Smoothening strata and compactified Jacobians: a survey announcement in the Bass case

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Abstract

This is an expository announcement based on a RIMS workshop talk. We explain how the local smoothening method for orbital integrals in the Bass case is reflected in the geometry of Hitchin fibers for GL_n. Under the double-point, integral-domain hypothesis on the spectral curve, the local Bass types globalize to smooth locally closed pieces of the Hitchin fiber, and these pieces agree with the usual stratification of the compactified Jacobian by partial normalizations. The rational-normalization case is mentioned only as an application of this comparison; the explicit cohomological calculation is not included here. The note records the structure of the results and omits detailed representability, smoothness, descent, and cohomological arguments.

1 Scope of this announcement

The purpose of this note is to record the main ideas of a comparison between two ways of organizing the same geometry: the local smoothening strata used in orbital-integral computations and the geometric strata of compactified Jacobians. The emphasis is on the structural picture rather than on technical details. In particular, this note does not contain the construction of the restricted Hitchin stacks, the infinitesimal lifting argument, or the verification of the cohomological calculation. These details are outside the scope of this workshop announcement.

Accordingly, the present text is written as a survey announcement: it states the main comparison results and explains the indexing mechanism, while avoiding detailed arguments, constructions, and numerical cohomological formulas.

2 Hitchin fibers in the Bass setting

Let k be either a finite field or an algebraic closure of a finite field, and assume that $\mathrm{char}(k) > n$. Let C/k be a smooth, geometrically connected, projective curve, and let

$$D = \sum_x d_x[x]$$

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be an even effective divisor of sufficiently large degree. The GL_n Hitchin stack $\mathbb{M}$ parametrizes pairs $(\mathcal{E}, \theta)$ consisting of a rank-n vector bundle $\mathcal{E}$ on C and a Higgs field

$$\theta : \mathcal{E} \longrightarrow \mathcal{E}(D).$$

The Hitchin base is

$$\mathbb{A} = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}_C(iD)),$$

and the Hitchin map sends $(\mathcal{E}, \theta)$ to the coefficients of the characteristic polynomial

$$\chi_\theta(X) = X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n.$$

Fix a point $\chi \in \mathbb{A}(k)$ in the elliptic locus, and let Y_χ be the associated spectral curve. The Beauville–Narasimhan–Ramanan correspondence identifies the Hitchin fiber with the compactified Jacobian of Y_χ:

$$\Phi^{-1}(\chi) \simeq \overline{\mathrm{Pic}^0}(Y_\chi), \tag{2.1}$$

where $\overline{\mathrm{Pic}^0}(Y_\chi)$ parametrizes rank-one torsion-free sheaves of degree zero on Y_χ; see [BNR89, Ngo10, Cha14].

The case considered here is the Bass case. Namely, at every singular point of Y_χ, the completed local ring is assumed to be an integral Bass order. For the spectral curves in this note, this is the same geometric condition as saying that all singularities are integral double points.

For a closed point $x \in C$, put

$$\chi_x(X) = \pi_x^{nd_x} \chi(X/\pi_x^{d_x}) \in \mathcal{O}_x[X], \tag{2.2}$$

where π_x is a uniformizer of the completed local ring $\mathcal{O}_x$ of C at x. Set

$$R_x = \mathcal{O}_x[X]/(\chi_x(X)), \qquad E_x = \mathrm{Frac}(\mathcal{O}_x)[X]/(\chi_x(X)).$$

After the standard translation of the Hitchin fiber, the nontrivial local data occur only at the finite set

$$\mathcal{S} = \{x \in C \mid nd_x + \mathrm{ord}_x(a_n) > 0\}.$$

The singular points of Y_χ correspond to

$$\mathcal{S}' = \{x \in \mathcal{S} \mid nd_x + \mathrm{ord}_x(a_n) > 1\}.$$

For $n \geq 3$ in the Bass range, this condition becomes $nd_x + \mathrm{ord}_x(a_n) = 2$.

3 From local smoothening to global types

We recall the local input only at the level needed for the global comparison. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$. The characteristic-polynomial morphism

$$\varphi : \mathfrak{gl}_{n,\mathfrak{o}} \longrightarrow \mathbb{A}^n_{\mathfrak{o}}$$

usually has singular fibers. For Bass orders, the smoothening method of [CKL, CHL] decomposes the relevant set of integral points into finitely many pieces indexed by local types. Each local type cuts out a representable subfunctor which is smooth along the fiber under consideration. In orbital-integral terms, the local integral is then expressed as a sum of contributions from these smooth pieces.

The type has two features. First, it records the cokernel of the corresponding lattice endomorphism. Secondly, at a double point it records a small amount of residual nilpotent data. For $n = 2$ this may be written in terms of decompositions

$$\mathfrak{o}/\pi^a \oplus \mathfrak{o}/\pi^b, \qquad 0 \leq a \leq b.$$

For $n \geq 3$ the double-point case involves a residual two-dimensional piece and an integer parameter. We do not spell out the exact functorial definition in this announcement.

A global type is obtained by taking the local Bass type at every point of $\mathcal{S}$ and the trivial type outside $\mathcal{S}$. Denote by $\mathcal{I}(\chi)$ the resulting finite set of admissible global types. The announced algebraic stratification is as follows.

Announced result A: algebraic smoothening. For each $T \in \mathcal{I}(\chi)$ there is a locally closed piece

$$(\Phi^T)^{-1}(\chi) \subset \Phi^{-1}(\chi)$$

which is smooth over k. Moreover, for every finite extension k'/k, and also for $k' = \bar{k}$, one has a disjoint decomposition of sets

$$\Phi^{-1}(\chi)(k') = \bigsqcup_{T \in \mathcal{I}(\chi)} (\Phi^T)^{-1}(\chi)(k').$$

The construction uses the cokernel of the Higgs field to impose the sheaf-theoretic part of the type, and then imposes the residual nilpotent condition locally. Smoothness is ultimately reduced to the local smooth integral models of [CKL, CHL]; the passage from local lifting to global lifting uses the adelic description of Hitchin bundles.

4 Comparison with partial normalizations

The compactified Jacobian of an integral curve with double points has a natural geometric stratification. If

$$\pi' : Y'_\chi \longrightarrow Y_\chi$$

is a partial normalization, then pushforward of line bundles gives a locally closed stratum

$$\mathrm{Pic}^0(Y'_\chi) \subset \overline{\mathrm{Pic}^0}(Y_\chi).$$

Gagné's description [Gag97] gives, after passing to finite extensions of k or to $\bar{k}$, a decomposition

$$\overline{\mathrm{Pic}^0}(Y_\chi)(k') = \bigsqcup_{\pi' : Y'_\chi \to Y_\chi} \mathrm{Pic}^0(Y'_\chi)(k'),$$

where the index runs over partial normalizations up to isomorphism.

The key point is that the two indexing systems are the same in the Bass case. A partial normalization is determined locally by overorders

$$R_x \subset R'_x \subset \mathcal{O}_{E_x}, \qquad x \in \mathcal{S}',$$

where $\mathcal{O}_{E_x}$ is the normalization of R_x. On the other hand, an adelic representative of a Hitchin point determines the stabilizing overorder of the corresponding local lattice,

$$(g_x\mathcal{O}_x^n : g_x\mathcal{O}_x^n) = \{a \in E_x \mid a(g_x\mathcal{O}_x^n) \subset g_x\mathcal{O}_x^n\}.$$

The Bass classification identifies this overorder with the local type.

Announced result B: equality of the two stratifications. For every partial normalization $\pi' : Y'_\chi \to Y_\chi$, there is a unique global type $T \in \mathcal{I}(\chi)$ such that, under the identification (2.1),

$$(\Phi^T)^{-1}(\chi) = \mathrm{Pic}^0(Y'_\chi)$$

as locally closed subvarieties of $\overline{\mathrm{Pic}^0}(Y_\chi)$.

Thus the smoothening strata are not merely a local matrix device; in the global Hitchin fiber they recover the intrinsic geometric strata of the compactified Jacobian.

5 A cohomological application

Assume now that $k = \mathbb{F}_q$ and that the normalization of Y_χ is isomorphic to $\mathbb{P}^1_k$. In this situation the comparison above gives a practical route from local Bass data to the compactly supported ℓ-adic cohomology of

$$\Phi^{-1}(\chi) \simeq \overline{\mathrm{Pic}^0}(Y_\chi).$$

The relevant indexing remains the same finite set of global types, or equivalently the same set of partial normalizations. Thus the local smoothening method does not merely give a decomposition of orbital integrals; it also organizes the global geometry in a way that is compatible with the cohomological calculation.

The explicit formula, together with the proof of the corresponding decomposition statement, is not recorded in this announcement. Its verification requires a stratum-wise analysis and a careful comparison with the product formula for Hitchin fibers. For the present note, the point is the structural one: in the rational-normalization case, the equality between smoothening strata and partial-normalization strata supplies the bridge between local integral models and the cohomology of the compactified Jacobian. Compare [Lau06, Yun16] for the relation between Hitchin fibers, compactified Jacobians, and affine Springer fibers.

6 Outlook

The Bass hypothesis has two separate roles. It makes the local smoothening types explicit, and it makes the corresponding geometric indexing set equal to the set of partial normalizations. Outside the Bass range, one still expects local smoothening strata whenever suitable local models are available, but the geometric indexing set should be richer than partial normalizations alone. The first natural case is $n = 3$ beyond the double-point range, where local smoothening already exists in [CKL].

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