HYPERBOLIZATION OF AFFINE LIE ALGEBRAS

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1. Introduction

We report our collaborative work [41] with Kaiwen Sun and Brandon Williams on the classification of singular automorphic products and Borcherds–Kac–Moody algebras.

Affine Lie algebras are the simplest class of infinite dimensional Kac–Moody Lie algebras and have modularity through their representation theory [31]. In 1983, Feingold and Frenkel [17] extended the affine Lie algebra of type A_1 to a hyperbolic Kac–Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and expressed its characters in terms of Siegel modular forms of genus 2 and even weight.

In 1988, Borcherds [2] introduced generalized Kac–Moody algebras, which are now usually called Borcherds–Kac–Moody or simply BKM algebras. These Lie algebras are also defined in terms of Chevalley–Serre generators and relations that are encoded in a generalized Cartan matrix, and they differ from Kac–Moody algebras mainly by allowing the diagonal entries of the Cartan matrix to be non-positive. In other words, simple roots are allowed to be imaginary, whereas simple roots in a Kac–Moody algebra are always real. The best-known example of a BKM algebra is the monster Lie algebra. In 1992, Borcherds [5] constructed this algebra as the BRST cohomology related to the monster vertex algebra [3, 18] by means of the no-ghost theorem from string theory. By considering the action of the monster group on the denominator identity of the monster Lie algebra, Borcherds proved the celebrated monstrous moonshine conjecture. Furthermore, he observed that the denominator functions of some BKM algebras are modular forms on orthogonal groups of signature $(l, 2)$. In 1995 and 1998 Borcherds [6, 8] developed the theory of singular theta lift to construct modular forms for arithmetic subgroups of $\mathrm{O}(l, 2)$ that have infinite product expansions. These are called automorphic products, or Borcherds products, and they are natural candidates for the denominator functions of BKM algebras.

In 1996, Gritsenko and Nikulin [20] constructed an automorphic correction of the rank-three hyperbolic Lie algebra considered previously by Feingold and Frenkel. More precisely, they extended this hyperbolic Lie algebra to a (hyperbolic) BKM algebra by adding infinitely many imaginary simple roots, so that the denominator of the resulting BKM algebra is exactly the Igusa cusp form of weight 35 on $\mathrm{Sp}_2(\mathbb{Z})$. Later, they constructed automorphic corrections of other hyperbolic Lie algebras in a series of papers [23, 25, 24, 21, 26]. A common feature of these corrections is that the resulting BKM algbera only has finitely many real simple roots, a Weyl chamber of finite volume and a Weyl vector of negative norm. Moreover, its denominator usually defines a cuspidal automorphic product on $\mathrm{O}(l,2)$. These corrections extend affine Lie algebras in a nice way, but they have several features that are not preferred from our perspective:

(1) for the automorphic correction $\mathfrak{G}$ of an affine Lie algebra $\hat{\mathfrak{g}}$, the multiplicity of an imaginary root of $\hat{\mathfrak{g}}$ in $\mathfrak{G}$ is strictly greater than its multiplicity in $\hat{\mathfrak{g}}$;
(2) the set of imaginary simple roots is very complicated, although the set of real simple roots is easy to describe;
(3) it is not clear what the vertex algebra does, nor how to construct $\mathfrak{G}$ naturally beyond simply listing generators and relations.

In our work, we extend affine Lie algebras to (hyperbolic) BKM algebras in a different way. Certain affine Lie algebras $\hat{\mathfrak{g}}$ will be extended to BKM algebras $\mathcal{G}_{\mathfrak{g}}$ that have infinitely many real simple roots and satisfy:

(a) for any root α of $\hat{\mathfrak{g}}$, the root multiplicities of α in $\hat{\mathfrak{g}}$ and $\mathcal{G}_{\mathfrak{g}}$ are the same, which ensures that $\mathcal{G}_{\mathfrak{g}}$ can be viewed as a graded module over $\hat{\mathfrak{g}}$ in some sense;
(b) the imaginary simple roots of $\mathcal{G}_{\mathfrak{g}}$ are negative integral multiples of the Weyl vector;
(c) the Lie algebras $\hat{\mathfrak{g}}$ and $\mathcal{G}_{\mathfrak{g}}$ are closely related to some exceptional vertex algebras, and in many cases $\mathcal{G}_{\mathfrak{g}}$ have natural constructions with known symmetry groups inherited from vertex algebras, like the monster Lie algebra.

Our main results are about the classification and construction of such extensions, which are connected with various types of modular forms. More precisely, there are exactly 81 affine Lie algebras $\hat{\mathfrak{g}}$ that extend to hyperbolic BKM algebras in such a nice way. These extensions correspond to three special types of vertex algebra, and we call them hyperbolizations of affine Lie algebras.

2. Denominators of affine Lie algebras and Jacobi forms

We first review the modularity of affine Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank r. The product side of the Macdonald–Weyl denominator identity of the associated affine Lie algebra $\hat{\mathfrak{g}}$ is identical to the holomorphic function

$$\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z}) = \eta(\tau)^r \prod_{\alpha\in\Delta_{\mathfrak{g}}^+} \frac{\vartheta(\tau,\langle\alpha,\mathfrak{z}\rangle)}{\eta(\tau)},$$

where $\Delta_{\mathfrak{g}}^+$ denotes a set of positive roots, the Dedekind eta function η and the odd Jacobi theta function ϑ are defined as follows:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad \tau\in\mathbb{H},\ q=e^{2\pi i\tau},$$

$$\vartheta(\tau,z) = -q^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^{n-1}\zeta)(1-q^n\zeta^{-1})(1-q^n), \quad z\in\mathbb{C},\ \zeta=e^{2\pi iz}.$$

Note that $\vartheta_{\mathfrak{g}}$ is an example of lattice-index Jacobi forms (see, e.g., [12, 27]). Such Jacobi forms are generalizations of the classical Jacobi forms introduced by Eichler and Zagier [16]. Let L be an even positive definite lattice. A Jacobi form of integral weight k and index L is a holomorphic function $\varphi : \mathbb{H}\times(L\otimes\mathbb{C})\to\mathbb{C}$ that is modular under $\mathrm{SL}_2(\mathbb{Z})$ and doubly quasi-periodic, namely

$$\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{\mathfrak{z}}{c\tau+d}\right)=(c\tau+d)^k\exp\left(i\pi t\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right)\varphi(\tau,\mathfrak{z}),\quad A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}),$$
$$\varphi(\tau,\mathfrak{z}+x\tau+y)=\exp\big(-i\pi t\big((x,x)\tau+2(x,\mathfrak{z})\big)\big)\varphi(\tau,\mathfrak{z}),\quad x,y\in L,$$

and whose Fourier expansion satisfies a boundary condition. In this context, $\vartheta_{\mathfrak{g}}$ is a holomorphic Jacobi form of weight $r/2$ and index $P_{\mathfrak{g}}^\vee(h_{\mathfrak{g}}^\vee)$ for some character, where $P_{\mathfrak{g}}^\vee$ is the dual of the root lattice and $h_{\mathfrak{g}}^\vee$ is the dual Coxeter number. Jacobi forms defined by a product expression similar to $\vartheta_{\mathfrak{g}}$ are called theta blocks following Gritsenko–Skoruppa–Zagier [27]. Characters of integrable highest weight representations of $\hat{\mathfrak{g}}$ can be calculated by the Weyl–Kac character formula and define certain vector-valued Jacobi forms of weight 0.

3. Singular automorphic products

We review orthogonal modular forms and the Borcherds theta lift. Let $l\geq 3$ be an integer. A modular form of weight $k\in\mathbb{Z}$ and character χ for an arithmetic subgroup $\Gamma<\mathrm{O}(l,2)$ is a holomorphic function on the associated type IV symmetric domain that satisfies

$$F(t\mathcal{Z})=t^{-k}F(\mathcal{Z}),\quad t\in\mathbb{C}^\times,$$
$$F(g\mathcal{Z})=\chi(g)F(\mathcal{Z}),\qquad g\in\Gamma.$$

Let M be an even lattice of signature $(l,2)$. The input into Borcherds' theta lift is a vector-valued modular form of weight $1-l/2$ with integral Fourier expansion for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ attached to the discriminant form M'/M, and the output is a meromorphic modular form for a certain subgroup of $\mathrm{O}(M)$ which has an infinite product expansion at any 0-dimensional cusp and whose divisors are linear combinations of hyperplanes.

Inspired by modularity of affine Lie algebras, it is natural to focus on hyperbolic BKM algebras whose denominators are modular. Let $\mathcal{G}$ be a BKM algebra whose denominator function coincides with the Fourier expansion of an $\mathrm{O}(l,2)$-modular form F at some 0-dimensional cusp. Since F has an infinite product expansion, by Bruinier's converse theorem [10, 11] one expects that it can be constructed by the Borcherds lift. In this case, the roots and their multiplicities are encoded in the Fourier expansion of the input. When F has singular weight, i.e., weight $l/2-1$, the Fourier expansion is supported only on isotropic vectors, often forcing the imaginary simple roots of $\mathcal{G}$ to be negative integral multiples of the Weyl vector. Moreover, in this particular case it is conjectured that $\mathcal{G}$ can be constructed as the BRST cohomology related to some vertex algebra, similarly to the monster Lie algebra. This suggests focusing on BKM algebras whose denominators are automorphic products of singular weight. The zeros of F that contain the cusp are actually hyperplanes orthogonal to the real roots of $\mathcal{G}$, hence F is anti-invariant under the reflections through these hyperplanes. It is natural to expect that F is anti-invariant under all reflections associated with zeros of F. This has been proven by [47], which yields the result that F is a reflective modular form.

A non-constant modular form on $\Gamma<\mathrm{O}(M)$ is called reflective if it vanishes only on mirrors of reflections fixing the lattice M. Reflective modular forms were introduced in 1998 by Borcherds [6, 8] and Gritsenko–Nikulin [24], and their classification has been an active project for the past thirty years (see [21, 38, 39, 32, 33, 46, 44]), because they have nice applications to hyperbolic reflection groups [9, 25], algebraic geometry [7, 22, 33] and free algebras of modular forms [43] in addition to infinite-dimensional Lie algebras.

4. Main results

BKM algebras whose denominator functions are reflective automorphic products of singular weight are exceptional. The main examples are the fake monster algebra [4] and their twists by the Conway group Co_0 [5, 37, 48]. Conjecturally, there are only finitely many such algebras. It is an open problem to classify and construct them. Many partial results have been achieved [5, 36, 1, 37, 38, 14, 48]. We provide some new results in this direction.

Let U be an even unimodular lattice of signature $(1,1)$ and let L be an even positive-definite lattice. The input of the Borcherds lift on lattices of type $2U \oplus L$ can be identified with certain Jacobi forms of weight 0 and index L. We will identify affine Lie algebras, which naturally extends to BKM algebras or superalgebras whose denominators or super-denominators are reflective Borcherds products of singular weight on lattices of type $2U \oplus L$.

Theorem 4.1. *Let F be a reflective Borcherds product of singular weight on $2U \oplus L$ whose Jacobi form input has Fourier expansion*

$$\phi(\tau, \mathfrak{z}) = \sum_{n\in\mathbb{Z}} \sum_{\ell\in L'} f(n,\ell) q^n \zeta^\ell$$

satisfying $f(0,\ell) \geq 0$ for all $\ell \in L'$. If L is the Leech lattice, then F is the denominator of the fake monster algebra and ϕ is the full character of the Leech lattice vertex operator algebra. Otherwise, the set

$$\mathcal{R} = \{\ell \in L' : \ \ell \neq 0, \ f(0,\ell) \neq 0\}$$

determines a finite-dimensional semi-simple Lie algebra

$$\mathfrak{g} = \bigoplus_{j=1}^{s} \mathfrak{g}_{j,k_j}$$

with the same rank as L such that the identity

$$C := \frac{\dim \mathfrak{g}}{24} - a = \frac{h_j^\vee}{k_j} \tag{4.1}$$

holds for any $1 \leq j \leq s$ and such that the leading Fourier–Jacobi coefficient of F at the 1-dimensional cusp determined by $2U$ coincides with the denominator of the associated affine Lie algebra $\hat{\mathfrak{g}}$. For any $1 \leq j \leq s$, $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$, k_j is a positive integer that indicates the level of $\mathfrak{g}_j$ and $h_j^\vee$ is the dual Coxeter number of $\mathfrak{g}_j$. The number a equals $f(-1,0)$, which must be 0 or 1. When $a = 0$, $k_j > 1$ for any $1 \leq j \leq s$. The cases $a = 0$ and $a = 1$ are called symmetric and anti-symmetric, respectively.

By [47], reflective Borcherds products of singular weight have only simple zeros and are anti-invariant under reflections associated with their zeros. Therefore, the nonzero coefficients $f(0,\ell)$ of the Jacobi form input must be 1 if $\ell \neq 0$, and $f(0,0)$ equals the rank of $\mathfrak{g}$ because the Borcherds product is of singular weight. The above theorem then follows by extending our previous argument to classify reflective modular forms [46, 43, 45].

If the Fourier expansion of F defines the (super)-denominator of a BKM (super)-algebra $\mathcal{G}$, then the (super)-denominator has the form

$$e^\rho \prod_{\alpha>0} \left(1 - e^{-\alpha}\right)^{f(nm,\ell)},$$

where ρ is the Weyl vector of F, and where (n, ℓ, m) are the coordinates of the positive roots $\alpha \in U \oplus L'$ with $n \in \mathbb{Z}$, $m \in \mathbb{N}$, $\ell \in L'$ and $\alpha^2 = \ell^2 - 2nm$. The above $\hat{\mathfrak{g}}$ is embedded into $\mathcal{G}$ as the sum of the root spaces associated with roots of type $\pm(n, \ell, 0)$. In this way, $\mathcal{G}$ can be regarded as a graded module over $\hat{\mathfrak{g}}$ graded by $m \in \mathbb{N}$. This leads us to define $\mathcal{G}$ as a *hyperbolization* of $\hat{\mathfrak{g}}$ and F as a *hyperbolization* of the denominator of $\hat{\mathfrak{g}}$.

In the antisymmetric case, Equation (4.1) has 221 solutions and we can exclude 152 of them. In the symmetric case, Equation (4.1) has 17 solutions and we can rule out 5 of them. As a result, there are only 81 affine Lie algebras with hyperbolization. Theorem 4.1 also shows that the central charge of the affine vertex operator algebra generated by $\hat{\mathfrak{g}}$ is

$$c_{\mathfrak{g}} = 24(C + a)/(C + 1).$$

In particular, $c_{\mathfrak{g}} = 24$ if $a = 1$, and $c_{\mathfrak{g}} = 12$ if $a = 0$ and $C = 1$. This motivates the three groupings in the following theorem.

Theorem 4.2. *There are* 81 *possibilities for the semi-simple Lie algebra* $\mathfrak{g}$ *in Theorem 4.1 and they fall into three categories:*

(1) 69 *make up Schellekens' list of semi-simple* V_1 *structures of holomorphic vertex operator algebras of central charge* 24*;*
(2) 8 *correspond to the* $\mathcal{N} = 1$ *structures of holomorphic vertex operator superalgebras* F_{24} *of central charge* 12 *composed of* 24 *fermions;*
(3) The remaining 4 *cases* $A_{1,16}$*,* $A_{1,8}^2$*,* $A_{1,4}^4$ *and* $A_{2,9}$ *possess an exceptional modular invariant that comes from a nontrivial automorphism of the fusion algebra.*

Case (1) is anti-symmetric, while Cases (2) and (3) are symmetric. Schellekens' list [40] was established using the solutions of Equation (4.1) with $a = 1$. Holomorphic vertex operator superalgebras of central charge 12 were classified by Creutzig, Duncan and Riedler [13], and the $\mathcal{N} = 1$ structures of F_{24} were determined in [28], corresponding to solutions of Equation (4.1) with $a = 0$ and $C = 1$. The exceptional modular invariants mentioned in (3) were discovered around the 1990s by Moore and Seiberg [35], Verstegen [42] and Gannon [19]. The 4 exotic cases satisfy Equation (4.1) with $a = 0$ and $C < 1$.

We now present hyperbolizations of these affine Lie algebras.

Theorem 4.3. *For any* $\mathfrak{g}$ *in Theorem 4.2 there exists a singular-weight reflective Borcherds product* $\Psi_{\mathfrak{g}}$ *on some lattice* $2U \oplus L_{\mathfrak{g}}$ *whose leading Fourier–Jacobi coefficient equals the denominator of* $\hat{\mathfrak{g}}$*. Moreover, the Jacobi form input* $\phi_{\mathfrak{g}}$ *can be expressed as a* $\mathbb{Z}$*-linear combination of full characters of the affine vertex operator algebra generated by* $\hat{\mathfrak{g}}$*.*

Note that these $L_{\mathfrak{g}}$ are chosen so that the root lattice of the resulting BKM superalgebra is $U \oplus L'_{\mathfrak{g}}$. We summarize the construction of hyperbolizations. If $\mathfrak{g}$ is in Schellekens' list, then $L_{\mathfrak{g}}$ is the orbit lattice in Höhn's construction [30] of the holomorphic VOA of central charge 24 with $V_1 = \mathfrak{g}$, and the Jacobi form input $\phi_{\mathfrak{g}}$ is the full character of the VOA. If $\mathfrak{g}$ is of symmetric type, $L_{\mathfrak{g}}$ is the maximal even sublattice of the coweight lattice of $\mathfrak{g}$. If $\mathfrak{g}$ determines an $\mathcal{N} = 1$ structure of F_{24}, then the Jacobi form input can be expressed in terms of the characters of F_{24} as

$$\phi_{\mathfrak{g}} = (\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}})/2.$$

Finally, we will explain the relation between Jacobi form inputs and exceptional modular invariants for the remaining four $\mathfrak{g}$. The Jacobi form input for $\mathfrak{g} = A_{1,16}$ can be written in terms of affine characters as

$$\phi_{A_{1,16}} = \chi^{A_{1,16}}_{2,\frac{1}{9}} + \chi^{A_{1,16}}_{14,\frac{28}{9}} - \chi^{A_{1,16}}_{8,\frac{10}{9}}.$$

The difference between the simple current modular invariant and the exceptional modular invariant [35] is given by $|\phi_{A_{1,16}}|^2$. Similar relations hold for the other three $\mathfrak{g}$.

In conclusion, affine Lie algebras, vertex algebras, and BKM (super)-algebras are therefore closely connected from the point of view of the attached modular forms. The connections are illustrated in Figure 1.

$$\begin{tikzcd}
\text{Affine Lie algebra} \arrow[r] \arrow[d, "\text{Denominator}"] \arrow[dr, "\text{Character}"] \arrow[rr, bend left, "\text{Hyperbolization}"] & \text{Vertex algebra} \arrow[r, "\text{BRST}"] \arrow[d, "\text{Character}"] & \text{BKM superalgebra} \arrow[d, "\text{Denominator}"] \\
\text{Theta block} \arrow[r] & \text{Weight 0 Jacobi form} \arrow[r, "\text{Lift}"] & \text{Borcherds product} \arrow[ll, bend left, "\text{Leading FJ coefficient}"]
\end{tikzcd}$$

FIGURE 1. Hyperbolization of affine Lie algebras

In the end, we formulate several remarks related to the main theorems.

(1) The monster Lie algebra can be regarded as a hyperbolization of the trivial Lie algebra, and the associated vertex algebra is the monster vertex operator algebra. This corresponds to the degenerate case $L = 0$ of Theorem 4.1. The fake monster Lie algebra is a hyperbolization of the abelian Lie algebra of dimension 24, and the associated vertex algebra is the Leech lattice vertex operator algebra. This corresponds to the special case of Theorem 4.1 where L is equal to the Leech lattice. These two cases are in some sense complementary to Theorem 4.2.
(2) Our hyperbolizations correspond to certain exceptional vertex algebras, and thus the BKM algebras are expected to have natural constructions through the BRST cohomology. This type of realization has been achieved in [4, 29, 34, 15, 28] under some technical assumptions for $\mathfrak{g}$ from Schellekens' list and for the $\mathcal{N} = 1$ structure of F_{24}. However, such a realization is completely open for $\mathfrak{g}$ related to the four exceptional modular invariants.
(3) Our 77 hyperbolizations of the first two types are also closely related to the twists of the fake monster algebra [5] with respect to the action of 19 certain conjugacy classes of the Conway group Co_0. We refer to [41, Remark 6.5, Theorem 7.5] for details.

(4) If the q^0-term of ϕ in Theorem 4.1 has negative Fourier coefficients, then $\mathcal{R}$ determines a semi-simple Lie superalgebra and the corresponding BKM superalgebra will have odd real roots. We will consider hyperbolizations of affine Lie superalgebras in a separate paper. In addition, it is worth extending these theorems to reflective Borcherds products of singular weight on general lattices of type $U(N) \oplus U \oplus L$. This will cover some interesting BKM superalgebras, including the fake monster Lie superalgebra that corresponds to the holomorphic SCFT V^{fE_8} of central charge 12.

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