Petersson norms of Jacobi-Eisenstein series and Gross-Kohnen-Zagier's formula

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We report our results on exact evaluation of regularized norms of Eisenstein series. This is in part a joint work with Shuichi Hayashida.

1 Introduction

Let $k \geq 4$ be an even integer and $\mathbb{H} = \{\tau = u + iv; v > 0\}$ the upper half-plane. We write M_k and S_k for the spaces of modular forms and cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, respectively.

Definition 1. *For $f, g \in S_k$, define*

$$\langle f, g \rangle_1 = \int_D f(\tau)\overline{g(\tau)} v^{k-2} du dv, \qquad D = \{\tau = u + iv \in \mathbb{H};\ |\tau| \geq 1,\ |u| \leq \tfrac{1}{2}\}.$$

This is called the Petersson inner product.

The Petersson inner product was introduced by Hans Petersson in 1939. An overview of its development and subsequent applications can be found in [4]. The Petersson inner product plays an indispensable role in the theory of modular forms. Among its many applications are the determination of generating systems of modular forms via Poincaré series, the existence of basis consisting of simultaneous eigenfunctions of Hecke operators, and trace formulas for Hecke operators through the construction of reproducing kernels.

1.1 L-functions and algebraicity

The Petersson norm also appears in other contexts. A typical example is the following. Let

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}$$

($q = e^{2\pi i \tau}$) denote Ramanujan's Δ function, and define the associated twisted symmetric-square L-function by

$$L_2(s, \Delta, \chi_5) = L(2s, \chi_5^2) \cdot \sum_{n=1}^{\infty} \chi_5(n) \tau(n^2) n^{-s-k+1} \qquad (\chi_5 = \left(\frac{5}{\cdot}\right) \in \{0, \pm 1\}).$$

For appropriate values of r, the special value $b(1,r) = \dfrac{L_2(r, \Delta, \chi_5)}{\langle \Delta, \Delta \rangle_1} \cdot \dfrac{5^2 \cdot 11!}{4^{12}\pi^{13}}$ at $s = r$ admits an explicit closed-form evaluation:

$$b(1,1) = \frac{2^{12} \cdot 3^6 \cdot 7}{5^8}, \quad b(1,3) = \frac{2^{12} \cdot 3^2 \cdot 7 \cdot 2851}{5^{12} \cdot 13}\pi^4, \quad b(1,5) = \frac{2^{15} \cdot 3 \cdot 1511599}{5^{17} \cdot 7 \cdot 13}\pi^8.$$

In this way, the special value is related to an algebraic quantity (indeed, a rational number in the present case) after division by the Petersson norm and a suitable power of π.

1.2 The Kohnen-Zagier Formula [9]

Let $k \geq 2$ be an integer. Let $f \in S_{2k}$ be a normalized Hecke eigenform and $g \in S^+_{k+\frac{1}{2}}$ a Hecke eigenform such that $T^+_{k+\frac{1}{2}}(p^2)g = a_f(p)g$ for all primes p. Here $S^+_{k+\frac{1}{2}}$ is the Kohnen space of half-integral weight cusp forms on $\Gamma_0(4)$ and $T^+_{k+\frac{1}{2}}(p^2)$ is the Hecke operator acting on it. Let D be any fundamental discriminant with $(-1)^k D > 0$. Then $\mathcal{S}^+_D(g) = c_g(|D|)f$ by the Shimura map $\mathcal{S}^+_D$ and one has

$$\frac{|c_g(|D|)|^2}{\frac{1}{6}\langle g, g\rangle_4} = \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(f, D, k)}{\langle f, f\rangle_1},$$

where $\chi_D = \left(\frac{D}{\cdot}\right)$ is the Kronecker symbol, $L(f, D, s) = \sum_{n=1}^{\infty}\chi_D(n)a_f(n)n^{-s}$ is the quadratic twist of the L-function associated with f, and $\langle g, g\rangle_4$ denotes the Petersson norm on $S^+_{k+\frac{1}{2}}$. For $g \in M^+_{k+\frac{1}{2}}$, the Shimura map is defined by

$$\mathcal{S}^+_D(g)(\tau) = \frac{L_D(1-k)}{2}c_g(0) + \sum_{n\geq 1}\left(\sum_{d|n}\chi_D(d)d^{k-1}c_g\left(\frac{n^2|D|}{d^2}\right)\right)e(n\tau) \in M_{2k},$$

where $L_D(s)$ is the Dirichlet L-function and $e(x) = e^{2\pi i x}$. Once again, the fundamental role of the Petersson norm is clearly visible in this formula.

2 Koblitz's Comment [7]

I had been interested in the following question: what should be regarded as the Eisenstein series analogue of the Kohnen-Zagier formula? In my talk, I reported some results obtained in connection with this problem.

The question originates from a remark in Koblitz's textbook. Let $k \geq 2$ be an integer. There are two Eisenstein series of interest: the classical Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and the Eisenstein series in the Kohnen space $M^+_{k+\frac{1}{2}}$ discovered by Cohen:

$$G_{2k} \in M_{2k}; \quad G_{2k}(\tau) = -\frac{B_{2k}}{4k}\cdot\frac{1}{2}\sum_{(c,d)=1}(c\tau+d)^{-2k},$$

$$G_{k+\frac{1}{2}} \in M^+_{k+\frac{1}{2}}; \quad G_{k+\frac{1}{2}}(\tau) = \zeta(1-2k) + \sum_{\substack{d\geq 1\\ (-1)^k d\equiv 0,1 \pmod 4}} L_{(-1)^k d}(1-k)e(d\tau),$$

where the Cohen function $L_D(s)$ is defined, for a complex variable s and any integer D, by

$$L_D(s) = \begin{cases} \zeta(2s-1), & \text{if } D = 0,\\ L(s, \chi_{d_K})\sum_{a|f}\mu(a)\chi_{d_K}(a)a^{-s}\sigma_{1-2s}(f/a), & \text{if } D \neq 0,\ D \equiv 0, 1 \pmod 4,\\ 0, & \text{if } D \equiv 2, 3 \pmod 4.\end{cases}$$

Here d_K is the discriminant of $K = \mathbb{Q}(\sqrt{D})$ or 1, $f \in \mathbb{N}$ is defined by $D = d_K f^2$, χ_{d_K} is the Kronecker symbol, μ is the Möbius function, $\zeta(s)$ is Riemann's zeta function, $L(s, \chi)$ is the Dirichlet L-function, $\sigma_s(n) = \sum_{d|n} d^s$. If $d_K = 1$ and $D = f^2$, then χ_{d_K} is the trivial character χ_1 and $L(s, \chi_1) = \zeta(s)$.

Under the same notation in §1.2, we have

$$\mathcal{S}^+_D(G_{k+\frac{1}{2}}) = L_D(1-k)\cdot G_{2k}.$$

Putting $g = G_{k+\frac{1}{2}}$, we obtain $c_g(|D|) = L_D(1-k)$ and $L(G_{2k}, D, s) = L_D(s)L_D(s+1-2k)$. Hence, up to a constant factor, we deduce $L(G_{2k}, D, k) = L_D(k)L_D(1-k) \risingdotseq L_D(1-k)^2 = c_g(|D|)^2$ by the functional equation. Thus, if one focuses only on the numerators appearing in the Kohnen-Zagier formula, one finds a striking analogy even in the Eisenstein series setting. Against this background, Koblitz [7, p. 218] makes the following remark.

The Kohnen-Zagier theorem does not include this case. One cannot even define the norms except for cusp forms. However, we may think of this result as a prototype for the theorem of Kohnen-Zagier.

The reason that the Petersson norm cannot be defined is that the integral $\int_D E_k(\tau)\overline{E_k(\tau)}v^{k-2}dudv$ defining the norm diverges. Therefore, some modification is necessary, and one must introduce a *regularized norm* $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$.

For Eisenstein series of integral weight, Zagier [13] gave both a definition and an explicit evaluation of the regularized norm. Consequently, if one could define and compute an analogous regularized norm for the half-integral weight Eisenstein series $G_{k+\frac{1}{2}}$ on $\Gamma_0(4)$, then an Eisenstein series analogue of the Kohnen-Zagier formula could be formulated.

To this end, let us first review Zagier's method.

3 Zagier's regularized norm [13]

According to Zagier, the regularized norm $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$ should be

$$\lim_{T\to\infty}\left(\int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^{k-2}\,dudv - \frac{T^{k-1}}{k-1}\right),$$

where $D = \{\tau = u+iv \in \mathbb{H}; |\tau| \geq 1, |u| \leq 1/2\}$, $D_T = \{u+iv \in D; v \leq T\}$ with a sufficiently large T. By $E_k(\tau)\overline{E_k(\tau)} = 1 + O(v^{-N})$ $(\forall N)$ as $v \to \infty$ and $\int_{T_0}^T v^{k-2}dv = \left[\frac{v^{k-1}}{k-1}\right]_{T_0}^T = \frac{T^{k-1}}{k-1} - \frac{T_0^{k-1}}{k-1}$, the correction term looks natural.

While this gives a satisfactory definition of the regularized norm, one would also like to determine its exact value. Both the existence of the limit and its explicit evaluation are consequences of Zagier's Rankin-Selberg method, which we now describe.

3.1 Zagier's Rankin-Selberg method and the computation of $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$

We now state Zagier's Rankin-Selberg method. For the general case where the integer k is not necessarily zero, we follow Tsuyumine [12] which treats a more general setting.

Theorem 1. *Let $k \geq 4$ be even. Suppose that $F : \mathbb{H} \to \mathbb{C}$ is continuous and satisfies*
(a) $F(\gamma\tau) = (c\tau+d)^k F(\tau)$ $(\tau \in \mathbb{H}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}))$,
(b) $F(\tau) = \psi(v) + O(v^{-N})$ $(\forall N)$ as $v \to \infty$, where

$$\psi(v) = \sum_{i=1}^{l} \frac{c_i}{n_i!} v^{\alpha_i} \log^{n_i} v \quad (l \in \mathbb{N}, c_i, \alpha_i \in \mathbb{C}, n_i \in \mathbb{N}\cup\{0\}).$$

For such an F, put

$$R(s, F) = \int_0^\infty \int_0^1 (F(\tau) - \psi(v))v^{s-2}dudv.$$

Then $R(s,F)$ converges absolutely for $\Re(s) \gg 0$, and

$$R(F,s) = \int_{D_T} F(\tau)E_{-k}(\tau,s)\ d\mu$$
$$+ \int_{D-D_T} (F(\tau)E_{-k}(\tau,s) - \psi(v)e(v,s))\ d\mu - h_T(s) + \phi(s)k_T(s).$$

Here $d\mu = dudv/v^2$, $D = \{\tau = u+iv \in \mathbb{H}; |u| \le 1/2, |\tau| \ge 1\}$, $D_T = \{u+iv \in D; v \le T\}$ and

$$h_T(s) = \int_0^T \psi(v)v^{s-2}\ dv, \qquad k_T(s) = \int_T^\infty \psi(v)v^{1-s+k-2}\ dv,$$

$$E_k(\tau,s) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{v^s}{(c\tau+d)^k|c\tau+d|^{2s}}, \quad e(v,s) = \int_0^1 E_{-k}(\tau,s)\ du = v^s + \phi(s)v^{1-s+k}.$$

Combined with explicit forms of $h_T(s), k_T(s)$ as rational functions of s and analytic properties of $E_k(\tau,s)$, the function $R(F,s)$ can be continued meromorphically to the whole s-plane and satisfies a functional equation.

To compute $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$, we let $k=0$ and take $F(\tau) = E_k(\tau)\overline{E_k(\tau)}v^k$ in Theorem 1. By

$$E_k(\tau) = \frac{1}{2}\sum_{(c,d)=1} \frac{1}{(c\tau+d)^k} = 1 - \frac{2k}{B_k}\sum_{n\ge1}\sigma_{k-1}(n)e(n\tau).$$

$$\psi(v) = v^k, \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}, \quad B_k : \text{the } k\text{-th Bernoulli number},$$

it follows from Theorem 1 that

$$\zeta^*(2s)R(s,F) = \int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k\zeta^*(2s)E_0(\tau,s)\frac{dudv}{v^2} - \zeta^*(2s)\frac{T^{s+k-1}}{s+k-1}$$
$$+ \int_{D-D_T} [E_k(\tau)\overline{E_k(\tau)}v^k\zeta^*(2s)E_0(\tau,s) - v^k\zeta^*(2s)e(v,s)]\frac{dudv}{v^2} - \zeta^*(2s-1)\frac{T^{k-s}}{k-s}$$

with $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Hence, we obtain the Rankin type formula of $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$ as

$$\begin{aligned}
&\mathrm{Res}_{s=1}\zeta^*(2s)R(s,F)\\
&= \frac{1}{2}\left(\int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k\frac{dudv}{v^2} + \int_{D-D_T}[E_k(\tau)\overline{E_k(\tau)}v^k - v^k]\frac{dudv}{v^2} - \frac{T^{k-1}}{k-1}\right)\\
&= \frac{1}{2}\lim_{T\to\infty}\left(\int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k\frac{dudv}{v^2} - \frac{T^{k-1}}{k-1}\right)\\
&= \frac{1}{2}\langle E_k, E_k\rangle_1^{\mathrm{reg}}.
\end{aligned}$$

In addition, by

$$\begin{aligned}
R(s,F) &= \frac{(2k)^2}{B_k^2}\sum_{n\ge1}\sigma_{k-1}(n)^2 n^{-s-k+1}\int_0^\infty e^{-4\pi v}v^{k+s-2}\ dv\\
&= \frac{(2k)^2}{B_k^2}\frac{\zeta(s)^2\zeta(s+k-1)\zeta(s-k+1)}{\zeta(2s)}(4\pi)^{-s-k+1}\Gamma(s+k-1),
\end{aligned}$$

one has

$$\langle E_k, E_k\rangle_1^{\mathrm{reg}} = \frac{1}{k-1}\frac{2\cdot k!}{B_k}\frac{\zeta(k-1)}{(4\pi)^{k-1}}.$$

4 Obstacle for half-integral weight case

Let $k \geq 2$ be an integer, D a fundamental discriminant with $(-1)^k D > 0$. We saw $L(G_{2k}, D, k) = L_D(k)L_D(1-k) \fallingdotseq L_D(1-k)^2 = c_{G_{k+\frac{1}{2}}}(|D|)^2$. The problem we want to consider is as follows: *does the following statement hold for some suitable and natural notion of $\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}}$?*

$$\frac{c_{G_{k+\frac{1}{2}}}(|D|)^2}{\frac{1}{6}\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}}} \overset{???}{=} \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}} \frac{L(G_{2k}, D, k)}{\langle G_{2k}, G_{2k} \rangle_1^{\mathrm{reg}}}.$$

In view of what we have seen so far, one possible approach is to ask whether Zagier's method can also be applied to the Eisenstein series $G_{k+\frac{1}{2}} \in M_{k+\frac{1}{2}}^+$ of half-integral weight. Zagier's method can be extended to congruence subgroups, such as $\Gamma_0(4)$. This makes it possible to define a regularized norm $\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}}$. Determining its exact value, however, is a different matter. If one wishes to follow Zagier's approach, it is necessary to obtain an explicit formula for the following Rankin zeta function in sufficient detail to permit analytic continuation and residue calculations at a suitable point α:

$$\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}} \rangle_4^{\mathrm{reg}} \fallingdotseq \mathrm{Res}_{s=\alpha} \left(\sum_{\substack{d \geq 1 \\ (-1)^k d \equiv 0,1 \pmod 4}} \frac{L_{(-1)^k d}(1-k)^2}{d^s} \right).$$

Since obtaining such an explicit formula seemed out of reach, my attempt remained at a standstill for a long time.

5 Tsuyumine's method [12]

Tsuyumine discovered the following method to compute the norm, which does not use any explicit formula for the Rankin zeta function. Recall the identity in Theorem 1: for $\Re(s) \gg 0$,

$$\begin{aligned} R(F, s) &= \int_0^\infty \int_0^1 (F(\tau) - \psi(v))v^{s-2} du dv \\ &= \int_{D_T} F(\tau)E_{-k}(\tau, s)\, d\mu - h_T(s) \\ &+ \int_{D-D_T} (F(\tau)E_{-k}(\tau, s) - \psi(v)e(v, s))\, d\mu + \phi(s)k_T(s). \end{aligned}$$

Let $F(\tau) = E_k(\tau) = \frac{1}{2}\sum_{(c,d)=1}(c\tau + d)^{-k}$ with $k \geq 4$ even. Since $\psi(v) = 1$, $\int_0^1 F(\tau)\, du = 1$ and $E_{-k}(\tau, s) = \overline{E_k(\tau, \overline{s} - k)}v^k$, one has

$$\begin{aligned} 0 &= R(F, s) \\ &= \int_{D_T} E_k(\tau)\overline{E_k(\tau, \overline{s} - k)}v^k\, d\mu - \frac{T^{s-1}}{s-1} \\ &+ \int_{D-D_T} (E_k(\tau)\overline{E_k(\tau, \overline{s} - k)}v^k - e(v, s))\, d\mu + \frac{\pi^{2s-k-1}\Gamma(1-s)\zeta(2-2s+k)}{\Gamma(s-k)\zeta(2s-k)} \frac{T^{k-s}}{s-k}. \end{aligned}$$

Replacing s by $s + k$, we have

$$\begin{aligned} 0 = R(F, s+k) &= \int_{D_T} E_k(\tau)\overline{E_k(\tau, \overline{s})}v^k\, d\mu - \frac{T^{s+k-1}}{s+k-1} \\ &+ \int_{D-D_T} (E_k(\tau)\overline{E_k(\tau, \overline{s})}v^k - e(v, s+k))\, d\mu + \frac{\pi^{2s+k-1}\Gamma(1-s-k)\zeta(2-2s-k)}{\Gamma(s)\zeta(2s+k)} \frac{T^{-s}}{s}. \end{aligned}$$

Letting $s = 0$, we get

$$0 = R(F,k) = \int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k \, d\mu - \frac{T^{k-1}}{k-1}$$
$$+ \int_{D-D_T} (E_k(\tau)\overline{E_k(\tau)}v^k - v^k) \, d\mu + \left.\frac{\pi^{k-1}\Gamma(1-s-k)\zeta(2-2s-k)}{\Gamma(s+1)\zeta(2s+k)}\right|_{s=0}$$

and

$$\int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k \, d\mu - \frac{T^{k-1}}{k-1}$$
$$= -\int_{D-D_T} (E_k(\tau)\overline{E_k(\tau)}v^k - v^k) \, d\mu - \left.\frac{\pi^{k-1}\Gamma(1-s-k)\zeta(2-2s-k)}{\Gamma(s+1)\zeta(2s+k)}\right|_{s=0}.$$

Letting $T \to \infty$, we obtain

$$\langle E_k, E_k\rangle_1^{\mathrm{reg}} = \lim_{T\to\infty}\left(\int\int_{D_T} E_k(\tau)\overline{E_k(\tau)}v^k \frac{dudv}{v^2} - \frac{T^{k-1}}{k-1}\right)$$
$$= -\left.\frac{\pi^{k-1}\Gamma(1-s-k)\zeta(2-2s-k)}{\Gamma(s+1)\zeta(2s+k)}\right|_{s=0} = \frac{1}{k-1}\frac{2\cdot k!}{B_k}\frac{\zeta(k-1)}{(4\pi)^{k-1}}.$$

This allows one to determine the value of the regularized norm without appealing to an explicit formula for the Rankin zeta function. This argument is remarkably elegant, and more importantly, this method can be extended to the setting of half-integral weight modular forms and Jacobi forms. We would like to mention that there is another approach based on the theory of harmonic Maass forms [10]. In both methods, the value of $\langle E_k, E_k\rangle_1^{\mathrm{reg}}$ arises from the constant term of the non-holomorphic Eisenstein series $e(v,s) = \int_0^1 E_{-k}(\tau,s)\, du = v^s + \phi(s)v^{1-s+k}$. The relevant quantity is the coefficient $\phi(s)$, which disappears when one restricts attention to holomorphic Eisenstein series. It is therefore quite remarkable that a quantity associated with a holomorphic object can only be detected by passing to its non-holomorphic counterpart. Furthermore, for various families of Eisenstein series, the constant terms of their non-holomorphic analogues exhibit a similar structure. This suggests that the phenomenon described above may hold in a much broader setting.

6 Kohnen-Zagier's formula for Eisenstein series [10]

Tsuyumine's method works for half-integral weight case. In fact, we can obtain the following result.

Theorem 2. *Let $k \geq 2$ be an integer, D a fundamental discriminant with $(-1)^k D > 0$. Then $\mathcal{S}_D^+(G_{k+\frac{1}{2}}) = L_D(1-k)\cdot G_{2k}$, $c_{G_{k+\frac{1}{2}}}(|D|) = L_D(1-k)$ and we have*

$$\frac{c_{G_{k+\frac{1}{2}}}(|D|)^2}{\frac{1}{6}\langle G_{k+\frac{1}{2}}, G_{k+\frac{1}{2}}\rangle_4^{\mathrm{reg}}} = \frac{(k-1)!}{\pi^k}|D|^{k-\frac{1}{2}}\frac{L(G_{2k},D,k)}{\langle G_{2k},G_{2k}\rangle_1^{\mathrm{reg}}}.$$

Here $L(G_{2k},D,s) = \sum_{n=1}^\infty \chi_D(n)\sigma_{2k-1}(n)n^{-s}$.

7 The case of Jacobi forms

What happens in the case of Jacobi forms? As motivation, let us mention the following two topics: (1) Gross-Kohnen-Zagier's formula, (2) a question posed by Böcherer and Das. A natural approach is to extend Zagier's method to Jacobi forms and then apply Tsuyumine's idea. The results presented below are joint work with Shuichi Hayashida.

Let $k, m \in \mathbb{N}$ and put

$$\mathbb{H}_T = \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \gamma D_T = \{\tau = u + iv \in \mathbb{H}; \max_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \Im(\gamma\tau) \le T\},$$

$$D = \{\tau = u + iv \in \mathbb{H}; |u| \le 1/2, |\tau| \ge 1\}, \qquad D_T = \{u + iv \in D; v \le T\}.$$

For Jacobi forms $\phi, \psi \in J_{k,m}$, consider the theta expansions

$$\phi(\tau, z) = v^{1/4-k/2} \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} f_\mu(\tau)\theta_{m,\mu}(\tau, z), \qquad \psi(\tau, z) = v^{1/4-k/2} \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} g_\mu(\tau)\theta_{m,\mu}(\tau, z).$$

Notice that

$$\int_{\mathrm{SL}_2(\mathbb{Z})^J \backslash (\mathbb{H}_T \times \mathbb{C})} \phi(\tau, z)\overline{\psi(\tau, z)} v^{k-3} e^{-4\pi m \frac{y^2}{v}}\, dudvdxdy = \frac{1}{4\sqrt{m}} \int_{D_T} \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} f_\mu(\tau)\overline{g_\mu(\tau)}\, \frac{dudv}{v^2}.$$

Definition 2. *We define $\langle \phi, \psi \rangle^{\mathrm{reg}}$ by*

$$\lim_{T \to \infty} \left(\int_{\mathrm{SL}_2(\mathbb{Z})^J \backslash (\mathbb{H}_T \times \mathbb{C})} \phi(\tau, z)\overline{\psi(\tau, z)} v^{k-3} e^{-4\pi m \frac{y^2}{v}}\, dudvdxdy - \frac{1}{4\sqrt{m}} \frac{T^{k-\frac{3}{2}}}{k - \frac{3}{2}} \sum_{\substack{\mu \in \mathbb{Z}/2m\mathbb{Z} \\ \mu^2 \equiv 0 \ (4m)}} a_\mu(0)\overline{b_\mu(0)} \right)$$

$$= \frac{1}{4\sqrt{m}} \lim_{T \to \infty} \left(\int_{D_T} \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} f_\mu(\tau)\overline{g_\mu(\tau)}\, \frac{dudv}{v^2} - \frac{T^{k-\frac{3}{2}}}{k - \frac{3}{2}} \sum_{\substack{\mu \in \mathbb{Z}/2m\mathbb{Z} \\ \mu^2 \equiv 0 \ (4m)}} a_\mu(0)\overline{b_\mu(0)} \right).$$

Here $a_\mu(0)v^{k/2-1/4}$ (resp. $b_\mu(0)v^{k/2-1/4}$) is the constant term of $f_\mu(\tau)$ (resp. $g_\mu(\tau)$).

This definition follows from Zagier's method. In fact, the integrand satisfies the assumption in Theorem 1.

Böcherer-Das pairing [2]. Let $k \ge 2$ be even, $m \in \mathbb{N}$. Böcherer-Das defined the paring $\{\phi, \psi\}$ for any $\phi, \psi \in J_{k,m}$. It has the form

$$\{\phi, \psi\} = \frac{1}{4\sqrt{m}} \int_D \sum_{\mu \in \mathbb{Z}/2m\mathbb{Z}} \mathbb{D}_{k-1/2}(f_\mu(\tau)\overline{g_\mu(\tau)})\, \frac{dudv}{v^2},$$

where f_μ, g_μ are theta coefficients and $\mathbb{D}_{k-1/2} = v^2(\partial_u^2 + \partial_v^2) - (k - 1/2)(k - 3/2)$.

They remak that "It seems to be a rather difficult question to determine whether the quantity $\{E_{k,1}, E_{k,1}\}$ is non-zero" for the Jacobi-Eisenstein series $E_{k,1} \in J_{k,1}$ in Eichler-Zagier [3]. By generalizing Tsuyumine's approach, we obtain the following answer.

Theorem 3. *For $\phi, \psi \in J_{k,m}$, we have $\{\phi, \psi\} = (3/2 - k)(k - 1/2) \cdot \langle \phi, \psi \rangle^{\mathrm{reg}}$. If $k \ge 4$ is even, then*

$$\langle E_{k,1}, E_{k,1} \rangle^{\mathrm{reg}} = -(-1)^{k/2} 2^{-k} \pi \frac{\zeta(2k-3)}{\zeta(2k-2)} \frac{\Gamma(k-3/2)}{\Gamma(k-1/2)}.$$

In the remaining of this report, we review the way to compute $\langle E_{k,1}, E_{k,1} \rangle^{\mathrm{reg}}$.

8 The way to computing $\langle E_{k,1}, E_{k,1}\rangle^{\rm reg}$

8.1 Notion of Jacobi forms

For $m \in \mathbb{N}$, $t \in \mathbb{Z}$, let

$$\theta_{m,t}(\tau, z) = \sum_{\lambda\in\mathbb{Z}} e^m \left(\tau\left(\lambda + t/(2m)\right)^2 + 2z\left(\lambda + t/(2m)\right)\right)$$

be the theta series, where $e^m(x) = e^{2\pi imx}$. Put

$$\Theta(\tau, z) = {}^t(\theta_{m,0}(\tau, z), \cdots , \theta_{m,r}(\tau, z), \cdots)_{r\in\mathbb{Z}/2m\mathbb{Z}}.$$

For $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, put

$$J(M, \tau) = c\tau + d, \quad M(\tau, z) = (M\tau, z/(c\tau + d)).$$

As is well-known, the theta formula

$$\Theta(M(\tau, z)) = e^m\left(cz^2/(c\tau + d)\right) J(M, \tau)^{1/2}U(M)\Theta(\tau, z)$$

holds, where $U(M)$ is a certain unitary matrix of size $2m$ and the branch of $z^\alpha = e^{\alpha\log z}$ for $z \in \mathbb{C}\setminus\{0\}$ is taken so that $\log z = \log|z| + i\arg z$ with $-\pi < \arg z \le \pi$.

A Jacobi form $\phi \in J_{k,m}$ has the theta expansion

$$\phi(\tau, z) = v^{1/4-k/2} \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} f_r(\tau)\theta_{m,r}(\tau, z).$$

The vector valued function

$$v^{1/4-k/2}F(\tau) = v^{1/4-k/2} \cdot {}^t(f_0(\tau), \cdots , f_r(\tau), \cdots)_{r\in\mathbb{Z}/2m\mathbb{Z}}$$

is holomorphic on $\mathbb{H}$ and the cusp $i\infty$, and satisfies

$$F(M\tau) = \overline{U(M)}\ j_M(\tau)F(\tau), \quad M \in \mathrm{SL}_2(\mathbb{Z}) \text{ with } j_M(\tau) = \left(\frac{J(M, \tau)}{|J(M, \tau)|}\right)^{k-1/2}.$$

8.2 Zagier's method for Jacobi forms

The next theorem is a Jacobi form analogue of Theorem 1.

Theorem 4. *Let $k \in \mathbb{Z}$, $m \in \mathbb{N}$. Suppose that $F(\tau) = {}^t(f_0(\tau), \cdots , f_r(\tau), \cdots)_{r\in\mathbb{Z}/2m\mathbb{Z}}$ is continuous on $\mathbb{H}$ satisfying that*

(a) $F(M\tau) = \overline{U(M)}j_M(\tau)F(\tau)$ ($\tau \in \mathbb{H}$, $M \in \mathrm{SL}_2(\mathbb{Z})$),

(b) $f_r(\tau) = \psi_r(v) + O(v^{-N})$ ($\forall N$) as $v = \Im(\tau) \to \infty$, where

$$\psi_r(v) = \sum_{j=1}^{l} \frac{c_{rj}}{n_{rj}!} v^{\alpha_{rj}} \log^{n_{rj}} v \quad (l \in \mathbb{N}, c_{rj}, \alpha_{rj} \in \mathbb{C}, n_{rj} \in \mathbb{N}\cup\{0\}).$$

For such a function F, put

$$\Psi(v) = {}^t(\psi_0(v), \cdots , \psi_r(v), \cdots)_{r\in\mathbb{Z}/2m\mathbb{Z}}.$$

For $r \in R^{\rm null}_{m,k}$ and a vector w_r defined below and for $\Re(s) \gg 0$, we define

$$R_r(F, s) = \int_0^\infty \int_0^1 {}^t(F(\tau) - \Psi(v))w_r v^{s-2}\, dudv.$$

Then $R_r(F,s)$ $(r \in R_{m,k}^{\mathrm{null}})$ *converges absolutely for* $\Re(s) \gg 0$, *and*

$$R_r(F,s) = \int_{D_T} {}^tF(\tau)\overline{E_r(\tau,\overline{s})}\, d\mu$$
$$+ \int_{D-D_T} ({}^tF(\tau)\overline{E_r(\tau,\overline{s})} - {}^t\Psi(v)\overline{e_r(v,\overline{s})})\, d\mu - h_{r,T}(s) + k_{r,T}(s).$$

Here $d\mu = dudv/v^2$, $E_r(\tau,s)$ *is the vector valued Eisenstein series defined below,* $e_r(v,s)$ *is the constant term of* $E_r(\tau,s)$,

$$h_{r,T}(s) = \int_0^T {}^t\Psi(v) w_r v^{s-2}\, dv, \quad k_{r,T}(s) = \int_T^\infty {}^t\Psi(v)\overline{\phi_r(\overline{s})} v^{1-s-2}\, dv.$$

Combined with explicit forms of $h_{r,T}(s), k_{r,T}(s)$ *as rational functions of* s *and analytic properties of* $E_r(\tau,s)$, *the function* $R_r(F,s)$ *can be continued meromorphically to the whole* s*-plane. Moreover, the vector* $\mathbf{R}(F,s) = (R_r(F,s))_{r\in R_{m,k}^{\mathrm{null}}}$ *satisfies the functional equation*

$$\mathbf{R}(F,1-s) = \mathbf{R}(F,s)\Phi(1-s).$$

After Arakawa [1], we explain more notation used in the above theorem. Put

$$e_r = {}^t(\delta_{r,j})_{j\in\mathbb{Z}/2m\mathbb{Z}} = {}^t(0,\cdots,0,1,0,\cdots,0) \qquad (r \in \mathbb{Z}/2m\mathbb{Z}),$$

$$R_m^{(2)} = \{r \in \mathbb{Z}/2m\mathbb{Z}; r \equiv -r \pmod{2m}\} = \{0 (\mathrm{mod}\ 2m),\ m (\mathrm{mod}\ 2m)\}.$$

Put $w_r = (e_r + (-1)^k e_{-r})/2$ if $r \in R_m^{(2)}$, and put $w_r = (e_r + (-1)^k e_{-r})/\sqrt{2}$ if $r \notin R_m^{(2)}$. Let $R_m^{\mathrm{null}} = \{r \in \mathbb{Z}/2m\mathbb{Z}; r^2 \equiv 0 \pmod{4m}\}$. For $r \in R_m^{\mathrm{null}}$ and $\Re(s) > 1$, define

$$E_r(\tau,s) = \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} j_M(\tau)^{-1}\Im(M\tau)^s\, \overline{U(M)}^{-1} w_r$$

$(\Gamma_\infty = \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right); n \in \mathbb{Z}\})$. Notice that $E_r(\tau,s) = (-1)^k E_{-r}(\tau,s)$ $(r \in R_m^{\mathrm{null}})$ and

$$E_r(M\tau,s) = \overline{U(M)} j_M(\tau) E_r(\tau,s) \quad (M \in \mathrm{SL}_2(\mathbb{Z})).$$

Take $R_m^\sim \subset \mathbb{Z}/2m\mathbb{Z}$ so that

$$\mathbb{Z}/2m\mathbb{Z} = R_m^{(2)} \cup R_m^\sim \cup (-R_m^\sim) \quad \text{(disjoint)},$$

where $-R_m^\sim = \{-r \in \mathbb{Z}/2m\mathbb{Z}; r \in R_m^\sim\}$. Put $R_{m,k} = R_m^\sim \cup R_m^{(2)}$ if k is even, and put $R_{m,k} = R_m^\sim$ if k is odd. Then $R_{m,k}^{\mathrm{null}}$ is defined by

$$R_{m,k}^{\mathrm{null}} = R_{m,k} \cap R_m^{\mathrm{null}}.$$

Let

$$e_r(v,s) = \int_0^1 E_r(\tau,s)\, du = \sum_{p\in R_{m,k}^{\mathrm{null}}} (\delta_{rp}v^s + v^{1-s}\varphi_{rp}(s)) w_p = v^s w_r + v^{1-s}\phi_r(s),$$

$$\Phi(s) = (\varphi_{rp}(s))_{r,p\in R_{m,k}^{\mathrm{null}}}, \quad \mathbf{E}(\tau,s) = (E_r(\tau,s))_{r\in R_{m,k}^{\mathrm{null}}}.$$

Arakawa proved that the matrices $\Phi(s)$, $\mathbf{E}(\tau,s)$ can be continued meromorhically to the whole s-plane and satify the functional equations

$$\mathbf{E}(\tau,1-s) = \mathbf{E}(\tau,s)\ {}^t\Phi(s), \quad \Phi(s)\Phi(1-s) = I_{\sharp(R_{m,k}^{\mathrm{null}})}, \quad \overline{{}^t\Phi(\overline{s})} = \Phi(s).$$

8.3 Real analytic Jacobi-Eisenstein series and its norm

Let $k \in \mathbb{Z}$, $m \in \mathbb{N}$, $\kappa = (1/2)(k - 1/2)$, $\Re(s) > 5/4$, $r \in R_m^{\mathrm{null}}$. Define

$$E_{k,m,r}^{\mathrm{A}}((\tau,z),s) = \sum_{M \in \Gamma_\infty^+ \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{q \in \mathbb{Z}} \Im(M\tau)^{s-\kappa} \frac{e^m\left(\left(q + \frac{r}{2m}\right)^2 M\tau + 2\left(q + \frac{r}{2m}\right)\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d}\right)}{J(M,\tau)^k}$$

with $\Gamma_\infty^+ = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right); n \in \mathbb{Z}\}$. Arakawa [1] proved a meromorphic continuation of $E_{k,m,r}^{\mathrm{A}}((\tau,z),s)$ to the whole s-plane. For an integer $k > 3$ and $t, r \in R_{m,k}^{\mathrm{null}}$, put

$$E_{k,m,t}^{\mathrm{A}}|_{s=\kappa} = E_{k,m,t}^{\mathrm{A}}((\tau,z),\kappa) \in J_{k,m}, \quad E_{k,m,r}^{\mathrm{A}}|_{s=\kappa} = E_{k,m,r}^{\mathrm{A}}((\tau,z),\kappa) \in J_{k,m}.$$

By Theorem 4 combined with Tsuyumine's idea, we obtain the following result.

Theorem 5. *Let $\epsilon_r = 2$ for $r \in R_m^{(2)}$ and $\epsilon_r = \sqrt{2}$ for $r \in \mathbb{Z}/2m\mathbb{Z} - R_m^{(2)}$. If $k > 3$, then*

$$\epsilon_t^{-1}\epsilon_r^{-1}\langle E_{k,m,t}^{\mathrm{A}}|_{s=\kappa}, E_{k,m,r}^{\mathrm{A}}|_{s=\kappa}\rangle^{\mathrm{reg}} = -\frac{1}{4\sqrt{m}} \left.\frac{\varphi_{rt}(s)}{s-\kappa}\right|_{s=\kappa}.$$

Therefore, the evaluation of the norm reduces to the computation of the quantities appearing on the right-hand side. If m is square-free, this computation was carried out in [1], from which the value of $\langle E_{k,1}, E_{k,1}\rangle^{\mathrm{reg}}$ in Theorem 3 follows. Let $m = ab^2$ $(a, b \in \mathbb{N}, \square \nmid a)$. Then $(1/2)E_{k,m,r}^{\mathrm{A}}((\tau,z),\kappa) = E_{k,m,r/(2ab)}^{\mathrm{EZ}}(\tau,z)$, where the right-hand side is the Jacobi-Eisenstein series defined in [3]. In particular, $E_{k,1} = (1/2)E_{k,1,0}^{\mathrm{A}}((\tau,z),\kappa) = E_{k,1,0}^{\mathrm{EZ}}(\tau,z)$.

8.4 Skoruppa-Zagier map and new spaces of Eisenstein series [11]

Let $k \geq 3$, $N \in \mathbb{N}$ and let $M_{2k}(N)^-$ be the subspace consisting of $f \in M_{2k}(N)$ satisfying that $f\left(-\frac{1}{N\tau}\right) = (-1)^{k+1}N^k\tau^{2k}f(\tau)$. For any $n_0, r_0 \in \mathbb{Z}$, assume that $D_0 = r_0^2 - 4Nn_0 < 0$ is a fundamental discriminant. Let $\phi \in J_{k+1,N}$ with

$$\phi(\tau,z) = \sum_{n,r \in \mathbb{Z},\ 4Nn - r^2 \geq 0} c(n,r)e(n\tau + rz).$$

The *Skoruppa-Zagier map* $\mathcal{S}_{k+1,N,(D_0,r_0)} : J_{k+1,N} \to M_{2k}(N)^-$ is then defined by

$$\mathcal{S}_{k+1,N,(D_0,r_0)}(\phi)(\tau) = \frac{L(1-k,\chi_{D_0})}{2}c(0,0) + \sum_{n \geq 1}\left(\sum_{d|n}\chi_{D_0}(d)d^{k-1}c\left(n_0\frac{n^2}{d^2}, r_0\frac{n}{d}\right)\right)e(n\tau).$$

For $N = F^2$ $(F \in \mathbb{N})$, the *new space of Jacobi-Eisenstein series* is defined by

$$\mathcal{E}_{k+1,F^2}^{J,\mathrm{new}} = \langle E_{k+1,F^2,1,\chi}^J;\ \chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times}, \mathrm{primitive}, \chi(-1) = (-1)^{k+1}\rangle_{\mathbb{C}},$$

$$E_{k+1,F^2,1,\chi}^J = \sum_{t \in \mathbb{Z}/F\mathbb{Z}} \chi(t)E_{k+1,F^2,t}^{\mathrm{EZ}} \in J_{k+1,F^2}, \qquad E_{k+1,F^2,t}^{\mathrm{EZ}}(\tau,z) = (1/2)E_{k+1,F^2,2Ft}^{\mathrm{A}}((\tau,z),\kappa).$$

The *new space of Eisenstein series* is defined by

$$\mathcal{E}_{2k}^{\mathrm{new},-} = \langle E_{2k}^{(\chi)};\ \chi \in \widehat{(\mathbb{Z}/F\mathbb{Z})^\times}, \mathrm{primitive}, \chi(-1) = (-1)^{k+1}\rangle_{\mathbb{C}},$$

$$E_{2k}^{(\chi)}(\tau) = \delta_{1,F}\frac{\zeta(1-2k)}{2} + \sum_{l \geq 1}\sigma_{2k-1}^{(\chi)}(l)e(l\tau) \in M_{2k}(F^2), \quad \sigma_{k-1}^{(\chi)}(l) = \sum_{d|l} d^{k-1}\overline{\chi(d)}\chi(l/d).$$

One has

$$\mathcal{S}_{k+1,F^2,(D_0,r_0)}\left(E^J_{k+1,F^2,1,\chi}\right)(\tau) = c_{k+1}(n_0,r_0)E^{(\chi)}_{2k}(\tau).$$

Here $c_{k+1}(n_0,r_0)$ is the (n_0,r_0)-th Fourier coefficients of $E^J_{k+1,F^2,1,\chi}$.

8.5 Gross-Kohnen-Zagier's formula for Eisenstein series

As an application, we have the following Gross-Kohnen-Zagier's formula for Eisenstein series.

Theorem 6. *Let $k \geq 3$ be an integer and F an odd square-free positive integer, $F^* = (-1)^{(F-1)/2}F$ the associated fundamental discriminant, and χ_{F^*} the Kronecker character associated with F^*. Assume that $\chi_{F^*}(-1) = (-1)^{k+1}$. Let n and r be integers such that $D = r^2 - 4F^2n < 0$ is a negative fundamental discriminant and $(D,F) = 1$. Then the following holds:*

$$\frac{c_{k+1}(n,r)^2}{\langle E^J_{k+1,F^2,1,\chi_{F^*}}, E^J_{k+1,F^2,1,\chi_{F^*}}\rangle^{\mathrm{reg}}} = \frac{(k-1)!}{2^{2k-1}\pi^k F^{2(k-1)}}|D|^{k-\frac{1}{2}}\frac{L(E^{(\chi_{F^*})}_{2k}, D, k)}{\langle E^{(\chi_{F^*})}_{2k}, E^{(\chi_{F^*})}_{2k}\rangle^{\mathrm{reg}}_{F^2}}.$$

Here $c_{k+1}(n,r)$ is the (n,r)-th Fourier coefficients of $E^J_{k+1,F^2,1,\chi_{F^*}}$, $\langle E^J_{k+1,F^2,1,\chi_{F^*}}, E^J_{k+1,F^2,1,\chi_{F^*}}\rangle^{\mathrm{reg}}$ and $\langle E^{(\chi_{F^*})}_{2k}, E^{(\chi_{F^*})}_{2k}\rangle^{\mathrm{reg}}_N$ are the regularized norms, $L(E^{(\chi_{F^*})}_{2k}, D, s) = \sum_{n=1}^{\infty}\chi_D(n)\sigma^{(\chi_{F^*})}_{2k-1}(n)n^{-s}$.

A general formula for the regularized norm was obtained in Theorem 5. Moreover, both the norm and the quantity $c_{k+1}(n,r)$ appearing on the left-hand side of Theorem 6 are directly related to Fourier coefficients of $E^{\mathrm{A}}_{k,F^2,2Ft}((\tau,z), s+\kappa)$. Therefore, in order to determine their exact values, it suffices to compute these Fourier coefficients. For this purpose, it is convenient to introduce the Jacobi-Eisenstein series twisted by a character defined by

$$E^{\chi}_{k,F^2}(\tau,z,s) = \frac{v^s}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\sum_{\lambda\in\mathbb{Z}}\chi(\lambda)\frac{e\left(\lambda^2\frac{a\tau+b}{c\tau+d} + 2F\lambda\frac{z}{c\tau+d} - F^2\frac{cz^2}{c\tau+d}\right)}{(c\tau+d)^k|c\tau+d|^{2s}}.$$

Here $(\tau,z) \in \mathbb{H}\times\mathbb{C}$, $2\Re(s)+k > 3$, χ is a Dirichlet character modulo F with $\chi(-1) = (-1)^k$, and integers a,b is taken so that $ad-bc = 1$ for each pair c,d.

Theorem 7. *Let F be an odd square-free positive integer, and let χ be a primitive Dirichlet character modulo F with $\chi(-1) = (-1)^k$. Then the Fourier coefficients of $E^{\chi}_{k,F^2}(\tau,z,s)$ can be described explicitly in closed form.*

For the precise statement, we refer the reader to [6]. In order to make use of the earlier computations in [8], we impose the assumptions on F and χ. A rough description of the Fourier coefficient formula for $E^{\chi}_{k,F^2}(\tau,z,s)$ can already be found in [3]. However, for the (n,r)-th Fourier coefficient, neither the proportionality constant nor the Euler p-factors for primes $p \mid F$ and $p \mid r$ are made explicit there. Theorem 7 fills this gap by determining these quantities explicitly.

The series $E^{\chi}_{k,F^2}(\tau,z,s)$ is better suited to the available techniques for computing Fourier coefficients. Since it is connected to $E^{\mathrm{A}}_{k,F^2,2Ft}((\tau,z),s+\kappa)$ by the following relation, the determination of the Fourier coefficients of the former yields that of the latter.

Lemma 1. *Let $k\in\mathbb{Z}$, $F\in\mathbb{N}$, $s\in\mathbb{C}$, $2\Re(s)+k>3$, $\kappa = (1/2)(k-1/2)$.*

(i) Let χ be a Dirichlet character modulo F with $\chi(-1) = (-1)^k$. Then

$$E^{\chi}_{k,F^2}(\tau,z,s) = \sum_{t\in\mathbb{Z}/F\mathbb{Z}}\chi(t)(1/2)E^{\mathrm{A}}_{k,F^2,2Ft}((\tau,z),s+\kappa).$$

(ii) Let t be an integer with $(F,t)=1$. Then

$$(1/2)E^{\mathrm{A}}_{k,F^2,2Ft}((\tau,z),s+\kappa)=\frac{1}{\varphi(F)}\sum_{\chi\in\widehat{(\mathbb{Z}/F\mathbb{Z})^\times}}\overline{\chi(t)}E^{\chi}_{k,F^2}(\tau,z,s),$$

where $\widehat{(\mathbb{Z}/F\mathbb{Z})}^\times$ is the character group and $\varphi(F)$ is the Euler function.

It is likely that Theorem 6 continues to hold for $k=2$; however, this case requires further investigation.

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