

CHARACTERIZATION OF THE UNIT OBJECT IN LOCALIZED QUANTUM UNIPOTENT CATEGORY

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1. INTRODUCTION

Let R be a quiver Hecke algebra and $R\text{-gmod}$ be the category of finite-dimensional graded R -modules. The localization procedure for monoidal categories was introduced in [3] and [11]. We let $\widetilde{R\text{-gmod}}[w]$ denote the localization of $R\text{-gmod}$ associated with an element w of Weyl group. In [8], Kashiwara and Nakashima showed that the set of equivalence classes of simple objects up to grading shifts in $\widetilde{R\text{-gmod}}[w]$ admits a crystal structure and is isomorphic to the so-called cellular crystal $\mathbb{B}_{\mathbf{i}}$ where $\mathbf{i}$ is a reduced word of w . This isomorphism of crystals enables us to define a function ε_i^* on $\mathbb{B}_{\mathbf{i}}$. In this article, we present explicit formulae for ε_i^* , and use them to give a characterization of the unit object of $\widetilde{R\text{-gmod}}[w_0]$ in the case of classical finite types. This is a survey of [17].

2. CRYSTALS

We briefly review the theory of crystals ([7]).

2.1. Preliminaries. Let I be the index set. An integer-valued square matrix $A = (a_{ij})$ is the symmetrizable generalized Cartan matrix if it satisfies

- (1) $a_{ii} = 2$ ($i \in I$),
- (2) $a_{ij} \leq 0$ if $i \neq j$,
- (3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.
- (4) there exists a diagonal matrix $D = \text{diag}(d_i \mid i \in I)$ with all d_i positive integers such that DA is symmetric.

The dual weight lattice $P^\vee$ is a free abelian group with a basis $\{h_i \mid i \in I\}$ and let $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. Then we define the weight lattice to be $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$. Let $\Pi^\vee = \{h_i \mid i \in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i \mid i \in I\}$ that satisfies $\langle h_i, \alpha_j \rangle = \alpha_j(h_i) = a_{ij}$. We call the elements of Π simple roots and the elements of $\Pi^\vee$ simple coroots.

Definition 2.1. A Cartan datum associated with the generalized Cartan matrix A is a quintuple $(A, P, P^\vee, \Pi, \Pi^\vee)$.

We call $Q = \oplus_{i \in I} \mathbb{Z} \alpha_i$ the root lattice and $Q_+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ the positive root lattice. For an element $\beta = \sum_i m_i \alpha_i \in Q_+$, we define $|\beta| = \sum_i m_i$, which is called the height of β . We also define a symmetric bilinear form $(\ , \)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i, \alpha_j) = d_i a_{ij}$ ($i, j \in I$) and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for any $\lambda \in \mathfrak{h}^*$ and $i \in I$.

Let $U_q(\mathfrak{g})$ be the quantum group associated with $(A, P, P^\vee, \Pi, \Pi^\vee)$, which is a $\mathbb{Q}$ -algebra generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) with certain relations. Let $U_q^-(\mathfrak{g})$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by f_i ($i \in I$).

Let $W = \langle s_i \rangle_{i \in I}$ be the Weyl group associated with P , where s_i is the simple reflection defined by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ ($\lambda \in P$). In the sequel, we assume $A = (a_{ij})$ and W are of finite type. For the numbering of Dynkin diagrams of Lie algebras, we adopt the ones in [1, Chap.4].

2.2. Language of crystal.

Definition 2.2. A 6-tuple $(B, \text{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i \in I}$ is a *crystal* if B is a set and there exists an element denoted by $0 \notin B$ and maps:

$$\begin{aligned} \text{wt} : B &\rightarrow P, \quad \varepsilon_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I) \\ \tilde{e}_i : B \sqcup \{0\} &\rightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\} \quad (i \in I), \end{aligned}$$

satisfying :

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$.
- (2) If $b, \tilde{e}_i b \in B$, then $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
- (3) If $b, \tilde{f}_i b \in B$, then $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.
- (4) For $b, b' \in B$ and $i \in I$, one has $\tilde{f}_i b = b'$ if only if $b = \tilde{e}_i b'$.
- (5) If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$ and $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Example 2.3 ([6]). There exists the crystal basis $(L(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})$ defined by

$$\begin{aligned} L(\infty) &:= \sum_{k \geq 0, i_1, \dots, i_k \in I} \mathbb{A} \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\infty, \\ B(\infty) &= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\infty \bmod qL(\infty) \mid k \geq 0, i_1, \dots, i_k \in I\} \setminus \{0\}, \\ \varepsilon_i(b) &= \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle, \end{aligned}$$

where $u_\infty = 1 \in U_q(\mathfrak{g})$, $\tilde{e}_i$ and $\tilde{f}_i \in \text{End}_{\mathbb{Q}(q)}(U_q^-(\mathfrak{g}))$ are the Kashiwara operators and $\mathbb{A}$ is a local subring of $\mathbb{Q}(q)$ at $q = 0$.

Example 2.4. For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and its crystal structure is given by

$$\begin{aligned} \text{wt}((n)_i) &= n\alpha_i, \quad \varepsilon_i((n)_i) = -n, \quad \varphi_i((n)_i) = n, \\ \varepsilon_j((n)_i) &= \varphi_j((n)_i) = -\infty \quad (i \neq j), \\ \tilde{e}_i((n)_i) &= (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i, \\ \tilde{e}_j((n)_i) &= \tilde{f}_j((n)_i) = 0 \quad (i \neq j). \end{aligned}$$

Definition 2.5. For crystals $\mathcal{B}_1, \mathcal{B}_2$, a *crystal morphism* $\Psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \rightarrow \mathcal{B}_2 \cup \{0\}$ such that

- (1) $\Psi(0) = 0$,
- (2) if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\text{wt}(\Psi(b)) = \text{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$,
- (3) if $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.

$\Psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is called an *embedding* if $\Psi : \mathcal{B}_1 \cup \{0\} \rightarrow \mathcal{B}_2 \cup \{0\}$ is an injection and called an *isomorphism* if a bijection.

Crystals have the tensor product, whose crystal structure is obtained by the tensor product rule of crystal bases.

Definition 2.6. The tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ whose crystal structure is given by

- (1) $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$,
- (2) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle\}$,
- (3) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle\}$,
- (4) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$
- (5) $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$

where we write $b_1 \otimes b_2$ for $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

2.3. Cellular crystal. For the tensor product of the crystals B_i (see Example 2.4), there exist isomorphisms, called braid type isomorphisms.

Proposition 2.7 ([18]). For $i \in I$, set $B_i = \{(z)_i \mid z \in \mathbb{Z}\}$ and let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix. Then, there exist the following isomorphisms of crystals $\phi_{ij}^{(k)}$ ($k = 0, 1, 2, 3$):

$$\begin{aligned}
\phi_{ij}^{(0)} : B_i \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i, (z_1)_i \otimes (z_2)_j \mapsto (z_2)_j \otimes (z_1)_i, \quad \text{if } a_{ij}a_{ji} = 0, \\
\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i &\xrightarrow{\sim} B_j \otimes B_i \otimes B_j, \quad \text{if } a_{ij}a_{ji} = 1, \\
(z_1)_i \otimes (z_2)_j \otimes (z_3)_j &\mapsto (\max(z_3, z_2 - z_1))_j \otimes (z_1 + z_3)_i \otimes (-\max(-z_1, z_3 - z_2))_j, \\
\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i, \quad \text{if } a_{ij} = -1, a_{ji} = -2, \\
(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j &\mapsto (\max(z_4, z_2 - 2z_1, 2z_3 - z_2))_j \otimes (\max(z_1 + z_4, z_2, z_1 - z_2 + 2z_3))_i \\
&\otimes (-\max(-z_3, -z_4 - 2z_1, -2z_2 + 2z_3 - z_4))_j \otimes (-\max(-z_3 + z_4, -z_1, z_3 - z_2))_i. \\
\phi_{ji}^{(2)} : B_j \otimes B_i \otimes B_j \otimes B_i &\xrightarrow{\sim} B_i \otimes B_j \otimes B_i \otimes B_j, \quad \text{if } a_{ij} = -1, a_{ji} = -2, \\
(z_1)_j \otimes (z_2)_i \otimes (z_3)_j \otimes (z_4)_i &\mapsto (\max(-z_2 + z_3, -z_1 + z_2, z_4))_i \otimes (\max(z_1 - 2z_2 + 2z_3, z_3, z_1 + z_1 + 2z_4))_j \\
&\otimes (-\max(-2z_2 + z_3 - z_4, -z_1 - z_4, -z_2))_j \otimes (-\max(-2z_2 + z_3, -z_1, z_3 - 2z_4))_i.
\end{aligned}$$

We call these $\phi_{ij}^{(k)}$ ($k = 0, 1, 2, 3$) the *braid-type isomorphisms* of B_i 's. We omit the formula of $\phi_{ji}^{(3)}$ since we do not use in this article. Note that $\phi_{ij}^{(k)}$ and $\phi_{ji}^{(k)}$ are inverse to each other.

By braid-type isomorphisms, for any $w \in W$ and its reduced words $i_1 \cdots i_l$ and $j_1 \cdots j_l$, we obtain

$$B_{i_1} \otimes \cdots \otimes B_{i_l} \cong B_{j_1} \otimes \cdots \otimes B_{j_l}.$$

Definition 2.8. We define a *braid-move* on the set of reduced words of $w \in W$ to be a composition of the following transformations induced from the braid relations:

$$\begin{aligned}
\cdots i j \cdots &\rightarrow \cdots j i \cdots (a_{ij}a_{ji} = 0), \quad \cdots i j i \cdots \rightarrow \cdots j i j \cdots (a_{ij}a_{ji} = 1), \\
\cdots i j i j \cdots &\rightarrow \cdots j i j i \cdots (a_{ij}a_{ji} = 2), \quad \cdots i j i j i j \cdots \rightarrow \cdots j i j i j i \cdots (a_{ij}a_{ji} = 3),
\end{aligned}$$

which are called 2-move, 3-move, 4-move, 6-move respectively.

Definition 2.9 (*Cellular crystal*). For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_k$ of $w \in W$, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a cellular crystal associated with $\mathbf{i}$.

For a sequence of indices $\mathbf{i} = (i_1, \dots, i_k) \in I^k$, we identify $\mathbb{B}_{\mathbf{i}}$ with $\mathbb{Z}^k$ as follows:

$$(x_1, \dots, x_m) \mapsto \tilde{f}_{i_1}^{x_1}(0)_{i_1} \otimes \cdots \otimes \tilde{f}_{i_m}^{x_m}(0)_{i_m} = (-x_1)_{i_1} \otimes \cdots \otimes (-x_m)_{i_m},$$

where if $n < 0$, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$.

By the tensor structure of crystals, we can describe the explicit crystal structure on $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_m}$ as follows: For $x = (x_1, \dots, x_m) \in \mathbb{B}_{\mathbf{i}}$, define

$$\sigma_k(x) := x_k + \sum_{j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j$$

and for $i \in I$ define

$$\begin{aligned}
\tilde{\sigma}^{(i)}(x) &:= \max\{\sigma_k(x) \mid 1 \leq k \leq m \text{ and } i_k = i\}, \\
\tilde{M}^{(i)} &= \tilde{M}^{(i)}(x) := \{k \mid 1 \leq k \leq m, i_k = i, \sigma_k(x) = \tilde{\sigma}^{(i)}(x)\}, \\
\tilde{m}_f^{(i)} &= \tilde{m}_f^{(i)}(x) := \max \tilde{M}^{(i)}(x), \quad \tilde{m}_e^{(i)} = \tilde{m}_e^{(i)}(x) := \min \tilde{M}^{(i)}(x).
\end{aligned}$$

The actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ and the functions ε_i, φ_i and wt are written explicitly:

$$\tilde{f}_i(x)_k := x_k + \delta_{k, \tilde{m}_f^{(i)}}, \quad \tilde{e}_i(x)_k := x_k - \delta_{k, \tilde{m}_e^{(i)}}, \quad (2.1)$$

$$\text{wt}(x) := - \sum_{k=1}^m x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \tilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \text{wt}(x) \rangle + \varepsilon_i(x). \quad (2.2)$$

3. QUIVER HECKE ALGEBRA AND ITS MODULES

We review the basics of the quiver Hecke algebras (cf.[14, 11, 4, 20]).

3.1. Definition of Quiver Hecke Algebra. For a finite index set I and a field $\mathbf{k}$, let $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I} \in \mathbf{k}[u, v]$ be polynomials satisfying:

- (1) $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$ for any $i, j \in I$.
- (2) $\mathcal{Q}_{i,j}(u, v)$ is in the form:

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \sum_{a(\alpha_i, \alpha_i) + b(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j)} t_{i,j;a,b} u^a v^b & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$.

For $\beta = \sum_i m_i \alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$, set $I^\beta := \{\nu = (\nu_1, \dots, \nu_m) \in I^m \mid \sum_{k=1}^m \alpha_{\nu_k} = \beta\}$.

Definition 3.1. For $\beta \in Q_+$, the *quiver Hecke algebra* $R(\beta)$ associated with a Cartan matrix A and polynomials $\mathcal{Q}_{i,j}(u, v)$ is the $\mathbf{k}$ -algebra generated by

$$\{e(\nu) \mid \nu \in I^\beta\}, \quad \{x_k \mid 1 \leq k \leq n\}, \quad \{\tau_i \mid 1 \leq i \leq n-1\}$$

with the following relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu, \nu'} e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \quad e(\nu)x_k = x_k e(\nu), \quad x_k x_l = x_l x_k, \\ \tau_l e(\nu) &= e(s_l(\nu)) \tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \text{ if } |k-l| > 1, \\ \tau_k^2 e(\nu) &= \mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\ (\tau_k x_l - x_{s_k(l)} \tau_k) e(\nu) &= \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) &= \begin{cases} \overline{\mathcal{Q}}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\overline{\mathcal{Q}}_{i,j}(u, v, w) = \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u-w} \in \mathbf{k}[u, v, w]$.

Proposition 3.2. The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a $\mathbb{Z}$ -graded algebra.

Definition 3.3. For $\beta, \gamma \in Q_+$, set $e(\beta, \gamma) = \sum_{\nu \in I^\beta, \nu' \in I^\gamma} e(\nu, \nu')$. We define an injective homomorphism $\xi(\beta, \gamma) : R(\beta) \otimes R(\gamma) \rightarrow e(\beta, \gamma)R(\beta + \gamma)e(\beta, \gamma)$ by

$$\begin{aligned} \xi(\beta, \gamma)(e(\nu) \otimes e(\nu')) &= e(\nu, \nu'), \\ \xi(\beta, \gamma)(x_k e(\beta) \otimes 1) &= x_k e(\beta, \gamma), \quad \xi(\beta, \gamma)(1 \otimes x_k e(\gamma)) = x_{k+|\beta|} e(\beta, \gamma), \\ \xi(\beta, \gamma)(\tau_k e(\beta) \otimes 1) &= \tau_k e(\beta, \gamma), \quad \xi(\beta, \gamma)(1 \otimes \tau_k e(\gamma)) = \tau_{k+|\beta|} e(\beta, \gamma). \end{aligned}$$

Definition 3.4. Let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a $\mathbb{Z}$ -graded $R(\beta)$ -module. Define a *grading shift functor* q on the category of graded $R(\beta)$ -modules $R(\beta)\text{-Mod}$ by

$$qM := \bigoplus_{k \in \mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$

For $M, N \in R(\beta)\text{-Mod}$, let $\text{Hom}_{R(\beta)}(M, N)$ be the space of degree preserving morphisms and define $\text{Hom}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{R(\beta)}(q^k M, N)$, which is a space of morphisms up to grading shift. We define $\deg(f) = k$ for $f \in \text{Hom}_{R(\beta)}(q^k M, N)$.

Definition 3.5. For $M \in R(\beta)\text{-Mod}$ and $N \in R(\gamma)\text{-Mod}$, define the *convolution product* $\circ$ by

$$M \circ N := R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N)$$

$\circ$ is an exact bifunctor, that is, $\circ$ is an exact functor with respect to both variables. We also define

$$M \nabla N := \text{hd}(M \circ N), \quad M \Delta N := \text{soc}(M \circ N),$$

where $\text{hd}(M \circ N)$ is the head of $M \circ N$ and $\text{soc}(M \circ N)$ is the socle of $M \circ N$.

Definition 3.6. For $M \in R(\beta)\text{-Mod}$, define the dual module $M^* := \text{Hom}_{\mathbf{k}}(M, \mathbf{k})$ with the $R(\beta)$ -module structure given by

$$(r \cdot f)(u) := f(\psi(r)u) \quad \text{for } r \in R(\beta), u \in M \text{ and } f \in M^*,$$

where ψ is an anti-automorphism on $R(\beta)$ that preserves all generators. if $M \cong M^*$ we call M is *self-dual*.

3.2. Categorification of quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$. Let $R(\beta)\text{-gmod}$ be the full subcategory of $R(\beta)\text{-Mod}$ whose objects are finite-dimensional graded $R(\beta)$ -modules and set $R\text{-gmod} = \bigoplus_{\beta \in Q_+} R(\beta)\text{-gmod}$. Define the functors

$$\begin{aligned} E_i &: R(\beta)\text{-gmod} \rightarrow R(\beta - \alpha_i)\text{-gmod} \\ M &\mapsto e(\alpha_i, \beta - \alpha_i)M, \\ F_i &: R(\beta)\text{-gmod} \rightarrow R(\beta + \alpha_i)\text{-gmod} \\ M &\mapsto F_i(M) = \langle i \rangle \circ M. \end{aligned}$$

where $e(\alpha_i, \beta - \alpha_i) := \sum_{v \in I^\beta, v_1 = i} e(v)$ and $\langle i \rangle := R(\alpha_i)/R(\alpha_i)x_1$ is a 1-dimensional simple $R(\alpha_i)$ -module. We also define E_i^* and F_i^* in the opposite manner. Note that the functors E_i, E_i^*, F_i and F_i^* are exact functors.

Since the grading shift functor q is exact, it induces a $\mathbb{Z}[q, q^{-1}]$ -module structure on $\mathcal{K}(R\text{-gmod})$ via $q[M] := [qM]$. Moreover, since the convolution product $\circ$ is an exact bifunctor, it induces a multiplication on $\mathcal{K}(R\text{-gmod})$ given by $[M][N] := [M \circ N]$. With this multiplication, $\mathcal{K}(R\text{-gmod})$ becomes a $\mathbb{Z}[q, q^{-1}]$ -algebra.

The following theorem is one of the fundamental properties of the quiver Hecke algebra, which states that the quiver Hecke algebra categorifies the quantum group.

Theorem 3.7 ([14, 20]). There exists an isomorphism of $\mathbb{Q}(q)$ -algebras

$$\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n}).$$

3.3. Categorification of the crystal $B(\infty)$.

Lemma 3.8 ([14]). For any simple $R(\beta)$ -module M , $\text{soc}(E_i M)$, $\text{hd}(E_i M)$ and $\text{hd}(F_i M)$ are all simple modules. Here we also have that $\text{soc}(E_i M) \cong \text{hd}(E_i M)$ up to grading shifts.

For $M \in R(\beta)\text{-gmod}$, define

$$\text{wt}(M) = -\beta, \quad \varepsilon_i(M) = \max\{n \in \mathbb{Z} \mid E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle, \quad (3.1)$$

$$\widetilde{E}_i M := q_i^{1-\varepsilon_i(M)} \text{soc}(E_i M) \cong q_i^{\varepsilon_i(M)-1} \text{hd}(E_i M), \quad \widetilde{F}_i M := q_i^{\varepsilon_i(M)} \text{hd}(F_i M). \quad (3.2)$$

Set $\mathbb{B} := \{S \mid S \text{ is a self-dual simple module in } R\text{-gmod}\}$. Then, it follows from Lemma 3.8 that $\widetilde{E}_i$ and $\widetilde{F}_i$ are well-defined on $\mathbb{B}$.

Theorem 3.9 ([16]). The 6-tuple $(\mathbb{B}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \text{wt}, \{\varepsilon_i\}, \{\varphi_i\})_{i \in I}$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B} \xrightarrow{\sim} B(\infty).$$

4. LOCALIZATION

We recall the construction of the localization. See [3], [11], [12] and [13] for more details.

4.1. Localization of monoidal caterogies via graded braider. We recall the subcategory $\mathcal{C}_w$ and its localization $\widetilde{\mathcal{C}}_w$. For $M \in R(\beta)\text{-gMod}$, we define

$$W(M) := \{\gamma \in Q_+ \cap (\beta - Q_+) \mid e(\gamma, \beta - \gamma)M \neq 0\},$$

For $w \in W$, we define the full monoidal subcategory of $R\text{-gmod}$ by

$$\mathcal{C}_w := \{M \in R\text{-gmod} \mid W(M) \subset Q_+ \cap wQ_-\}.$$

We note that for finite type $\mathfrak{g}$ and the longest element $w = w_0$ in W , $\mathcal{C}_{w_0} = R\text{-gmod}$. For a Weyl group element w , let $s_{i_1} \cdots s_{i_l}$ be its reduced expression. For a dominant weight $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l} \Lambda \rangle. \quad (k = 1, \dots, l)$$

We define the *determinantal module* associated with w and Λ by

$$\mathbf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l} \mathbf{1},$$

where $\mathbf{1}$ is a trivial $R(0)$ -module. We have $\mathbf{M}(w\Lambda, \Lambda) \in \mathcal{C}_w$.

Let Λ be a $\mathbb{Z}$ -lattice, $(\mathcal{T} = \oplus_{\lambda \in \Lambda} \mathcal{T}_\lambda, \otimes)$ be a $\mathbf{k}$ -linear Λ -graded monoidal category, q be the grading shift functor on $\mathcal{T}$ and $\mathbf{1} \in \mathcal{T}_0$ be the unit object of $\mathcal{T}$.

Definition 4.1 ([11]). A *graded braider* is a triple (C, R_C, ϕ) , where $C \in \mathcal{T}$, $\mathbb{Z}$ -linear map $\phi : \Lambda \rightarrow \mathbb{Z}$ and a morphism

$$R_C : C \otimes X \rightarrow q^{\phi(\lambda)} X \otimes C \quad (X \in \mathcal{T}_\lambda),$$

which is functorial in $X \in \mathcal{T}$ such that satisfying the following commutative diagram:

$$\begin{array}{ccc} C \otimes X \otimes Y & \xrightarrow{R_C(X) \otimes Y} & q^{\phi(\lambda)} X \otimes C \otimes Y \\ & \searrow R_C(X \otimes Y) & \downarrow X \otimes R_C(Y) \\ & & q^{\phi(\lambda+\mu)} (X \otimes Y) \otimes C, \end{array} \quad \begin{array}{ccc} C \otimes \mathbf{1} & \xrightarrow{R_C(\mathbf{1})} & \mathbf{1} \otimes C \\ & \searrow \cong & \downarrow \cong \\ & & C. \end{array}$$

The triple

$$\{M(w\Lambda_i, \Lambda_i, R_{M(w\Lambda_i, \Lambda_i)}, \phi_{M(w\Lambda_i, \Lambda_i)})_{i \in I} \quad (4.1)$$

becomes a non-degenerate real commuting family of braiders. See [11] for the definition of a real commuting family of braiders.

We apply the localization procedure to $R\text{-gmod}$ and $\mathcal{C}_w$. Let $\widetilde{\mathcal{C}}_w := \mathcal{C}_w[M(w\Lambda_i, \Lambda_i)^{\circ-1}; i \in I]$ denote the localization of $\mathcal{C}_w$ by (4.1). We also let $\widetilde{R\text{-gmod}}[w] := R\text{-gmod}[M(w\Lambda_i, \Lambda_i)^{\circ-1}; i \in I]$ denote the localization of $R\text{-gmod}$ by the same graded braider. We set $C_\Lambda := M(w\Lambda, \Lambda)$, in particular, $C_i := M(w\Lambda_i, \Lambda_i)$. We summarize the propertise of $\widetilde{R\text{-gmod}}[w]$.

Proposition 4.2 ([11]). Let $Q_w : R\text{-gmod} \rightarrow \widetilde{R\text{-gmod}}[w]$ be the canonical functor. Then,

- (1) $\widetilde{R\text{-gmod}}[w]$ is an abelian category and the functor Q_w is exact.
- (2) For any simple object $S \in R\text{-gmod}$, $Q_w(S)$ is simple or 0 in $\widetilde{R\text{-gmod}}[w]$.
- (3) $\widetilde{C}_i := Q_w(C_i)$ ($i \in I$) is invertible central graded braider in $\widetilde{R\text{-gmod}}[w]$.

For $\mu \in P$, define $\widetilde{C}_\mu$ such that $\widetilde{C}_\mu := Q_w(C_\mu)$ for $\mu \in P_+$, $\widetilde{C}_{-\Lambda_i} = C_i^{\circ-1}$ and $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_\lambda \circ \widetilde{C}_\mu$ for $\lambda, \mu \in P$ up to grading shifts.

- (4) Any simple object in $\widetilde{R\text{-gmod}}[w]$ is isomorphic to $\widetilde{C}_\Lambda \circ Q_w(S)$ for some simple module $S \in R\text{-gmod}$ and $\Lambda \in P$.

Note that in (4) $\Lambda \in P$ and $S \in R\text{-gmod}$ are not necessarily unique.

For any $w \in W$, we have a quasi-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_w & \xrightarrow{\quad \iota_w \quad} & R\text{-gmod} \\ \Phi_w \downarrow & & \downarrow Q_w \\ \widetilde{\mathcal{C}}_w & \xrightarrow{\quad \widetilde{\iota}_w \quad} & \widetilde{R\text{-gmod}[w]}. \end{array}$$

where Φ_w and Q_w denote the localization functors and $\widetilde{\iota}_w$ is a functor induced from the inclusion functor ι_w , which gives an equivalence of categories [13].

4.2. Crystal Structure on localized quantum coordinate rings. We define a crystal structure of $\widetilde{R\text{-gmod}[w]}$.

Definition 4.3. Let X, Y be objects in a monoidal category $\mathcal{T}$, and $\varepsilon : X \otimes Y \rightarrow 1$ and $\eta : 1 \rightarrow Y \otimes X$ morphisms in $\mathcal{T}$. We say that a pair (X, Y) is *dual pair* or X is a *left dual* to Y or Y is a *right dual* to X if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\text{id} \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes \text{id}} 1 \otimes X \simeq X, \quad Y \simeq 1 \otimes Y \xrightarrow{\eta \otimes \text{id}} Y \otimes X \otimes Y \xrightarrow{\text{id} \otimes \varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathcal{D}(X)$ and a left dual to X by $\mathcal{D}^{-1}(X)$.

Theorem 4.4 ([11], [12]). $\widetilde{\mathcal{C}}_w$ is *rigid*, i.e., every object in $\widetilde{\mathcal{C}}_w$ has left and right duals.

Definition 4.5 ([8]). Let $(\mathcal{C}, \otimes)$ be a graded monoidal category and $M, N \in \mathcal{C}$ be simple objects. When $\dim \text{Hom}(M \otimes N, N \otimes M) = 1$, the pair (M, N) is said to be Λ -*definable*. A non-zero morphism $\mathbf{r}_{M,N}$ in this space is called the R -*matrix*, and we set $\Lambda(M, N) := \deg(\mathbf{r}_{M,N})$.

Let $\text{Irr}(\widetilde{\mathcal{C}}_w)$ be the set of equivalence classes of simple objects in $\widetilde{\mathcal{C}}_w$ up to grading shifts. We define a crystal structure on $\text{Irr}(\widetilde{\mathcal{C}}_w)$.

Theorem 4.6. [8] For $X \in \text{Irr}(\widetilde{\mathcal{C}}_w)$, we define the operators by

$$\begin{aligned} \varepsilon_i(X) &= d_i^{-1} \widetilde{\Lambda}(Q_w(\langle i \rangle), X), \quad \varepsilon_i^* = d_i^{-1} \widetilde{\Lambda}(X, Q_w(\langle i \rangle)), \\ \varphi_i(X) &= \varepsilon_i(X) + \langle h_i, \text{wt}(X) \rangle, \quad \varphi_i^*(X) = \varepsilon_i^*(X) + \langle h_i, \text{wt}(X) \rangle, \\ \widetilde{F}_i X &= q_i^{\varepsilon_i(X)} Q_w(\langle i \rangle) \nabla X, \quad \widetilde{F}_i^* = q_i^{\varepsilon_i^*(X)} X \nabla Q_w(\langle i \rangle), \\ \widetilde{E}_i X &= q_i^{\varphi_i(X)+1} X \nabla \mathcal{D} Q_w(\langle i \rangle), \quad \widetilde{E}_i^* X = q_i^{\varphi_i^*(X)+1} \mathcal{D}^{-1} Q_w(\langle i \rangle) \nabla X, \end{aligned}$$

where $d_i = (\alpha_i, \alpha_i)/2$ and $\widetilde{\Lambda}(M, N) := (\Lambda(M, N) + (\text{wt}(M), \text{wt}(N)))/2$. Then these operators define a crystal structure on $\text{Irr}(\widetilde{\mathcal{C}}_w)$.

Let $\mathbf{i} = i_1 i_2 \cdots i_l$ be a reduced word, $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ be a reduced expression of w and $\mathbb{B}_{\mathbf{i}}$ be the cellular crystal associated to $\mathbf{i}$. For $M \in \mathcal{C}_w$, we define $K_{\mathbf{i}}(M) \in \mathbb{Z}^l$ by

$$\begin{aligned} M_l &= M, \quad c_k = \varepsilon_{i_k}^*(M_k) \text{ for } 1 \leq k \leq l, \\ M_{k-1} &= (\widetilde{E}_{i_k}^*)^{c_k}(M_k), \quad K_{\mathbf{i}}(M) = (c_1, \dots, c_l). \end{aligned}$$

We regard $K_{\mathbf{i}}(M)$ as an element of $\mathbb{B}_{\mathbf{i}}$ by $(c_1, \dots, c_l) \mapsto \widetilde{f}_{i_1}^{c_1}(0)_{i_1} \otimes \cdots \otimes \widetilde{f}_{i_l}^{c_l}(0)_{i_l}$. We extend $K_{\mathbf{i}} : \mathcal{C}_w \rightarrow \mathbb{Z}^l$ to $K_{\mathbf{i}} : \widetilde{\mathcal{C}}_w \rightarrow \mathbb{Z}^l$ by

$$\widetilde{C}_{\Lambda}^{-1} \circ Q_w(M) \mapsto K_{\mathbf{i}}(M) - K_{\mathbf{i}}(C_{\Lambda})$$

for $\Lambda \in P_+$ [8].

The following theorem is necessary to define ε_i^* on $\mathbb{B}_{\mathbf{i}}$,

Theorem 4.7 ([8]). Let $w \in W$ and $i \in I$ satisfying $w' = ws_i < w$. Define $\mathbf{E}_{w',w}^* : \text{Irr}(\widetilde{\mathcal{C}}_w) \rightarrow \text{Irr}(\widetilde{\mathcal{C}}_{w'})$ by

$$(\widetilde{C}_{\Lambda})^{-1} \circ Q_w(M) \mapsto Q_{w'}(\widetilde{E}_i^{*\max} C_{\Lambda})^{-1} \circ Q_{w'}(\widetilde{E}_i^{*\max} M).$$

Then we have a bijective map

$$\Psi_{w',w} : \text{Irr}(\widetilde{\mathcal{C}}_w) \longrightarrow \text{Irr}(\widetilde{\mathcal{C}}_{w'}) \times \mathbb{Z} \quad (X \mapsto (\mathbf{E}_{w',w}^*(X), \varepsilon_i^*(X)))$$

and an isomorphism of crystals $K_{\mathbf{i}} : \text{Irr}(\widetilde{\mathcal{C}}_w) \rightarrow \mathbb{Z}^l \simeq \mathbb{B}_{\mathbf{i}}$.

5. CHARACTERIZATION OF THE UNIT OBJECT

In the rest of this article, we restrict ourselves to the case where $\mathfrak{g}$ is of classical finite type. As an application of Theorem 4.7, we characterize the unit object of $R\text{-}\widetilde{\text{gmod}}[w]$. Let $w = w_0$ be the longest element of the Weyl group W , and fix a reduced expression $\mathbf{i}$ of w_0 , which we call a *reduced longest word*. Then any letter i can be moved to the rightmost position via braid moves, each inducing a braid-type isomorphism $\phi_{ij}^{(k)}$ ($k = 0, 1, 2$). This yields a crystal isomorphism

$$\phi : \mathbb{B}_{\mathbf{i}} \rightarrow \mathbb{B}_{\mathbf{i}'} \otimes B_i \quad (x \mapsto \phi(x) = b' \otimes (a)_i),$$

where ϕ is a composition of such isomorphisms and $\mathbf{i}, \mathbf{i}'$ are reduced expressions of w . For $x \in \mathbb{B}_{\mathbf{i}}$, we define $\varepsilon_i^*(x) = -a$. This definition is motivated by $\Psi_{w',w}$ in Theorem 4.7, and well-defined since it is independent of both the choice of reduced longest word and the sequence of braid moves. We write ε_i^* for $\varepsilon_i^*(x)$ when no confusion arises.

To exhibit explicit formulae of $\varepsilon_i^*(x)$, we express $x \in \mathbb{B}_{\mathbf{i}}$ as

$$x = (z_{1,i_1})_{i_1} \otimes \dots \otimes (z_{j,i_k})_{i_k} \otimes \dots \otimes (z_{s,i_n})_{i_n},$$

where the double index (j, i_k) of z_{j,i_k} indicates the j -th occurrence of i_k reading from left to right. For example, for $\mathbf{i} = 1(21)(321)$, which is a longest word of A_3 , we express

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes (z_{1,3})_3 \otimes (z_{2,2})_2 \otimes (z_{3,1})_1.$$

5.1. Main results. We now provide explicit formulae for $\varepsilon_i^*(x)$ for a fixed reduced longest word $\mathbf{i}$ in each type and a characterization of the unit object in $R\text{-}\widetilde{\text{gmod}}[w]$.

Proposition 5.1. Let W be the Weyl group, $\mathbf{i} = i_1 \dots i_l$ be a reduced longest word and $\mathbb{B}_{\mathbf{i}}$ be the cellular crystal associated $\mathbf{i}$. In what follows, we understand $z_{j,k} = 0$ whenever $k = 0$.

- (1) Type A_n : Fix $\mathbf{i} = (1)(21) \dots (n-1 \dots 21)(n \dots 21)$ as a reduced longest word. For $x \in \mathbb{B}_{\mathbf{i}}$ expressed as

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes \dots \otimes (z_{1,n})_n \otimes \dots \otimes (z_{n-1,2})_2 \otimes (z_{n,1})_1,$$

$\varepsilon_i^*(x)$ is given by

$$\varepsilon_i^*(x) = \max_{1 \leq k \leq i} (z_{n-i+2,k-1} - z_{n-i+1,k}) \quad (1 \leq i \leq n).$$

- (2) Type B_n : Fix $\mathbf{i} = (12 \dots n)^n$ as a reduced longest word. For $x \in \mathbb{B}_{\mathbf{i}}$ expressed as

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes \dots \otimes (z_{1,n})_n \otimes (z_{2,1})_1 \otimes \dots \otimes (z_{n,n-1})_{n-1} \otimes (z_{n,n})_n,$$

$\varepsilon_i^*(x)$ is given by

$$\varepsilon_i^* = \max_{1 \leq k \leq n-i-1} (-\zeta_i, z_{i+k+1,n-k} - z_{i+k+1,n-k-1}) \quad (1 \leq i \leq n-2),$$

$$\zeta_i = -\max(-z_{i+1,n-1} + z_{i+1,n}, -\eta_i, z_{i+1,n-1} - z_{i,n}) \quad (1 \leq i \leq n-1),$$

$$\eta_i = -\max_{1 \leq k \leq n-1} (z_{i+1,k-1} - z_{i,k}) \quad (i \leq i \leq n-1),$$

$$\varepsilon_{n-1}^* = -\zeta_{n-1}, \quad \varepsilon_n^* = -z_{n,n}.$$

- (3) Type C_n : Fix $\mathbf{i} = (12 \dots n)^n$ as a reduced longest word. $x \in \mathbb{B}_{\mathbf{i}}$ has the same expression as in type B_n . $\varepsilon_i^*(x)$ and $\eta_i(x)$ are the same as those in type B_n , while ζ_i is given by

$$\zeta_i(x) = -\max(-2z_{i,n} + z_{i+1,n-1}, -\eta_i, z_{i+1,n-1} - 2z_{i+1,n}) \quad (1 \leq i \leq n-1).$$

(4) Type D_n : Fix $\mathbf{i} = (12 \dots n)^{n-1}$ as a reduced longest word. For $x \in \mathbb{B}_{\mathbf{i}}$ expressed as

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes \dots \otimes (z_{1,n})_n \otimes (z_{2,1})_1 \otimes \dots \otimes (z_{n-1,n-1})_{n-1} \otimes (z_{n-1,n})_n,$$

$\varepsilon_i^*(x)$ is given by

$$\begin{aligned} \varepsilon_i^* &= \max_{1 \leq k \leq n-i-2} (-\kappa_i, z_{i+k+1,n-k-1} - z_{i+k+1,n-k-2}) \quad (1 \leq i \leq n-3), \\ \kappa_i &= -\max \left(\begin{array}{l} -\theta_i, z_{i+1,n-1} - z_{i,n}, z_{i+1,n-2} - z_{i,n-1} - z_{i,n}, \\ z_{i+1,n} - z_{i,n-1}, z_{i+1,n} + z_{i+1,n-1} - z_{i+1,n-2} \end{array} \right) \quad (1 \leq i \leq n-2), \\ \theta_i &= -\max_{1 \leq k \leq n-2} (z_{i+1,k-1} - z_{i,k}) \quad (1 \leq i \leq n-2), \\ \varepsilon_{n-2}^* &= -\kappa_{n-2}, \quad \varepsilon_{n-1}^* = -z_{n-1,n-1}, \quad \varepsilon_n^* = -z_{n-1,n}. \end{aligned}$$

Using these explicit formulae for ε_i^* , we obtain the following result.

Theorem 5.2. Let W be the Weyl group of classical finite type. $\mathbf{i} = i_1 \dots i_l$ be a reduced longest word, and $\mathbb{B}_{\mathbf{i}}$ be the cellular crystal associated $\mathbf{i}$. For $x \in \mathbb{B}_{\mathbf{i}}$, the following are equivalent.

- (1) $\text{wt}(x) = 0$ and for all $i \in I$, $\varepsilon_i^*(x) = 0$.
- (2) $x = (0)_{i_1} \otimes \dots \otimes (0)_{i_l} \in \mathbb{B}_{\mathbf{i}}$.

As an immediate consequence of Theorem 5.2, we obtain the following characterization of the unit object in $R\text{-gmod}[w]$.

Corollary 5.3. For $X \in \text{Irr}(\widetilde{\mathcal{C}}_w)$, the following are equivalent.

- (1) $\text{wt}(X) = 0$ and for all $i \in I$, $\varepsilon_i^*(X) = 0$.
- (2) $X = \mathbf{1}$.

5.2. Outline of proof. We outline the proof of Proposition 5.1 and Theorem 5.2 for type A_n . Fix $\mathbf{i} = (1)(21) \dots (n-1 \dots 21)(n \dots 21)$ as a reduced longest word. For this word, there is an algorithm to move a letter i to the right position, as described below.

- (Step1) By applying 2-moves, move the i -th occurrence of the letter 1 (reading from **right to left**) as far to the right as possible. Then we obtain the subword 121 and apply a 3-move : $121 \mapsto 212$.
- (Step2) By applying 2-moves, move the right 2 appeared in (Step1) as far to the right as possible. Then we obtain the subword 232 and apply a 3-move : $232 \mapsto 323$.
- (Step3) Iterate (Step 2) successively for $3, 4, \dots, i-1$ until the subword $i(i-1)i$ appears.
- (Step4) Since there only exist letters strictly less than $i-1$ on the right side of the right i appeared in (Step3), one can move the letter i to the rightmost position by applying 2-moves.

This algorithm induces an crystal isomorphism $\phi : \mathbb{B}_{\mathbf{i}} \rightarrow \mathbb{B}_{\mathbf{i}'i}$ where $\mathbf{i}$ and $\mathbf{i}'i$ are reduced words of w . Via this isomorphism, $\varepsilon_i^*(x)$ is given as in Proposition 5.1 (1), which can be proven by induction on n , the rank of A_n . It is then easy to see that the only solution to $\text{wt}(x) = 0$ and $\varepsilon_i^*(x) = 0$ ($\forall i \in I$) is $x = (0)_{i_1} \otimes \dots \otimes (0)_{i_l} \in \mathbb{B}_{\mathbf{i}}$.

The proof is best illustrated by considering a small-rank case.

Example 5.4. Let $n = 4$ and fix $\mathbf{i} = 1(21)(321)(4321)$ as a reduced longest word. We move 3 to the rightmost position as an example. The procedure in this case is

$$\begin{aligned} 1213214321 &\xrightarrow{\text{(Step1)}} 1231214321 \xrightarrow{\text{(Step1)}} 123\mathbf{2}124321 \xrightarrow{\text{(Step2)}} 1232142321 \\ &\xrightarrow{\text{(Step2)}} 123214\mathbf{3}231 \xrightarrow{\text{(Step4)}} 123214321\mathbf{3} \end{aligned}$$

Then this procedure induce the following composition of isomorphisms

$$\begin{aligned}
& (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes (z_{1,3})_3 \otimes (z_{2,2})_2 \otimes (z_{3,1})_1 \otimes (z_{1,4})_4 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1 \\
& \mapsto (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{1,3})_3 \otimes (z_{2,1})_1 \otimes (z_{2,2})_2 \otimes (z_{3,1})_1 \otimes (z_{1,4})_4 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1 \\
& \mapsto (z_{1,3})_3 \otimes (X)_2 \otimes (Y)_1 \otimes (-\max(-z_{2,1}, z_{3,1} - z_{2,2}))_2 \otimes (z_{1,4})_4 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1 \\
& \mapsto (z_{1,3})_3 \otimes (X)_2 \otimes (Y)_1 \otimes (z_{1,4})_4 \otimes (-\max(-z_{2,1}, z_{3,1} - z_{2,2}))_2 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1 \\
& \mapsto (S)_3 \otimes (T)_2 \otimes (-\max(\max(-z_{2,1}, z_{3,1} - z_{2,2}), z_{3,2} - z_{2,3}))_3 \otimes (z_{4,1})_1 \\
& \mapsto (S)_3 \otimes (T)_2 \otimes (z_{4,1})_1 \otimes (-\max(\max(-z_{2,1}, z_{3,1} - z_{2,2}), z_{3,2} - z_{2,3}))_3.
\end{aligned}$$

Hence, we have

$$\varepsilon_3^* = \max(\max(-z_{2,1}, z_{3,1} - z_{2,2}), z_{3,2} - z_{2,3}) = \max(-z_{2,1}, z_{3,1} - z_{2,2}, z_{3,2} - z_{2,3}).$$

Similarly, we obtain

$$\varepsilon_1^* = z_{4,1}, \quad \varepsilon_2^* = \max(-z_{3,1}, z_{4,1} - z_{3,2}), \quad \varepsilon_4^* = \max(-z_{1,1}, z_{2,1} - z_{1,2}, z_{2,2} - z_{1,3}, z_{2,3} - z_{1,4}).$$

Assuming $\text{wt} = 0$ and $\varepsilon_i^* = 0$ for all $i \in I$ yields the following system of inequalities

$$\begin{aligned}
& z_{1,1}, z_{2,1}, z_{3,1}, z_{4,1} \geq 0, \quad z_{2,1} \leq z_{1,2}, \quad z_{3,1} \leq z_{2,2} \leq z_{1,3}, \quad z_{4,1} \leq z_{3,2} \leq z_{2,3} \leq z_{1,4}, \\
& z_{1,1} + z_{2,1} + z_{3,1} + z_{4,1} = z_{1,2} + z_{2,2} + z_{3,2} = z_{1,3} + z_{2,3} = z_{1,4} = 0.
\end{aligned}$$

Then it is easy to see $z_{i,j} = 0$ for all i, j .

Remark 5.5. Although we assumed that $\mathfrak{g}$ is of classical finite type, there is no mathematical reason to assume so. We conjecture that Theorem 5.2 holds for all finite types of Lie algebras. Indeed, we have computed explicit formulae for ε_i^* , and using the description, we have verified that the theorem holds for type G_2, F_4 and E_6 .

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