iota(T^{\#}(F)).$$

We may call $\mathcal{S}_{\widetilde{T(F)}}$ the *local packet group* or *$\mathcal{S}$-group*, since it plays a role analogous to the usual $\mathcal{S}$-group in the Langlands correspondence for a non-covering linear group. Let

$$\partial(T(F))^{\perp} = \{h \in \mathrm{H}^1(F, (Y/Y^{\#})(1)) \mid \forall t \in T(F),\ B(h \cup \partial t) = 0\}$$

denote the annihilator of $\partial(T(F))$ in the pairing

$$\mathrm{H}^1(F, (Y/Y^{\#})(1)) \times \mathrm{H}^1(F, (Y/Y^{\#})(1)) \to \mathrm{H}^2(F, (\mathbb{Z}/n)(2)) = \mu_n.$$

Then we see that $\mathrm{Z}^{\dagger}(F) = \partial^{-1}(\partial(T(F))^{\perp})$, and that

$$\mathcal{S}_{\widetilde{T(F)}} \cong \partial(\mathrm{Z}^{\dagger}(F)) = \partial(T(F))^{\perp} \cap \partial(T(F)).$$

Next, let M be a maximal abelian subgroup of $\widetilde{T(F)}$. Then the image $M^{\dagger} \subset T(F)$ of M is maximally isotropic for the bilinear map

$$\mathrm{comm}\colon T(F) \times T(F) \to \mu_n.$$

By the definition of $\mathrm{Z}^{\dagger}(F) \subset T(F)$, this map induces a non-degenerate bilinear map

$$T(F)/\,\mathrm{Z}^{\dagger}(F) \times T(F)/\,\mathrm{Z}^{\dagger}(F) \to \mu_n.$$

Note that the order of $T(F)/\,\mathrm{Z}^{\dagger}(F)$ is finite by Proposition 3.4. Further the restriction $T(F) \times M^{\dagger} \to \mu_n$ induces a non-degenerate bilinear map $T(F)/M^{\dagger} \times M^{\dagger}/\,\mathrm{Z}^{\dagger}(F) \to \mu_n$. This implies that the index

$$[T(F) : \mathrm{Z}^{\dagger}(F)] = [T(F) : M^{\dagger}][M^{\dagger} : \mathrm{Z}^{\dagger}(F)] = [T(F) : M^{\dagger}]^2$$

is a square.

4. Global correspondence for $\widetilde{T(\mathbb{A})}$

In this section, we formulate the global Langlands correspondence for an n-fold covering group $\widetilde{T(\mathbb{A})} \to T(\mathbb{A})$ of a torus T defined over a number field F, and state our main result on the global $\mathcal{S}$-group. Here $\mathbb{A} = \mathbb{A}_F$ is the adele ring of F, and $\widetilde{T(\mathbb{A})}$ is assumed to be a cover defined by Brylinski and Deligne.

For simplicity, any representation is assumed to be unitary throughout this section. Further, any representation of a covering group is assumed to be genuine throughout this note. To be precise, we fix an embedding

$$j\colon \mu_n \to \mathbb{C}^{\times}$$

of the cyclic group of order n. Recall that a representation of a covering group $\mu_n \to \widetilde{G} \to G$ of a topological group is *genuine* if $\mu_n \subset \widetilde{G}$ acts as multiplication by scalars in $\mathbb{C}$ via the embedding j. That is, for any $\zeta \in \mu_n$ and any vector v in the representation space, one has

$$\zeta \cdot v = j(\zeta)v.$$

A *genuine character* $\chi \colon \widetilde{G} \to \mathbb{C}^\times$ is a character that is genuine as a representation, which means $\chi|_{\mu_n} = j$.

We should note that the covering group $\widetilde{T(\mathbb{A})}$ satisfies an analogue of the Stone-von Neumann theorem [11, 4.1]. Thus, to establish a correspondence of irreducible representations π in the style of Langlands, it suffices to parametrize the central characters

$$\omega_\pi \colon \mathrm{Z}(\widetilde{T(\mathbb{A})}) \to \mathbb{C}^1.$$

Here, we write $\mathrm{Z}(\widetilde{T(\mathbb{A})})$ for the center of the cover, and $\mathbb{C}^1$ for the unit circle in the complex plane $\mathbb{C}$.

We prefer to work on the image

$$\mathrm{Z}^\dagger(\mathbb{A}) \subset T(\mathbb{A})$$

of the center $\mathrm{Z}(\widetilde{T(\mathbb{A})})$. Indeed, the parametrization of genuine characters of $\mathrm{Z}(\widetilde{T(\mathbb{A})})$ is reduced to that of characters of $\mathrm{Z}^\dagger(\mathbb{A})$, if we fix one of the genuine characters as a base point. At a place v of the number field F, let T_v denote the local torus $T(F_v)$, and

$$\mathrm{Z}^\dagger_v = \mathrm{Z}^\dagger(F_v)$$

the image of the center $\mathrm{Z}(\widetilde{T}_v)$ of the cover via $\widetilde{T}_v \to T_v$. By the construction of the Brylinski-Deligne cover $\widetilde{T(\mathbb{A})}$, the image $\mathrm{Z}^\dagger(\mathbb{A})$ of the center is determined place-wise. Thus the group $\mathrm{Z}^\dagger(\mathbb{A})$ decomposes into the restricted direct product $\prod'_v \mathrm{Z}^\dagger_v$ with respect to the family $\{\mathrm{Z}^\dagger_v \cap T(\mathcal{O}_v)\}_v$ of intersections.

Then our Brylinski-Deligne cover $\widetilde{T(\mathbb{A})}$ defines an isogeny $T^\# \xrightarrow{\iota} T$ of tori, whose image $\iota(T^\#(\mathbb{A}))$ is included in the group $\mathrm{Z}^\dagger(\mathbb{A})$, with quotient

$$\mathrm{Z}^\dagger(\mathbb{A})/\iota(T^\#(\mathbb{A})) = \prod_v \mathcal{S}_v.$$

At a place v, the factor $\mathcal{S}_v = \mathcal{S}_{\widetilde{T(F_v)}}$ is the local $\mathcal{S}$-group in the local Langlands correspondence for the cover $\widetilde{T(F_v)}$. Thus the isogenous torus $T^\#$ approximates the group $\mathrm{Z}^\dagger(\mathbb{A})$ with finite range of difference at each place. Then we reduce the parametrization to the Langlands correspondence for the torus $T^\#$.

In an automorphic representation, the group of rational points over the number field F acts trivially through the diagonal embedding. Note that [1, §10] our Brylinski-Deligne cover $\widetilde{T(\mathbb{A})}$ admits a canonical splitting over the diagonal embedding of $T(F)$. By this splitting, we regard $T(F)$ as a subgroup of the cover $\widetilde{T(\mathbb{A})}$. To parametrize automorphic representations of $\widetilde{T(\mathbb{A})}$, we need to modify the preceding correspondence, as follows.

Proposition 4.1 (analogue of the Stone-von Neumann theorem [11, 4.2])**.** *Let $\widetilde{T(\mathbb{A})} \to T(\mathbb{A})$ be an n-fold Brylinski-Deligne covering group of an adelic torus. Then the genuine unitary automorphic representations π of the group $\widetilde{T(\mathbb{A})}$ have genuine unitary central characters ω_π. Moreover, this correspondence is bijective onto the set of genuine unitary characters of the center $\mathrm{Z}(\widetilde{T(\mathbb{A})})$ that are trivial on $\mathrm{Z}(\widetilde{T(\mathbb{A})}) \cap T(F)$.*

Then, the parametrization of such genuine characters is again reduced to that of characters of $\mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A}) \cap T(F))$, if we fix one of the genuine characters as a base point. Here the group $\mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A}) \cap T(F))$ is approximated by the inclusion

$$\iota(T^\#(\mathbb{A})/T^\#(F)) \subset \mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A}) \cap T(F))$$

with quotient $(\prod_v \mathcal{S}_v)/\mathcal{S}_{\widetilde{T(\mathbb{A})}}$, where the group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ is defined as follows.

Definition 4.2 (global $\mathcal{S}$-group)**.** For a Brylinski-Deligne covering group $\widetilde{T(\mathbb{A})} \to T(\mathbb{A})$, we define the abelian group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ as the quotient

$$\mathcal{S}_{\widetilde{T(\mathbb{A})}} = (\mathrm{Z}^\dagger(\mathbb{A}) \cap T(F))/(\iota(T^\#(\mathbb{A})) \cap T(F)) \,.$$

This may be called the *global packet group* or *$\mathcal{S}$-group*. Compare with the definition of the local $\mathcal{S}$-group in 3.5.

In the rest of this section, we explain the role and the finiteness of this group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$, which are analogous with those of a usual global $\mathcal{S}$-group for a linear group. Note that the sequences in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \iota(T^\#(\mathbb{A})) \cap T(F) & \to & \mathrm{Z}^\dagger(\mathbb{A}) \cap T(F) & \to & \mathcal{S}_{\widetilde{T(\mathbb{A})}} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \iota(T^\#(\mathbb{A})) & \to & \mathrm{Z}^\dagger(\mathbb{A}) & \to & \prod_v \mathcal{S}_v & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \iota(T^\#(\mathbb{A})/T^\#(F)) & \to & \mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A}) \cap T(F)) & \to & (\prod_v \mathcal{S}_v)\big/\mathcal{S}_{\widetilde{T(\mathbb{A})}} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

are all exact. The bottom sequence fits into another diagram

$$\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \left(\prod_v \mathcal{S}_v\right)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}} & & & & & & \\
 & & \uparrow & & & & & & \\
1 & \leftarrow & \mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A})\cap T(F)) & \leftarrow & \mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\Big/\left(\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\cap T(F)\right) & \leftarrow & \mu_n & \leftarrow & 1 \\
 & & \uparrow & & & & & & \\
1 & \leftarrow & \iota(T^\#(\mathbb{A})/T^\#(F)) & \overset{\iota}{\twoheadleftarrow} & T^\#(\mathbb{A})/T^\#(F) & & & & \\
 & & \uparrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

of exact sequences. We take the Pontryagin dual $(\ )^\vee = \mathrm{Hom}(\ ,\mathbb{C}^1)$ of these groups, noting the following fact.

Proposition 4.3 (Global Langlands correspondence for a torus [5, Théorème 6.4])**.** *Let* W_F *be the Weil group of the algebraic number field* F*. Then we have a natural surjection*

$$\mathrm{GLC}_{T^\#}\colon\ \mathrm{H}^1(\mathrm{W}_F,(Y^\#)^\vee)\to \mathrm{Hom}(T^\#(\mathbb{A})/T^\#(F),\mathbb{C}^1)$$

with finite kernel. Here, the Pontryagin dual $(Y^\#)^\vee = \mathrm{Hom}(Y^\#,\mathbb{C}^1))$ *is the maximal compact subgroup of the Langlands dual* $\widehat{T^\#}$ *of the torus* $T^\#$*.*

Thus, we obtain the following frame of the global Langlands correspondence for the covering group $\widetilde{T(\mathbb{A})}\to T(\mathbb{A})$.

Proposition 4.4. *Recall that* $j\colon \mu_n\hookrightarrow\mathbb{C}^1$ *is the fixed embedding to define genuine representations. Then the objects introduced above fit in the diagram*

$$\begin{array}{cccccc}
 & 1 & & \left\{\begin{array}{c}\textit{genuine unitary automorphic}\\ \textit{representations of } \widetilde{T(\mathbb{A})}\end{array}\right\} & & \\
 & \downarrow & & {\scriptstyle\omega_{(\ )}}\downarrow{\scriptstyle 1:1} & & \\
 & \left(\left(\prod_v \mathcal{S}_v\right)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}\right)^\vee & & \{\textit{genuine characters}\} & \to & \{j\} \\
 & \downarrow & & \cap & & \cap \\
1\to & \left(\mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A})\cap T(F))\right)^\vee & \to & \left(\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\Big/\left(\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\cap T(F)\right)\right)^\vee & \to & \mu_n^\vee\to 1 \\
 & \downarrow{\scriptstyle (\ )|\iota(T^\#(\mathbb{A})/T^\#(F))} & & & & \\
 & \left(\iota(T^\#(\mathbb{A})/T^\#(F))\right)^\vee & \overset{\iota^*}{\hookrightarrow} & (T^\#(\mathbb{A})/T^\#(F))^\vee & & \\
 & \downarrow & & \wr\Vert & & \\
 & 1 & & \mathrm{H}^1(\mathrm{W}_F,(Y^\#)^\vee)\big/\operatorname{Ker}\mathrm{GLC}_{T^\#}\ , & &
\end{array}$$

where the middle horizontal sequence and the left vertical one are exact.

By definition, the group $\left(\left(\prod_v \mathcal{S}_v\right)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}\right)^\vee$ acts simply transitively on the set of genuine unitary characters of $\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\Big/\left(\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\cap T(F)\right)$.

Thus, fixing such a character χ as a base point defines a bijection

$$\begin{array}{c}\left\{\text{genuine unitary characters of } \mathrm{Z}\Big(\widetilde{T(\mathbb{A})}\Big)\Big/\Big(\mathrm{Z}\Big(\widetilde{T(\mathbb{A})}\Big)\cap T(F)\Big)\right\}\\ \Big\downarrow{\scriptstyle b_\chi}\\ \left(\mathrm{Z}^\dagger(\mathbb{A})/\left(\mathrm{Z}^\dagger(\mathbb{A})\cap T(F)\right)\right)^\vee\end{array}$$

By Proposition 4.4, the composed map $\mathrm{GLC}_{T^\#}^{-1}\circ\iota^*\circ(\,)|_{\iota(T^\#(\mathbb{A})/T^\#(F))}$:

$$\left(\mathrm{Z}^\dagger(\mathbb{A})/(\mathrm{Z}^\dagger(\mathbb{A})\cap T(F))\right)^\vee\to \mathrm{H}^1(\mathrm{W}_F,(Y^\#)^\vee)/\operatorname{Ker}\mathrm{GLC}_{T^\#}$$

has kernel $\left((\prod_v\mathcal{S}_v)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}\right)^\vee$.

Definition 4.5. Fix a genuine unitary character χ of the quotient group $\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\Big/\left(\mathrm{Z}\left(\widetilde{T(\mathbb{A})}\right)\cap T(F)\right)$. Then the *global Langlands correspondence* for the Brylinski-Deligne cover $\widetilde{T(\mathbb{A})}\to T(\mathbb{A})$ is the composition

$$\begin{array}{c}\left\{\text{genuine unitary automorphic representations of } \widetilde{T(\mathbb{A})}\right\}\\ {\scriptstyle b_\chi(\omega_{(\,)})}\Big\downarrow{\scriptstyle 1:1}\\ \left(\mathrm{Z}^\dagger(\mathbb{A})/\left(\mathrm{Z}^\dagger(\mathbb{A})\cap T(F)\right)\right)^\vee\\ {\scriptstyle \mathrm{GLC}_{T^\#}^{-1}\circ\iota^*\circ(\,)|_{\iota(T^\#(\mathbb{A})/T^\#(F))}}\Big\downarrow\\ \mathrm{H}^1(\mathrm{W}_F,(Y^\#)^\vee)/\operatorname{Ker}\mathrm{GLC}_{T^\#}\end{array}$$

Note that some packets, or fibers, may be empty. We prefer to call each fiber an automorphic packet, since it is the subset of a usual global packet consisting of only automorphic representations. By definition, the non-empty automorphic packets are described by the group $(\prod_v\mathcal{S}_v)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}$, as follows.

Proposition 4.6 (role of $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$)**.** *In the global Langlands correspondence defined above, each non-empty automorphic packet* Π*, which is a subset of a usual global packet, is just an orbit under the Pontryagin dual group* $\left((\prod_v\mathcal{S}_v)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}\right)^\vee$. *Thus, Choosing a base point in the automorphic packet* Π *defines a one-to-one correspondence between* Π *and* $\left((\prod_v\mathcal{S}_v)\Big/\mathcal{S}_{\widetilde{T(\mathbb{A})}}\right)^\vee$.

Therefore, a unitary character of $\prod_v\mathcal{S}_v$ corresponds to a genuine unitary automorphic representation if and only if it is trivial on the global packet group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$. Then, the following main result states that $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ is a finite group.

Theorem 4.7 (N. [7])**.** *Let* F *be an algebraic number field containing a full set* μ_n *of* n*-th roots of unity,* $\mathbb{A}$ *the adele ring, and* T *an algebraic* *torus defined over F. Let $\widetilde{T(\mathbb{A})} \to T(\mathbb{A})$ be a Brylinski-Deligne covering group of the adelic torus. Then, the group $\mathcal{S}_{\widetilde{T(\mathbb{A})}}$ has finite order.*

The proof is written in our preprint [7] on L-indistinguishability for covering groups of tori. Here, we only mention that the following proposition is the essential part of our proof. Recall that at any place v of the number field F, the symbol $\mathrm{Z}^\dagger(F_v) \subset T(F_v)$ denotes the image of the center $\mathrm{Z}(\widetilde{T(F_v)})$ of the cover $\widetilde{T(F_v)}$.

Proposition 4.8 (N. [6, Corollary 5.3])**.** *We fix a finite Galois extension L/F where the torus T splits over L. Let $v \nmid n$ be a finite place of F such that the ramification index e of L/F at v is relatively prime to the degree n of the cover. Then we have an inclusion*

$$\mathrm{Z}^\dagger(F_v) \subset \iota(T^\#(F_v)) \cdot T(\mathcal{O}_v).$$

Acknowledgements

The author would like to express gratitude to Professor Tamotsu Ikeda for the insightful discussions. This work was supported by JSPS KAKENHI Grant Number JP23KJ1298.

References

[1] J.-L. Brylinski and P. Deligne. Central extensions of reductive groups by $\mathbf{K}_2$. *Publ. Math. Inst. Hautes Études Sci.*, (94):5–85, 2001.
[2] D. Bump and D. Ginzburg. Symmetric square L-functions on $\mathrm{GL}(r)$. *Ann. of Math. (2)*, 136(1):137–205, 1992.
[3] W. T. Gan and F. Gao. The Langlands-Weissman program for Brylinski-Deligne extensions. *Astérisque*, (398):187–275, 2018. L-groups and the Langlands program for covering groups.
[4] A. C. Kable. The tensor product of exceptional representations on the general linear group. *Ann. Sci. École Norm. Sup. (4)*, 34(5):741–769, 2001.
[5] J.-P. Labesse. Cohomologie, L-groupes et fonctorialité. *Compositio Math.*, 55(2):163–184, 1985.
[6] Y. Nakata. Local Langlands correspondence for covering groups of tori, and the packet-indexing groups. *J. Number Theory*, 289:132–159, 2026.
[7] Y. Nakata. L-indistinguishability for covering groups of algebraic tori, preprint.
[8] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume 341 of *Lecture Notes in Math.*, pages 85–147. Springer, Berlin-New York, 1973.
[9] S. Takeda. The twisted symmetric square L-function of $\mathrm{GL}(r)$. *Duke Math. J.*, 163(1):175–266, 2014.
[10] M. H. Weissman. Metaplectic tori over local fields. *Pacific J. Math.*, 241(1):169–200, 2009.
[11] M. H. Weissman. Covers of tori over local and global fields. *Amer. J. Math.*, 138(6):1533–1573, 2016.
[12] M. H. Weissman. L-groups and parameters for covering groups. *Astérisque*, (398):33–186, 2018. L-groups and the Langlands program for covering groups.

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