ISOMORPHISM BETWEEN JACOBI FORMS OF INDEX D_{2n+1} AND ELLIPTIC MODULAR FORMS OF LEVEL 2

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1. Introduction

Jacobi forms appear in the study of modular forms of several variables like modular forms on symplectic groups and orthogonal groups. In this article we consider Jacobi forms as holomorphic functions on $\mathfrak{H} \times \mathbb{C}^n$, where $\mathfrak{H}$ denotes the Poincaré upper half-plane.

The main purpose of this article is to explain an isomorphism between holomorphic Jacobi forms of lattice index D_{2n+1} and elliptic modular forms of level 2 as Hecke modules (Thm. 3.1. Cusp forms case is explained in Thm. 4.2). This isomorphism was conjectured by Mocanu [9] and is a kind of generalization of the isomorphism between Jacobi forms of integer index N and elliptic modular forms of level N. In particular, in the proof of the main theorem we show that Jacobi forms of lattice index D_{2n+1} correspond to certain modular forms of half-integral weight of level 8 and certain powers of the Dedekind eta-function (Thm. 3.5). Finally, by virtue of the Shimura correspondences which were given by Ueda-Yamana [12] and Yang [14], we obtain the main theorem. In sections 4 to 7 we give some applications of the main theorem.

2. Jacobi forms of lattice index

Let L be a free $\mathbb{Z}$-module of finite rank n, equipped with a $\mathbb{Z}$-valued symmetric bilinear form β. By abuse of language, we put $\beta(x) := \frac{1}{2}\beta(x,x)$ for $x \in L$. Such a pair $\underline{L} = (L, \beta)$ is called a lattice. We extend $\beta(x,y)$ as a $\mathbb{C}$-valued symmetric bilinear form on $L \otimes \mathbb{C}$.

For example, $L = \mathbb{Z}^n$, $\beta(x,y) = x(2M)^t y$ for a half-integral symmetric matrix M. Then $\underline{L} = (L, \beta)$ is a lattice.

Definition 2.1. *Let $k \in \mathbb{N}$, and let $\underline{L} = (L, \beta)$ be an even lattice and positive-definite (i.e. $\beta(x) \in \mathbb{Z}_{>0}$ for any $x \in L - \{0\}$).*

A holomorphic function $\phi : \mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a Jacobi form of weight k of index $\underline{L}$, if ϕ satisfies

(1) $^\forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, $(\tau, z) \in \mathfrak{H} \times (L \otimes \mathbb{C})$, *then*

$$(\phi|A)(\tau,z) := \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) e\left(\frac{-c\beta(z)}{c\tau+d}\right)(c\tau+d)^{-k} = \phi(\tau,z).$$

(2) $^\forall x, y \in L$, $(\tau, z) \in \mathfrak{H} \times (L \otimes \mathbb{C})$, *then*

$$(\phi|(x,y))(\tau,z) := \phi(\tau, z+\tau x+y)e(\beta(x)\tau + \beta(x,z)) = \phi(\tau,z).$$

(3) $\phi(\tau, z) = \sum_{n' \in \mathbb{Z}} \sum_{\substack{r' \in L^{\#} \\ n' \geq \beta(r')}} c_\phi(n', r') e(n'\tau + \beta(r', z))$, *where*

$$L^{\#} := \{y \in L \otimes \mathbb{Q} : \beta(x, y) \in \mathbb{Z} \text{ for } {}^\forall x \in L\}.$$

Here we put $e(*) := e^{2\pi i *}$ *and* $q := e(\tau)$.

A Jacobi form ϕ *is called a Jacobi cusp form if* $c_\phi(n', r') = 0$, *unless* $n' > \beta(r')$ *in the condition* (3).

We denote by $J_{k,\underline{L}}$ (resp. $J_{k,\underline{L}}^{\mathrm{cusp}}$) the space of all Jacobi forms (resp. Jacobi cusp forms) of weight k and index $\underline{L}$.

For any half-integral symmetric matrix M, we denote by $J_{k,M}$ the space of Jacobi forms of weight k and index M. (For the definition of Jacobi forms of matrix index, see [16].) If $\underline{L} = (\mathbb{Z}^n, (x, y) \mapsto x(2M)^t y)$, then $J_{k,M} \cong J_{k,\underline{L}}$ via a bijection $\mathbb{C}^n \cong L \otimes \mathbb{C}$.

Denote by $M_k(N) := M_k(\Gamma_0(N))$ the space of elliptic modular forms of weight k with respect to $\Gamma_0(N)$ ($k \in \mathbb{Z}_{\geq 0}, N \in \mathbb{Z}_{>0}$). Let $M_k^{\mathrm{new}}(N)$ be the space of new forms in $M_k(N)$. For $\epsilon \in \{+, -\}$ we set

$$M_k^{\mathrm{new},\epsilon}(N) := \{f \in M_k^{\mathrm{new}}(N) \ : \ f\left(\frac{-1}{N\tau}\right) = \epsilon i^{-k} N^{k/2} \tau^k f(\tau)\}.$$

Remark that if $f \in M_k^{\mathrm{new},\epsilon}(N)$, then the completed L-function $L^*(f, s)$ of f satisfies $L^*(f, s) = \epsilon L^*(f, k - s)$ (cf. [11, P. 136]). We put $M_{2k}^{\epsilon_1}(1) := M_{2k}^{\mathrm{new},\epsilon_1}(1)$. Then, $M_{2k}^{\epsilon}(1) = \{0\}$ unless $(-1)^k = \epsilon$.

Assume $n = 1$, $N \in \mathbb{Z}_{>0}$, $\beta(x, y) = 2Nxy$ for $x, y \in \mathbb{Z}$ and $\underline{L} = (\mathbb{Z}, \beta)$. Then $J_{k,\underline{L}} \cong J_{k,N}$, where $J_{k,N}$ is the space of Jacobi forms of integer index N. The following theorem was obtained in [11] by calculating trace formulas for Jacobi forms.

Theorem 2.2 (Skoruppa-Zagier (1988))**.** *For any* $k, N \in \mathbb{Z}_{>0}$, *we have*

$$J_{k+1,N}^{new} \cong M_{2k}^{new,-}(N)$$

as Hecke modules. Here $J_{k+1,N}^{new}$ *denotes the space of Jacobi new forms* (*see* [11] *for the definition of Jacobi new forms*).

It is expected that such isomorphism also holds for any lattice index $\underline{L}$. In particular, it is expected

$$J_{k,\underline{L}} \cong \{\text{certain elliptic modular form of weight } k_1\} \tag{2.1}$$

as Hecke modules, where $k_1 = k - \frac{r}{2}$ if rank r of L is even, and $k_1 = 2k - 1 - r$ if rank r of L is odd (see [1, p. xv].

In this article we show the isomorphism (2.1) for the case $\underline{L} = D_r$, where D_r is a root lattice with odd rank r.

2.1. **Discriminant module (form, group) and Jacobi forms.** Let $\underline{L} = (L, \beta)$ be an even lattice (i.e. $\beta(L) \subset \mathbb{Z}$). Then $L^\sharp/L$ is a finite abelian group and the reduction of β modulo $\mathbb{Z}$ induces a bilinear form on $L^\sharp/L$. This module is called the discriminant module (or discriminant form, discriminant group) of $\underline{L}$. We set the discriminant module

$$\mathbb{D}_{\underline{L}} := (L^\sharp/L, x + L \mapsto \beta(x) + \mathbb{Z}).$$

Proposition 2.3 (Boylan-Skoruppa(2014), Ajouz(2015)). *Let $\underline{L_1}$, $\underline{L_2}$ be even positive-definite lattices. If $\mathbb{D}_{\underline{L_1}} \cong \mathbb{D}_{\underline{L_2}}$, then*
$$J_{k+\lceil \frac{r_1}{2} \rceil, \underline{L_1}} \cong J_{k+\lceil \frac{r_1}{2} \rceil, \underline{L_2}}$$
as Hecke modules. Here r_i is the rank of L_i.

The bijection of this isomorphism was given in [2, Thm. 2.5] and the compatibility with Hecke operators was shown in [1, Thm. 4.2.4].

2.2. **Lattice D_r.** We treat the case $\underline{L} = D_r$. For $r \in \mathbb{Z}_{>0}$, put the set
$$D_r := \left\{ (x_1, ..., x_r) \in \mathbb{Z}^r \ : \ \sum_{i=1}^{r} x_i \in 2\mathbb{Z} \right\}$$
and the lattice D_r is equipped with the standard Euclidean bilinear form $\beta(x, y) = x^t y$ for $x, y \in D_r$. Then we have
$$D_r^\sharp = \left\{ x \in \left(\frac{1}{2}\mathbb{Z}\right)^r \ : \ x \in \mathbb{Z}^r \text{ or } x \in \left(\frac{1}{2} + \mathbb{Z}\right)^r \right\}.$$
The set $\{0, \frac{e_1+\cdots+e_r}{2}, e_r, \frac{e_1+\cdots+e_{r-1}-e_r}{2}\}$ gives a set of representatives of $D_r^\sharp/D_r$, where $\{e_1, ..., e_r\}$ denotes a standard basis of $\mathbb{Z}^r$. We have
$$D_r^\sharp/D_r \cong \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{if } r \equiv 1 \pmod 2, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } r \equiv 0 \pmod 2) \end{cases}$$
as finite abelian groups. We remark that $\beta(0) = 0$, $\beta(e_r) = \frac{1}{2}$ and $\beta(\frac{e_1+\cdots+e_r}{2}) = \beta(\frac{e_1+\cdots-e_r}{2}) = \frac{r}{8}$.

For any lattice $\underline{L}$, the smallest positive-integer N which satisfies $N\beta(x) \in \mathbb{Z}$ $(\forall x \in L)$ is called the level of $\underline{L}$. If $r \equiv 1 \pmod 2$, then the level of D_r is 8.

From now on, we assume $r \equiv 1 \pmod 2$. Then
$$D_r^\sharp/D_r \cong \left(\mathbb{Z}/4\mathbb{Z}, x + 4\mathbb{Z} \mapsto \frac{rx^2}{8} + \mathbb{Z} \right)$$
as a finite quadratic module. Therefore, by virtue of Prop. 2.3, the Hecke module $J_{k+\frac{r+1}{2}, D_r}$ is determined by r modulo 8.

For $r \equiv 1 \pmod 8$, by using Thm. 2.2, we have
$$J_{k+\frac{r+1}{2}, D_r} \cong J_{k+1, D_1} \cong J_{k+1,2} \cong M_{2k}^{\text{new},-}(2) \oplus M_{2k}^{-}(1).$$

For $r \in \{1, 3, 5, 7\}$, we have isomorphisms between Jacobi forms of lattice index and matrix index:
$$J_{k+\frac{r+1}{2}, D_r} \cong J_{k+\frac{r+1}{2}, M_r},$$
where M_r is a Gram matrix of D_r. For example, $M_1 = 2$, $M_3 = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$, $M_5 = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}$ and $M_7 = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 1 \end{pmatrix}$.

Remark that $|D_r^\sharp/D_r| = |(2M_r)^{-1}\mathbb{Z}^r/\mathbb{Z}^r| = \det(2M_r) = 4$.

3. Main theorem

A. Mocanu [9] posed the following conjecture. Some numerical examples of Euler factors and dimension formulas supported the conjecture.

Conjecture 1 (Mocanu (2019)). *If $r \equiv 1 \pmod 2$ and $k \geq 2$, then*

$$J_{k+\frac{r+1}{2},D_r} \cong M_{2k}^{new,\epsilon_2}(2) \oplus M_{2k}^{\epsilon_1}(1)$$

as Hecke algebra module, where $\epsilon_i \in \{+,-\}$ are given by

$r \pmod 8$	ϵ_2	ϵ_1
1	$-$	$-$
3	$-$	$+$
5	$+$	$-$
7	$+$	$+$

Remark that the case $r \equiv 1 \pmod 8$ is implied in [11]. *We obtain another proof for the case $r \equiv 1 \pmod 8$.*

The above signatures ϵ_i are written by using the Kronecker symbols:

$$\epsilon_1 = -\left(\frac{-4}{r}\right), \quad \epsilon_2 = -\left(\frac{-8}{r}\right).$$

The main result of this article is

Theorem 3.1. *Conj. 1 is true.*

Roughly speaking, the proof of this theorem is obtained by using three isomorphisms from $M_{2k}^{\mathrm{new},\epsilon_2}(2)$ to $J_{k+\frac{r+1}{2},D_r}$ via scalar-valued modular forms and vector-valued modular forms of half-integral weight:

$$M_{2k}^{\mathrm{new},\epsilon_2}(2) \leftrightarrow \left\{\begin{array}{c}\text{certain scalar-valued} \\ \text{modular form} \\ \text{of weight } k+\frac{1}{2}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{certain vector-valued} \\ \text{modular form} \\ \text{of weight } k+\frac{1}{2}\end{array}\right\} \to J_{k+\frac{r+1}{2},D_r}.$$

The image of this composition from $M_{2k}^{\mathrm{new},\epsilon_2}(2)$ to $J_{k+\frac{r+1}{2},D_r}$ can be regarded as new forms in Jacobi forms. The orthogonal complement of this image with respect to the Petersson inner product is $J^{\mathrm{old}}_{k+\frac{r+1}{2},D_r}$ (the space of Jacobi old forms) which corresponds to $M_{2k}^{\epsilon_1}(1)$. The space $J^{\mathrm{old}}_{k+\frac{r+1}{2},D_r}$ is obtained as the image of the composition of the Ikeda lifting and a map given by the Fourier-Jacobi expansion:

$$M_{2k}^{\epsilon_1}(1) \xrightarrow{\text{Ikeda lift}} \left\{\begin{array}{c}\text{Siegel modular form} \\ \text{of weight } k+\frac{r+1}{2} \in 2\mathbb{N}, \\ \text{degree } r+1 \text{ and level } 1\end{array}\right\} \xrightarrow{\text{F-J expansion}} J_{k+\frac{r+1}{2},M_r} \cong J_{k+\frac{r+1}{2},D_r},$$

where $J_{k+\frac{r+1}{2},M_r}$ is the space of Jacobi forms of weight $k+\frac{r+1}{2}$ and matrix index M_r.

3.1. **Half-integral weight.** We put $\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$. Let $N, k \in \mathbb{Z}$ such that $4|N$. A holomorphic function f on $\mathfrak{H}$ is called a modular form of weight $k+\frac{1}{2}$ and level N, if f satisfies

$$f(A\tau) = \left(\frac{\theta(A\tau)}{\theta(\tau)}\right)^{2k+1} f(\tau)$$

for any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and if f^2 satisfies the cusp condition, where we set $A\tau := (a\tau+b)/(c\tau+d)$. Denote by $M_{k+\frac{1}{2}}(N)$ the set of modular forms of weight $k+\frac{1}{2}$ and level N. For any $N \in \mathbb{Z}_{>0}$ such that $4|N$, we put

$$M^{+}_{k+\frac{1}{2}}(N) := \left\{ f(\tau) = \sum_{n\in\mathbb{Z}_{\geq 0}} a_f(n)q^n \in M_{k+\frac{1}{2}}(N) \ : \ a_f(n) = 0 \text{ unless } n \equiv 0, (-1)^k \mod 4 \right\}.$$

We treat the case $N = 8$. For $r \in \mathbb{Z}$ which satisfies $-r \equiv (-1)^k \mod 4$ (it is equivalent to the condition $k + \frac{r+1}{2} \equiv 0 \pmod 2$), we put

$$M^{+,-r}_{k+\frac{1}{2}}(8) := \left\{ f \in M^{+}_{k+\frac{1}{2}}(8) \ : \ a_f(n) = 0 \text{ unless } n \equiv 0, 4, -r \mod 8 \right\}.$$

In particular, if $a_f(n) = 0$ for all n such that $n \equiv 4 - r \pmod 8$, then $f \in M^{+}_{k+\frac{1}{2}}(8)$ belongs to $M^{+,-r}_{k+\frac{1}{2}}(8)$.

For example, we have $\theta \in M^{+,-7}_{1/2}(8)$.

The following Proposition follows from [12].

Proposition 3.2 (Ueda-Yamana (2010))**.**

$$M^{+}_{k+\frac{1}{2}}(8) = \begin{cases} M^{+,-3}_{k+\frac{1}{2}}(8) \oplus M^{+,-7}_{k+\frac{1}{2}}(8), & \textit{if } k \textit{ is even}, \\ M^{+,-1}_{k+\frac{1}{2}}(8) \oplus M^{+,-5}_{k+\frac{1}{2}}(8), & \textit{if } k \textit{ is odd}. \end{cases}$$

For a holomorphic function $f(\tau) = \sum_{n\in\mathbb{Z}_{\geq 0}} c_f(n)q^n$ on $\mathfrak{H}$ and for any prime p, we define

$$f|T(p^2) := \sum_{n=0}^{\infty} \left(c_f(p^2 n) + p^{k-1} \left(\frac{(-1)^k n}{p}\right) c_f(n) + p^{2k-1} c_f(n/p^2) \right) q^n.$$

If $p \nmid N$, then $T(p^2)$ is a Hecke operator on $M_{k+\frac{1}{2}}(N)$.

For any odd prime p, the Hecke operator $T(p^2)$ preserves the space $M^{+,-r}_{k+\frac{1}{2}}(8)$. The following theorem follows from [12].

Theorem 3.3 (Ueda-Yamana (2010))**.** *If $k + \frac{r+1}{2}$ is even and $k \geq 1$, then*

$$M^{+,-r}_{k+\frac{1}{2}}(8) \cong M^{new,\epsilon_2}_{2k}(2) \oplus M^{\epsilon_1}_{2k}(1)$$

as Hecke modules. It means that if $h \in M^{+,-r}_{k+\frac{1}{2}}(8)$ is an eigenform, and $h|T(p^2) = \lambda_p h$, then there is an eigenform $f \in M_{2k}(*)$ which satisfies $f|T(p) = \lambda_p f$, where $T(p)$ is the usual Hecke operator.

3.2. **half-integral weight of η-type.** We put $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$.

For $r \in \{1,3,5,7\}$, We consider the space

$$\eta^{24-3r}M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1) = \left\{\eta(\tau)^{24-3r}f(\tau) \,:\, f \in M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1)\right\}.$$

Let $g \in \eta^{24-3r}M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1)$. We take the Fourier expansion $g(\tau) = \sum_{n=1}^{\infty} c_g(n)q^{n/8}$. We remark that $c_g(n) = 0$ unless $n \equiv -r \pmod 8$. For any odd prime p and for $g \in \eta^{24-3r}M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1)$, we define a Hecke operator

$$g|\tilde{T}(p^2) := \left(\frac{-4}{p}\right)\sum_{n=1}^{\infty}\left(c_g(p^2n) + p^{k-1}\left(\frac{(-1)^k n}{p}\right)c_g(n) + p^{2k-1}c_g(n/p^2)\right)q^{n/8}. \tag{3.1}$$

Remark that we put the Legendre symbol $\left(\frac{-4}{p}\right)$ on the RHS.

The following theorem follows from [14].

Theorem 3.4 (Yang (2014)). *For $r \in \{1,3,5,7\}$ and for any integer $k \geq 1$ such that $k + \frac{r+1}{2} \equiv 1 \pmod 2$, we have*

$$\eta^{24-3r}M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1) \cong M_{2k}^{new,\epsilon_2}(2)$$

as Hecke modules. Here the action of the Hecke operators on the LHS is defined in (3.1).

3.3. **Hecke operators for Jacobi forms.** Hecke operators for Jacobi forms of lattice index were introduced in [1]. These were also introduced in [10] in the framework of Jacobi forms of matrix index. The following follows from [10]. We assume that r is odd. For

$$\phi(\tau,z) = \sum_{n'\in\mathbb{Z}}\sum_{\substack{r'\in D_r^{\#}\\ n'\geq\beta(r')}} c_\phi(n',r')e(n'\tau+\beta(r',z)) \in J_{k+\frac{r+1}{2},D_r}$$

and for any odd prime p, a Hecke operator $T^J(p^2)$ is defined by

$$(\phi|_{k+\frac{r+1}{2},\underline{L}}T^J(p^2))(\tau,z) := \sum_{n'\in\mathbb{Z}}\sum_{\substack{r'\in D_r^{\#}\\ n'\geq\beta(r')}} c_\phi^*(n',r')e(n'\tau+\beta(r',z)),$$

where

$$\begin{aligned} c_\phi^*(n',r') &:= c_\phi(p^2n',pr') + p^{k-1}\left(\frac{(-1)^{\frac{r+1}{2}}8(n'-\beta(r'))}{p}\right)c_\phi(n',r') \\ &\quad + p^{2k-1}\sum_{\lambda\in D_r/pD_r} c_\phi\left(\frac{1}{p^2}(n'-\beta(r',\lambda)+\beta(\lambda)), \frac{1}{p}(r'-\lambda)\right). \end{aligned}$$

3.4. **Theta decomposition.** Jacobi forms correspond to vector valued modular forms as follows. Let $\phi \in J_{k+\frac{r+1}{2},D_r}$ and denote by $c_\phi(n',r')$ the Fourier coefficient of ϕ. Then

$$\phi(\tau,z) = \sum_{x\in D_r^\sharp/D_r} f_x(\tau)\theta_{D_r,x}(\tau,z),$$

where $\theta_{D_r,x}(\tau,z) := \sum_{y\in x+D_r} e(\tau\beta(y)+\beta(y,z))$ and $f_x(\tau) := \sum_{n'\in\mathbb{Z}} c_\phi(n',x)e((n'-\beta(x))\tau)$.

The vector valued function $(f_x)_{x\in D_r^\sharp/D_r}$ satisfies:

$$f_x(\tau+1) = e(-\beta(x))f_x(\tau)$$

and

$$f_x(-\tau^{-1}) = 2^{-1}\tau^{k+\frac{1}{2}}e(r/8)\sum_{y \pmod 4} e(xyr/4)f_y(\tau).$$

Such holomorphic function is called a vector-valued modular form of weight $k+\frac{1}{2}$.

Recall that $D_r^\sharp/D_r \cong \mathbb{Z}/4\mathbb{Z}$ as finite abelian groups. We have $f_{-x} = (-1)^{k+\frac{r+1}{2}}f_x$. In particular, if $k+\frac{r+1}{2}$ is odd, then $f_{-x} = -f_x$ and $f_x = 0$ for $2x \in D_r$. We put

$$(\mathcal{J}(\phi))(\tau) := \begin{cases} \sum\limits_{x\in D_r^\sharp/D_r} f_x(8\tau), & \text{if } k+\frac{r+1}{2} \text{ is even,} \\ f_{v_1}(\tau), & \text{if } k+\frac{r+1}{2} \text{ is odd,} \end{cases}$$

where we set $v_1 := (e_1+\cdots+e_r)/2$.

Theorem 3.5. *The above defined map $\mathcal{J}$ gives the following isomorphism as Hecke modules:*

$$J_{k+\frac{r+1}{2},D_r} \cong \begin{cases} M^{+,-r}_{k+\frac{1}{2}}(8), & \textit{if } k+\frac{r+1}{2} \textit{ is even,} \\ \eta^{24-3r}M_{k+\frac{1}{2}-\frac{24-3r}{2}}(1), & \textit{if } k+\frac{r+1}{2} \textit{ is odd.} \end{cases}$$

4. Proof of main Theorem (Thm. 3.1) and cusp forms (Thm. 4.2)

4.1. **proof of the main Theorem.** Remark that if $k+\frac{r+1}{2}$ is odd, then $M^{\epsilon_1}_{2k}(1) = \{0\}$. By combining Theorems 3.3, 3.4 and 3.5, we obtain

$$J_{k+\frac{r+1}{2},D_r} \cong M^{\mathrm{new},\epsilon_2}_{2k}(2) \oplus M^{\epsilon_1}_{2k}(1) \tag{4.1}$$

as Hecke modules for any $r \in \{1,3,5,7\}$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, Conjecture 1 is true.

Moreover, due to Proposition 2.3, this isomorphism (4.1) is valid for any odd $r \in \mathbb{Z}_{>0}$.

4.2. **Isomorphisms between cusp forms.** The space of Jacobi forms has the following decomposition.

Lemma 4.1. *If $k \in \mathbb{Z}_{>0}$, then $J_{k+\frac{r+1}{2},D_r}$ has the decomposition*

$$J_{k+\frac{r+1}{2},D_r} = J^{new}_{k+\frac{r+1}{2},D_r} \oplus J^{old}_{k+\frac{r+1}{2},D_r}$$

where we defined the subspace $J^{old}_{k+\frac{r+1}{2},D_r} \subset J_{k+\frac{r+1}{2},D_r}$ such that $J^{old}_{k+\frac{r+1}{2},D_r} \cong M^{\epsilon_1}_{2k}(1)$, and where $J^{new}_{k+\frac{r+1}{2},D_r}$ is defined as the orthogonal complement of $J^{old}_{k+\frac{r+1}{2},D_r}$ in $J_{k+\frac{r+1}{2},D_r}$ with respect to the Petersson inner product. The map $M^{\epsilon_1}_{2k}(1) \to J^{old}_{k+\frac{r+1}{2},D_r}$ is obtained by the composition of three maps: the Ikeda lift, Fourier-Jacobi expansion and the isomorphism between Jacobi forms of lattice index and of matrix index.

Moreover, if $k \geq 2$, then we have $J^{new}_{k+\frac{r+1}{2},D_r} = J^{cusp,new}_{k+\frac{r+1}{2},D_r} := \left(J^{cusp}_{k+\frac{r+1}{2},D_r} \cap J^{new}_{k+\frac{r+1}{2},D_r}\right)$, where J^{cusp}_{k,D_r} is the space of Jacobi cusp forms in J_{k,D_r}.

We set $S^{**}_*(N) := \{\text{cusp form in } M^{**}_*(N)\}$, where $M^{**}_*(N)$ is $M^{\mathrm{new},+,-r}_{k+\frac{1}{2}}(8)$ or $M^{\mathrm{new},\epsilon_2}_{2k}(2)$. Remark that $M^{\mathrm{new},+,-r}_{k+\frac{1}{2}}(8)$ is isomorphic to $M^{\mathrm{new},\epsilon_2}_{2k}(2)$ by Theorem 3.3.

Theorem 4.2. *We have the following isomorphisms as Hecke modules.*
If $k \geq 2$ is even, then
$$J^{cusp,new}_{k+2,D_3} \cong S^{new,+,-3}_{k+\frac{1}{2}}(8) \cong S^{new,-}_{2k}(2) \cong \eta^{21} M_{k-10}(1) \cong J^{cusp,new}_{k+1,D_1}$$
and
$$J^{cusp,new}_{k+4,D_7} \cong S^{new,+,-7}_{k+\frac{1}{2}}(8) \cong S^{new,+}_{2k}(2) \cong \eta^{9} M_{k-4}(1) \cong J^{cusp,new}_{k+3,D_5}.$$
If $k \geq 3$ is odd, then
$$J^{cusp,new}_{k+1,D_1} \cong S^{new,+,-1}_{k+\frac{1}{2}}(8) \cong S^{new,-}_{2k}(2) \cong \eta^{15} M_{k-7}(1) \cong J^{cusp,new}_{k+2,D_3}$$
and
$$J^{cusp,new}_{k+3,D_5} \cong S^{new,+,-5}_{k+\frac{1}{2}}(8) \cong S^{new,+}_{2k}(2) \cong \eta^{3} M_{k-1}(1) \cong J^{cusp,new}_{k+4,D_7}.$$

5. The case $k = 1$

5.1. **Isomorphisms.** Conj. 1 is also true for the case $k = 1$. We have the following.

Theorem 5.1. *We obtain*
$$\dim J^{new}_{4,D_5} = \dim M^{+,-5}_{3/2}(8) = \dim M^{new,+}_{2}(2) = \dim \eta^3 M_0(1) = \dim J^{cusp,\ new}_{5,D_7} = 1$$
and the following one dimensional spaces are isomorphic as Hecke modules:
$$\mathbb{C}E_{4,D_5} \cong \mathbb{C}(\theta^3 | U_1(4)) \cong \mathbb{C}E^{(2)}_2 \cong \mathbb{C}\psi_{5,D_7} \cong \mathbb{C}\eta^3.$$
where $E_{4,D_5} \in J^{new}_{4,D_5}$ and $\psi_{5,D_7} \in J^{cusp,new}_{5,D_7}$ are Jacobi forms, and where we put
$$E^{(2)}_2 := 2E_2(2\tau) - E_2(\tau) = 1 + 24 \sum_n \sigma_1(n) q^n - 48 \sum_n \sigma_1(n) q^{2n} \in M^{new,+}_2(2),$$
and where $\theta^3 | U_1(4)$ is defined by
$$(\theta^3 | U_1(4))(\tau) := \sum_{\substack{n \in \mathbb{Z}_{\geq 0} \\ n \equiv 0,3 \pmod 4}} r_3(n) q^n \in M^{+,-5}_{3/2}(8),$$
and $r_3(n) := \#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = n\} = r_3(4n)$.

Remark that E_{4,D_5}, $\theta^3 | U_1(4)$ and $E^{(2)}_2$ are not (Jacobi) cusp forms, but these forms are isomorphic to (Jacobi) cusp forms by Theorem 5.1.

It is well-known that the function η^3 has the Fourier expansion
$$\eta(\tau)^3 = \sum_{n \in \mathbb{Z}_{>0}} \left(\frac{-4}{n}\right) n q^{n^2/8}.$$

It is straightforward to calculate that $\eta^3 | \tilde{T}(p^2) = (1+p)\eta^3$ for any odd prime p (see (3.1) for the definition of $\tilde{T}(p^2)$). Since E_{4,D_5} has the non-zero constant term and since $\dim J_{4,D_5} = 1$ (see [9, P.76]), it is not difficult to check the identity $E_{4,D_5} | T^J(p^2) = (1+p) E_{4,D_5}$ for any odd prime p.

5.2. **Zagier–Eisenstein series of weight** $3/2$**.** The Zagier–Eisenstein series (cf. [15]) of weight $3/2$ is defined by

$$\mathscr{F}(\tau) := \mathscr{H}_1(\tau) + \frac{1}{16\pi\sqrt{v}} \sum_{n\in\mathbb{Z}} \alpha(n^2 y) q^{-n^2},$$

where $v = \mathrm{Im}(\tau)$ and $\alpha(t) := \int_1^\infty e^{-4\pi u t} u^{-3/2}\, du$, and where

$$\mathscr{H}_1(\tau) := -\frac{1}{12} + \sum_{n\in\mathbb{Z}_{>0}} H(n) q^n,$$

and $H(n)$ denotes the Hurwitz class number ($H(0) = -1/12$, $H(3) = 1/3$, $H(4) = 1/2$, $H(7) = 1$ and so on). Note that $H(n) = 0$ for $n \equiv 1, 2 \pmod 4$. The function $\mathscr{F}$ is not holomorphic on $\mathfrak{H}$, but $\mathscr{F}$ satisfies the transformation formula $\mathscr{F}(A\tau) = \left(\frac{\theta(A\tau)}{\theta(\tau)}\right)^3 \mathscr{F}(\tau)$ for any $A \in \Gamma_0(4)$.

We put $(\mathscr{H}_1|U_1(4))(\tau) := \sum_{\substack{n\in\mathbb{Z}_{\geq 0}\\ n\equiv 0,3 \,(\mathrm{mod}\ 4)}} H(4n) q^n$. The identity

$$r_3(n) = 12(H(4n) - 2H(n)) \tag{5.1}$$

holds for any $n \in \mathbb{Z}_{\geq 0}$ (cf. [5, P. 3084]). This identity was essentially obtained by Gauss and Hermite (see [4, P.274]).

Due to the identity (5.1), we obtain the following lemma.

Lemma 5.2. *We have*

$$12(\mathscr{H}_1|U_1(4) - 2\mathscr{H}_1) = \theta^3|U_1(4) \in M^{+,-5}_{3/2}(8).$$

5.3. **Application to arithmetic functions.** We have $(\theta^3|U_1(4))\theta \in M_2(8)$. In particular, we obtain

$$(\theta^3|U_1(4))\theta = \frac{1}{12}E_2^{(2)}(\tau) - \frac{1}{12}E_2^{(2)}(2\tau) + E_2^{(2)}(4\tau).$$

By compering the Fourier coefficients of both sides, we obtain the following.

Corollary 5.3. *For any natural number* N*, we have*

$$\frac{1}{2} \sum_{\substack{s\in\mathbb{Z}\\ N-s^2\equiv 0,3 \mod 4}} r_3(N-s^2) = \sigma_1(N) - 3\,\sigma_1\!\left(\frac{N}{2}\right) + 14\,\sigma_1\!\left(\frac{N}{4}\right) - 24\,\sigma_1\!\left(\frac{N}{8}\right). \tag{5.2}$$

Here $\sigma_1(m)$ *denotes* 0*, if* m *is not a natural number.*

For example, we obtain $\sigma_1(1) = r_3(0)$, $\sigma_1(3) = \frac{1}{2}r_3(3)$, $\sigma_1(5) = r_3(4)$, $\sigma_1(7) = r_3(3)$, $\sigma_1(9) = r_3(8) + r_3(0)$, $\sigma_1(11) = \frac{1}{2}r_3(11)$, $\sigma_1(13) = r_3(12) + r_3(4)$ and so on.

Remark that the fact $\theta^4 \in M_2(4)$ induces the well-known formula

$$\frac{1}{2}\sum_{s\in\mathbb{Z}} r_3(N-s^2) = 4\sigma_1(N) - 16\,\sigma_1\!\left(\frac{N}{4}\right). \tag{5.3}$$

Through the identity (5.1), the identities (5.2) and (5.3) also follow from Kronecker–Hurwitz type class number relations. The Kronecker-Hurwitz class number relation is

$$\sum_{s\in\mathbb{Z}} H(4N-s^2)+\sum_{d|n}\min(d,N/d)=\begin{cases}-\dfrac{1}{12} & \text{if } N=0,\\ 2\sigma_1(N) & \text{if } N>0.\end{cases}$$

The Kronecker-Hurwitz class number relation was generalized by several authors. Recently, higher moments of Hurwitz class numbers in arithmetic progressions were obtained by Kane-Pujahari-Yang (see [7, Thm. 1.7]). The identity (5.2) is also related to classical formulas obtained by Liouville (see [13]).

6. Cohen type Eisenstein series of level 8

6.1. **Level** 4. Let $k\ge 2$. The Cohen Eisenstein series of weight $k+\frac{1}{2}$ of level 4 is defined by

$$\mathscr{H}_k(\tau):=\zeta(1-2k)+\sum_{N>0}H(k,N)q^N,$$

where we put

$$H(k,N):=H(k,|D|)\sum_{d|f}\mu(d)\left(\frac{D}{d}\right)d^{k-1}\sigma_{2k-1}(f/d),$$

and where D is a fundamental discriminant and $f\in\mathcal{N}$, such that $(-1)^kN=Df^2$, and we define $H(k,|D|):=L(1-k,\left(\frac{D}{*}\right))$. Here μ denotes the Möbius function.

Then, it is known that $\mathscr{H}_k$ belongs to $M^+_{k+\frac{1}{2}}(4)$ (cf. [4]).

6.2. **Level** 8. For $k\ge 2$ such that $(-1)^k\equiv -r \mod 4$, we introduce a Cohen-type Eisenstein series of level 8 by

$$\mathscr{H}^*_{r,k}(\tau):=\zeta(1-2k)+\sum_{N>0}H^*_r(k,N)q^N,$$

where we put

$$H^*_r(k,N):=H(k,N)+\frac{H(k,N)-H(k,4N)}{2^{k-1}\left(\left(\frac{8}{r}\right)+2^k\right)}.$$

Lemma 6.1. *For $k\ge 2$ and for $r\in\{1,3,5,7\}$ such that $(-1)^k\equiv -r \mod 4$, we have*

$$M^{+,-r}_{k+\frac{1}{2}}(8)=\mathbb{C}\mathscr{H}^*_{r,k}\oplus S^{+,-r}_{k+\frac{1}{2}}(8).$$

The Cohen type Eisenstein series $\mathscr{H}^*_{r,k}$ of level 8 corresponds to Jacobi–Eisenstein series of weight $k+\frac{r+1}{2}$ and index D_r by Thm. 3.5.

6.3. **Fourier coefficients of Jacobi–Eisenstein series of index D_r.**

Definition 6.2. *For $k\ge 2$ and odd integer $r>0$ such that $k+\frac{r+1}{2}$ is even, we define the Jacobi–Eisenstein series of weight $k+\frac{r+1}{2}$ and index D_r by*

$$E_{k+\frac{r+1}{2},D_r}(\tau,z):=\sum_{A\in SL(2,\mathbb{Z})/\Gamma_\infty}\sum_{\lambda\in D_r}1|(\lambda,0)|A,$$

where $\Gamma_\infty:=\{\pm\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\,:\,n\in\mathbb{Z}\}$.

The convergence of $E_{k+\frac{r+1}{2},D_r}$ is known in [1, Prop. 3.3.6] and [9, Thm. 2.6]. An explicit formula of the Fourier coefficients for Jacobi–Eisenstein series of lattice index was obtained in [8, Thm. 5.1]. By virtue of Thm. 3.5, we obtain another expression of the Fourier coefficients for the case of lattice D_r as follows.

Proposition 6.3. *Let $r > 0$ be an odd integer and $k + \frac{r+1}{2} \in 2\mathbb{Z}_{>0}$ with $k \geq 2$. The Jacobi-Eisenstein series $E_{k+\frac{r+1}{2},D_r} \in J_{k+\frac{r+1}{2},D_r}$ has the Fourier expansion:*

$$E_{k+\frac{r+1}{2},D_r}(\tau,z) \;=\; \sum_{n'\in\mathbb{Z}} \sum_{r'\in D_r^\sharp} e_{r,k}(n',r') e(n'\tau + \beta(r',z)),$$

where $N = 8(n' - \beta(r'))$ and

$$e_{r,k}(n',r') \;=\; \begin{cases} 1, & \text{if } N = 0, \\ \dfrac{H_r^*(k,N)}{\zeta(1-2k)}, & \text{if } N > 0 \text{ and } N \in 4\mathbb{Z}, \\ \dfrac{H_r^*(k,N)}{2\zeta(1-2k)}, & \text{if } N > 0 \text{ and } N \in -r + 8\mathbb{Z}. \end{cases}$$

7. Maps from Jacobi forms to elliptic modular forms

Let $k \in \mathbb{Z}_{>0}$, and let $k + \frac{r+1}{2}$ be an even integer. For $\phi \in J^{cusp}_{k+\frac{r+1}{2},D_r}$ we set

$$A(N) := \begin{cases} c_\phi(n',r'), & \text{if } N \equiv 0,4 \mod 8, \\ 2c_\phi(n',r'), & \text{if } N \equiv -r \mod 8, \end{cases}$$

where $N = 8(n' - \beta(r'))$ and $c_\phi(n',r')$ denotes the (n',r')-th Fourier coefficient of ϕ for $(n',r') \in \mathbb{Z} \times D_r^\sharp$.

Due to Thm. 3.5, we have

$$\sum_{\substack{N\in\mathbb{Z}_{>0} \\ N\equiv 0,4,-r \pmod 8}} A(N)q^N \in S^{+,-r}_{k+\frac{1}{2}}(8).$$

Moreover, by virtue of Thm. 3.3, we obtain the following.

Proposition 7.1. *For any fundamental discriminant $(-1)^k d_0$ such that $d_0 > 0$ and $d_0 \equiv 0$ mod 4, the map S_{d_0} defined by*

$$S_{d_0}(\phi) \;:=\; \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{(-1)^k d_0}{d} \right) d^{k-1} A\left(\frac{n^2}{d^2} d_0 \right) \right) q^n$$

gives a linear map $S_{d_0} : J^{cusp}_{k+\frac{r+1}{2},D_r} \to S^{new,\epsilon_2}_{2k}(2) \oplus S^{\epsilon_1}_{2k}(1)$. Moreover, this map is compatible with the action of Hecke operators.

Note that we have similar maps for the case of odd integral weight $k + \frac{r+1}{2}$ (see [6] for the detail). We also note that such maps for the case of lattice index D_r with $r \equiv 1 \pmod 8$ follows from [3] in the framework of Jacobi forms of matrix index.

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