HERMITIAN MODULAR FORMS AND ALGEBRAIC MODULAR FORMS ON $SO(6)$ WITH A COMMENT ON THETA SERIES

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In the workshop, I reported a joint work with Brandon Williams on our recent work on a conjectural correspondence a la Langlands between hermitian modular forms of degree two and algebraic modular forms of $SO(6)$. Since the paper has been already published in [5], it seems there is no point to explain the result here in details. Since our claim is essentially on those forms on $SU(4)$, after a short report on the paper [5], we give a concrete isogeny between $SU(4)$ and $SO(6)$ (the existence is classically well known but there is almost no reference for a concrete description except for [13]. Here we added a natural description by using the Clifford algebra and the Clifford group.) Then we explain hermitian modular forms as theta series associated with pluri-harmonic polynomials on $SU(4)$ and ask a basis problem.

1. Introduction for hermitian modular forms and algebraic modular forms

We denote by $\mathcal{H}_n$ the Hermitian upper half space of degree n:

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbb{C}); X = \overline{X^t}, Y = \overline{Y^t} > 0\}$$

where $\overline{*}$ is the complex conjugation and $Y > 0$ means Y is a positive definite hermitian matrix. We write

$$SU(n,n) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2n}(\mathbb{C}); \det(M) = 1, (M^t)\, J\overline{M} = J \right\},$$

where $J = i\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. The group $SU(n,n)$ acts on $\mathcal{H}_n$ by $Z \to MZ := (aZ + b)(cZ + d)^{-1}$. For any imaginary quadratic field K over $\mathbb{Q}$, we denote by O_K the ring of integers of K and we put

$$\Gamma_{n,K} = M_n(O_K) \cap SU(n,n).$$

For $n \geq 2$, a holomorphic function $f(Z)$ on $\mathcal{H}_n$ is called a hermitian modular form of weight k, if $f(MZ) = \det(cZ + d)^k F(Z)$ for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n,K}$. If F vanishes at cusps of $\Gamma_{n,K}$, then F is called a hermitian cusp form. From here we assume that $n = 2$. We write $\Gamma_{2,K} = \Gamma_K$ and denote the space of cusp forms of degree two by $S_k(\Gamma_K)$.

Now we define a corresponding algebraic modular forms. The compact twist of $SU(2,2)$ is $SU(4) = Spin(6)$, and $SU(4)$ is isogenous to $SO(6)$, so we would like to consider algebraic modular forms of $SO(6)$ with respect to a genus of senary lattices with characters. For a positive definite senary quadratic space (V, Q) where Q is a quadratic form over $\mathbb{Q}$, we put

$$SO(V) = \{g \in SL(V); Q(gv) = Q(v) \text{ for all } v \in V\}.$$

Let $SO(V)_A$ is the adelization of $SO(V)$ and L be a lattice in V. We put

$$SO(L) = \{g = (g_v) \in SO(V)_A; L_p g_p = L_p \text{ for every prime } p\},$$

where we put $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For any finite dimensional representation (ρ, W) of $SO(V)_\infty$, we extend this to a representation ρ of $SO(V)_A$ by

$$\rho : SO(V)_A \to SO(V)_\infty \to GL(W).$$

We denote by $\mathcal{L}$ the genus of quadratic forms containing L, Then the space of algebraic modular forms of weight ρ is defined to be

$$\mathfrak{M}_\rho(\mathcal{L}) = \{f \in SO(V)_A \to W; f(uga) = \rho(u)f(g) \\ \text{for any } u \in U(L), g \in SO(V)_A, a \in SO(V)\}.$$

We take a double coset decomposition

$$SO(V)_A = \coprod_{i=1}^{h} U(L) g_i SO(V)$$

and put $\Gamma_i = g_i^{-1} U(L) g_i \cap SO(V)$. Then we see easily that

$$\mathfrak{M}_\rho(\mathcal{L}) \cong \oplus_{i=1}^{h} W^{\Gamma_i}$$

where W^{Γ_i} is the vector subspace of Γ_i-invariant elements of W. For any $\{\pm 1\}$-valued character χ of $SO(V)$, we write

$$\mathfrak{M}_\rho(\mathcal{L}, \chi) = \oplus_{i=1}^{h} W^{\Gamma_i, \chi},$$

where

$$W^{\Gamma_i, \chi} = \{w \in W; \rho(\gamma_i) w = \chi(\gamma_i) w \text{ for all } \gamma_i \in \Gamma_i\}.$$

Now we must define concrete ρ, L, and χ in out setting. We define ρ to be the so-called spherical representation of degree ν of $SO(6)$. This corresponds with the Young diagram (or dominant integral weight) $(\nu, 0, 0)$, and the representation space can be taken as harmonic polynomials of 6 variables of homogeneous degree ν. To define L or a genus containing L, we consider the associated discriminant form of L given by $(L'/L, q)$ where L' is the dual of L and q is a quadratic form on L'/L induced by Q. If we consider the case where $(L'/L, q) \cong (O_K'/O_K, -N_{K/Q})$ where O_K' is the dual of O_K (the inverse of the different of O_K) and $-N_{K/\mathbb{Q}}$ is the minus of the norm, then the genus of even positive definite latticee of rank 6 whose discrminant form is isometric to $(O'_K/O_L.-N_{K/Q})$ is uniquely determined ([9]). This genus is locally given by

$$L_p = U \oplus U \oplus O_{K_p}(-1),$$

where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $O_{K_p}(-1)$ is the rank 2 lattice O_{K_p} whose norm is given by $-N_{K/\mathbb{Q}}$. We want to consider the algebraic modular forms associated with $Spin(V)$ corresponding to the spinor norm one subgroup of $SO(V)$. Although the group of spinor norms of $SO(V)$ is the infinite group $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, the spinor norms of elements in $SO(L_i)$ form a finite group modulo $(\mathbb{Q}^\times)^2$. The discriminant Δ of L is a product $d_1 \cdots d_r$ of distinct prime discriminant d_i (i.e. discriminant whose prime factor is unique). For any product d_0 of the fundamental discriminant $d_0|\Delta$. we define ν_{d_0} as

$$\nu_{d_0}(p) = \begin{cases} -1 & \text{if } p|d_0 \\ 1 & \textit{otherwise} \end{cases}$$

For $a \in SO(V)$, we define $spin_{d_0}(a) = \nu_{d_0}(N(a))$ where $N(a)$ is the spinor norm of a. Then algebraic modular forms with respect to $Spin(V)$ is nothing but algebraic modular forms with respect to

$$U_0(L) = \{i = (u_p) \in U(L); spin_{d_0}(u_p) = 1 \text{ for all } d_0|\Delta\}.$$

We denote this space by $\mathfrak{M}_\nu(Spin(\mathcal{L}))$. Then we have

$$\mathfrak{M}_\nu(Spin(\mathcal{L})) \cong \oplus_{d_0|\Delta} \oplus_{i=1}^h W^{\Gamma_i, spin_{d_0}}.$$

2. Conjecture

For the usual Hecke operators, we can define Euler factors of degree 6 and associated L functions of hermitian forms as in [10]. Also for $\mathfrak{M}_\nu(Spin(\mathcal{L}))$, we can define Hecke operators isomorphic to those of $S_K(\Gamma_K)$ at least for good Euler factors, and we may define L functions whose Euler factors are of degree 6.

Conjecture 2.1. *For any integer $k \geq 4$, there should exist a bijective Hecke equivariant correspondence between the space $S_k^0(\Gamma_K)$ and $\mathfrak{M}_{k-4}^0(Spin(\mathcal{L}))$ where superscript 0 means the subspace orthogonal to the lifting parts explained below.*

Here the lifting part is as follows.
(0) For $\mathfrak{M}_0(Spin(\mathcal{L}))$, the constant functions are algebraic modular forms and these should be excluded.
(1) There exists an injective Maass-Sugano lift from Jacobi forms of weight $k = \nu+4$ (or equivalently certain ellliptic cusp forms) to $S_k(\Gamma_K)$ and experimentally the same kind of lift exists to $\mathfrak{M}_\nu(Spin(\mathcal{L}))$, so this part need not be excluded.
(2) There exists the Ikeda-Miyawaki type lift by [2] fo $S_k(\Gamma_K)$ from $S_k(SL_2(\mathbb{Z})) \times S_{k-2}(SL_2(\mathbb{Z}))$, though there is no corresponding lift to $\mathfrak{M}_\nu(Spin(\mathcal{L}))$ experimentally, so this part should be excluded.

(3) There exists a Yoshida-type lift to $\mathfrak{M}_k(Spin(\mathcal{L}))$, that is,
(i) There should exist a lift of a triple cusp forms f, g, h such that $(f,g) \in S_{k-1}(SL_2(\mathbb{Z})) \times S_{k-1}(SL_2(\mathbb{Z}))$ and CM eigenform of weight 3 of level $\Gamma_0(D_K)$ for the prime discriminant D_K, to $\mathfrak{M}_{k-4}(Spin(\mathcal{L}))$, where D_K is the discriminant of $K/\mathbb{Q}$. The L function should be $L(f\otimes g,s)L(h,s-\nu-1)$.
(ii) Decompose $D_K = d_1d_2$ as a fundamental discriminant $d_1 > 0$ and d_2. For a CM elliptic cusp form of weight 3 of $\Gamma_0(|d_2|)$ and a Hilbert modular form F of pararel weight $(\nu+3,\nu+3)$, there should exist a lift from (h,F) such that

$$L^{Asai}(F,s)L(h,s-\nu-1)$$

is the L functions. Here $L^{Asai}(F,s)$ is the Asai L function defined in [1]. We refrain from making a general conjecture as to which Yoshida lift $Y_{F,h}$ occurs in $\mathfrak{M}_\nu(Spin(\mathcal{L}))$.

Now we explain various evidences of the conjecture.
(1) We know explicit structures of graded rings of hermitian modular forms for the case $D_K = -3, -4, -7, -8, -11$. (This is due e.g. to H. Aoki, T. Dern. A. Krieg, T. Ibukiyama, S. Nagaoka, B. Williams.) On the other hand, if we know Γ_i explicitly, then by the Molien series, it is easy to calculate exact dimensions of $\mathfrak{M}_\nu(Spin(\mathcal{L}))$ for any ν. So we can compare the dimensions of (conjectural) non-lifting part exactly. These all match for these five discriminants.
(2) We know explicit graded rings of $M_k(\Gamma_K)$ for such discriminants, and also we can explicitly calculate corresponding algebraic modular forms for small weights. The Euler factors of small primes all match.
(3) The main term (leading term for k) of the dimension of hermitian modular forms is essentially the volume of the fundamental domain of Γ_k and this is known. Also the main term of $\dim \mathcal{M}_\nu(Spin(\mathcal{L}))$ can be calculated by density formulas of quadratic forms. Both main terms of $M_k(\Gamma_K)$ and $\mathfrak{M}_{k-4}(Spin(\mathcal{L}))$ match with each other.
(4) We can generalize conjectures to make the space a bit smaller, taking the maximal discrete extension $\widetilde{\Gamma}_K$ of Γ_K and $SO(L)$ instead. This case also, the dimensions for $D_K = -3$, -4, -7, -8 and -11 match, and besides, the main terms for these subspaces for general D_K also match.

3. An isogeny from $SU(4)$ to $SO(6)$

In the above, we used harmonic polynomials of 6 variables to describe algebraic modular forms of $SO(6)$ and described the difference between $SU(4) = Spin(6)$ and $SO(6)$ by using the spinor charcters. But it is desirable to describe algebraic modular forms directly on $SU(4)$. We try to do this. It is well known that $SU(4)/\{\pm 1_4\} \cong SO(6)$, but it seems that the reference to give an explicit isomorphism between them is rare. The only reference we know is [13]. The isomorphism in [13] and the proof is very explicit, but I have an impression that the theoretical background is not so clear. Here we review [13] and also give an interpretation by the even Clifford group. For a positive definite quadratic space W over $\mathbb{R}$ with $\dim W = 6$, it is well known that $\Gamma_0(W)/\mathbb{Q}^\times \cong SO(6)$, where $\Gamma_0(W)$ is the even Clifford group. Later we give the even Clifford algebra $C_0(W)$ explicitly and compare Yokota's result with the isomorphism from the even Clifford group.

First we start from Yokota's description. Define a vector space V over $\mathbb{R}$ by

$$V = \left\{ \begin{pmatrix} 0 & z & -y & x \\ -z & 0 & -\overline{x} & -\overline{y} \\ y & \overline{x} & 0 & -\overline{z} \\ -x & \overline{y} & \overline{z} & 0 \end{pmatrix} ; x, y, z \in \mathbb{C} \right\}.$$

We identify this space with a six dimensional vector space over $\mathbb{R}$ by regarding x, y, $z \in \mathbb{C}$ as elements in $\mathbb{R}^6$ by

$$(1) \quad \left(\frac{1}{2}(z+\overline{z}), \frac{1}{2i}(z-\overline{z}), \frac{1}{2}(x+\overline{x}), \frac{1}{2i}(x-\overline{x}), \frac{1}{2}(y+\overline{y}), \frac{1}{2i}(y-\overline{y}) \right).$$

Let A be an element of $SU(4) = \{A \in SL_4(\mathbb{C}); AA^* = 1_4\}$, where $A^* = \overline{A}^t$. For $v \in V$, we put

$$N(v) = \frac{1}{4} Tr(vv^*) = x\overline{x} + y\overline{y} + z\overline{z},$$

and by this, V can be regarded as six dimensional quadratic space over $\mathbb{R}$. We define a mapping $f(A)$ of V to $V \otimes_{\mathbb{R}} \mathbb{C}$ by

$$f(A)v = AvA^t.$$

At this stage, there is no apriori reason that the image of $f(A)$ is in V, but later we prove it. Since

$$N(f(A)v) = \frac{1}{4} Tr(Av(A^t)(A^{t*})v^*A^*) = \frac{1}{4} Tr(Avv^*A^*) = N(v),$$

so at least $f(A)$ preserves metric and $f(A)$ can be regarded as an element in $SO(6, \mathbb{C})$. To show that $f(A) \in SO(6)$, we must show that $AvA^t \in V$ and $\det(f(A)) = 1$. This is not trivial at all. To show this, in [13], he uses an action of the Lie algebra of $SU(4)$ on V. Here we show more directly. We identify $V \otimes \mathbb{C}$ with the vector space $\{(z, \overline{z}, x, \overline{x}, y, \overline{y})\}$ such that z, $\overline{z}$, x, $\overline{x}$, y, $\overline{y}$ are independente variables and write $f(A)v = (Z, \overline{Z}, X, \overline{X}, Y, \overline{Y})$. We write this as

$$(Z, \overline{Z}, X, \overline{X}, Y, \overline{Y}) = (z, \overline{z}, x, \overline{x}, y\overline{y})R(A).$$

We will show that $R(A)$ induces real transformation on the underlying real vector space V under the identification (1).

For 4×4 matrix $A = (a_{ij})_{1 \le i,j \le 4}$, let $K = (a, b)$ and $J = (c, d)$ and denote by A_{KJ} the K-J minor of A, that is, the minor consisting of

a-th and b-th rows and c-th and d-th columns. For example we put $A_{(1,2),(3,4)} = \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$. By a direct calculation, we see that

$$R(A) =$$
$$\begin{pmatrix} A_{(12),(12)} & -A_{(34),(12)} & A_{(14),(12)} & -A_{(23),(12)} & -A_{(13),(12)} & -A_{(24),(12)} \\ -A_{(12),(34)} & A_{(34),(34)} & -A_{(14).(34)} & A_{(23),(34)} & A_{(13),(34)} & A_{(24),(34)} \\ A_{(12),(14)} & -A_{(34),(14)} & A_{(14),(14)} & -A_{(23),(14)} & -A_{(13),(14)} & -A_{(24),(14)} \\ -A_{(12),(23)} & A_{(34),(23)} & -A_{(14),(23)} & A_{(23),(23)} & A_{(13),(23)} & A_{(24),(23)} \\ -A_{(12),(13)} & A_{(34),(13)} & -A_{(14),(13)} & A_{(23),(13)} & A_{(13),(13)} & A_{(24),(13)} \\ -A_{(12),(24)} & A_{(34),(24)} & -A_{(14),(24)} & A_{(23),(24)} & A_{(13),(24)} & A_{(24),(24)} \end{pmatrix}.$$

We can directly show that this induces a real transformation. We explain this by an example. We have

$$\begin{aligned} Z &= A_{(12),(12)}z - A_{(12),(34)}\overline{z} + \cdots \\ \overline{Z} &= -A_{(34),(12)}z + A_{(34),(34)}\overline{z} + \cdots \end{aligned}$$

So

$$\begin{aligned} Z + \overline{Z} &= (A_{(12),(12)} - A_{(34),(12)})z + (-A_{(12),(34)} + A_{(34),(34)})\overline{z} + \cdots . \\ &= (A_{(12),(12)} - A_{(34),(12)} - A_{(12),(34)} + A_{(34),(12)})(z + \overline{z})/2 \\ &\quad + (A_{(12),(12)} - A_{(34),(12)} + A_{(12),(34)} - A_{(34),(12)})(z - \overline{z})/2 + \cdots \end{aligned}$$

Here we must show that $A_{(12),(12)} - A_{(34),(12)} - A_{(12),(34)} + A_{(34),(12)}$ is real, for example. By definition, we have $\det(A) = 1$ and $AA^* = 1_4$, so $A^* = A^{-1} = \det(A)^{-1}\widetilde{A} = \widetilde{A}$, where $\widetilde{A}$ is the cofactor matrix. We write components of A by $(a_{ij})_{1\le i,j\le 4}$ and write

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad A^* = \begin{pmatrix} A_1^* & A_3^* \\ A_2^* & A_4^* \end{pmatrix}$$

by 2×2 block matrices A_i and A_i^*. Then by the definition of $\widetilde{A}$, we have $A_4^* = \begin{pmatrix} \tilde{a_{33}} & \tilde{a_{43}} \\ \tilde{a_{34}} & \tilde{a_{44}} \end{pmatrix}$, where $\tilde{a_{ij}}$ is the (i,j)-cofactor. So, by the so-called Jacobi's identity, we have

$$\overline{A_{(34),(34)}} = \overline{\det(A_4)} = \det(A_4^*) = \det(A)A_{(12),(12)} = A_{(12),(12)}.$$

In the same way, we have $\overline{A_{(12),(34)}} = A_{(34),(12)}$. For the identification (1), we have

$$\begin{aligned} Z + \overline{Z} &= (A_{(12),(12)} - A_{(34),(12)} - A_{(12),(34)} + A_{(34),(34)})\frac{(z+\overline{z})}{2} \\ &\quad + i(A_{(12),(12)} + A_{(12),(34)} - A_{(34),(12)} - A_{(34),(34)})\frac{(z-\overline{z})}{2i} \\ &\quad + \cdots \end{aligned}$$

For example, we have

$$\begin{aligned}&A_{(12),(12)} - A_{(34),(12)} - A_{(12),(34)} + A_{(34),(34)}\\&= 2Re(A_{(12),(12)}) - 2Re(A_{(12),(34)})\\&i(A_{(12),(12)} + A_{(12),(34)} - A_{(34),(12)} - A_{(34),(34)})\\&= -2Im(A_{(12),(12)}) - 2Im(A_{(12),(34)})\end{aligned}$$

so the coefficient of $(z+\overline{z})$ and $(z-\overline{z})/i$ is real. In the same way, we can show that all the components of the transformation matrix on V over $\mathbb{R}^6$ of $f(A)$ is real and $\det(f(A)) = 1$, so we can regard $f(A) \in SO(6)$ after a conjugation. More concretely, if we put $P = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. then we have

$(Re(Z), Im(Z), Re(X), Im(X), Re(Y), Im(Y)) =$
$(Re(z), Im(z), Re(x), Im(x), Re(y), Im(y))((P^{-1} \otimes 1_3)R(A)(P \otimes 1_3)).$

This is surely a proof but a bit clumsy. More theoretical proof can be given later, but before that, using this isomorphism, we compare the representation of $SU(4)$ corresponding to the Young diagram $(\nu, \nu, 0, 0)$ with that of $SO(6)$ corresponding to $(\nu, 0, 0)$. For any $n \times m$ matrix B, and for any $I \subset \{1, \ldots, n\}$ and $J \subset \{1, \ldots, m\}$ with $|I| = |J|$, we denote by B_{IJ} the (I, J) minor of B, that is, the determinant of submatrix of B consisting of $i \in I$ rows and $j \in J$ columns of B. By $U = (u_{ij}) \in M_{2,4}$, we denote 2×4 matrix of variables. For any $J \subset \{1, 2, 3, 4\}$ with $|J| = 2$, we write $U_{(12),J} = U_J$. For ν subsets J_1, …, J_ν, we consider a polynomial $p(U) = U_{J_1} \cdots U_{J_\nu}$ and consider the vector space V_ν spanned by all these polynomials. For $A \in GL_4(\mathbb{C})$, we define the left action on V_ν by

$$p(U) \to p(UA).$$

This is a representation of $GL_4(\mathbb{C})$ on V_ν corresponding to $(\nu, \nu, 0, 0)$. By seeing our calculation, we see that this representation is equivalent to the action of $SO(6)$ on V. (Note that $R(A)$ is a representation from the right and we should take the transpose.) For example, when $\nu = 1$, V_1 is spanned by $U_{(12)}$, $U_{(34)}$, U_{14}, U_{23}, U_{13}, U_{24} and we may put

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) = \big(&-U_{12} + U_{34}, i(U_{12} + U_{34}),\\&- U_{14} + U_{23}, i(U_{14} + U_{23}), U_{13} + U_{24}, (-U_{13} + U_{24})\big)/2\end{aligned}$$

to identify the representation space. The polynomials $U_{(ij)}$ have the Plücker relations. So we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = -U_{(12)}U_{34} - U_{14}U_{23} + U_{13}U_{24} = 0.$$

By the standard decomposition of polynomials of 6 variables modulo harmonic polynomials, this gives an isomorphism between polynomials generated by $U_{(ij)}$ and harmonic polynomials of 6 variables.

Now we come back to the Clifford algebra. We denote by $C(W)$ the Clifford algebra of a positive definite six dimentional quadratic space W over $\mathbb{R}$. We denote by $C_0(W)$ the even Clifford subalgebra of $C(W)$. We denote by $e_1, \ldots, e_6$ the orthogonal normal basis of W and we will give a concrete identification of $C_0(W)$ with $M_4(\mathbb{C})$ such that $W = \mathbb{R}e_1 + \cdots + \mathbb{R}e_6$ is somehow identified with the space V defined before, though W itself is not equal to V. We put $c = e_1e_2e_3e_4e_5e_6$. This is an element in the center of $C_0(W)$ (but not in the center of $C(W)$) since $e_ic = -ce_i$. We put $W_3 = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$. We can identify the even Clifford algebra $C_0(W_3)$ with

$$\mathbb{H} = \begin{pmatrix} x_1 + x_2 i & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - x_2 i \end{pmatrix}$$

for $x = x_1 + x_2e_1e_2 + x_3e_2e_3 + x_4e_1e_3$. Here the algebra $\mathbb{H}$ is the division quaternion algebra over $\mathbb{R}$. If we put $W_4 = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3 + \mathbb{R}e_4$, then the even Clifford algebra $C_0(W_4)$ is identified with $\mathbb{H} \oplus \mathbb{H}$. We write may this as

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} ; x, y \in H \right\} \subset M_4(\mathbb{C})$$

where $C_0(W_3)$ is embedded in $C_0(W_4)$ by $x \to \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. More concretely we may identify

$$e_1e_2e_3e_4 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}.$$

We have

$$e_3e_4 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad e_1e_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_2e_4 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

It is easily seen that these elements span $\mathbb{H} + \mathbb{H}$. Even Clifford algebra of $W_5 = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3 + \mathbb{R}e_4 + \mathbb{R}e_5$is isomorphic to $M_2(H) \subset M_4(\mathbb{C})$. We write this down (Similar realization is in [6].) We put

$$e_4e_5 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Then we have

$$e_1e_5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad e_2e_5 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$e_3e_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad e_1e_2e_4e_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$e_1e_3e_4e_5 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad e_2e_3e_4e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Next we obtain $C_0(W)$ just by adding $c = e_1e_2e_3e_4e_5e_6 = i \cdot 1_6$ to this. Missing generators are given by

$$e_1e_6 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad e_2e_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3e_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad e_4e_6 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$e_5e_6 = \begin{pmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{pmatrix} \quad e_1e_2e_3e_6 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$e_1e_2e_4e_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & - \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad e_1e_2e_5e_6 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_1e_3e_4e_6 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad e_1e_3e_5e_6 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$e_1e_4e_5e_6 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad e_2e_3e_4e_6 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$e_2e_3e_5e_6 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad e_2e_4e_5e_6 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_3e_4e_5e_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

These generate $C_0(W) \cong M_4(\mathbb{C})$. These basis consist of two kind of elements x such that $x \in iGL_4(\mathbb{Q})$ and in $x \in GL_4(\mathbb{Q})$. We put $j = e_1e_4e_5$ and $x = e_{i_1}e_{i_2}\cdots e_{i_{2n}}$. Then we have

$$e_kxe_k = \begin{cases} x & \text{if } k \text{ is contained in } \{i_1, \ldots, i_{2n}\}, \\ -x & \text{if } k \text{ is contained in } \{i_1, \ldots, i_{2n}\}. \end{cases}$$

So by seeing the generators of $C_0(W)$ given above, we see that $jxj^{-1}x = -x$ if $x \in iGL_4(\mathbb{Q})$ and $jxj^{-1}x = x$ if $x \in GL_4(\mathbb{Q})$. That is, the conjugation by j means complex conjugation of the components of $M_4(\mathbb{C})$. On the other hand, we have $(e_1e_2e_4e_5)j = e_2$, $(e_1e_3e_4e_5)j = e_3$, $(e_1e_4e_5e_6)j = e_6$, $(e_1e_4)j = -e_5$, $(e_4e_5)j = -e_1$, $(e_1e_5)j = e_4$. Here e_1e_4, e_4e_5, e_1e_5, $e_1e_1e_4e_5$, $e_1e_3e_4e_5$, $e_1e_4e_5e_6$ spans V defined before. So by multiplying j from the right, the vector space $W = \oplus_{i=1}^{6}\mathbb{R}e_i$ can be bijectively embedded in $V \subset M_4(\mathbb{C})$. Now it is well known that the even Clifford group in this case is given by

$$\Gamma_0(W) = \{g \in M_4(\mathbb{C})^\times; gg^* = n(g)1_6\}.$$

It is well known that the isomorphism of $Spin(6)$ to $SU(4)$ is induced by the mapping $W \ni w \to gwg^* = gwg^{-1} \in W$. Now the vector space V defined before is not a subspace of $C_0(V)$ itself but we have $V = Wj$. So for $w \in W$ and $A \in Spin(6)$, writing $wj = v$, we have

$$Aw\overline{A}^t = A(wj)(j^{-1}\overline{A}^t j) = AvA^t.$$

This is the action of $Spin(6)$ on W and also the action on V defined before. This induces the isomorphism $SU(4)/\pm 1 \cong Spin(6)/\{\pm 1_6\} \cong SO(6)$

4. Theta series with pluri-harmonic polynomials

Cohen and Resnikoff [4] proved that we can define hermitian modular forms by using a special kind of hermitian lattices. It is well known that we can construct modular forms of one variable or Siegel modular forms by attaching pluri-harmonic polynomials with quadratic lattices. In the same way, we can define theta series of hermitian lattices with pluri-harmonic polynomials to construct hermitian modular forms. (Here a general definition of pluri-harmonic polynomials is given in Clerc [3].)

For elliptic modular forms, by the result of Eichler, theta series of harmonic polynomials of four variables generates new elliptic modular forms. This means that there should exist an bijection between the span of theta series defined by so-called Brandt matrices and algebraic modular forms. This is much deeper than the fact that elliptic modular forms correspond abstractly with an irreducible subspace of harmonic polynomials of four variables.

On the other hand, in general tube domain (including Siegel modular cases of degree not less than 2), the theta series of this kind does not correspond directly with algebraic modular forms of the compact twist of $Sp(n, \mathbb{R})$. For example, when $n = 2$, the compact twist of $Sp(2, \mathbb{R})$ is the compact symplectic group $Sp(2)$ and the algebraic modular forms of weight 0 should correspond with Siegel modular forms of weight 3. But it seems there is no natural quadratic lattices whose theta series are of weight 3 matching to algebraic modular forms of weight 0.

In the hermian case of degree 2, we have a way to construct theta series from hermitian lattices of rank 4 (i.e. lattices corresponding to weight 0 algebraic modular forms without pluri-harmonic polynomials) of weight 4 as in [4]. So it is very natural to ask if there should exist a correspondence between algebraic modular form of weight ν and hermitian modular forms of degree 2 of weight $\nu + 4$.

First we explain hermitian lattices in [4], and then will give a construction of hermitian modular forms from pluri-harmonic polynomials in this case. Let H be a $d \times d$ positive definite hermitian matrix. Let K be an imaginary quadratic field with discriminant D.

Lemma 4.1 ([4])**.** *Assume that* $\det(H) = 2^d/D^{d/2}$ *and* $\{xHx^*; x \in O_K\} \subset 2\mathbb{Z}$*. Then there exists such* H *if and only if* $4|d$*.*

The above assumption means that the diagonal component of H belongs to $2\mathbb{Z}$ and the other components are in $(2/\sqrt{-D})O_K$.

The following claim is written in [4] for $d = 4$. By using [7], we can show the same claim for any d with $4|d$.

Lemma 4.2. *For any finite place* v*, the above hermitian matrix is isometric locally over* $GL_d(O_v)$ *to* $\perp_{i=1}^{d/2} H_0$ *where*

$$H_0 = \begin{pmatrix} 0 & -2/\sqrt{D} \\ 2/\sqrt{-D} & 0 \end{pmatrix}.$$

We can write down a lot of concrete hermitian lattices of this type.

Theorem 4.3 ([4], see also [12])**.** *For any* H *as above and* $Z \in \mathcal{H}$*, put*

$$\Theta_H(Z) = \sum_{N \in M_{n,d}(O_K)} e\left(\frac{1}{2}Tr(NHN^*Z)\right).$$

Then this is a hermitian modular forms of weight d *with respect to* $\Gamma_n(K)$

For polynomials such that $P(AW)=\det(A)^{\nu}P(W)$, a part of pluri-harmonic polynomials in the sense of [3], we may define

$$\Theta_{H,P}(Z)=\sum_{N\in M_{n,d}(O_K)}P(NS)e\left(\frac{1}{2}Tr(NHN^*Z)\right),$$

where S is the unique positive definite hermitian matrix such that $S^2=H$. This gives a hermitian modular forms of weight $d+\nu$. This is generalized to more general pluri-harmonic polynomials including the vector valued weight case. The details will be announced elsewhere.

We have the following interesting question.

Question 4.4. *Any hermitian modular form of degree* 2 *belonging to* Γ_K *is spanned by the above kind of theta series?*

As in the classical case of Eichler, this is something different from the correspondence between algebraic modular forms and hermitian modular forms obtained by the mere comparison of the trace formula.

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