EXPLICIT FOURIER COEFFICIENTS OF THE EISENSTEIN SERIES AND THETA LIFT ON $\mathrm{SO}(3, n+1)$

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Abstract. We survey the results in [6, 7] concerning explicit Fourier coefficients of Eisenstein series and the theta lift on $\mathrm{SO}(3, n+1)$.

1. Introduction

We use the notations in [6]. Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$. For an integer $n \geq 3$, let A be a positive definite integral matrix of size $n-2$. Assume A is even if n is even while $A = \mathrm{diag}(1, A')$ if n is odd where A' is a positive definite integral matrix of size $n-3$. Let $J_{1,n} = \mathrm{antidiag}(1, -A, 1)$, $J_{2,n} = \mathrm{antidiag}(1, 1, -A, 1, 1)$ and $J_{3,n+1} = \mathrm{antidiag}(1, 1, 1, -A, 1, 1, 1)$ which are of size $n+2$, $n+3$ and $n+4$ respectively. Let $G = \mathrm{SO}(3, n+1)$ be the special orthogonal group over $\mathbb{Q}$ associated to the symmetric pairing defined by $J_{3,n+1}$. Note that if n is even, G does not have discrete series. Assume G splits everywhere at finite places and it induces $n \equiv 2 \bmod 8$ if n even and $n \equiv 3 \bmod 8$ if n odd. In [13], Pollack developed a theory of modular forms on $\mathrm{SO}(3, n+1)$, and defined the Eisenstein series E_l on $G(\mathbb{A})$ of weight l with respect to $G(\widehat{\mathbb{Z}})$, and computed unramified and archimedean part of Fourier coefficients. He showed the algebraicity of the Fourier coefficients. However, he did not compute the Siegel series explicitly which come from the bad finite places. The computation of the Siegel series is complicated and it is done in [6]. One corollary of our explicit computation is that we can show that Fourier coefficients of Eisenstein series have bounded denominator.

By using the explicit Siegel series, we can write down an Ikeda type formal series [3, 4] from a normalized Hecke eigen cusp form of weight on $\mathrm{SL}_2(\mathbb{Z})$. In [7], we showed that the formal series is the theta lift to $\mathrm{SO}(3, n+1)$, using the degenerate Whittaker functions. One consequence is

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that we have an explicit Fourier expansion of the theta lift. In this note we survey the results from [6, 7].

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2. Automorphic forms on $G = \mathrm{SO}(3, n+1)$

Recall $G = \mathrm{SO}(V)$. The maximal compact subgroup K of $G(\mathbb{R})$ is $S(\mathrm{O}(3) \times \mathrm{O}(n+1))$. Let $K_0 = \mathrm{SO}(3) \times \mathrm{SO}(n+1)$. We have a map $S(\mathrm{O}(3) \times \mathrm{O}(n+1)) \to \mathrm{O}(3) = (\mathrm{SU}(2)/\mu_2) \rtimes \{\pm 1\}$. For each $\lambda \in \mathbb{C}$, we consider the normalized induced representation

$$I(\lambda) := \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} |\nu|^{\lambda - \frac{n}{2} - 1}.$$

By the Peter-Weyl theorem, we have

$$\mathrm{Ind}_{K_0 \cap P(\mathbb{R})}^{K_0} \mathbf{1} \simeq C^\infty((S^2 \times S^n)/\{\pm 1\}) \simeq \bigoplus_{\substack{a,b \geq 0 \\ a+b:\, \mathrm{even}}} \mathcal{H}^a(\mathbb{C}^3) \boxtimes \mathcal{H}^b(\mathbb{C}^{n+1})$$

where S^k stands for the k-dimensional sphere and $\mathcal{H}^i(\mathbb{C}^j)$ stands for the harmonic polynomials on $\mathbb{C}^j$ of degree i. Using this, we have

$$I(\lambda)|_K = \bigoplus_{\substack{a,b \geq 0 \\ a+b:\, \mathrm{even}}} \mathrm{Ind}_{K_0}^{K} (\mathcal{H}^a(\mathbb{C}^3) \boxtimes \mathcal{H}^b(\mathbb{C}^{n+1})). \tag{2.1}$$

In particular, if $b = 0$ (hence a is even, say $a = 2l$), then $\mathcal{H}^{2l}(\mathbb{C}^3) \boxtimes \mathbf{1}$ is naturally extended to a representation of $\mathrm{O}(3)$ and it appears in $I(\lambda)|_K$ with multiplicity one.

Definition 2.1. Let $l \geq 0$ be an integer. Let $\mathbb{V}_l = \mathrm{Sym}^{2l}(\mathbb{C}^2)$ be the $(2l+1)$-dimensional representation of K that factors through $\mathrm{O}(3)$ and we regard it with $\mathcal{H}^{2l}(\mathbb{C}^3) \boxtimes \mathbf{1}$ as a K-type of $I(l+1)$. We fix a basis $\{[X]^{l+v}[Y]^{l-v}\}_{v=-l}^{l}$ of $\mathbb{V}_l$ where $[X^k] = \frac{X^k}{k!}$ and $[Y^k] = \frac{Y^k}{k!}$ for $k \geq 0$. Modular forms on G of weight l are $\mathbb{V}_l$-valued automorphic functions ϕ on $G(\mathbb{A})$ that satisfy:

(1) $\phi(gk) = k^{-1}\phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K$.
(2) ϕ is annihilated by a special differential operator $\mathcal{D}_l$.
(3) As a $(\mathfrak{g}, K)$-module, ϕ generates an irreducible constituent of $I(l+1)$.

Proposition 2.2. *Assume $l > \frac{n}{2}$ is an even integer. Let F be a non-zero modular form on G of weight l and Π_∞ be the corresponding irreducible admissible representation of $G(\mathbb{R})$. Then, as a $(\mathfrak{g}, K)$-module, Π_∞ is isomorphic to a unique irreducible component of $\Pi_{l,0}^{3,n+1}|_{G(\mathbb{R})}$ containing $\mathcal{H}^{2l}(\mathbb{C}^3) \boxtimes \mathbf{1}$ as a minimal K-type. (see* [11, Proposition 2.1] *for $\Pi_{l,0}^{3,n+1}$).*

Theorem 2.3. [13] *Suppose ϕ is a modular form of weight $l \geq 1$ on G. Then*

$$\phi(g) = \phi_0(g) + \sum_{\substack{\eta \in V'(\mathbb{Q}) \\ q(\eta) \geq 0, \eta \neq 0}} a_\phi(\eta)(g_{\mathbf{f}})\mathcal{W}_{2\pi\eta}(g_\infty),$$

for $g = g_{\mathbf{f}} g_\infty$ in $G(\mathbb{A}_{\mathbf{f}}) \times G(\mathbb{R})$, where $a_\phi(\eta) : G(\mathbb{A}_{\mathbf{f}}) \longrightarrow \mathbb{C}$ is a locally constant function. Moreover,

$$\phi_0(m) = \Phi(m)X^{2l} + \beta(m_f)X^l Y^l + \Phi'(m)Y^{2l},$$

where Φ is an automorphic function associated to a holomorphic modular form of weight l on M, β is a locally constant function on $M(\mathbb{A}_{\mathbf{f}})$, and Φ' is a certain $(K \cap M)$-right translate of Φ.

Here $\mathcal{W}_\eta(g) : G(\mathbb{R}) \longrightarrow \mathbb{V}_l$ is a generalized Whittaker function of type η satisfying

(1) $\mathcal{W}_\eta(n(x)g) = e^{i(\eta,x)}\mathcal{W}_\eta(g)$;
(2) $\mathcal{W}_\eta(gk) = k^{-1}\mathcal{W}_\eta(g)$;
(3) $D_l\mathcal{W}_\eta(g) = 0$;
(4) Suppose $\eta \in V'(\mathbb{R})$, $\eta \neq 0$, and $(\eta, \eta) \geq 0$. For $t \in GL_1(\mathbb{R}), m \in \mathrm{SO}(V')(\mathbb{R})$, set

$$u_\eta(t, m) = it\sqrt{2}(\eta, m(iv_1 - v_2)), \tag{2.2}$$

where

$$v_1 := (1, \overbrace{0, \ldots, 0}^{n}, 1),\ v_2 := (0, 1, \overbrace{0, \ldots, 0}^{n-2}, 1, 0).$$

Then

$$\mathcal{W}_\eta(t, m) = t^l|t| \sum_{-l \leq v \leq l} \left(\frac{|u_\eta(t, m)|}{u_\eta(t, m)}\right)^v K_v(|u_\eta(t, m)|)[X^{l+v}][Y^{l-v}], \tag{2.3}$$

where K_v is the K-Bessel function defined by

$$K_v(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^v \, \frac{dt}{t}.$$

We have the following criterion of cusp forms on G.

Proposition 2.4. *Let ϕ be a modular form of weight $l \geq 1$ and be of level one, hence it is fixed by $G(\widehat{\mathbb{Z}})$. If it has the Fourier expansion of the form*

$$\phi(g) = \sum_{\substack{\eta \in V'(\mathbb{Q}) \\ q(\eta) > 0}} a_\phi(\eta)(g_f)\mathcal{W}_{2\pi\eta}(g_\infty),$$

then ϕ is a cusp form.

3. Eisenstein series on SO$(3, n+1)$

Let P be the Siegel parabolic subgroup of G with the Levi decomposition $P = MN$ where $M = \{\mathrm{diag}(t, m, t^{-1}) \mid t \in \mathrm{GL}_1,\ m \in \mathrm{SO}(2,n)\} \simeq \mathrm{GL}_1 \times \mathrm{SO}(2,n)$ and $N \simeq \mathbb{G}_a^{n+2}$ where $\mathrm{SO}(2,n)$ is defined by $J_{2,n}$.

For any $l > n+1$ and a Schwartz-function $\Phi_{\mathbf{f}}$ on $V(\mathbb{A}_{\mathbf{f}})$, we define a section f_l of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} |\nu|^{s-\frac{n+2}{2}}$ (normalized induction) by

$$f_l(g, \Phi_{\mathbf{f}}, s) = f_{\mathrm{fte}}(g_{\mathbf{f}}, \Phi_{\mathbf{f}}, s) f_{l,\infty}(g_\infty, s),\ g = g_{\mathbf{f}} g_\infty \in G(\mathbb{A})$$

where

$$f_{\mathrm{fte}}(g_{\mathbf{f}}, \Phi_{\mathbf{f}}, s) = \int_{\mathrm{GL}_1(\mathbb{A}_{\mathbf{f}})} |t|^s \Phi_{\mathbf{f}}(t g_{\mathbf{f}}^{-1} e) dt,\ g_{\mathbf{f}} \in G(\mathbb{A}_{\mathbf{f}})$$

and $f_{l,\infty}(g_\infty, s)$ is the $V_l = \mathrm{Sym}^{2l}(\mathbb{C}^2)$-valued, K-equivalent section of $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} |\nu|^{s-\frac{n+2}{2}}$. Then, we define the associated Eisenstein series as

$$E_l(g, \Phi_{\mathbf{f}}, s) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_l(\gamma g, \Phi_{\mathbf{f}}, s),\ g \in G(\mathbb{A}).$$

By definition, $E_l(g, \Phi_{\mathbf{f}}, s) = E_l(g_\infty, g_{\mathbf{f}} \cdot \Phi_{\mathbf{f}}, s)$ where $g_{\mathbf{f}} \cdot \Phi_{\mathbf{f}}$ is defined by $g_{\mathbf{f}} \cdot \Phi_{\mathbf{f}}(v) = \Phi_{\mathbf{f}}(g_{\mathbf{f}}^{-1} v)$ for $v \in V(\mathbb{A})$. It converges absolutely for $Re(s) > n+2$. When $s = l+1$ and $l+1 > n+2$, $E_l(g, \Phi_{\mathbf{f}}, s = l+1)$ gives rise to a modular form of weight l, and it has the Fourier expansion as

$$E_l(g) = E_0(g) + \sum_{\substack{\eta \in V'(\mathbb{Q}) \\ q(\eta) \geq 0, \eta \neq 0}} a_{E_l}(\eta)(g_{\mathbf{f}}) \mathcal{W}_{2\pi\eta}(g_\infty),$$

for $g = g_{\mathbf{f}} g_\infty$ in $G(\mathbb{A}_{\mathbf{f}}) \times G(\mathbb{R})$, where $\mathcal{W}_{2\pi\eta}$ is Pollack's spherical function associated to the additive character $e^{2\pi\sqrt{-1}(\eta,*)_{V'}}$ on $V'(\mathbb{R})$, and $a_{E_l}(\eta) : G(\mathbb{A}_{\mathbf{f}}) \longrightarrow \mathbb{C}$ is a locally constant function. Moreover, for $m \in M$,

$$E_0(m) = f_l(g, s) + E_l^M(m) X^{2l} + E_l^{'M}(m) Y^{2l},$$

where $f_l(g, s)$ is the inducing section, and E_l^M is an Eisenstein series on M which is a holomorphic modular form of weight l on M, and $E_l^{'M}$ is a certain $(K \cap M)$-right translate of E_l^M.

If η is of rank one, i.e., $\eta \neq 0$ and $q(\eta) = 0$, then the rank one Fourier coefficient of $E_l(g, \Phi_{\mathbf{f}})$ is given by

$$2l! i^l (2\pi)^l \sigma_l(\eta) \mathcal{W}_{2\pi\eta}(g_\infty),$$

where $\sigma_l(\eta)=\prod_p \sigma_{l,p}(\eta)$, and $\sigma_{l,p}(\eta)=\sum_{i=0}^{v_p(\eta)} p^{il}$. Here given $\eta\in V'(\mathbb{Z})$, we define $v_p(\eta)=\max\{n:\eta\in p^nV'(\mathbb{Z}_p)\}$.

If η is of rank two, i.e., $q(\eta)>0$, then it is given as follows:

When n is even ($n>2$), for each $\eta\in V'(\mathbb{Q})$ with $q(\eta)>0$ and each rational prime p, there exists a Laurent polynomial $\widetilde{Q}_{\eta,\Phi_p}(X_p)=\widetilde{Q}_{\eta,p}(X_p)\in\mathbb{Z}[X_p,X_p^{-1}]$ depending on η and the unramified Schwartz function $\Phi_p=\mathrm{char}_{V'(\mathbb{Z}_p)}\in\mathcal{S}(V'(\mathbb{Q}_p))$ satisfying $\widetilde{Q}_{\eta,p}(X_p)=\widetilde{Q}_{\eta,p}(X_p^{-1})$ such that

$$a_{E_l}(\eta)=C_{l,n}q(\eta)^{\frac{l-\frac{n}{2}}{2}}\prod_p \widetilde{Q}_{\eta,p}(p^{\frac{l-\frac{n}{2}}{2}})$$

for the constant $C_{l,n}$.

When n is odd, we need a slight modification. Let $l>n+1$ be an even integer. Write $q(\eta)=\mathfrak{d}_\eta\mathfrak{f}_\eta^2$ such that $\mathfrak{d}_\eta$ is the absolute discriminant of $\mathbb{Q}\left(\sqrt{\epsilon q(\eta)}\right)/\mathbb{Q}$, where $\epsilon=\begin{cases}1, & \text{if } q(\eta)_1\equiv 1 \pmod 4\\ -1, & \text{if } q(\eta)_1\equiv 3 \pmod 4\end{cases}$, and $q(\eta)=2^{v_2(\eta)}q(\eta)_1$, $q(\eta)_1$ odd. Let χ_η be the primitive Dirichlet character corresponding to $\mathbb{Q}\left(\sqrt{\epsilon q(\eta)}\right)/\mathbb{Q}$. Then for each rational prime p, there exists a Laurent polynomial $\widetilde{Q}_{\eta,\Phi_p}(X_p)=\widetilde{Q}_{\eta,p}(X_p)\in\mathbb{Z}[X_p,X_p^{-1}]$ depending on η and $\Phi_p=\mathrm{char}_{V'(\mathbb{Z}_p)}$ satisfying $\widetilde{Q}_{\eta,p}(X_p)=\widetilde{Q}_{\eta,p}(X_p^{-1})$ such that

$$a_E(\eta)=C'_{l,n}L(\tfrac{n+1}{2}-l,\chi_\eta)\mathfrak{f}_\eta^{l-\frac{n}{2}}\prod_p \widetilde{Q}_{\eta,p}(p^{l-\frac{n}{2}}),$$

for some constant $C'_{l,n}$. These formulas immediately imply that Fourier coefficients have a bounded denominator.

4. Fourier expansion of the theta lift on SO$(3,n+1)$

Let $l>n+1$ be an even integer. Let $f(z)=\sum_{m=1}^{\infty}a_f(m)m^{\frac{k-1}{2}}q^m$ be a normalized Hecke eigen cusp form of weight $k=\begin{cases}l-\frac{n+2}{2} & \text{if } n \text{ is even,}\\ 2l-n+1 & \text{if } n \text{ is odd}\end{cases}$, with respect to $\mathrm{SL}_2(\mathbb{Z})$. For each prime p, let $a_f(p)=\alpha_p+\alpha_p^{-1}$ with $\alpha_p\in\mathbb{C}^1$. Let $\pi=\pi_f=\otimes'\pi_p\otimes\pi_\infty$ be the unitary cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f. For each prime p, we have $\pi_p\simeq\pi_p(\mu_p,\mu_p^{-1})$ so that $\mu_p(p)=\alpha_p$. Let τ be any fixed cuspidal component of $\pi|_{\mathrm{SL}_2(\mathbb{A})}$ when n is even or τ be the Shimura-Waldspurger lift of π to $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ when n is odd. Let $\Pi_f=\otimes'_p\Pi_{f,p}\otimes\Pi_{f,\infty}$ be any component of the restriction to G of the theta lift of τ to $\mathrm{O}(3,n+1)$. A well-known standard method shows that Π_f is a non-zero square integrable automorphic representation of $G(\mathbb{A})$. Further, by [1] for non-archimedean places and [9], [11, Proposition 2.1] for the archimedean place, we see that $\Pi_{f,p}$ is a unique spherical quotient of the degenerate principal series

$$\begin{cases} \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mu_p^2\circ\nu & \text{if } n \text{ is even,} \\ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mu_p\circ\nu & \text{if } n \text{ is odd} \end{cases}$$

and $\Pi_{f,\infty}$ is a unique irreducible component of $\Pi^{3,n+1}_{l,0}|_{G(\mathbb{R})}$ containing $\mathcal{H}^{2l}(\mathbb{C}^3)\boxtimes\mathbf{1}$ as a minimal K_∞-type. Here $\Pi^{3,n+1}_{l,0}$ is the unitarizable, irreducible non-tempered, cohomological admissible representation of $\mathrm{O}(V)(\mathbb{R})$ given in [11, Proposition 2.1]. Let $\varepsilon=\begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd}\end{cases}$. By using the degenerate Whitaker functions (cf. [5], [8]), for each prime $p<\infty$ and $\eta\in V'(\mathbb{Q}_p)$ with $q(\eta)\neq 0$, we can define a non-zero functional

$$w^{\mu_p}_\eta \in \mathrm{Wh}(\Pi_{f,p}) := \mathrm{Hom}_{N(\mathbb{Q}_p)}(\Pi_{f,p}, \psi_p((\eta,*)_{V'})$$

where ψ_p is the standard additive character of $\mathbb{Q}_p$. Let us fix an intertwining map

$$\Pi_f \hookrightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A})), \quad \phi\mapsto F_\phi.$$

By multiplicity one for the Whittaker spaces and [13, p.621, Theorem 3.2.4], we can expand $F_\phi=(F_\phi)_0+(F_\phi)_+$ as follows for each distinguished vector $\phi=\otimes'_p\phi_p\otimes\phi^{\mathrm{Po}}_\infty=\phi_{\mathbf{f}}\otimes\phi^{\mathrm{Po}}_\infty$ with the Pollack's section $\phi^{\mathrm{Po}}_\infty\in\Pi_{f,\infty}\simeq\Pi^{3,n+1}_{l,0}$. The initial term $(F_\phi)_0$ is the constant term of F_ϕ along N, and it is the Oda-Rallis-Schiffmann lift [12, 14] to $\mathrm{SO}(2,n)$ from f.

The second term is given by

$$(F_\phi)_+(g)=\sum_{\substack{\eta\in V'(\mathbb{Q})\\ q(\eta)>0}} c_\eta(\phi)w^{\mu^\varepsilon_{\mathbf{f}}}_\eta(g_{\mathbf{f}}\cdot\phi_{\mathbf{f}})\mathcal{W}_{2\pi\eta}(g_\infty),\ c_\eta(\phi)\in\mathbb{C},\ g=g_{\mathbf{f}}g_\infty\in G(\mathbb{A})$$

where $w^{\mu^\varepsilon_{\mathbf{f}}}_\eta(g_{\mathbf{f}}\cdot\phi_{\mathbf{f}}):=\prod_{p<\infty} w^{\mu^\varepsilon_p}_\eta(g_p\cdot\phi_p)$ for $g_{\mathbf{f}}=(g_p)_p$. We show that any index $0\neq\eta\in V'(\mathbb{Q})$ with $q(\eta)=0$ does not appear in the expansion of the second term $(F_\phi)_+$. We first determine $w^{\mu_p}_\eta(\phi_p)$ for an unramified vector ϕ_p by explicitly computing the Siegel series for Eisenstein series. We then compute $c_\eta(\phi_{\mathbf{f}})$ using Fourier–Jacobi expansions for an unramified vector $\phi_{\mathbf{f}}$.

We have our main results.

Theorem 4.1 (n even ($n>2$))**.** *Let $A_f(\eta)(g_{\mathbf{f}})=q(\eta)^{\frac{k-1}{2}}\prod_p \widetilde{Q}_{\eta,g_p\cdot\Phi_p}(\alpha_p)$ for $g_{\mathbf{f}}=(g_p)_p\in G(\mathbb{A}_{\mathbf{f}})$. Then, for $\phi\in\Pi_f$ with $\phi_{\mathbf{f}}$ is the unramified distinguished vector and $\phi_\infty=\phi^{\mathrm{Po}}_\infty$, there exists an* *absolute non-zero constant C such that*

$$F_\phi(g) = (F_\phi)_0 + C \sum_{\substack{\eta\in V'(\mathbb{Q})\\ q(\eta)>0}} A_f(\eta)(g_{\mathbf{f}})\mathcal{W}_{2\pi\eta}(g_\infty),\ g = g_{\mathbf{f}} g_\infty \in G(\mathbb{A})$$

and it is a non-zero Hecke eigen square integrable automorphic form of weight l with respect to $G(\widehat{\mathbb{Z}})$.

When n is odd, we need a slight modification. Let $l > n+1$ be an even integer. Let f be a normalized Hecke eigen cusp form of weight $k := 2l - n + 1$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. Let $f(z) = \sum_{m=1}^{\infty} a_f(m) m^{\frac{k-1}{2}} q^m$. Let $a_f(p) = \alpha_p + \alpha_p^{-1}$. Let $h(z) = \sum_{\substack{m>0\\ (-1)^{\frac{k}{2}} m \equiv 0,1 \ (\mathrm{mod}\ 4)}} c(m) q^m \in S_{l-\frac{n}{2}+1}(\Gamma_0(4))$ be the one corresponding to f by the Shimura correspondence.

Theorem 4.2 (n odd ($n \geq 3$))**.** *Let $A_f(\eta)(g_{\mathbf{f}}) = c(\mathfrak{d}_\eta)\mathfrak{f}_\eta^{\frac{k-1}{2}} \prod_{p|q(\eta)} \widetilde{Q}_{\eta, g_p\cdot\Phi_p}(\alpha_p)$ for $g_{\mathbf{f}} = (g_p)_p \in G(\mathbb{A}_{\mathbf{f}})$. Then, for $\phi \in \Pi_f$ with $\phi_{\mathbf{f}}$ is the unramified distinguished vector and $\phi_\infty = \phi_\infty^{\mathrm{Po}}$, there exists an absolute non-zero constant C such that*

$$F_\phi(g) = (F_\phi)_0 + C \sum_{\substack{\eta\in V'(\mathbb{Q})\\ q(\eta)>0}} A_f(\eta)(g_{\mathbf{f}})\mathcal{W}_{2\pi\eta}(g_\infty),\ g = g_{\mathbf{f}} g_\infty \in G(\mathbb{A})$$

and it is a non-zero Hecke eigen square integrable automorphic form of weight l with respect to $G(\widehat{\mathbb{Z}})$.

4.1. **L-functions of the theta lift.** Let $f = \sum_{m=1}^{\infty} a_f(m) m^{\frac{k-1}{2}} q^m$ be a Hecke eigen cusp form of weight $k = \begin{cases} l - \frac{n}{2} + 1 & \text{for } n \text{ even,} \\ 2l - n + 1 & \text{for } n \text{ odd} \end{cases}$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. Let $a_f(p) = \alpha_p + \alpha_p^{-1}$. Let $L(s, \pi_f) = \prod_p L(s, \pi_p)$, where $L(s, \pi_p) = (1 - \alpha_p p^{-s})^{-1}(1 - \alpha_p^{-1} p^{-s})^{-1}$.

4.1.1. *n even.* Let $n = 8a + 2$ ($n > 2$). Let $\Pi_f = \otimes' \Pi_{f,p} \otimes \Pi_{f,\infty}$ be the theta lift.

When p is unramified, $G(\mathbb{Q}_p) \simeq \mathrm{SO}(4a+3, 4a+3)$, and $\Pi_{f,p} = \mathrm{Ind}_P^G |\ |^{2s_p} \rtimes 1_{\mathrm{SO}}$, where $p^{s_p} = \alpha_p$ and 1_{SO} is the trivial representation of $\mathrm{SO}(4a+2, 4a+2)$, i.e., the quotient of $\mathrm{Ind}_B^{\mathrm{SO}(4a+2,4a+2)} |\ |^{4a+1} \otimes |\ |^{4a} \otimes \cdots \otimes |\ |^0$, where B is the Borel subgroup of $\mathrm{SO}(4a+2, 4a+2)$. Notice $2s_p$ since we replace $p^{\frac{l-\frac{n}{2}}{2}}$ by α_p. If n is odd, we replace $p^{l-\frac{n}{2}}$ by α_p.

Consider $\widetilde{G} = \mathrm{SO}(4a+4, 4a+4)$, and $R = M'N'$, $M' \simeq \mathrm{GL}_1 \times \mathrm{SO}(4a+3, 4a+3)$. Then Langlands-Shahidi theory applies to $(\widetilde{G}, M', \Pi_p)$, and we have

Theorem 4.3. *The standard L-function of Π_f is given by*

$$L(s,\Pi_f,\mathrm{St})=L(s,\mathrm{Sym}^2\pi_f)\zeta(s+\frac{n}{2})\zeta(s+\frac{n}{2}-1)\cdots\zeta(s+1)\zeta(s)\zeta(s-1)\cdots\zeta(s-\frac{n}{2}).$$

4.1.2. *n odd.* Let $n=8a+3$. When p is unramified, $G(\mathbb{Q}_p)\simeq \mathrm{SO}(4a+3,4a+4)$, and $\Pi_{f,p}=\mathrm{Ind}_P^G|\ |^{s_p}\rtimes 1_{\mathrm{SO}}$, where $p^{s_p}=\alpha_p$ and 1_{SO} is the trivial representation of $\mathrm{SO}(4a+2,4a+3)$, i.e., the quotient of $\mathrm{Ind}_B^{\mathrm{SO}(4a+2,4a+3)}|\ |^{4a+\frac{3}{2}}\otimes|\ |^{4a+\frac{1}{2}}\otimes\cdots\otimes|\ |^{\frac{1}{2}}$.

Consider $\widetilde{G}=\mathrm{SO}(4a+4,4a+5)$, and $R=M'N'$, $M'\simeq \mathrm{GL}_1\times\mathrm{SO}(4a+3,4a+4)$. Then Langlands-Shahidi theory applies to $(\widetilde{G},M',\Pi_p)$, and we have

Theorem 4.4. *The standard L-function of Π_f is given by*

$$L(s,\Pi_f,\mathrm{St})=L(s,\pi_f)\zeta(s+\frac{n}{2})\zeta(s+\frac{n}{2}-1)\cdots\zeta(s+\frac{1}{2})\zeta(s-\frac{1}{2})\cdots\zeta(s-\frac{n}{2}).$$

4.2. **Theta lift as residual spectrum.** We noted that Π_f is a non-zero square integrable automorphic representation of $G(\mathbb{A})$. Since it is not cuspidal, it should be in the residual spectrum. We make it more precise.

4.2.1. *n even.* Let $n=8a+2$. Let Π_f^O be the Oda-Rallis-Schiffmann lift of f to $G'=\mathrm{SO}(2,n)$. Let $\Pi_f^O=\Pi_\infty\otimes\otimes_p'\Pi_p$. Then by [10] or [1], the Satake parameter of Π_p is given by

$$diag(p^{4a},...,p,1,\alpha_p^2,\alpha_p^{-2},1,p^{-1},...,p^{-4a}).$$

Now let $G=\mathrm{SO}(3,n+1)$ and $P=MN$, $M\simeq GL_1\times\mathrm{SO}(2,n)$, and consider the global induced representation

$$I(s,\Pi_f)=Ind_P^G|\ |^s\otimes\Pi_f^O.$$

One can form the Eisenstein series $E(s,f_s)$ for a section $f_s\in I(s,\Pi_f)$. It has a simple pole at $s=\frac{n}{2}$, and the residue generates the residual automorphic representation

$$J(\frac{n}{2},\Pi_f)=J(\frac{n}{2},\Pi_\infty)\otimes\otimes_p'J(\frac{n}{2},\Pi_p).$$

Then $J(\frac{n}{2},\Pi_f)$ is the theta lift Π_f to $\mathrm{SO}(3,n+1)$. The Satake parameter of $J(\frac{n}{2},\Pi_p)$ is

$$diag(p^{4a+1},p^{4a},...,p,1,\alpha_p^2,\alpha_p^{-2},1,p^{-1},...,p^{-4a},p^{-4a-1}),$$

so that the L-function is

$$L(s,Sym^2\pi_f)\zeta(s+\frac{n}{2})\cdots\zeta(s+1)\zeta(s)\zeta(s-1)\cdots\zeta(s-\frac{n}{2}),$$

which agrees with the result in Section 4.1. We can also show that at the infinity place, $J(\frac{n}{2},\Pi_\infty)\simeq\Pi_{l,0}^{3,n+1}$.

4.2.2. *n odd.* Let n be odd, i.e., $n=8a+3$. Then the Satake parameter of Π_p is

$$diag(p^{4a+\frac{1}{2}},...,p^{\frac{1}{2}},\alpha_p,\alpha_p^{-1},...,p^{-4a-\frac{1}{2}}).$$

Now consider the global induced representation

$$I(s,\Pi_f)=Ind_P^G|\ |^s\otimes\Pi_f^O.$$

One can form the Eisenstein series $E(s,f_s)$ for a section $f_s\in I(s,\Pi_f)$. It has a pole at $s=\frac{n}{2}$, and the residue generates the residual automorphic representation

$$J(\frac{n}{2},\Pi_f)=J(\frac{n}{2},\Pi_\infty)\otimes\otimes_p' J(\frac{n}{2},\Pi_p).$$

Then $J(\frac{n}{2},\Pi_f)$ is the theta lift Π_f of h to SO$(3,n+1)$, where h is the weight $\frac{k+1}{2}$ holomorphic form corresponding to f under the Shimura correspondence. The Satake parameter of $J(\frac{n}{2},\Pi_p)$ is

$$diag(p^{4a+\frac{3}{2}},...,p^{\frac{1}{2}},\alpha_p,\alpha_p^{-1},p^{-\frac{1}{2}},...,p^{-4a-\frac{3}{2}}),$$

so that the L-function is

$$L(s,\pi_f)\zeta(s+\frac{n}{2})\cdots\zeta(s+\frac{1}{2})\zeta(s-\frac{1}{2})\cdots\zeta(s-\frac{n}{2}),$$

which agrees with the result in Section 4.1. We can also show the matching at the infinity place.

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