
SEIKO
effectivenes
[GenEl
[Belyi]
non-critical
Belyi map
Vojta
Vojta
concretely
computable
IUTchI ~ IV

effectiveness

[GenEll] (current)

[Belyi] non-critical Belyi map

Vojta general case

Vojta special case $\mathbb{P}^1 \setminus \{0,1,\infty\}$

cpt'ly bdd domain

concretely computable

IUTchI~IV

uniformity

w.r.t.

$[F:\mathbb{Q}]$

$= d$

$d^{4+\epsilon}$

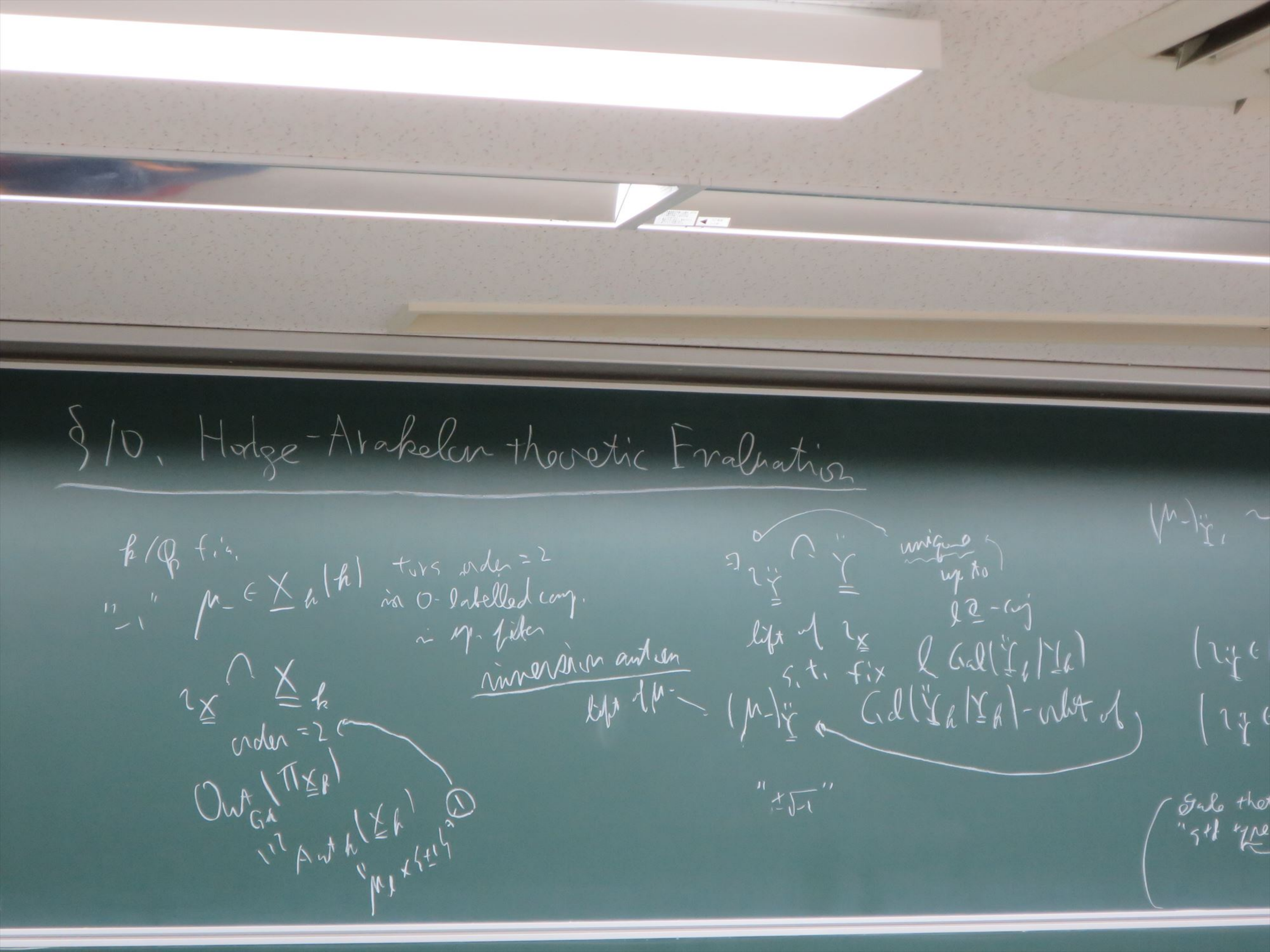
§10.

$k/($

$"$

$"$

-1



§10. Hodge-Arakelov theoretic Evaluation

$k/\mathbb{Q}_p$ fin.

"-1" $\mu_- \in \underline{X}_k(k)$ tors order $= 2$

in 0-labelled comp.

≈ sp. fiber

$\iota_{\underline{X}} \curvearrowright \underline{\underline{X}}_k$

order $= 2$

$\mathrm{Out}_{G_k}(\Pi_{\underline{X}_k})$

$\supseteq \mathrm{Aut}_k(\underline{X}_k)$

$\cong \mu_l \times \{\pm 1\}$ ①

inversion autom.

lift of μ_- → $(\mu_-)_{\ddot{\underline{Y}}}$

$\exists\ \iota_{\ddot{\underline{Y}}} \curvearrowright \ddot{\underline{\underline{Y}}}$ unique up to $l\underline{\mathbb{Z}}$-conj

lift of $\iota_{\underline{X}}$

s.t. fix & $\mathrm{Gal}(\ddot{\underline{Y}}_k/\underline{\underline{Y}}_k)$

$\mathrm{Gal}(\ddot{\underline{Y}}_k/\underline{\underline{Y}}_k)$-orbit of

"$\pm\sqrt{-1}$"

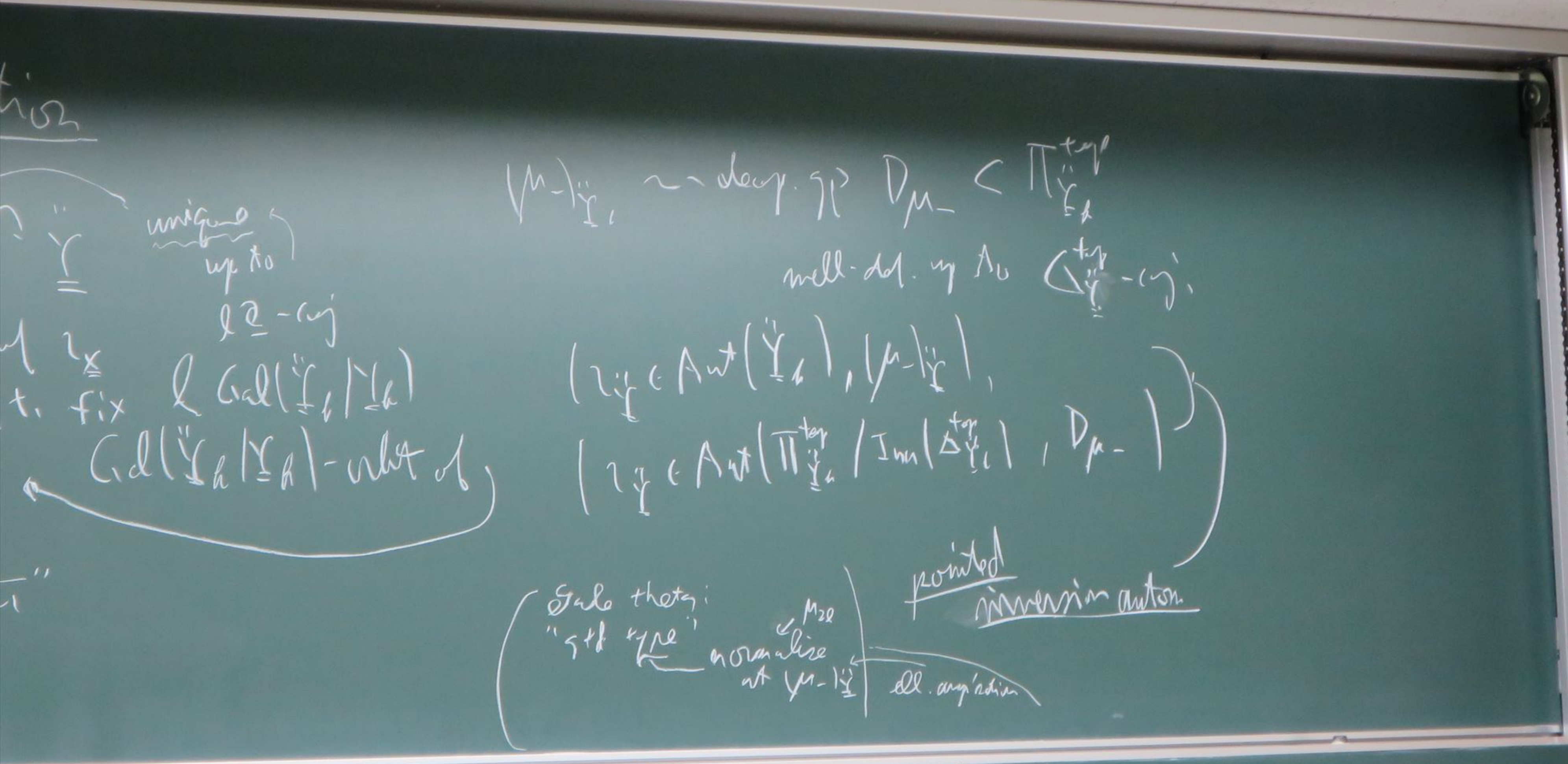
$(\mu_-)_{\ddot{\underline{Y}}},\ \sim$

$(\iota_{\ddot{\underline{Y}}} \in$

$(\iota_{\ddot{\underline{Y}}} \in$

Galois ther

"s.t. "$\underline{k}$ne



tion

$\ddot{\underline{\underline{Y}}}$ unique up to $\ell\underline{\mathbb{Z}}$-conj

$\iota_{\underline{\underline{X}}}$

t. fix & $\mathrm{Gal}(\ddot{\underline{Y}}_k / \underline{Y}_k)$

$\mathrm{Gal}(\ddot{\underline{Y}}_k / \underline{\underline{Y}}_k)$-orbit of

$(\mu_-)_{\ddot{\underline{Y}}}$, ⇝ decomp. gp $D_{\mu_-} \subset \Pi^{\mathrm{temp}}_{\ddot{\underline{Y}}_k}$

well-def. up to $\Delta^{\mathrm{temp}}_{\ddot{\underline{Y}}}$-conj.

$(\iota_{\ddot{\underline{Y}}} \in \mathrm{Aut}(\ddot{\underline{Y}}_k), (\mu_-)_{\ddot{\underline{Y}}})$,

$(\iota_{\ddot{\underline{Y}}} \in \mathrm{Aut}(\Pi^{\mathrm{temp}}_{\ddot{\underline{Y}}_k}) / \mathrm{Inn}(\Delta^{\mathrm{temp}}_{\ddot{\underline{Y}}_k}), D_{\mu_-})$

pointed inversion autom.

Étale theta: "std type" normalize at $(\mu_-)_{\ddot{\underline{Y}}}$ $\mu_{2\ell}$

ell. cusp'ization

10. Hodge-Arakelov theoretic Evaluation

[IUTch II, Rem 1.4.1]

$k/\mathbb{Q}_p$ fin.

"-1" $\mu_- \in \underline{X}_k(k)$ tors order $= 2$ in 0-labelled comp.

$\sim$ up. [illegible]

inversion autom

$\iota_{\underline{X}} \curvearrowright \underline{\underline{X}}_k$

order $= 2$

$\mathrm{Out}_{G_k}(\Pi_{\underline{X}_k})$ ①

$(\underline{X}_k)$

$\exists\ \iota_{\ddot{\underline{Y}}} \curvearrowright \ddot{\underline{\underline{Y}}}$

unique up to $\pm_0$

& $\underline{\Delta}$-conj

lift of $\iota_{\underline{X}}$

s.t. fix

& $\mathrm{Gal}(\ddot{\underline{Y}}_k | \underline{Y}_k)$

$\mathrm{Gal}(\ddot{\underline{Y}}_k | \underline{Y}_k)$-orbit of

lift of μ_- $\longrightarrow$ $(\mu_-)_{\ddot{\underline{Y}}}$

"$\pm\sqrt{-1}$"

§10.1 Radial Environment

Def 10.1 [IUTch II, Ex 1.7]

(1). $(\mathcal{R}, \mathcal{C}, \Phi)$: radial environment

$\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{R}, \mathcal{C}$: cat's s.t. $\forall$ hom's are isom's

$\Phi : \mathcal{R} \longrightarrow \mathcal{C}$ (alg'mically def'd) functor s.t. ess. surj.

$\mathcal{R}$: radial data

$\mathcal{C}$: coric data

Φ : radial alg'm

ment

om's are isom's

ally def'd) functor

s. surj.

(2). $(\mathcal{R}, \mathcal{C}, \Phi)$: multiradial env.

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ Φ : full

$(\mathcal{R}, \mathcal{C}, \Phi)$: uniradial

$\underset{\text{def}}{\Longleftrightarrow}$ $(\mathcal{R}, \mathcal{C}, \Phi)$: not multiradial

(3). $(\mathcal{R}, \mathcal{C}, \Phi)$, $({}^{\dagger}\mathcal{R}, {}^{\dagger}\mathcal{C}, {}^{\dagger}\Phi)$: rad. env.'s

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\sim"', "\Phi_{\mathcal{R}}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowright"] & {}^{\dagger}\mathcal{R} \arrow[d, "{}^{\dagger}\Phi"] \\
\mathcal{C} \arrow[r, "\sim"', "\Phi_{\mathcal{C}}"] & {}^{\dagger}\mathcal{C}
\end{tikzcd}$$

given $\Phi_{\mathcal{R}}$: $\underline{\text{multiradially defined}}$

$\underset{\text{def}}{\Longleftrightarrow}$ $(\mathcal{R}, \mathcal{C}, \Phi)$: mult. env.

e.g.

radial d

$\mathcal{R}$ Ob

H

r.

ltiradial

rad. envs

multiradially defined

$, \rho, \Phi)$: mult. env.

e.g. $(k/\mathbb{Q}_p^{\text{fin.}})$

radial data

$\mathcal{R}$ Obj (Π, G, α)

$\Pi \rightsquigarrow \Delta$

$\exists \cong \Pi^{\text{top}}_{\underline{X}_k}$ $\exists \cong G_k$ $\alpha : \Pi/\Delta \overset{\alpha}{\xrightarrow{\sim}} G$

full poly

Hom $(\Pi, G, \alpha) \xrightarrow{\sim} (\Pi^*, G^*, \alpha^*)$

pair isoms $\begin{pmatrix} \Pi \xrightarrow{\sim} \Pi^* \\ G \xrightarrow{\sim} G^* \end{pmatrix}$

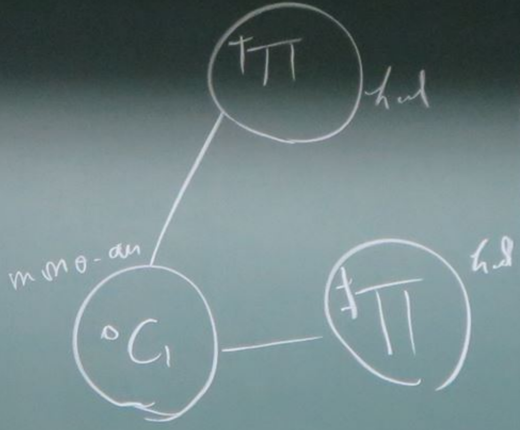
Coric data

$\mathcal{C}$ Obj $G \overset{\exists}{\cong} G_\ell$

Hom $G \underset{\text{isom.}}{\cong} G^*$

$\Phi(\Pi, G, \alpha) := G$

$\rightsquigarrow \Phi$: full $\rightsquigarrow (\mathcal{R}, \mathcal{C}, \Phi)$: multirad. env.



cf. $\Pi \overset{\Phi}{\longmapsto} \Pi/\Delta$

not full

$\rightsquigarrow$ unirad.

cf. $\mathrm{Aut}(\bar{k}/k) \underset{\neq}{\subset} \mathrm{Out}(G_k)$ $(p \geq 2)$

real an

hol

$\widehat{\Pi}^{\text{hol}}$

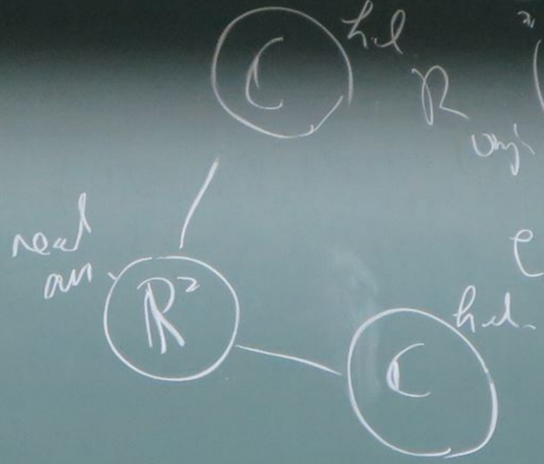
cf. $\Pi \xmapsto{\Phi} \Pi/\Delta$

not full

⇝ unirad.

cf. $\mathrm{Aut}(\bar{k}/k) \underset{\neq}{\subset} \mathrm{Out}(G_k)$

$(p>2)$



$R_{\text{unif.}}\ \overset{\mathbb{C}}{\wr\ \shortparallel}\ (C, V, \alpha)$ ← full poly

$\downarrow$

$e \quad \mathbb{R}^2$

$G \overset{\partial}{\curvearrowright} G_k$

$\mathrm{Isomet}(G)$

cp

preserved by $G_1 \cong G_2$

$\searrow$

$G \longmapsto \mathrm{Isomet}($

$\mathrm{Isomet}(-$

$, \alpha)$ full poly

$G \curvearrowright G_k \rightsquigarrow \Theta^{\times\mu}(G) \curvearrowleft \sim \Theta^{\times}/\mu$

$\mathrm{Isomet}(G) := \{$ G-isometries of $\Theta^{\times\mu}(G)$

i.e. $\alpha : \Theta^{\times\mu}(G) \xrightarrow{\sim} \Theta^{\times\mu}(G)$ G-equiv.

s.t. $G \supset^{\forall} H$ open, preserves $\mathrm{Im}\{\Theta^{\times}(G)^H \to \Theta^{\times\mu}(G)^H\}$ integral str. $\}$

cpt, top. closed

preserved by $G_1 \cong G_2$

$G \mapsto \mathrm{Isomet}(G)$

$\mathrm{Isomet}(-) \rightsquigarrow \mathrm{Isomet}$

$\mathrm{Im}(\hat{\mathcal{O}}^{\times}) = \mathrm{Im}\{\hat{\mathcal{O}}^{\times} \to \mathcal{O}^{\times}_{\mu} \subset \mathrm{Isomet}(G)\}$

$\mathrm{Ism}({}^{\dagger}\mathfrak{F}^{\vdash}, {}^{\ddagger}\mathfrak{F}^{\vdash})$

$\downarrow$

$\mathrm{Ism}({}^{\dagger}\mathfrak{S}^{\vdash}, {}^{\ddagger}\mathfrak{S}^{\vdash})$

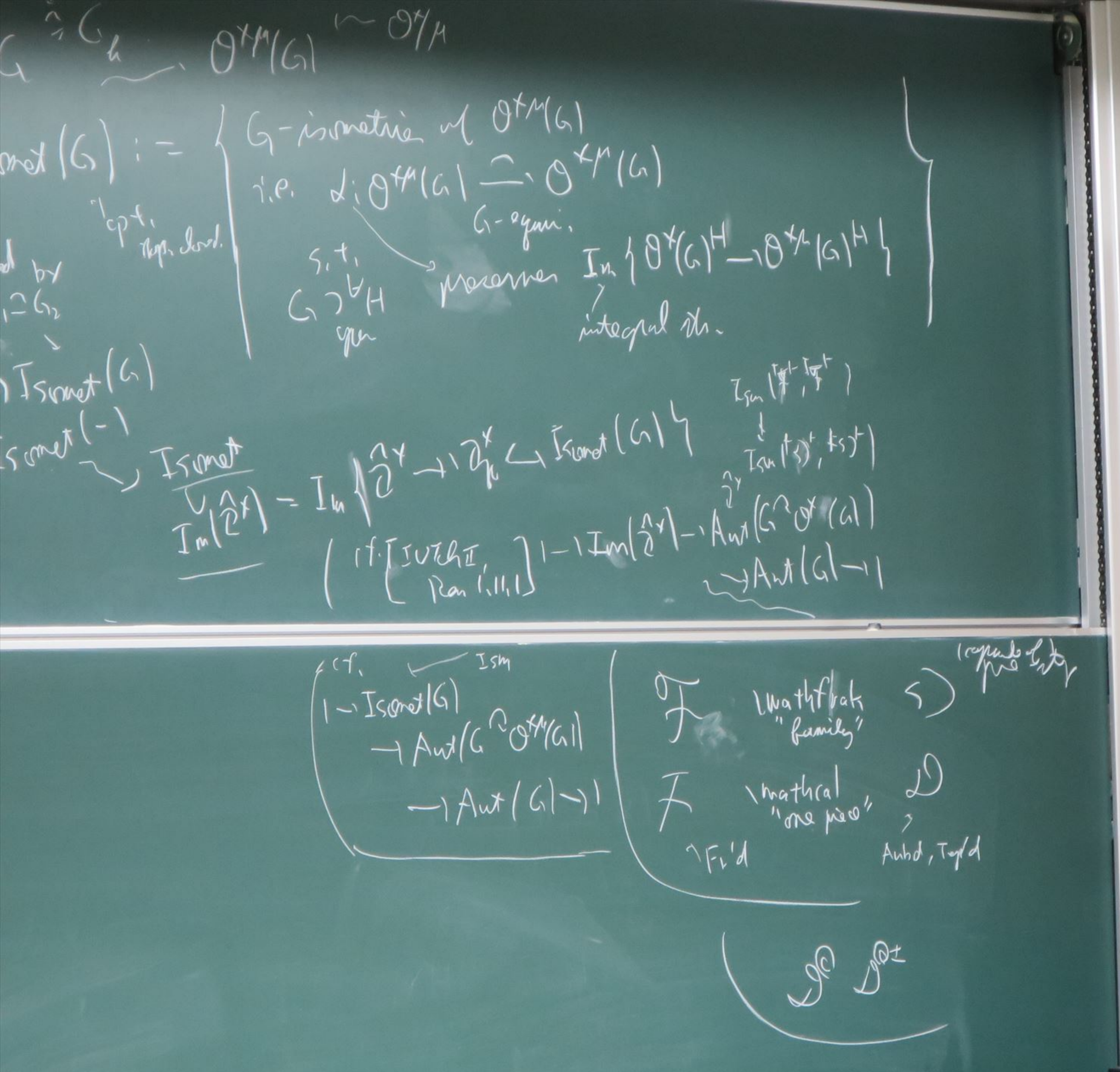
(cf. [IUTchII, Rem 1.11.1]) $1 \to \mathrm{Im}(\hat{\mathcal{O}}^{\times}) \to \mathrm{Aut}(G \curvearrowright \mathcal{O}^{\times}(G)) \to \mathrm{Aut}(G) \to 1$

Ism

$(\ \mathrm{Isomet}(G)$

$(\ \mathfrak{F}$ \mathfrak $\mathfrak{S})$

(regarded ... pre ...)



$G \xrightarrow{\sim} G_k \rightsquigarrow \mathcal{O}^{\times\mu}(G) \sim \mathcal{O}^{\times}/\mu$

smet(G) := { G-isometries of $\mathcal{O}^{\times\mu}(G)$

i.e. $\alpha : \mathcal{O}^{\times\mu}(G) \xrightarrow{\sim} \mathcal{O}^{\times\mu}(G)$ G-equiv.

s.t. $G \supset {}^{\forall}H$ open, α preserves $\mathrm{Im}\{\mathcal{O}^{\times}(G)^H \to \mathcal{O}^{\times\mu}(G)^H\}$ }

integral str.

cpt. top. closed

d by ${}_1 \xrightarrow{\sim} G_2$

Ismet(G)

smet$(-)$

$\rightsquigarrow \dfrac{\mathrm{Ismet}}{\cup \; \mathrm{Im}(\hat{\partial}^{\times})} = \mathrm{Im}\{\hat{\partial}^{\times} \to \partial^{\times}_{\mu} \subset \mathrm{Ismet}(G)\}$

Ism$(\Pi^{\times\mu}_{F}, \Pi^{\times\mu}_{F})$ ↓ Ism$(\ldots, \ldots)$

(cf. [IUTchII, Rem 1.11.1]) $1 \to \mathrm{Im}(\hat{\partial}^{\times}) \to \mathrm{Aut}(G \curvearrowright \mathcal{O}^{\times}(G)) \to \mathrm{Aut}(G) \to 1$

cf. ← Ism

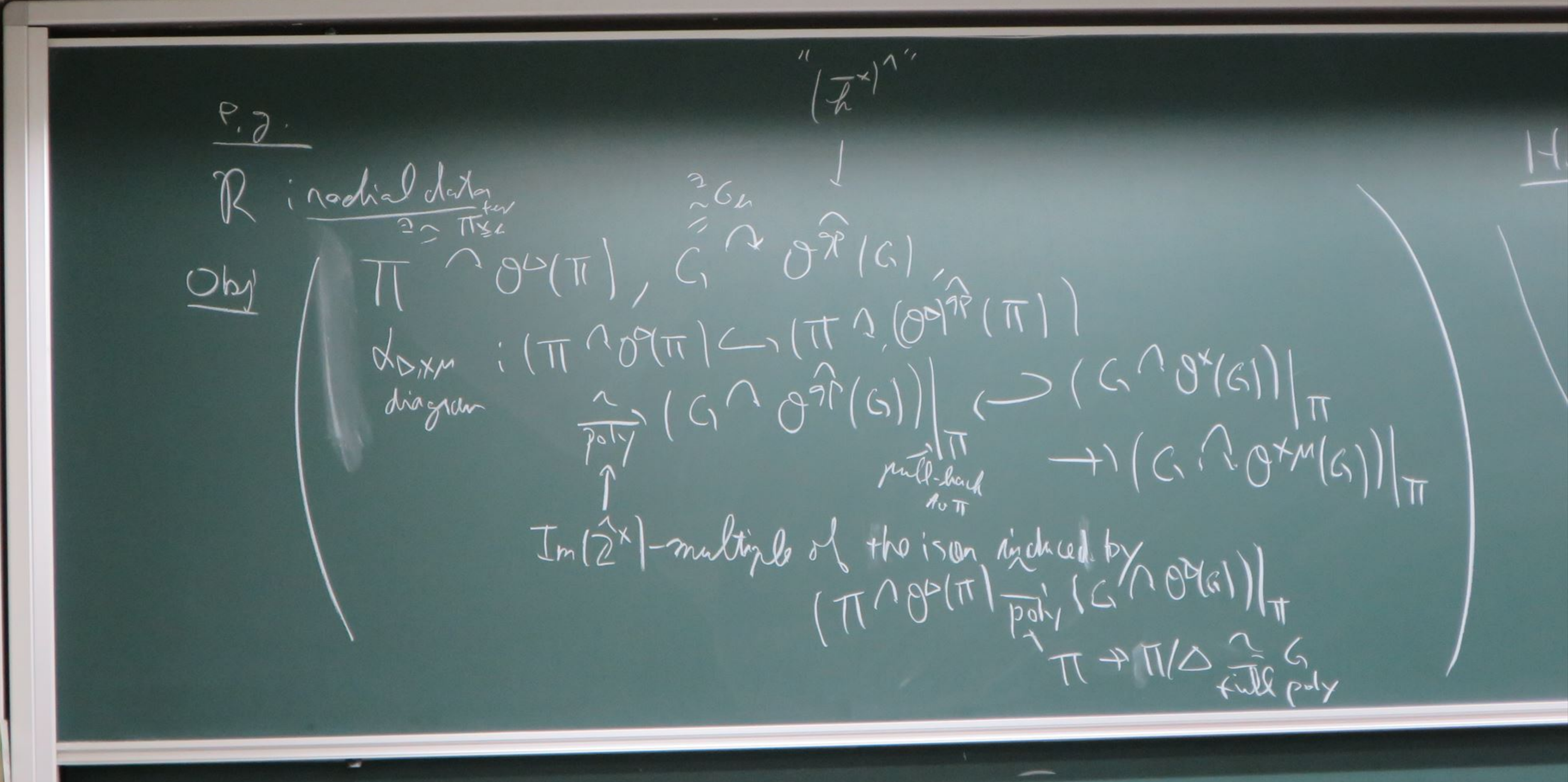
$1 \to \mathrm{Ismet}(G) \to \mathrm{Aut}(G \curvearrowright \mathcal{O}^{\times\mu}(G)) \to \mathrm{Aut}(G) \to 1$

$\mathfrak{F}$ \mathfrak "family" $\supset$) (regarded as pre-Frobenioids)

$\mathcal{F}$ \mathcal "one piece" $\mathcal{D}$

Frd   Anbd, Tpd

$\mathcal{D}^{\circledcirc}$ $\mathcal{D}^{\circledcirc\pm}$



e.g.

$\mathcal{R}$: radial data$_{\text{tor}}$ $\supseteq \Pi_{\leq k}$

"$(\bar{k}^{\times})^{\wedge}$"

$\downarrow$

Obj $\left(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi),\ \underset{\wr\!\wr}{\overset{\cong G_k}{}} G \curvearrowright \mathcal{O}^{\widehat{gp}}(G),\ (\Pi \curvearrowright (\mathcal{O}^{\triangleright})^{\widehat{gp}}(\Pi)) \right.$

$\alpha_{\triangleright,\times\mu}$ diagram : $(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi)) \hookrightarrow (\Pi \curvearrowright (\mathcal{O}^{\triangleright})^{\widehat{gp}}(\Pi))$

$\xrightarrow[\text{poly}]{\sim} (G \curvearrowright \mathcal{O}^{\widehat{gp}}(G))|_{\Pi} \hookleftarrow (G \curvearrowright \mathcal{O}^{\times}(G))|_{\Pi}$

pull-back to Π

$\twoheadrightarrow (G \curvearrowright \mathcal{O}^{\times\mu}(G))|_{\Pi}$

$\uparrow$

$\mathrm{Im}(\hat{\mathbb{Z}}^{\times})$-multiple of the isom induced by

$(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi)) \xrightarrow[\text{poly}]{\sim} (G \curvearrowright \mathcal{O}^{\triangleright}(G))|_{\Pi}$

$\Pi \twoheadrightarrow \Pi/\Delta \xrightarrow[\text{full poly}]{\sim} G$ $\left.\right)$

H.

$))$

$\to (G \curvearrowright \vartheta^{\times}(G))|_{\Pi}$

$\longmapsto (G \curvearrowright \vartheta^{\times\mu}(G))|_{\Pi}$

ced by

$(G \curvearrowright \theta^{\Delta}(G))|_{\Pi}$

$\Pi \twoheadrightarrow \Pi/\Delta \xrightarrow{\sim} G$

Hom

$\cdot\ \Pi \overset{\text{isom}}{\xrightarrow{\sim}} \Pi^*$

} induces

$(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi)) \xrightarrow{\sim} (\Pi^* \curvearrowright \mathcal{O}^{\triangleright}(\Pi^*))$

$\cdot\ G \underset{\text{isom}}{\xrightarrow{\sim}} G^*$ induces $(G \curvearrowright \mathcal{O}^{\widehat{\triangleright}}(G)) \xrightarrow{\sim} (G^* \curvearrowright \mathcal{O}^{\widehat{\triangleright}}(G^*))$

$\mathrm{Im}(\hat{\mathbb{Z}}^{\times})$-multiple of

$\vdash \in$ conic data

Obj $(G \overset{\supseteq G_k}{\curvearrowright} \mathcal{O}^{\times\mu}(G))$

Hom $G \xrightarrow{\sim} G^*$ induces $(G \curvearrowright \mathcal{O}^{\times\mu}(G)) \xrightarrow{\sim} (G^* \curvearrowright \mathcal{O}^{\times\mu}(G^*))$

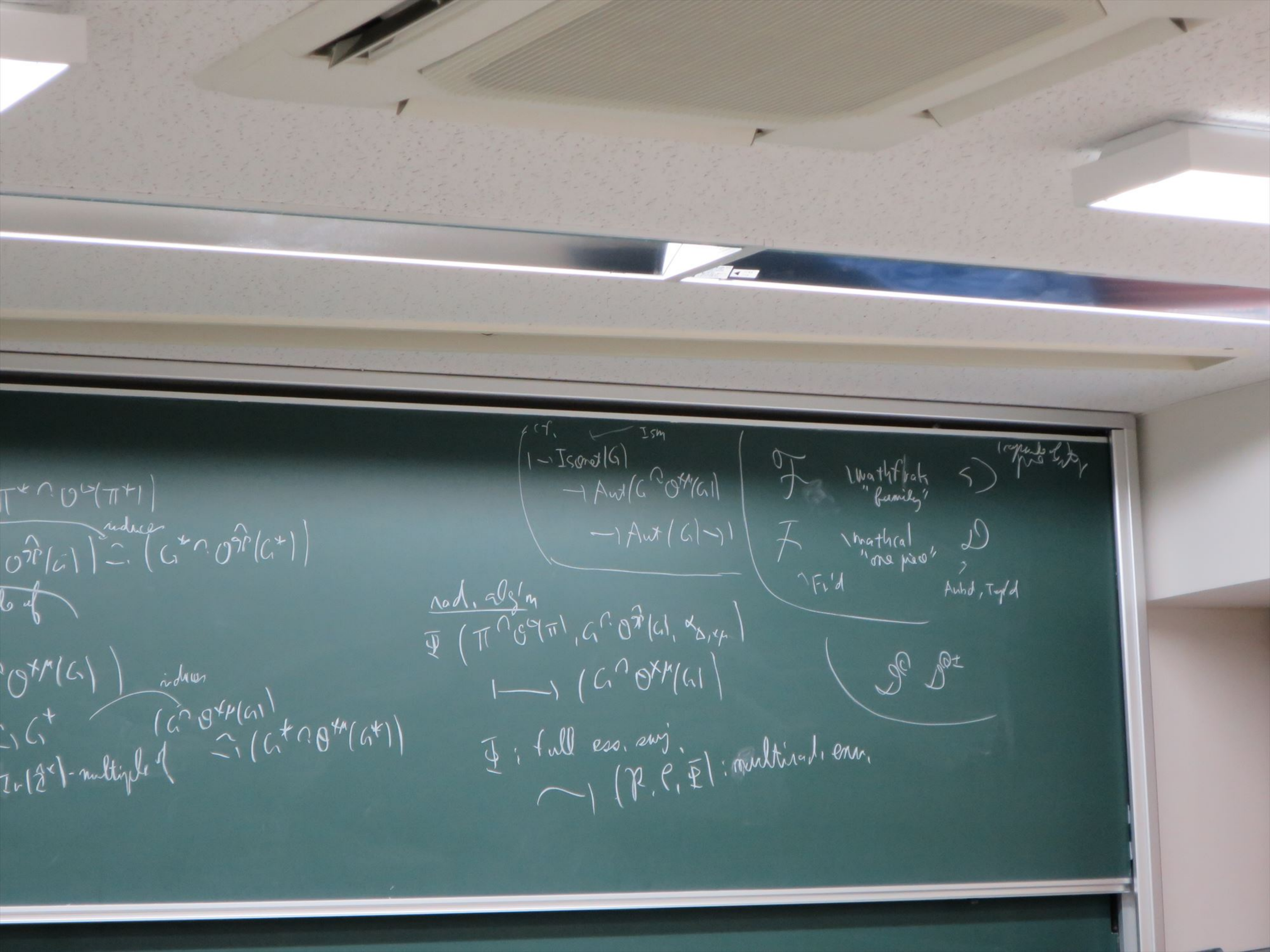
$\mathrm{Im}(\hat{\mathbb{Z}}^{\times})$-multiple of

rad. alg'm

$\Phi\ (\Pi \curvearrowright \mathcal{O}^{\triangleright}$

$\longmapsto$

Φ : f



$\Pi^{*} \curvearrowright \mathcal{O}^{\triangleright}(\Pi^{*}))$

induces

$\mathcal{O}^{\hat{\pi}}(G)) \xrightarrow{\sim} (G^{*} \curvearrowright \mathcal{O}^{\hat{\pi}}(G^{*}))$

b of

$\mathcal{O}^{\times\mu}(G))$

induces

$\xrightarrow{\sim} G^{*}$ $(G \curvearrowright \mathcal{O}^{\times\mu}(G))$

$\xrightarrow{\sim} (G^{*} \curvearrowright \mathcal{O}^{\times\mu}(G^{*}))$

$\mathrm{Ir}(\hat{\mathbb{Z}}^{\times})$-multiple of

cf. ← Ism

$1 \to \mathrm{Isomet}(G) \to \mathrm{Aut}(G \curvearrowright \mathcal{O}^{\times\mu}(G)) \to \mathrm{Aut}(G) \to 1$

$\mathfrak{F}$ \mathfrak "family" ⊂) (corresponds [illegible])

$\mathcal{F}$ \mathcal "one piece" $\mathcal{D}$

Fr'd

Auhd, Top'd

$\mathcal{D}^{\circledcirc}$ $\mathcal{D}^{\circledcirc\pm}$

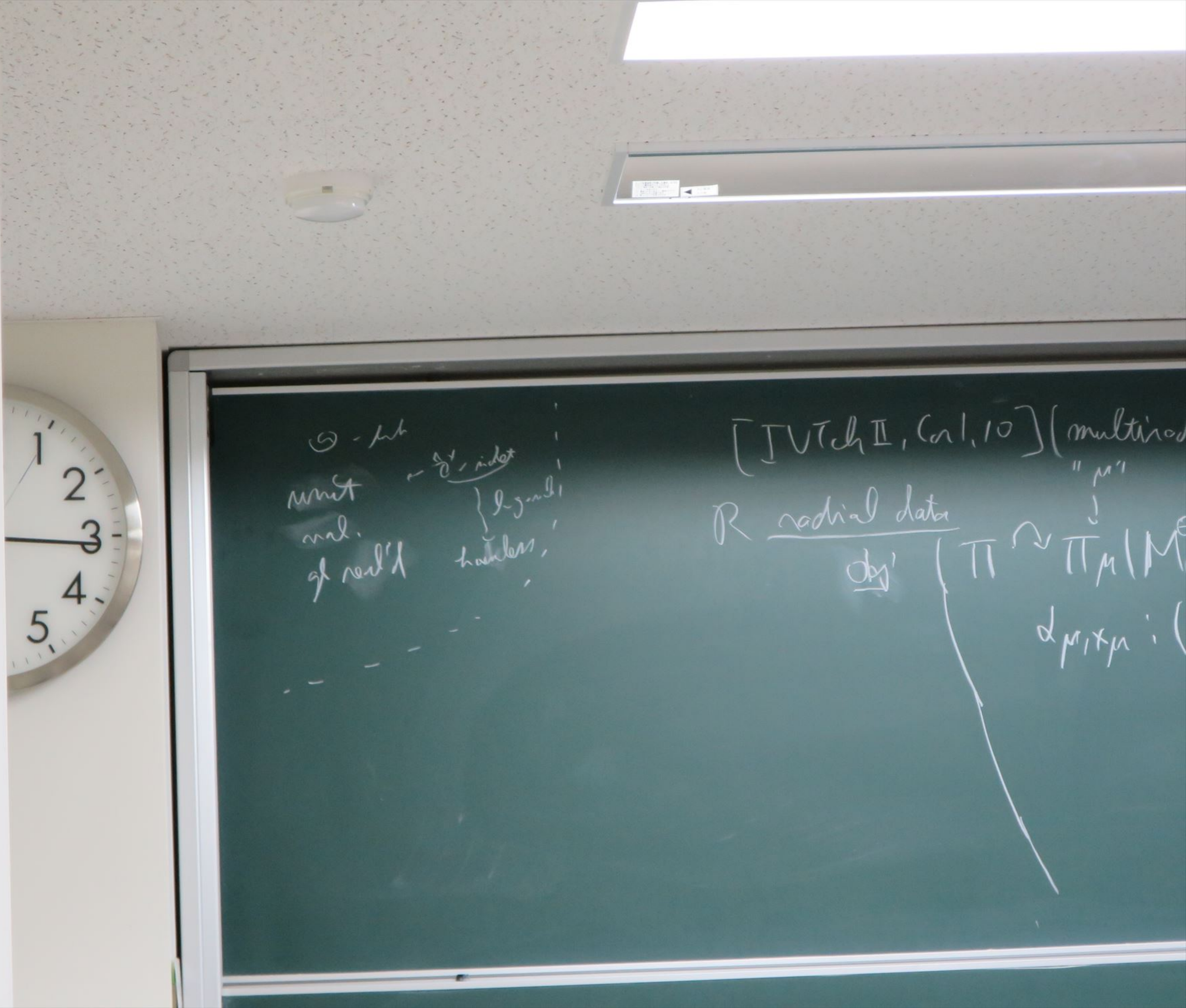
rad. alg'm

$\Psi(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi), G \curvearrowright \mathcal{O}^{\hat{\pi}}(G), \alpha_{\Delta,\times\mu})$

$\longmapsto (G \curvearrowright \mathcal{O}^{\times\mu}(G))$

Ψ: full ess. surj.

$\rightsquigarrow (\mathcal{R}, \rho, \bar{\Psi})$: multirad. env.

[IUTch II, Cor 1.10] (multirad
"μ"
R radial data
obj'
$\Pi \curvearrowright \Pi_\mu | M$
α_{μ}, x_μ : (
unit
nal.
gl real'd
hashless,

[IUTch II, Cor 1.10] (multirad. mono-theta cycl. rig.)

"μ" system w.r.t. $(\text{mod } N)_{N \geq 1}$ of mono-theta env.

R <u>radial data</u>

<u>obj'</u>
$$\left(\Pi \curvearrowright \Pi_{\mu}(M^{\Theta}_{\circledast}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z},\ G \curvearrowright \mathcal{O}^{\times\mu}(G) \right).$$

$$\alpha_{\mu, \times\mu} : \left(\Pi \curvearrowright \Pi_{\mu}(M^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \right) \xrightarrow{\text{poly}} \left(G \curvearrowright \mathcal{O}^{\times\mu}(G) \right) \Big|_{\Pi}$$

induced by

$$\begin{tikzcd} \Pi/\Delta \arrow[r, "\sim", "\text{full poly}"'] & G & & \\ \mathcal{O}^{\mu}(\Pi) \arrow[r, phantom, "\subset"] \arrow[rrr, bend right, "\text{trivi}"'] & \mathcal{O}^{\times}(\Pi) \arrow[rr, "\sim", "\text{poly}"'] & & \mathcal{O}^{\times}(G) \arrow[d] \\ \text{"}\mu\text{"} & & & \mathcal{O}^{\times\mu}(G) \end{tikzcd}$$

Ind($\widehat{\mathbb{Z}}^{\times}$)-orbit of isom induced by

$$\mu \xrightarrow{\text{triv}} \mathcal{O}^{\times\mu}$$

Hom $\Pi \cong$

$G \cong$

<u>coric data</u>

$\Psi(\Pi \curvearrowright$

duced
)

Thm $\Pi \xrightarrow{\sim} \Pi^*$ ⇝ induces ....

$G \xrightarrow{\sim} G^*$ ⇝ induces ....

$\mathrm{Im}(\hat{\partial}^\times)$-multiple

Coric data $\mathcal{C}$ obj $(G \curvearrowright \mathcal{O}^{\times\mu}(G))$

Thm $\mathrm{Im}(\hat{\partial}^\times)$-multiple of isom. induced by $G \xrightarrow{\sim} G^*$

$\Phi(\Pi \curvearrowright \ldots, G \curvearrowright \ldots, \alpha_{\mu,x_\mu}) \mapsto (G \curvearrowright \mathcal{O}^{\times\mu}(G))$

$\mathcal{C}^\dagger := \mathcal{C}$

$\mathcal{R}^\dagger$ **obj**

objs of $\mathcal{R}$

w/

(Cycl. Rig. Mono-Th.)

$(l\Delta_\Theta)(\Pi) \xrightarrow{\sim} \Pi_\mu(\mathcal{M}^\Theta_*(\Pi))$

$(\xrightarrow{\sim} (l\Delta_\Theta)(\mathcal{M}^\Theta_*(\Pi)))$

$(\mathcal{R} \underset{\text{functor}}{\xrightarrow{\Phi}} \mathcal{F}$

⇝

$\mathcal{R}^\dagger :=$ graph$(\mathcal{R} \to \mathcal{F})$

obj $(A, \Phi(A)))$

$$\mathcal{R} \xrightarrow{\text{functor}} \mathcal{R}^{\dagger}$$

$$(\Pi^{\Omega} \cdots, C^{\Omega} \cdots, \alpha_{\mu, x_\mu})$$

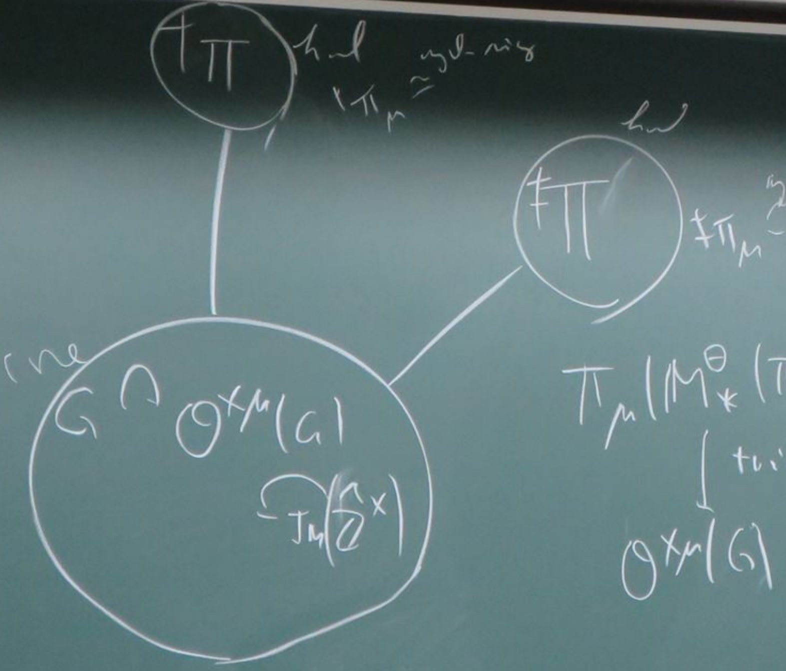
$$\mapsto (\Pi^{\Omega} \cdots, C^{\Omega} \cdots, \alpha_{\mu, x_\mu}, (\text{Gal. Rig Mono-Th}))$$

multiradially defined

(ne

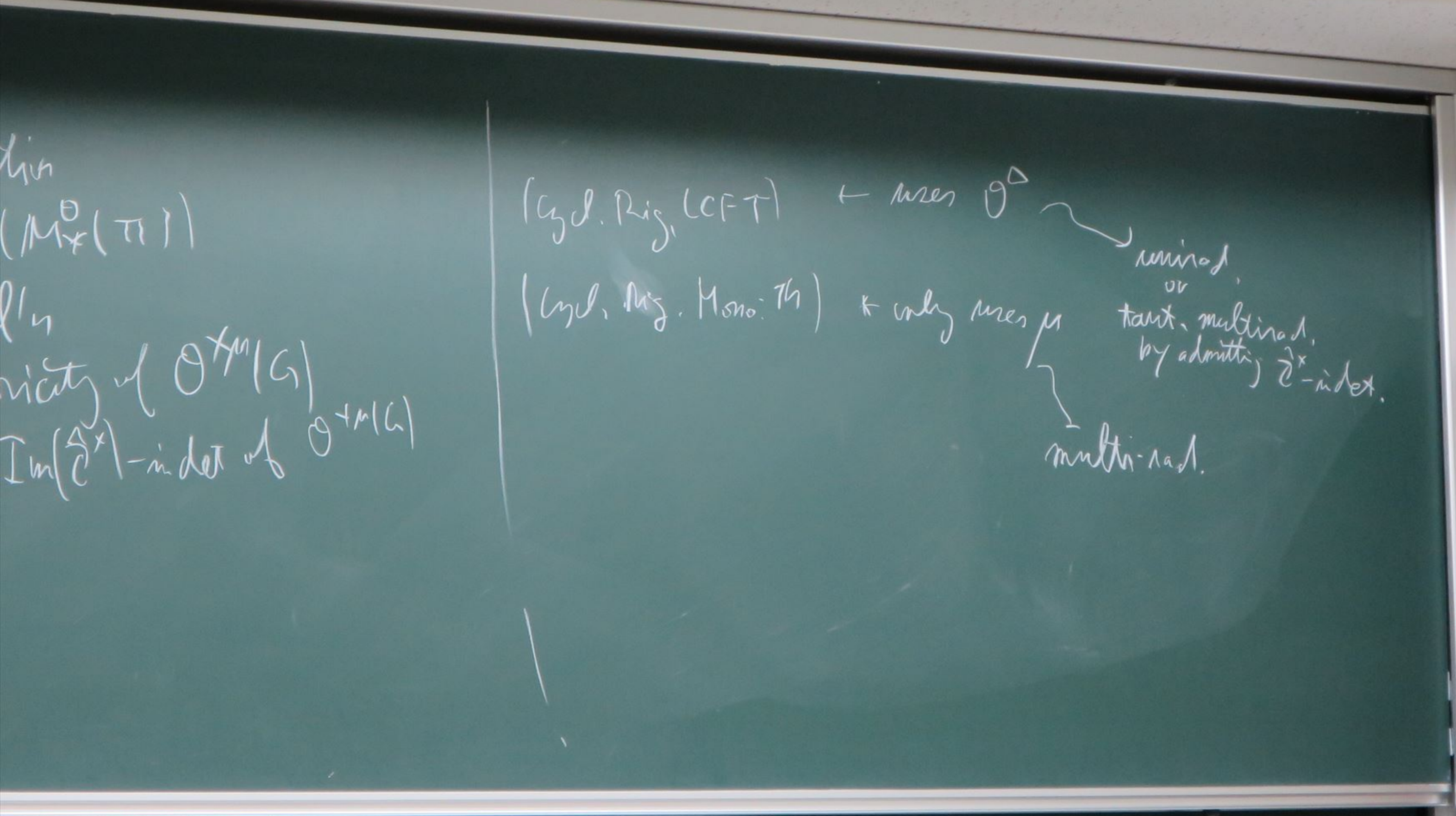
G

-Th)



$$\begin{tikzcd}
\Pi_{\mu}(M^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \arrow[d, "\text{trivial}"] \\
\mathcal{O}^{\times\mu}(G)
\end{tikzcd}$$

any rigidification
on $\Pi_{\mu}(M^{\Theta}_{*}(\Pi))$
has no rel'n
w/ $\begin{cases} \text{the coricity of } \mathcal{O}^{\times\mu}(G) \\ \text{the } \mathrm{Im}(\hat{\mathbb{C}}^{\times})\text{-indet of } \mathcal{O}^{\times\mu}(G) \end{cases}$



lin

($\mathbb{M}^{\Theta}_{*}(\pi)$)

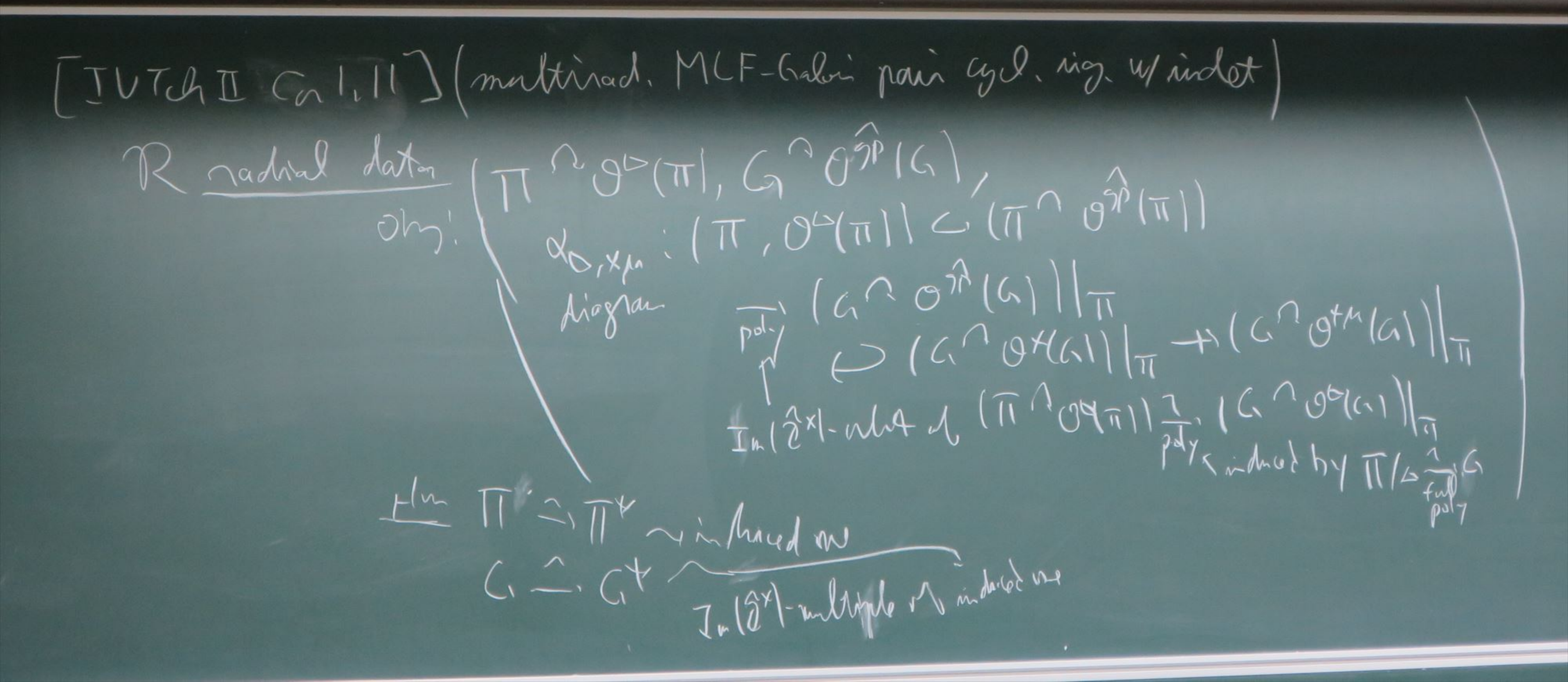
l'n

ricity of ($\Theta^{\times\mu}(G)$)

$\mathrm{Im}(\hat{\mathbb{Z}}^{\times})$-indet of $\Theta^{\times\mu}(G)$

(Cycl. Rig. LCFT) ← uses Θ^{Δ} $\rightsquigarrow$ unirad. or tamt. multirad. by admitting $\hat{\mathbb{Z}}^{\times}$-indet.

(Cycl. Rig. Mono. Th) ← only uses μ $\rightsquigarrow$ multi-rad.



[IUTch II Cor 1.11] (multirad. MLF-Galois pair cycl. rig. w/ indet)

R radial datum

Obj: $\Big(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi),\ G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G),$

$\alpha_{\Delta, x_\mu} : (\Pi, \mathcal{O}^{\triangleright}(\Pi)) \subset (\Pi \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(\Pi))$

diagram $\xrightarrow[\text{poly}]{\sim} (G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G))|_{\Pi}$

$\hookleftarrow (G \curvearrowright \mathcal{O}^{\times}(G))|_{\Pi} \twoheadrightarrow (G \curvearrowright \mathcal{O}^{\times\mu}(G))|_{\Pi}$

$\mathrm{Im}(\widehat{\mathbb{Z}}^{\times})$-orbit of $(\Pi \curvearrowright \mathcal{O}^{\triangleright}(\Pi)) \xrightarrow[\text{poly}]{\sim} (G \curvearrowright \mathcal{O}^{\triangleright}(G))|_{\Pi} \Big)$

induced by $\Pi/\Delta \xrightarrow[\text{full poly}]{\sim} G$

Thm $\Pi \xrightarrow{\sim} \Pi^{\dagger}$ ~ induced one

$G \xrightarrow{\sim} G^{\dagger}$ ~ $\mathrm{Im}(\widehat{\mathbb{Z}}^{\times})$-multiple of induced one

$(G))\|_{\Pi}$

$))\|_{\Pi}$

$\Pi/\Delta \xrightarrow[\text{full poly}]{\sim} G$

$\mathcal{E}$ obj $(G \curvearrowright \mathcal{O}^{\times\mu}(G))$

Hom Ind$(\hat{\partial}^{\times})$-orbit of isom induced by $G \xrightarrow{\sim} G^{*}$

$\mathcal{E}^{\dagger} := \mathcal{E}$

$\mathcal{R}^{\dagger}$ = graph of

$$\mathcal{R} \longrightarrow \mathcal{E}$$

$(\Pi^{\curvearrowright}\cdots, G^{\curvearrowright}\cdots, \alpha_{\triangleright, x_\mu}) \longmapsto$ (Ind$(\hat{\partial}^{\times})$-orbit of (Cycl. Rig. (cf 7)) $\mu_{\hat{\partial}}(G) \xrightarrow{\sim} \mu_{\hat{\partial}}(\mathcal{O}^{\times}(G))$,

Aut(G)-orbit of (Cycl. Rig. (cf 7)) $\mu_{\hat{\partial}}(G) \xrightarrow{\sim} (\ell\Delta_{\Theta})(\Pi)$

full poly $\Pi/\Delta \xrightarrow{\sim} G$

& $\mu_{\hat{\partial}}(G) \xrightarrow{\sim} \mu_{\hat{\partial}}(\Pi)$)

$\mathcal{R} \longrightarrow \mathcal{R}^{\dagger}$

is multiradially defined

uses $\mathcal{O}^{\triangleright}$

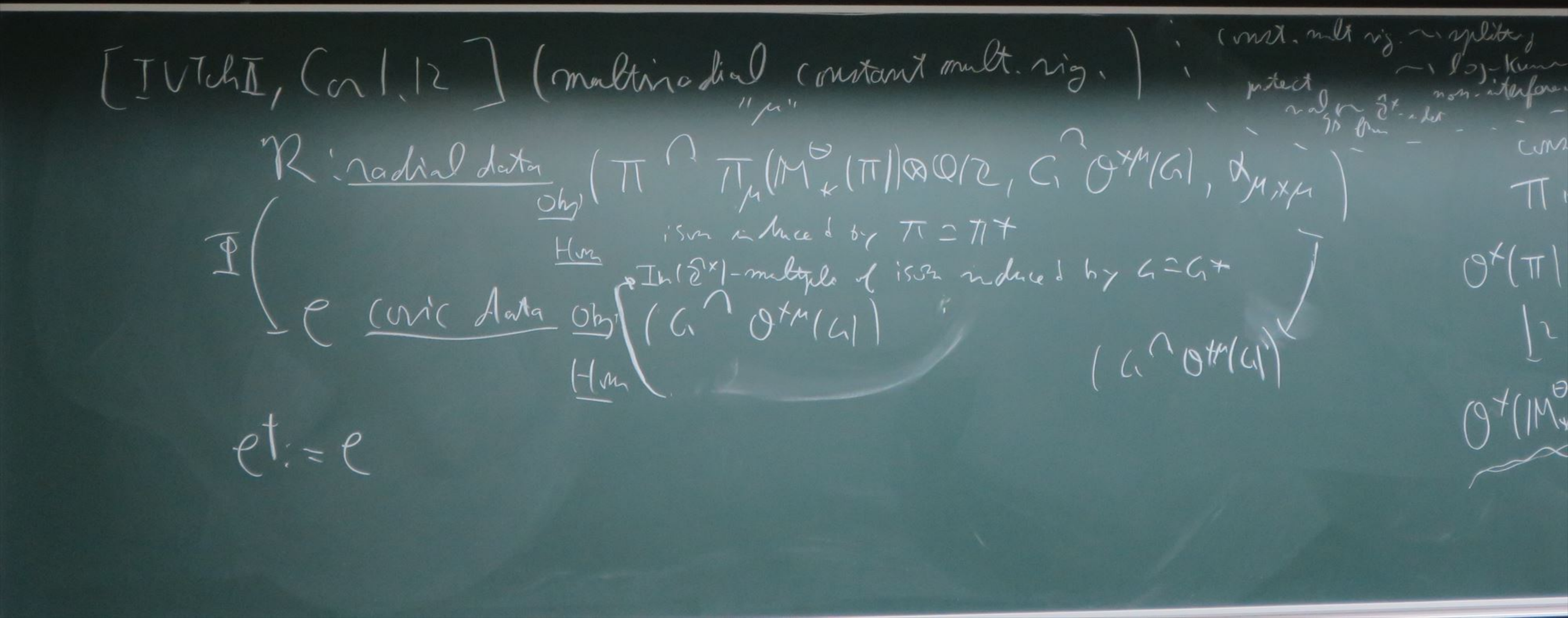
$\mathcal{O}^{\times\mu}$

!

shared in the core

only after admitting indet,

it becomes multirad.



[IUTchII, Cor 1.12] (multiradial constant mult. rig.) ; const. mult rig. ⇝ splitting ⇝ log-Kumm

"μ"

protect ⇝ val ⇝ $\hat{\mathcal{O}}^\times$-indet from

non-interfere

$\mathcal{R}$: radial data Obj: $(\Pi \curvearrowright \Pi_\mu(\mathbb{M}^\Theta_*(\Pi))\otimes\mathbb{Q}/\mathbb{Z},\ G \curvearrowright \mathcal{O}^{\times\mu}(G),\ \alpha_{\mu,\times\mu})$

Hom: isom induced by $\Pi \cong \Pi^*$

Φ

$\mathcal{C}$ coric data Obj: $(G \curvearrowright \mathcal{O}^{\times\mu}(G))$ $(G \curvearrowright \mathcal{O}^{\times\mu}(G))$

Hom: $\mathrm{Im}(\hat{\mathbb{Z}}^\times)$-multiple of isom induced by $G \cong G^*$

$\mathcal{C}^\dagger := \mathcal{C}$

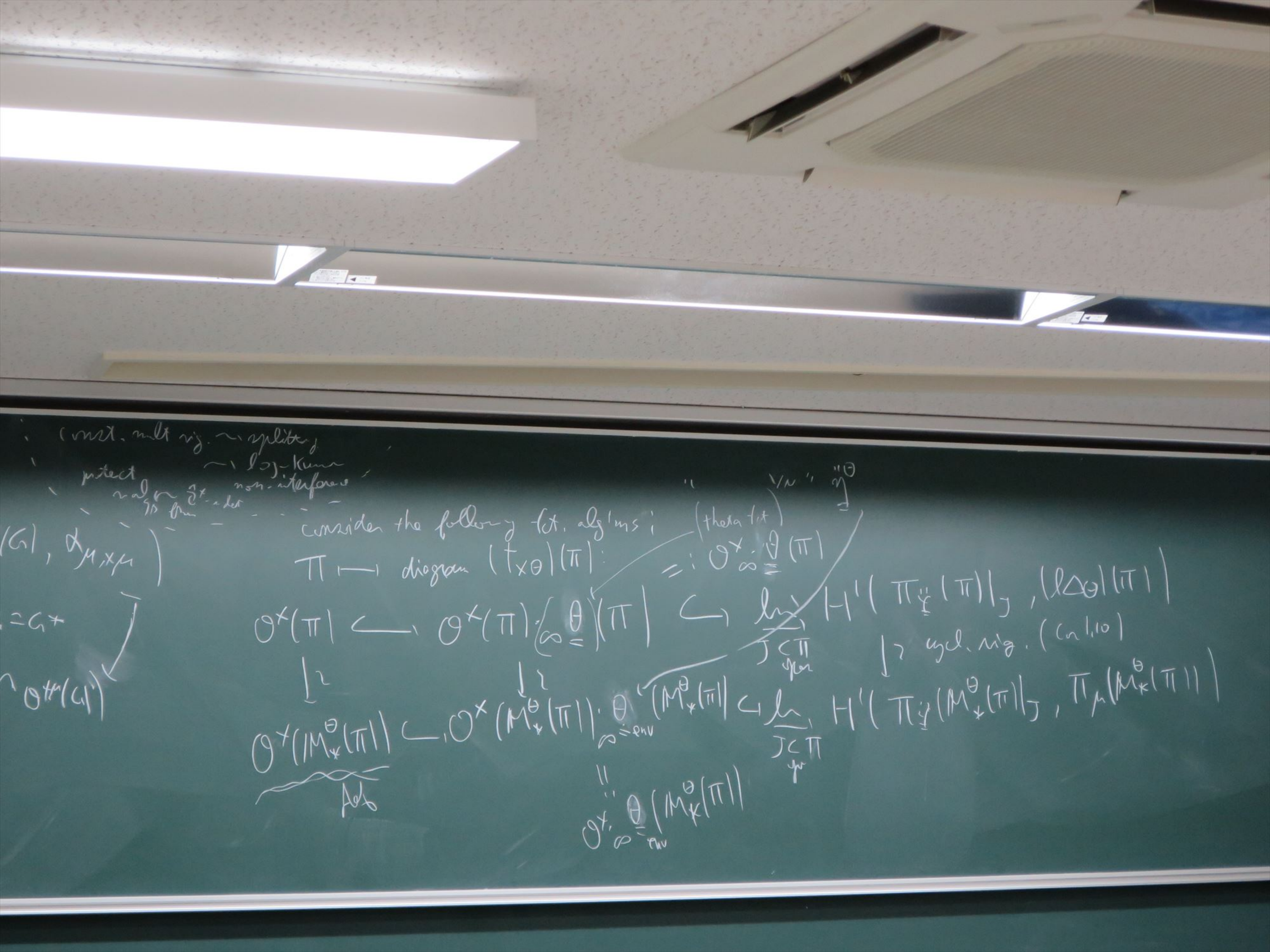
const

Π

$\mathcal{O}^\times(\Pi)$

$\wr$

$\mathcal{O}^\times(\mathbb{M}^\Theta_*$



const. mult. rig. ⇝ splitting

⇝ log-Kummer

protect

non-interference

val. on $\hat{\mathcal{O}}^{\times}$-det

gp. form

$(G), \alpha_{\mu, \times\mu})$

$= G*$

$\mathcal{O}^{\times\mu}(G)$

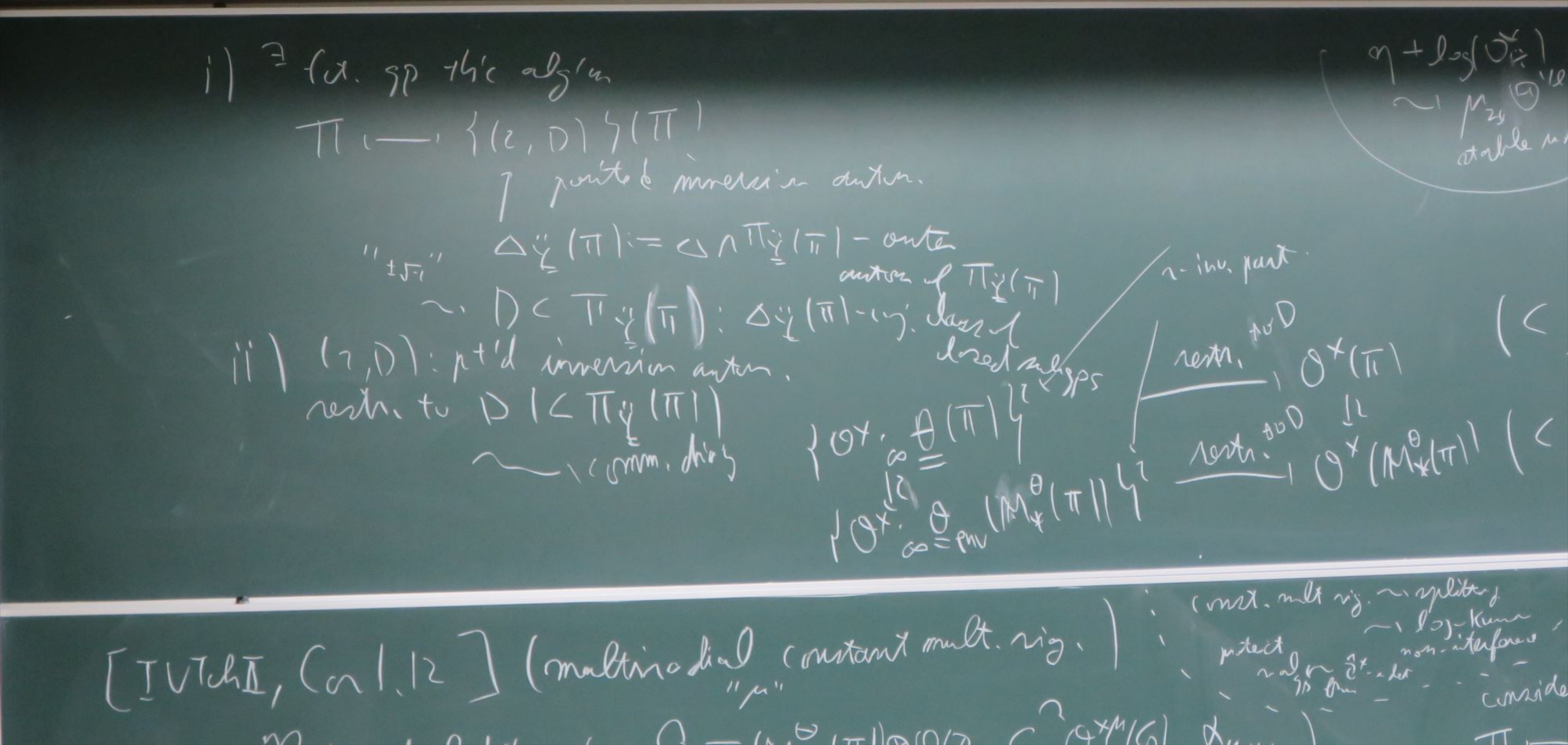
consider the following fct. alg'ms:

$\Pi \longmapsto$ diagram $(\dagger_{\times\Theta})(\Pi)$:

"(theta fct)$^{1/N}$" $\ddot{\eta}^{\Theta}$

$=: \mathcal{O}^{\times}_{\infty} \cdot \underline{\underline{\theta}}(\Pi)$

$$\begin{tikzcd}
\mathcal{O}^{\times}(\Pi) \arrow[r, hook] \arrow[d, "\wr"] & \mathcal{O}^{\times}(\Pi) \cdot {}_{\infty}\underline{\underline{\theta}}(\Pi) \arrow[r, hook] \arrow[d, "\wr"] & \varinjlim_{J \subset \Pi \text{ open}} H^1(\Pi_{\ddot{Y}}(\Pi)|_J, (l\triangle\Theta)(\Pi)) \arrow[d, "\wr", "\text{cycl. rig. (Cor 1.10)}"'] \\
\underbrace{\mathcal{O}^{\times}(M^{\Theta}_{*}(\Pi))}_{\text{Aut}} \arrow[r, hook] & \mathcal{O}^{\times}(M^{\Theta}_{*}(\Pi)) \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}}(M^{\Theta}_{*}(\Pi)) \arrow[r, hook] \arrow[d, equal] & \varinjlim_{J \subset \Pi \text{ open}} H^1(\Pi_{\ddot{Y}}(M^{\Theta}_{*}(\Pi))|_J, \Pi_{\mu}(M^{\Theta}_{*}(\Pi))) \\
 & \mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}}(M^{\Theta}_{*}(\Pi)) &
\end{tikzcd}$$



i) ∃ fct. gp th'c alg'm

$\Pi \longmapsto \{(\iota, D)\}(\Pi)$

↑ pointed inversion autom.

"$\pm\sqrt{-1}$" $\Delta_{\underline{\underline{\ddot{Y}}}}(\Pi) := \iota \curvearrowright \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ - outer autom. of $\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$

$\rightsquigarrow D \subset \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$: $\Delta_{\underline{\underline{\ddot{Y}}}}(\Pi)$-conj. class of closed subgps

ii) (ι, D) : pt'd inversion autom.

restr. to D $(\subset \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi))$

$\rightsquigarrow$ comm. diag

$$\begin{tikzcd}
\{\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}(\Pi)\}^\iota \arrow[r, "\text{restr. to } D"] \arrow[d, phantom, "\wr"] & \mathcal{O}^\times(\Pi) \arrow[d, phantom, "\wr"] & (\subset \\
\{\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}_{\text{env}}(\mathcal{M}^\theta_*(\Pi))\}^\iota \arrow[r, "\text{restr. to } D"] & \mathcal{O}^\times(\mathcal{M}^\theta_*(\Pi)) & (\subset
\end{tikzcd}$$

ι-inv. part

$\eta + \log(\mathcal{O}^\times_{\ddot{\ }})$ ...'le

$\rightsquigarrow \mu_{2\ell}$ (Θ) 'le

stable ...

[IUTchII, Cor 1.12] (multiradial constant mult. rig. "μ") ; const. mult. rig. ⇝ splitting ⇝ log-Kummer non-interference, protect ... consider

$\eta + \log(\mathcal{O}^{\times}_{\ddot{K}})$ stable under 2

$\rightsquigarrow \mu_{2l}(\Theta)^{1/l}$

stable under 2

the inverse image

of $\mathcal{O}^{\mu}(-)$

$= {}_{\infty}\underline{\theta}(\Pi)^{2}$, ${}_{\infty}\underline{\theta}_{env}$

([illegible] of std e

...tion.

outer

...tion of $\Pi_{\underline{Y}}(\Pi)$

...nj. class of closed subgps

$\mathbb{Z}$-inv. part

$\xrightarrow{\text{restr. to } D} \mathcal{O}^{\times}(\Pi)$ $\left(\subset \varinjlim_{J \subset \Pi \text{ open}} H^1(J, (l\Delta_{\Theta})(\Pi)) \right)$

${}_{\infty}\underline{\theta}(\Pi)\}^{\mathbb{Z}}$

$\wr\!|$ cycl. rig

$\wr\!|$

${}_{\infty}\underline{\theta}_{env}(M^{\Theta}_{*}(\Pi))\}^{\mathbb{Z}}$ $\xrightarrow{\text{restr. to } D} \mathcal{O}^{\times}(M^{\Theta}_{*}(\Pi))$ $\left(\subset \varinjlim_{J \subset \Pi \text{ open}} H^1(J, \Pi_{\mu}(M^{\Theta}_{*}(\Pi))) \right)$

In particular,

$(t_{\mu\Theta})(\Pi)$

$\rightsquigarrow$ splitting

...t mult. rig.) ;

(const. mult rig. $\rightsquigarrow$ splitting,

$\rightsquigarrow$ log-Kummer non-interference,

protect

"(theta fct)$^{1/n}$" $\ddot{\eta}^{\Theta}$

stable ... len 2

e

len 2

$\varinjlim_{J \subset \Pi \text{ open}} H^1(J, (\ell\Delta_\Theta)(\Pi))$

$J \subset \Pi$ open

[illegible] char. [illegible]

$\varinjlim_{J \subset \Pi \text{ open}} H^1(J, \Pi_\mu(M^\Theta_*(\Pi)))$

the inverse image
of $\mathcal{O}^\mu(-)$
$= {}_\infty\underline{\underline{\theta}}(\Pi)^\iota, {}_\infty\underline{\underline{\theta}}_{env}(M^\Theta_*(\Pi))^\iota$
(def. of std [illegible]

Θ-labelled
splitting

$0 \to \mathcal{O}^\times \to \mathcal{O}^\triangleright \to \mathbb{Z}_{\geq 0} \to 0$
s . . .)

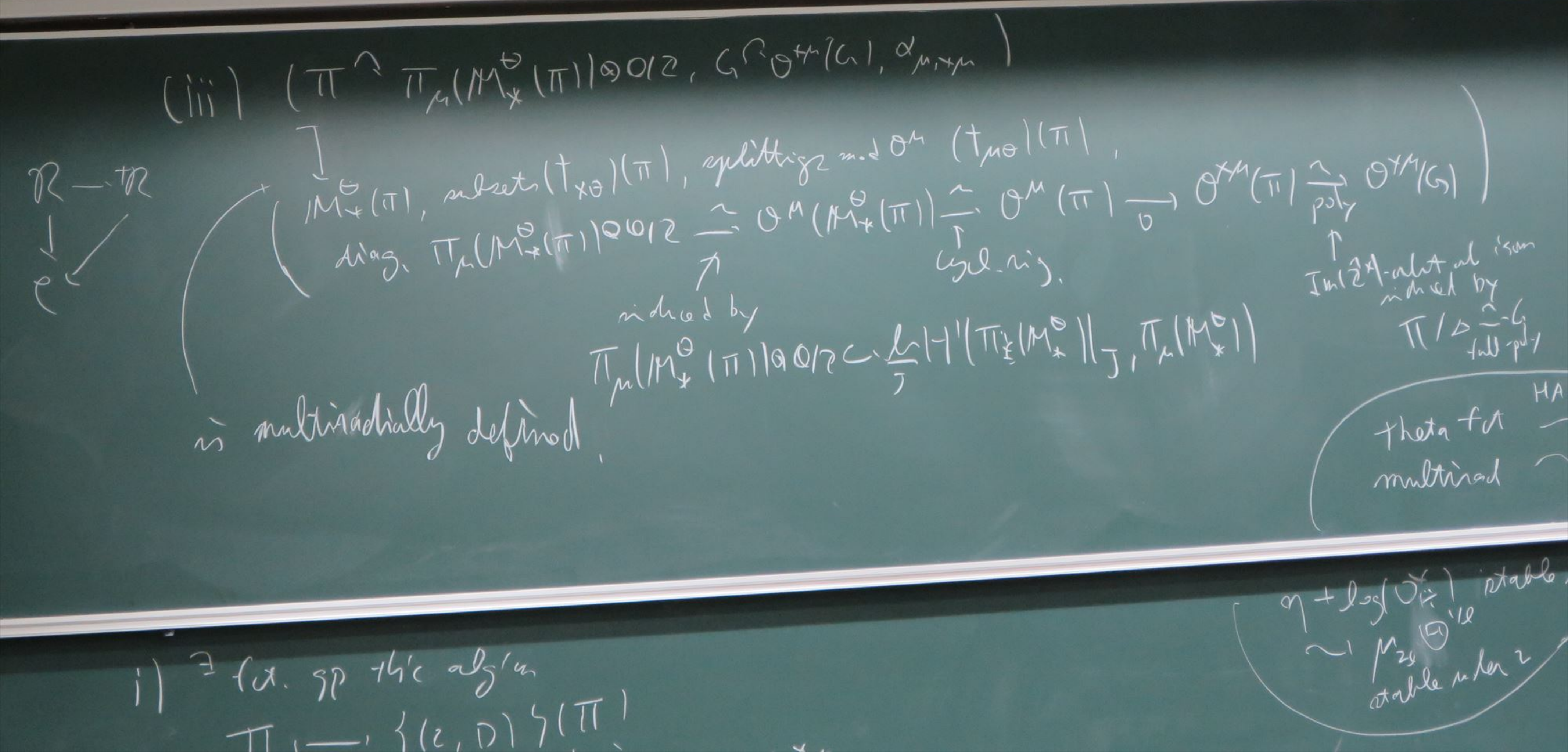
In particular, we obtain a Kd alg'm.

$(t_{\mu\Theta})(\Pi)$ ⇝ splitting $\begin{cases} \mathcal{O}^{\times\mu}(\Pi) \times \{{}_\infty\underline{\underline{\theta}}(\Pi)^\iota / \mathcal{O}^\mu(\Pi)\} \\ \mathcal{O}^{\times\mu}(M^\Theta_*(\Pi)) \times \{{}_\infty\underline{\underline{\theta}}_{env}(M^\Theta_*(\Pi))^\iota / \mathcal{O}^\mu(M^\Theta_*(\Pi))\} \end{cases}$

coric ⟵|⟶ "purely radial"

important splitting

"Θ



(iii) $(\Pi \curvearrowright \Pi_\mu(\mathcal{M}^\Theta_*(\Pi)) \otimes \mathbb{Q}/\mathbb{Z},\ G \curvearrowright \mathcal{O}^{\times\mu}(G),\ \alpha_{\mu,\times\mu})$

$$\begin{tikzcd} \mathcal{R} \arrow[r, mapsto] \arrow[d] & {}^\dagger\mathcal{R} \arrow[dl] \\ e & \end{tikzcd}$$

$]$

$\Big($ $\mathcal{M}^\Theta_*(\Pi)$, subsets $(\dagger_{\times\Theta})(\Pi)$, splittings mod $\mathcal{O}^\mu$ $(\dagger_{\mu\Theta})(\Pi)$,

diag. $\Pi_\mu(\mathcal{M}^\Theta_*(\Pi))\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mathcal{O}^\mu(\mathcal{M}^\Theta_*(\Pi)) \xrightarrow{\sim} \mathcal{O}^\mu(\Pi) \xrightarrow[0]{} \mathcal{O}^{\times\mu}(\Pi) \xrightarrow[\text{poly}]{\sim} \mathcal{O}^{\times\mu}(G)$ $\Big)$

↑ cycl. rig.

↑ $\mathrm{Im}(\hat{\mathbb{Z}}^\times)$-orbit of isom induced by $\Pi/\Delta \xrightarrow[\text{full poly}]{\sim} G$

↗ induced by

$\Pi_\mu(\mathcal{M}^\Theta_*(\Pi))\otimes\mathbb{Q}/\mathbb{Z} \hookrightarrow \varinjlim_J H^1(\Pi_{\underline{X}}(\mathcal{M}^\Theta_*)|_J, \Pi_\mu(\mathcal{M}^\Theta_*))$

is multiradially defined.

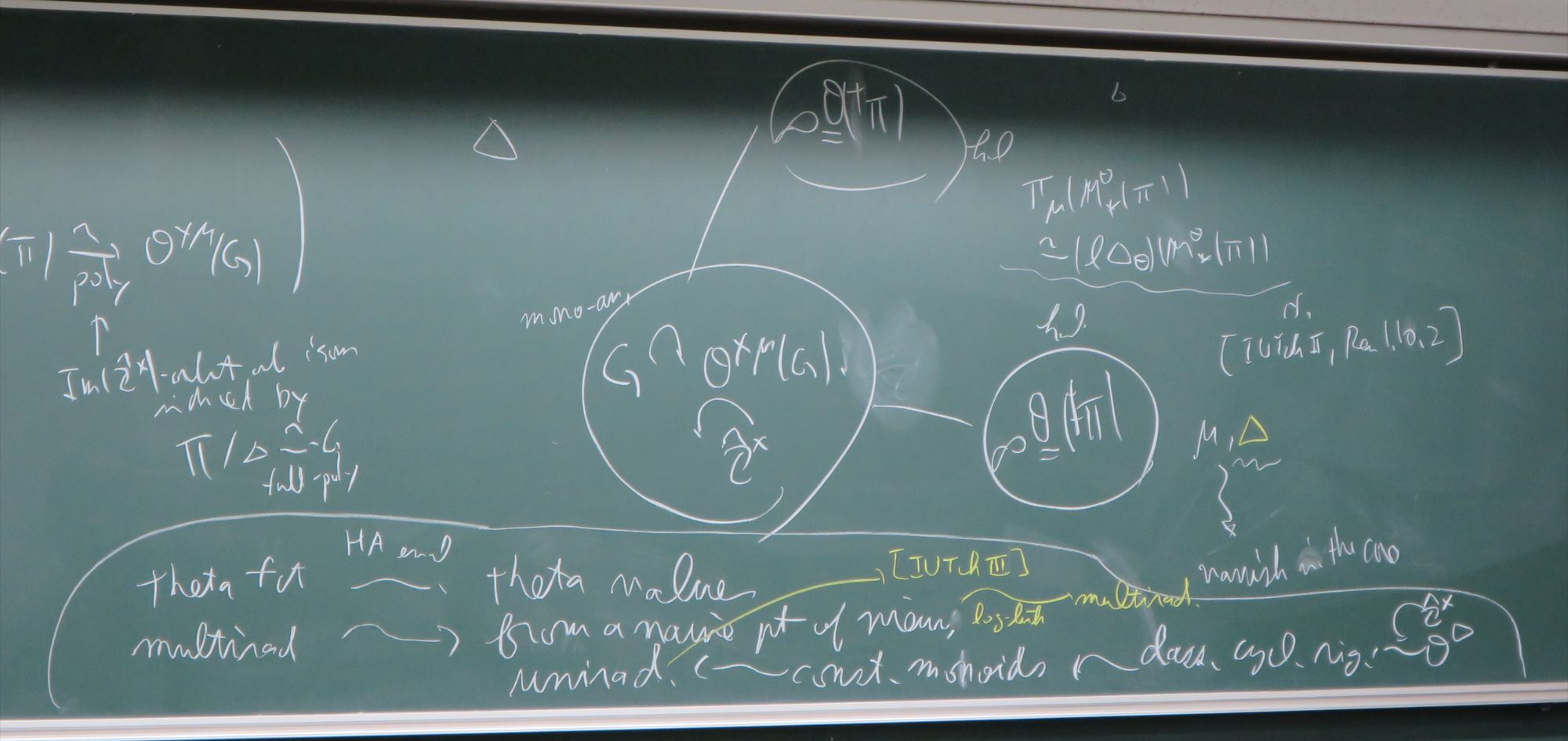
HA

theta fct
multirad

$\eta + \log(\mathcal{O}^\times_{\ddot{\cdot}})$ stable,
$\rightsquigarrow \mu_{2l}$ Θ'le
stable under 2

i) ∃ fct. gp th'c alg'm

$\Pi \longmapsto \{(\mathbb{Z}, D)\}(\Pi)$



$\triangle$

$(\Pi) \xrightarrow[\text{poly}]{\sim} \mathcal{O}^{\times\mu}(G)$

$\uparrow$

$\mathrm{Im}(\hat{\mathbb{Z}}^{\times})$-orbit of isom induced by

$\Pi/\Delta \underset{\text{full-poly}}{\xrightarrow{\sim}} G$

$\underline{\underline{\theta}}(^{\dagger}\Pi)$ hol

mono-an.

$G \curvearrowright \mathcal{O}^{\times\mu}(G)$.

$\hat{\mathbb{Z}}^{\times}$

hol.

$\underline{\underline{\theta}}(^{\ddagger}\Pi)$

$\Pi_{\mu}(M^{\Theta}_{*}(\Pi))$

$\cong (l\Delta_{\Theta})(M^{\Theta}_{*}(\Pi))$

cf. [IUTch II, Rem 1.10.2]

$\mu, \triangle$

vanish in the coo

theta fct $\overset{\text{HA emb}}{\rightsquigarrow}$ theta values → [IUTch III]

multirad ⟶ from a naive pt of view, $\overset{\text{log-link}}{\frown}$ multirad.

unirad. ⟵ const. monoids ⟵ class. cycl. rig. $\hat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}^{\triangleright}$

[IUTch II, Prop 2.1,

§10.2 Bad Places

$\underline{v} \in \underline{\mathbb{V}}^{bad}$

$$\begin{tikzcd}
\Pi^{tp}_{\ddot{\underline{\underline{Y}}}_{\underline{v}}} \arrow[r, hook, "\mu_2"] \arrow[d, "\Delta_\Theta \otimes \mathbb{Z}"] & \Pi^{tp}_{\underline{\underline{Y}}_{\underline{v}}} \arrow[r, hook, "l\underline{Z}"] \arrow[d, "\Delta_\Theta \otimes \mathbb{Z}"] & \Pi^{tp}_{\underline{\underline{X}}_{\underline{v}}} =: \Pi_{\underline{v}} \arrow[d, "\Delta_\Theta \otimes \mathbb{Z}"] \\
\Pi^{tp}_{\ddot{Y}_{\underline{v}}} \arrow[r, hook, "\mu_2"] & \Pi_{Y_{\underline{v}}} \arrow[r, hook, "l\underline{Z}"'] & \Pi^{tp}_{\underline{X}_{\underline{v}}}
\end{tikzcd}$$

indep

$\iota_{\underline{X}} \sim \iota_{\underline{X}}$

$\iota_{\ddot{\underline{Y}}} \sim \iota_{\ddot{Y}}$

dual graph of special fiber

$$\begin{tikzcd}
\Gamma_{\ddot{\underline{Y}}} \arrow[r] \arrow[d] & \Gamma_{\underline{Y}} \arrow[r] \arrow[d] & \Gamma_{\underline{X}} \arrow[d] \\
\Gamma_{\ddot{Y}} \arrow[r] & \Gamma_{Y} \arrow[r] & \Gamma_{\underline{X}}
\end{tikzcd}$$

[IUTch II, Prop 2.1, $\overset{\text{Rem}}{2.1.1}$, $\overset{\text{Prop}}{2.2}$]

; $\Pi_{\underline{\underline{X}}}$

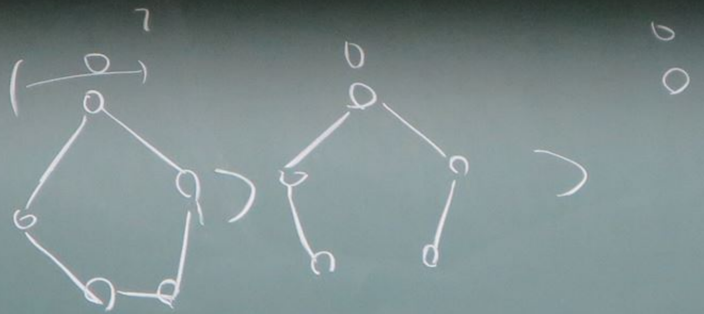
∞

indu

$\iota_{\underline{\underline{X}}} \rightsquigarrow \iota_{\underline{X}}$

$\iota_{\underline{\underline{\ddot{Y}}}} \rightsquigarrow \iota_{\ddot{Y}}$

$$\begin{tikzcd} {} \arrow[r] & \Gamma_{\underline{\underline{X}}} \arrow[d] \\ {} \arrow[r, hook] & \Gamma_{\underline{Y}} \end{tikzcd}$$



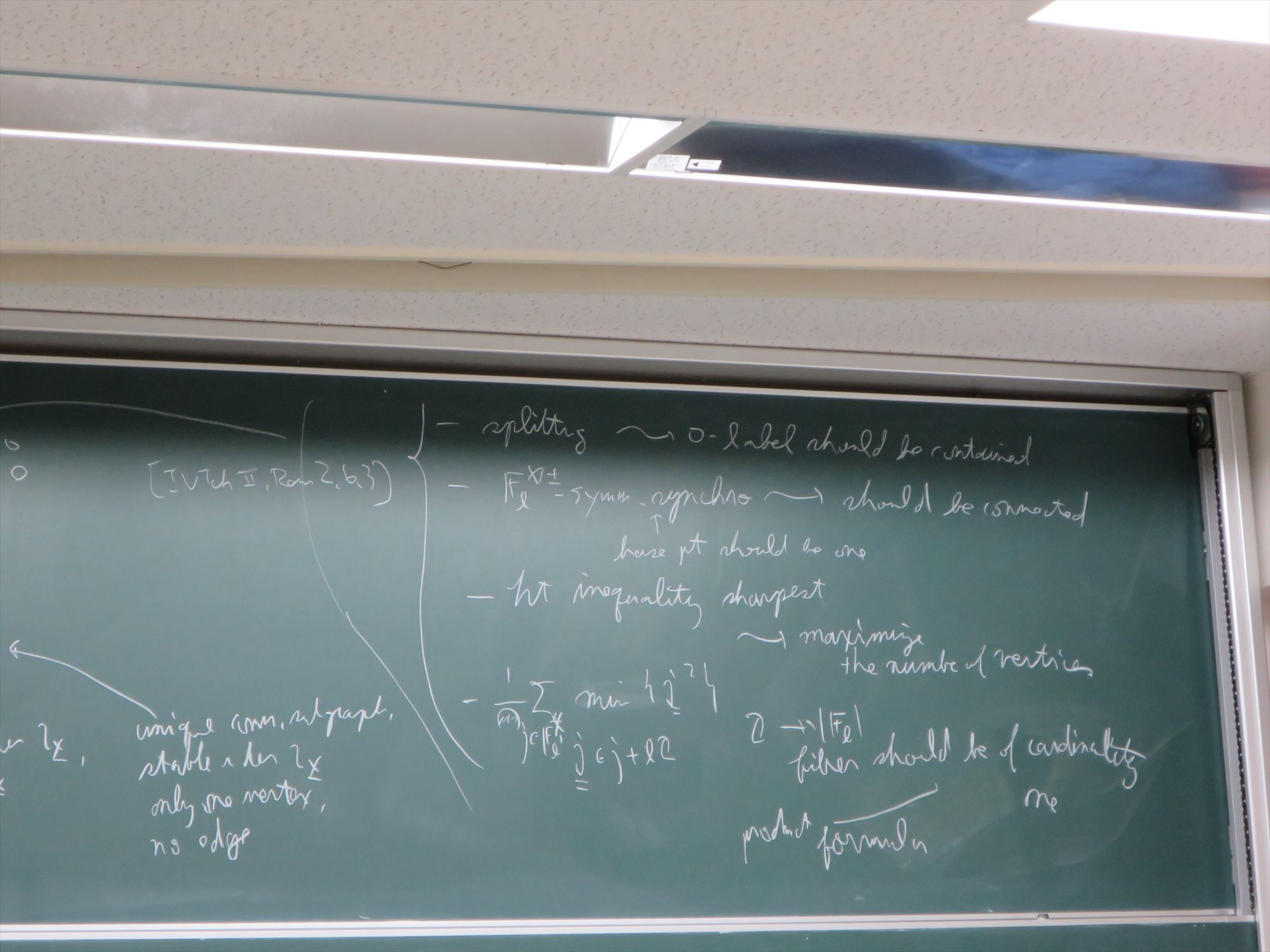
$\Gamma_{\underline{X}} \supset \Gamma^{\blacktriangle}_{\underline{X}} \supset \Gamma^{\bullet}_{\underline{X}}$

$\uparrow$

unique conn. subgp tree, stable under $\iota_{\underline{X}}$, containing vertices of $\Gamma_{\underline{X}}$

unique conn. subgraph,
stable under $\iota_{\underline{X}}$
only one vertex,
no edge

$\underline{\underline{\theta}}(\pi)$



o

o

[IUTch II, Rem 2.6])

- splitting ⇝ 0-label should be contained
- $\mathbb{F}_l^{\rtimes\pm}$-symm. synchro ⟶ should be connected
  ↑ base pt should be one
- ht inequality sharpest
  ⇝ maximize the number of vertices
- $\frac{1}{(l^*)}\sum_{j\in|\mathbb{F}_l^*|}\min\{\underline{j}^2\}$ $\underline{j}\in j+l\mathbb{Z}$
  $\mathbb{Z}\to|\mathbb{F}_l|$ fibers should be of cardinality one
  product formula

...er $l_{\underline{X}}$,

unique conn. subgraph,
stable under $l_{\underline{X}}$
only one vertex,
no edge

$\subseteq$: small

$\supseteq$: large

[IUTch II, Def 2.3]

(i)

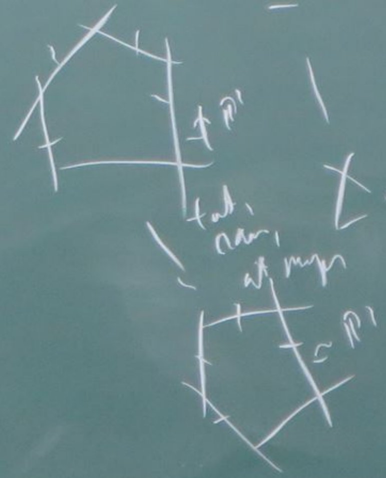
$$\begin{array}{ccccc}
\Delta_{\underline{v}} := \Delta^{tp}_{\underline{\underline{X}}_{\underline{v}}} & \subset & \Delta^{\pm}_{\underline{v}} := \Delta^{tp}_{\underline{X}_{\underline{v}}} & \subset & \Delta^{cor}_{\underline{v}} := \Delta^{tp}_{C_{\underline{v}}} \\
\cap & & \cap & & \cap \\
\Pi_{\underline{v}} & \subset & \Pi^{\pm}_{\underline{v}} := \Pi^{tp}_{\underline{X}_{\underline{v}}} & \subset & \Pi^{cor}_{\underline{v}} := \Pi^{tp}_{C_{\underline{v}}}
\end{array}$$

$$\begin{array}{ccc}
\Pi_{\subseteq} & := & \Pi_{\underline{v}} \text{ or } \Pi^{\pm}_{\underline{v}} \text{ or } \hat{\Pi}_{\underline{v}} \text{ or } \hat{\Pi}^{\pm}_{\underline{v}} \\
\cap & & \\
\Pi_{\supseteq} & := & \Pi^{\pm}_{\underline{v}} \quad \text{or} \quad \hat{\Pi}^{\pm}_{\underline{v}}
\end{array}$$

$\underline{\underline{X}}_{\underline{v}}$

$|$

$\underline{X}_{\underline{v}}$



(iii) $\pm$-cusp la

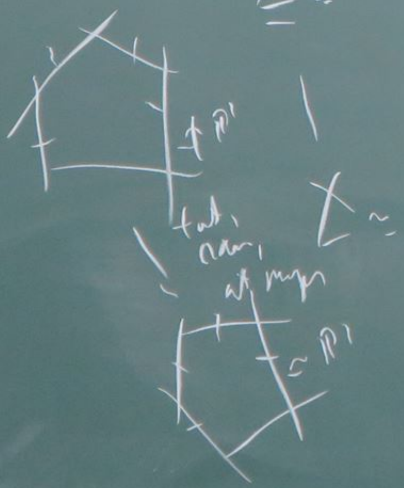
a Π

$\leq$: small
$\geq$: large

$:= \langle\ \rangle^{\text{trg}}_{C_{\geq}}$

$\cap$

$\Pi^{\text{cor}}_{\geq} := \Pi^{\text{trg}}_{C_{\geq}}$



(iii) $\underline{\pm\text{-cusp label}}$ of $\Pi_{\leq}$

a $\Pi_{\leq}$-conj class of the cuspidal decomp. gps of $\Pi_{\leq}$
s.t. the commensurator in $\Pi_{\geq}$
is a $\Pi_{\geq}$-conj. class
of cuspidal decomp. gp

$\mathrm{LabCusp}^{\pm}(\Pi_{\leq})$

H ⊂ G closed subgp
$C_G(H) := \{ g \in G \mid gHg^{-1} \cap H \subset H, gHg^{-1}$ of fin. index $\}$

$\Pi_{\leq} \cong \Pi_{\geq} \Rightarrow \mathrm{LabCu}$

$\mathbb{F}_l^{\times}$

+

(iv) $t \in \mathrm{LabCusp}^{\pm}($

$\rightsquigarrow$ uni

cuspidal

of the decomp. gps of $\Pi_{\underline{\underline{v}}}$

commensurator in $\Pi_{\underline{v}}$

$\Pi_{\underline{v}}$-conj. class

of cuspidal decomp. gp

closed subgp

$H \subset G$

$C_G(H) := \{ g \in G \mid gHg^{-1} \cap H \subset H, gHg^{-1}$ of finite index $\}$

$\Pi_{\underline{\underline{v}}} \cong \Pi_{\underline{v}} \Rightarrow \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{v}}}) \cong \mathrm{LabCusp}(^{\dagger}\mathcal{D}_{\underline{v}})$

$\mathbb{F}_l^{\times} \curvearrowright \quad \cup$

$^{\dagger}\eta^{0}_{\underline{v}}$: zero elt

$^{\dagger}\eta^{\pm}_{\underline{v}}$: ±-can. gen. well-def up to ±1

(iv) $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$

$\rightsquigarrow$ unique vertex of $\Gamma^{\blacktriangleright}_{\underline{X}}$

$\vee$

$\supseteq$: $\Gamma^{\bullet *}_{\underline{X}}$ no edges

$\rightsquigarrow \Pi_{\underline{v}\bullet *} \subseteq \Pi_{\underline{v}\blacktriangleright} \subseteq \Pi_{\underline{v}}$

decomp. gp well-def up to $\Pi_{\underline{v}\blacktriangleright}$-conj.

decomp. sp of $\mathcal{H}$ → $\Delta^{\text{trop}}_{X} \times \Delta^{\text{trop}}_{\mathcal{H}}$ ← mod. part

$\Delta^{\text{trop}}_{\mathcal{G}}$ sub. graph

$\Sigma = \{l\}$

$\not\ni$ p

mod. part Brim

special fiber of the stable model of X

$H \underset{\text{out}}{\subset} G$, $I \underset{\text{out}}{\subset} G$,

$\Uparrow$ $\not\Downarrow$

$I \underset{\text{out}}{\subset} H \underset{\text{out}}{\subset} G$ factorization

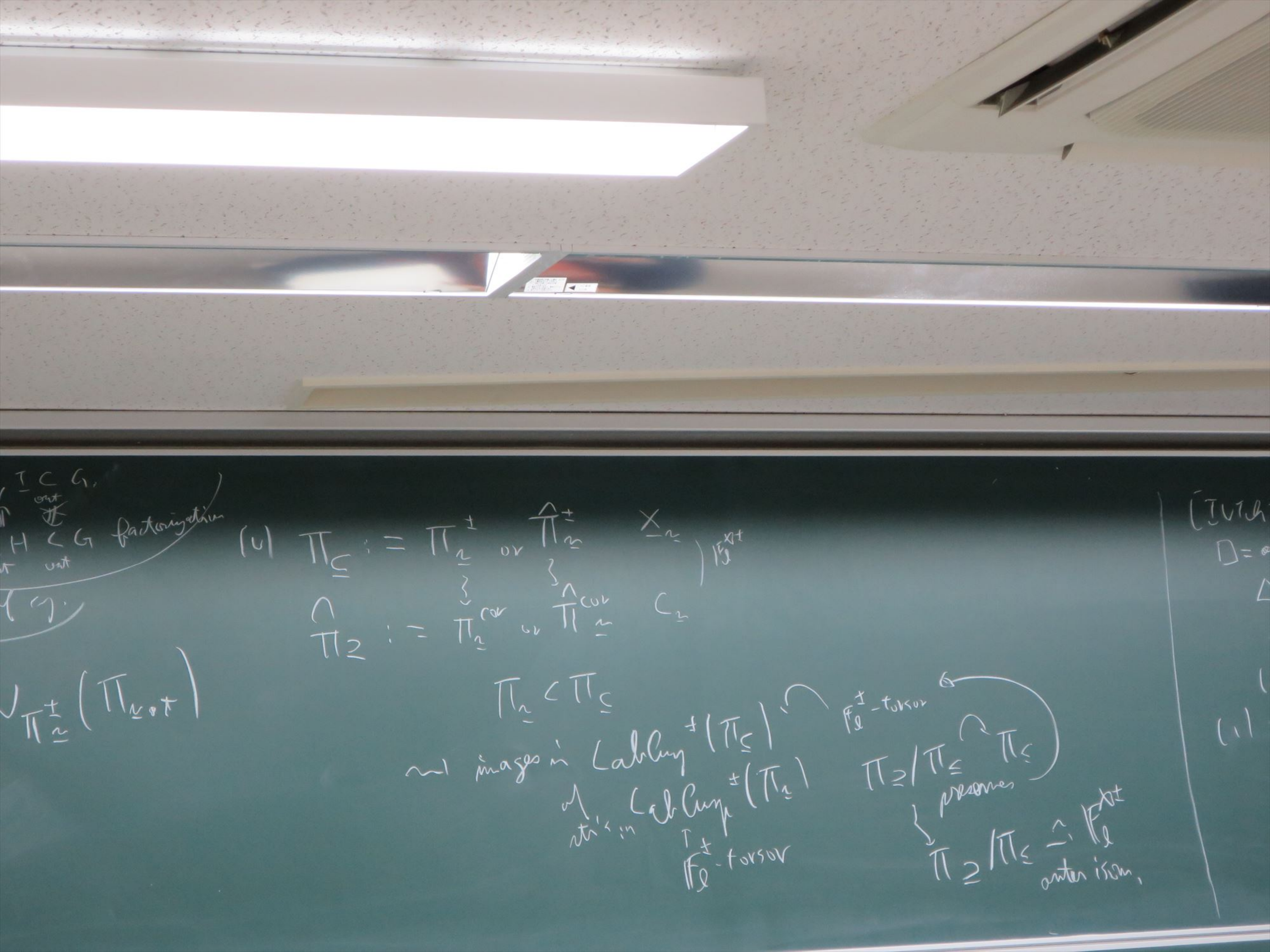
trop cy vs prof cy.

$$\Pi^{\pm}_{\underline{v}\bullet t} := N_{\Pi^{\pm}_{\underline{\sim}}}(\Pi_{\underline{v}\bullet t})$$

(4) $\Pi_{\underline{\subseteq}}$;

$\cap$

$\Pi_{\underline{2}}$



$I \subset G$,
out
$H < G$ factorization
out out
$\{ G$.

$\Pi^{\pm}_{\simeq}(\Pi_{\simeq \otimes \dagger})$

(ii) $\Pi_{\subseteq} := \Pi^{\pm}_{\simeq}$ or $\hat{\Pi}^{\pm}_{\simeq}$ $\quad X_{\simeq}$) $\mathbb{F}_\ell^{\times\pm}$

$\cap \qquad \wr \qquad \wr$

$\Pi_{\geq} := \Pi^{\rm cor}_{\simeq}$ or $\hat{\Pi}^{\rm cor}_{\simeq}$ $\quad C_{\simeq}$

$\Pi_{\simeq} < \Pi_{\subseteq}$

⇝ images in $\mathrm{LabCusp}^{\pm}(\Pi_{\subseteq}) \curvearrowleft \mathbb{F}_\ell^{\pm}$-torsor

$\cap$ str's in $\mathrm{LabCusp}^{\pm}(\Pi_{\simeq})$

$\mathbb{F}_\ell^{\pm}$-torsor

$\Pi_{\geq}/\Pi_{\subseteq} \curvearrowright \Pi_{\subseteq}$

preserves

$\Pi_{\geq}/\Pi_{\subseteq} \xrightarrow{\sim} \mathbb{F}_\ell^{\times\pm}$
outer isom.

[IUTch
$D =$
(1)

$\mathbb{F}_l^{\pm}$-torsor

$\Pi_{\geq}/\Pi_{\leq} \curvearrowleft \Pi_{\leq}$

preserves

$\Pi_{\geq}/\Pi_{\leq} \overset{\sim}{\to} \mathbb{F}_l^{\rtimes\pm}$

outer isom.

[IUTch II, Cor 2.4]

$\square = \dagger$ or $\ddagger$

$\Delta^{(\pm)}_{\simeq\square} := \Delta^{(\pm)}_{\simeq} \cap \Pi^{(\pm)}_{\simeq\square}$

$\Pi^{\ddot{}}_{\simeq\square} := \Pi_{\simeq\square} \cap \Pi^{\ddot{Y}}_{\simeq}$

$(\Delta \qquad (\Delta)$

(1) $\dagger \in \mathrm{LabCusp}^{\pm}(\Pi_{\simeq})$

$I_{\dagger} \subseteq \Pi_{\simeq}$ cuspidal inertia

$(\Delta_{\simeq\square}$

$\boxed{\pm}$

Consider the following sets of conj. classes:

$$\{ I_A^{\gamma_1} \}_{\gamma_1 \in \hat{\Pi}^{\pm}_{\underline{n}}} = \{ I_*^{\gamma_1} \}_{\gamma_1 \in \hat{\Delta}^{\pm}_{\underline{n}}} ,$$

$$\{ \Pi^{\gamma_2}_{\underline{n}\,\square} \}_{\gamma_2 \in \hat{\Pi}^{\pm}_{\underline{n}}} = \{ \Pi^{\gamma_2}_{\underline{n}\,\square} \}_{\gamma_2 \in \hat{\Delta}^{\pm}_{\underline{n}}} ,$$

$$\{ \Pi^{\pm\gamma_3}_{\underline{n}\,\square} \}_{\gamma_3 \in \hat{\Pi}^{\pm}_{\underline{n}}} = \{ \Pi^{\pm\gamma_3}_{\underline{n}\,\square} \}_{\gamma_3 \in \hat{\Delta}^{\pm}_{\underline{n}}}$$

the following

(a). $\gamma' \in$

(b). I

(c).

$H \underset{\text{out}}{\subset} G$, $\underline{I} \underset{\text{out}}{\subset} G$.

[illegible]

$\boxed{\pm}$

llowing

lasses;

$^{\pm}_{\simeq} = \{ I_{*}^{\gamma_1} \}_{\gamma_1 \in \hat{\Delta}^{\pm}_{\simeq}}$,

$\in \hat{\Pi}^{\pm}_{\simeq} = \{ \Pi^{\gamma_2}_{\simeq\square} \}_{\gamma_2 \in \hat{\Delta}^{\pm}_{\simeq}}$,

$_3 \in \hat{\Pi}^{\pm}_{\simeq} = \{ \Pi^{\pm\gamma_3}_{\simeq\square} \}_{\gamma_3 \in \hat{\Delta}^{\pm}_{\simeq}}$

the following is equiv:

(a). $\gamma' \in \Delta^{\pm}_{\simeq\square}$

(b). $I_{A}^{\gamma\gamma'} \subset \Pi^{\gamma}_{\simeq\square}$

(c). $I_{*}^{\gamma\gamma'} \subset (\Pi^{\pm}_{\simeq\square})^{\gamma}$

(ii) $\delta := \gamma\gamma'$

Assume $I_{*}^{\delta} = I_{*}$

$\rightsquigarrow$ ∃A def's

(a). decomp. Π

(b). decomp. Π

of μ_{-}

(c). decomp.

is equiv:

$\triangle^{\pm}_{v\square}$

$\gamma\gamma' \subset \Pi^{\gamma}_{v\square}$

$I^{\gamma\gamma'}_{*} \subset (\Pi^{\pm}_{v\square})^{\gamma}$

(ii) $\delta := \gamma\gamma' \in \hat{\triangle}^{\pm}_{v}$

Assume $I^{\delta}_{*} = I^{\gamma\gamma'}_{*} \subseteq \Pi^{\gamma}_{v\square} = \Pi^{\gamma}_{v\square}$

$\rightsquigarrow$ ∃ det'd

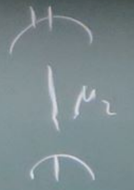
(a). decomp. gp $D^{\delta}_{*} := N_{\Pi^{\delta}_{v}}(I^{\delta}_{*}) \subseteq \Pi^{\delta}_{v\ddot{\square}}$

(b). decomp. gp of μ_--transl. $D^{\delta}_{\mu_-} \subset \Pi^{\delta}_{v\ddot{\square}}$

well-def up to $(\Pi^{\pm}_{v\square})^{\delta}$-conj.

(c). decomp. gp $D^{\delta}_{*,\mu_-} \subset \Pi^{\delta}_{v\ddot{\square}}$

well-def up to $(\Pi^{\pm}_{v\square})^{\delta}$-conj.

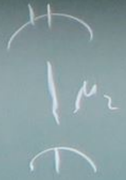


(iii)

con

[IUTch II, Co

$-\delta) \subseteq \Pi^{\delta}_{\underline{v}\ddot{\square}}$
-1



$-\delta$

$\underline{v}\ddot{\square}$

ℓ of ⋆o $(\Pi^{\pm}_{\underline{v}\square})^{\delta}$-conj.

$\Pi^{-\delta}_{\underline{v}\ddot{\square}}$

.dl of Ao $(\Pi^{\pm}_{\underline{v}\square})^{\delta}$-conj.

(iii) $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry

one base pt

$\mathbb{F}_\ell^{\rtimes\pm}$-synchro

"good diag."

hol. conic
ue obj
$\{0, \langle\ \rangle\}$
$= \Delta$

$\square = a \star$

consth. of (ii)(a), (c)

is compat. w/.

conj. by $\delta \in \widehat{\Pi}^{cn}_{\underline{v}}$

$$\left(\widehat{\Pi}^{cn}_{\underline{v}} / \widehat{\Pi}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_\ell^{\rtimes\pm} \right)$$

( temp. conj vs μ-ℓ conj
Th 6.6)

[IUTch II, Cor 2.5] (Π-thic Θ-eval) (Θ-eval "Π" "$\mathbb{M}$" "$\mathcal{F}$" Π. $I_{\mathbb{M}} \leftarrow x$

(i) $\Pi_{\underline{v}} \xmapsto{\text{sp thic}} (l\Delta_{\Theta})(\Pi_{\underline{v}^{\circledcirc}})$

$\hookrightarrow \Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}})$,

$\Pi_{\underline{v}^{\circledcirc}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}^{\circledcirc}})$)

(ii') $I_A^{\delta} = I_A^{\Psi'} \subseteq \Pi^{\delta}_{\underline{v}^{\circledcirc}} \subseteq \Pi^{\dashv}_{\underline{v}^{\circledcirc}} = \Pi^{\delta}_{\underline{v}^{\circledcirc}}$

$\dashv$-inv. sets

μ_{2l}

$(\mu^-$

(Ω)-equal "Π" "$\mathbb{M}$" "$\overline{X}$"

$$\begin{tikzcd} \Pi \arrow[dr] & & \overline{X} \arrow[dl] \\ & \mathbb{M} & \end{tikzcd}$$

γ-inv. sets $\underline{\underline{\Theta}}^2(\Pi^\gamma_{\underline{2}})$, $\omega\,\underline{\underline{\Theta}}^2(\Pi^\gamma_{\underline{2}})$

$\Pi^\delta_{\underline{v}\Theta}$

restr. to $\Pi^\gamma_{\underline{v}\ddot{\Theta}}$ $(\subseteq \Pi_{\underline{Y}}(\Pi^\gamma_{\underline{v}}))$

obtain

$\mu_{2\ell}$-orbit $\underline{\underline{\Theta}}^2(\Pi^\gamma_{\underline{v}\ddot{\Theta}}) \subseteq_\sim \underline{\underline{\Theta}}^1(\Pi_{\underline{v}\ddot{\Theta}}) \subset \varinjlim_{J} H^1(\Pi^\gamma_{\underline{v}\ddot{\Theta}}|_{\hat{J}}, (\ell\Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\Theta}}))$

$(\mu$-$)$ $\mu_{2\ell}$ μ $\hat{J} \underset{\text{open}}{\subset} \hat{\Pi}_{\underline{v}}$

$(-x)^2 = x^2$

& restr. to $D^\delta_{x,\mu}$

$\underline{\underline{\Theta}}$

$\mu_{2\ell}$-

depends only on $|x| \in |\mathbb{F}$

write (∞)

&

with $A_{\circledcirc}$ D^{δ}_{t,μ_-}

$t \in \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}})$

$\xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$

$\Pi^{\gamma}_{\underline{v}}))$

$\subset \varinjlim_{\hat{J}} H^1(\Pi^{\gamma}_{\underline{v}\circledcirc}|_{\hat{J}}, (l\Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\circledcirc}))$

$\hat{J} \subset \hat{\Pi}_{\underline{v}}$ open

$(-t)^2 = t^2$

$\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\circledcirc}) \subset \underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\circledcirc})$

μ_{2l}- / aliA

∞, $\subset \varinjlim_{J_G} H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\circledcirc})|_{J_G}, (l\Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\circledcirc}))$

$J_G \subset G(\Pi^{\gamma}_{\underline{v}\circledcirc})$ open

μ

depends only on $|t| \in |\mathbb{F}_l|$

write $_{(\infty)}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\circledcirc})$

(iii) $\Pi^{\gamma}_{\underline{v}\circledcirc}$: $\hat{\Delta}^{\pm}_{\underline{v}}$-conj. of $\Pi_{\underline{v}\circledcirc}$

$I^{\delta}_{\ast}$: $\hat{\Delta}^{\pm}_{\underline{v}}$-conj. of $I_{\ast} \subset \Pi^{\gamma}_{\underline{v}\circledcirc}$ s.t.

$\rightsquigarrow \underline{\underline{\Theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\circledcirc})$

$\rightsquigarrow$ collection of μ_{2l}-(μ-)orbits

$$\{{}_{(\infty)}\underline{\underline{\Theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\circledcirc})\}_{|t|\in|\mathbb{F}_l|}$$

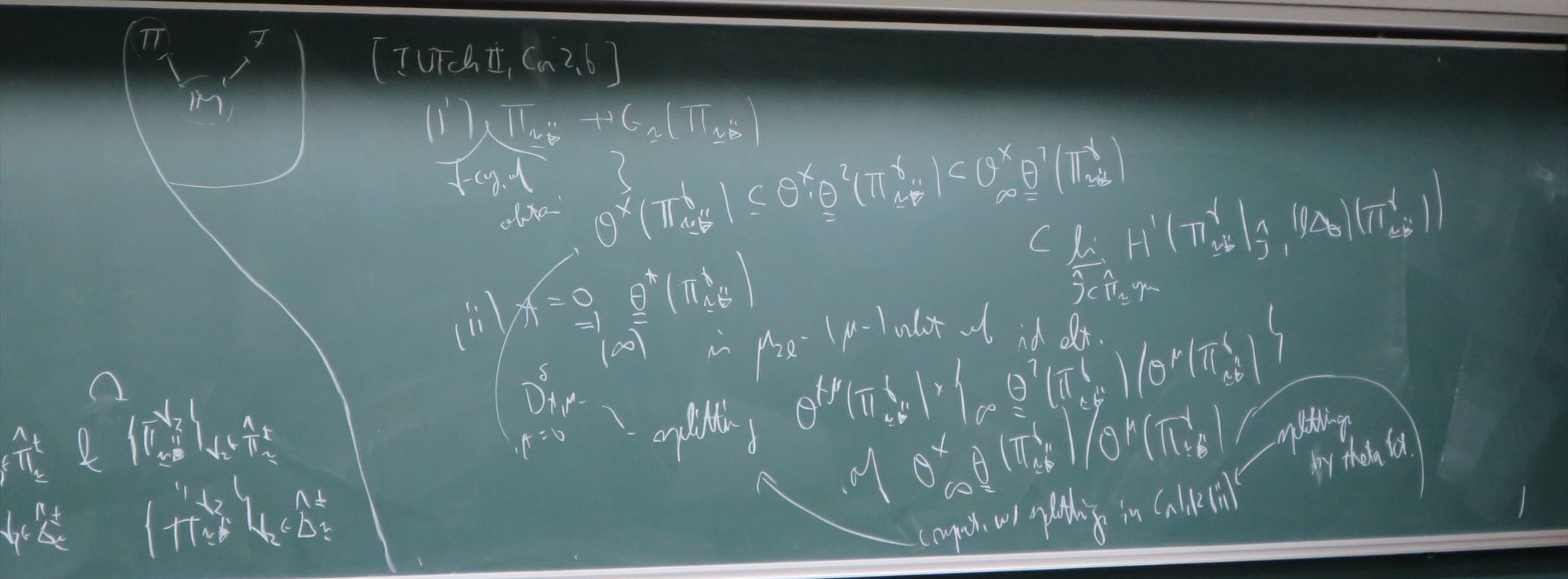
factorial w.r.t. $\Pi_{\underline{v}}$
compat w/ indep. conj. actions of $\hat{\Delta}^{\pm}_{\underline{v}}$ $\curvearrowright$ $\overset{\text{set}}{\{I^{\gamma_1}_{\ast}\}_{\gamma_1\in\hat{\Pi}^{\pm}_{\underline{v}}}}$ & $\overset{\curvearrowright}{\{\Pi^{\gamma_2}_{\underline{v}\circledcirc}\}_{\gamma_2\in\hat{\Pi}^{\pm}_{\underline{v}}}}$

$$\begin{array}{ccc} \{I^{\gamma_1}_{\ast}\}_{\gamma_1\in\hat{\Pi}^{\pm}_{\underline{v}}} & & \{\Pi^{\gamma_2}_{\underline{v}\circledcirc}\}_{\gamma_2\in\hat{\Pi}^{\pm}_{\underline{v}}} \\ \| & & \| \\ \{I^{\gamma_1}_{\ast}\}_{\gamma_1\in\hat{\Delta}^{\pm}_{\underline{v}}} & & \{\Pi^{\gamma_2}_{\underline{v}\circledcirc}\}_{\gamma_2\in\hat{\Delta}^{\pm}_{\underline{v}}} \end{array}$$

[IUTch

(1')

Cor 2.8 [IUM-th'



[IUTch II, Cor 2.6]

(i') $\Pi_{\underline{v}\ddot{\triangleright}} \to G_{\underline{v}}(\Pi_{\underline{v}\ddot{\triangleright}})$

l-cyc. of ... obtain

$\theta^{\times}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}}) \subseteq \theta^{\times} \cdot \underline{\underline{\theta}}^{\gamma}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}}) \subset \theta^{\times}_{\infty}\underline{\underline{\theta}}^{\gamma}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})$

$\subset \varinjlim_{\hat{J}\subset\hat{\Pi}_{\underline{v}}\text{ open}} H^{1}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}}|_{\hat{J}}, (l\Delta_{\Theta})(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}}))$

(ii) $A = \underline{\underline{\theta}}_{(\infty)}$ $\underline{\underline{\theta}}^{\star}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})$

in μ_{2l}- (μ-) orbit of id elt.

$D^{\delta}_{A,\mu-}$, $A = b$ $\rightsquigarrow$ splittings

$\theta^{\times\mu}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}}) \times \{{}_{\infty}\underline{\underline{\theta}}^{\gamma}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})/\theta^{\mu}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})\}$

of $\theta^{\times}_{\infty}\underline{\underline{\theta}}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})/\theta^{\mu}(\Pi^{\dagger}_{\underline{v}\ddot{\triangleright}})$

splittings by theta fct.

compat. w/ splittings in Cor 1.12 (iii)

$\cap$

$\hat{\Pi}^{\pm}_{\underline{v}}$ & $\{\Pi^{\dagger_2}_{\underline{v}\ddot{\triangleright}}\}_{\dagger_2\in\hat{\Pi}^{\pm}_{\underline{v}}}$

$\dagger_1 \in \hat{\Delta}^{\pm}_{\underline{v}}$ $\{\Pi^{\dagger_2}_{\underline{v}\ddot{\triangleright}}\}_{\dagger_2\in\hat{\Delta}^{\pm}_{\underline{v}}}$

[IUTch II, Cor 2.8] ($\mathbb{M}$-th'c Θ-eval)

$\Pi_{\underline{\underline{X}}}^{\gamma} \rightsquigarrow$ mono-theta env. $(\mathbb{M}^{\Theta}_{*})^{\gamma}$

(i') cycl rig. $(l\Delta_{\Theta})((\mathbb{M}^{\Theta}_{*\ddot{D}})^{\gamma}) \xrightarrow{\sim} \Pi_{\mu}((\mathbb{M}^{\Theta}_{*\ddot{D}})^{\gamma})$

$(l\Delta_{\Theta})((\mathbb{M}^{\Theta}_{*})^{\gamma}) \xrightarrow{\sim} \Pi_{\mu}((\mathbb{M}^{\Theta}_{*})^{\gamma})$

replace $(l\Delta_{\Theta})(-)$ by $\Pi_{\mu}(-)$

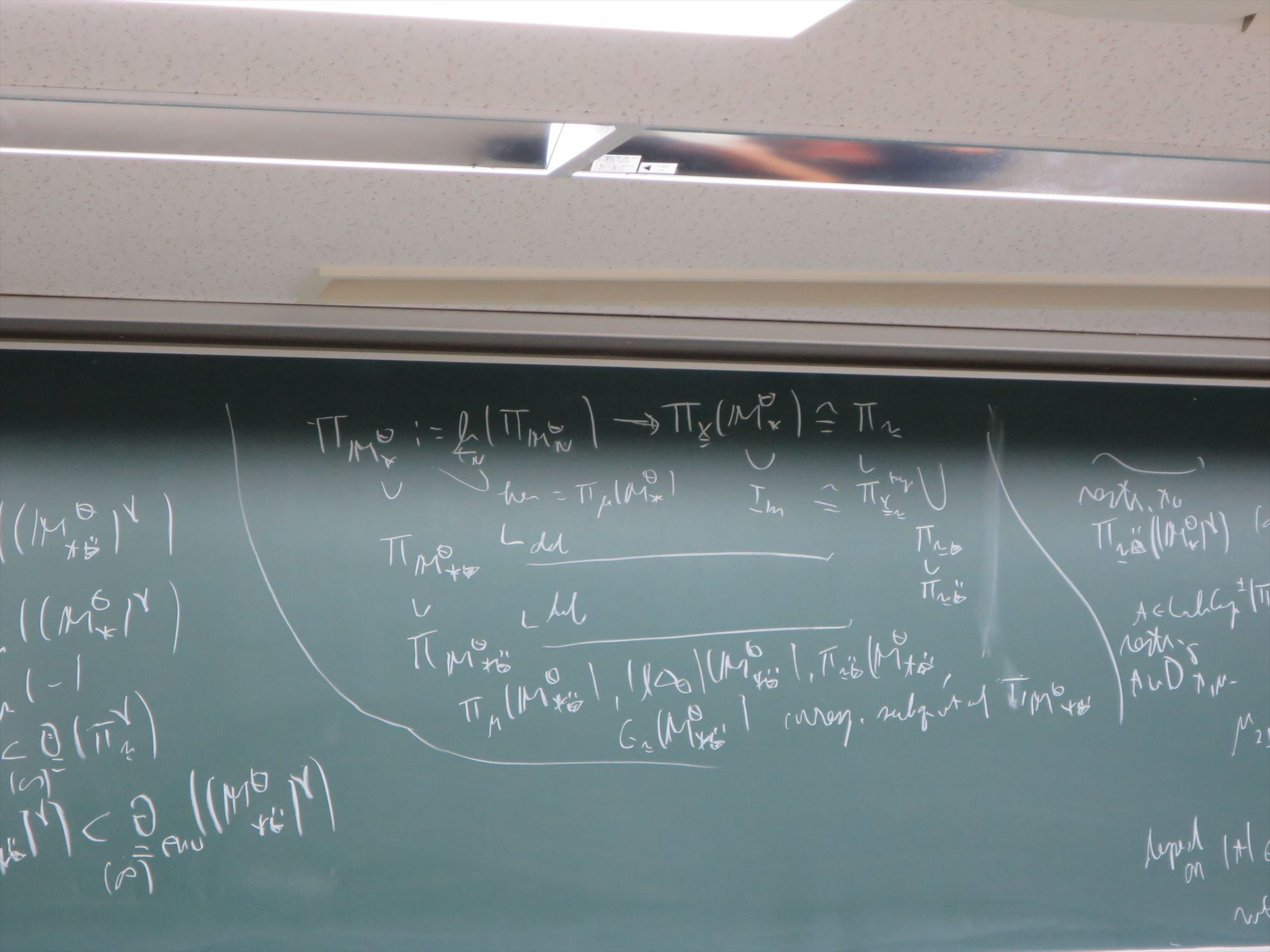
$\rightsquigarrow$ ι^{γ}-inv. subset $\underline{\underline{\Theta}}^{\iota}_{(\omega)}(\Pi^{\gamma}_{\underline{\underline{X}}}) \subset \underline{\underline{\Theta}}_{(\omega)}(\Pi^{\gamma}_{\underline{\underline{X}}})$

$\overset{\text{def}}{\rightsquigarrow}$ $\underline{\underline{\Theta}}^{\iota}_{(\omega)\,\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{D}})^{\gamma}) \subset \underline{\underline{\Theta}}_{(\omega)\,\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{D}})^{\gamma})$

$\Pi_{\mathbb{M}^{\Theta}_{*}}$

$\cup$

Π



$\Pi_{\mathcal{M}^{\Theta}_{*}} := \varprojlim_{N} (\Pi_{\mathcal{M}^{\Theta}_{N}}) \longrightarrow \Pi_{\underline{X}}(\mathcal{M}^{\Theta}_{*}) \cong \Pi_{v}$

$\cup$ ker $= \Pi_{\mu}(\mathcal{M}^{\Theta}_{*})$ $\cup$ $I_m \cong \Pi^{\mathrm{tp}}_{\underline{X}_{v}}$ $\cup$

$\Pi_{\mathcal{M}^{\Theta}_{*\circledcirc}}$ ⊂ dd

$\Pi_{\underline{v}\circledcirc}$ ⊃ $\Pi_{\underline{v}\ddot{\circledcirc}}$

$\cup$ ⊂ Ad

$\Pi_{\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}}}$

$\Pi_{\mu}(\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})$, $(l\Delta_{\Theta})(\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})$, $\Pi_{\underline{v}\ddot{\circledcirc}}(\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})$, $G_{v}(\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})$ corresp. subquot of $\Pi_{\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}}}$

$((\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})^{\gamma})$

$((\mathcal{M}^{\Theta}_{*})^{\gamma})$

$(-)$

$\subset \underline{\theta}(\Pi^{\gamma}_{v})$ (a)

$)^{\gamma}) \subset \underline{\theta}_{\mathrm{env}}((\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}})^{\gamma})$ (a)

restr. to

$\Pi_{\underline{v}\ddot{\circledcirc}}((\mathcal{M}^{\Theta}_{*})^{\gamma})$ (c

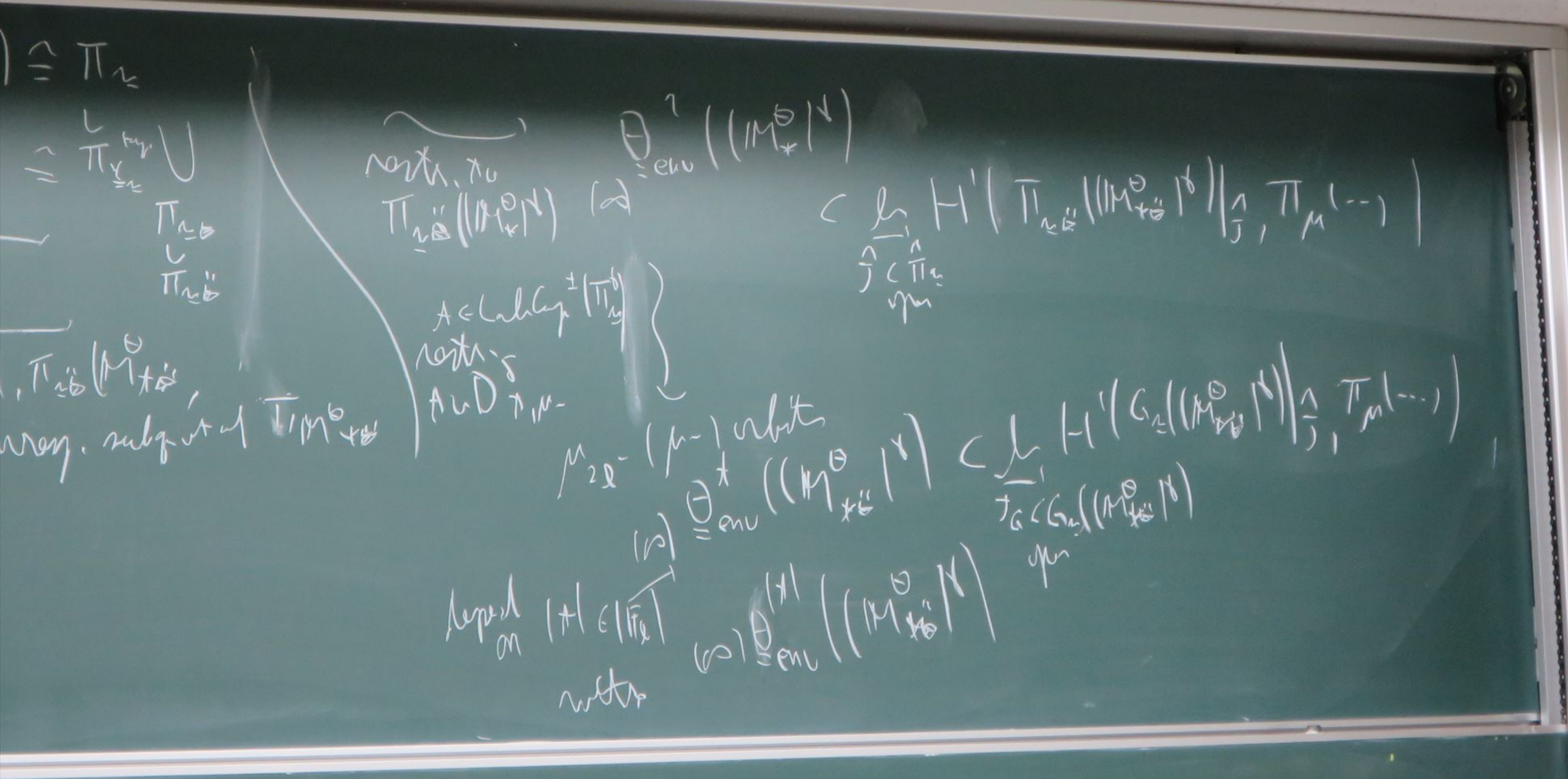
$A \in \mathrm{LocLin}^{\pm}(\Pi$

restr. δ

$A \cup D$ ↑, N-

μ_{2}

depend on $|A|$



$) \cong \Pi_{\simeq}$

$\cup$

$\cong \Pi_{Y_{\simeq}}^{\text{tmp}} \cup$

$\Pi_{\simeq\circledast}$

$\cup$

$\Pi_{\simeq\ddot{\circledast}}$

$, \Pi_{\simeq\ddot{\circledast}}(\mathbb{M}^{\Theta}_{*\ddot{\circledast}},$

wrong. subquot of $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\circledast}}}$

$\widetilde{\qquad}$,

vertic. π_0

$\Pi_{\simeq\ddot{\circledast}}((\mathbb{M}^{\Theta}_{*}|^{\gamma})$ (∞)

$\underset{=}{\Theta}{}^{?}_{\text{env}}((\mathbb{M}^{\Theta}_{*}|^{\gamma})$

$\subset \varinjlim_{\hat{J} \subset \hat{\Pi}_{\simeq} \text{ open}} H^1(\Pi_{\simeq\ddot{\circledast}}((\mathbb{M}^{\Theta}_{*\ddot{\circledast}}|^{\gamma})|_{\hat{J}}, \Pi_{\mu}(\ldots))$

$A \in \text{LabCusp}^{\pm}(\Pi^{\Theta}_{\ldots})$

vertic. δ

$A \cup D_{\pm, \mu-}$

μ_{2l}-(μ-) orbits

$(\infty)\, \underset{=}{\Theta}{}^{\pm}_{\text{env}}((\mathbb{M}^{\Theta}_{*\ddot{\circledast}}|^{\gamma}) \subset \varinjlim_{J_G \subset G_{\simeq}((\mathbb{M}^{\Theta}_{*\ddot{\circledast}}|^{\gamma}) \text{ open}} H^1(G_{\simeq}((\mathbb{M}^{\Theta}_{*\ddot{\circledast}}|^{\gamma})|_{\hat{J}}, \Pi_{\mu}(\ldots))$

depend on $|t| \in |\mathbb{F}_l|$

$(\infty)\, \underset{=}{\Theta}{}^{|t|}_{\text{env}}((\mathbb{M}^{\Theta}_{*\ddot{\circledast}}|^{\gamma})$

vertic.

(ii) $\Pi_{\underline{\mu}\circledcirc}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma})$: $\hat{\Delta}^{\pm}_{\sim}$-conj. of $\Pi_{\underline{\mu}\circledcirc}(\mathbb{M}^{\Theta}_{*\circledcirc})$

I^{δ}_{A} : $\hat{\Delta}^{\pm}_{\sim}$-conj. of $I_{A} \subset \Pi^{\gamma}_{\underline{\mu}\circledcirc}(\ldots)$

$\rightsquigarrow$ $\{\underline{\underline{\Theta}}^{|A|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma})\}_{|A|\in|\mathbb{F}_{\ell}|}$

functorial (∞)

w.r.t. $\mathbb{M}^{\Theta}_{*}$

compat. w/ $\hat{\Delta}^{\pm}_{\sim}$-conj. $\curvearrowright \{I^{\gamma_1}_{A}\}_{\gamma_1\in\hat{\Delta}^{\pm}_{\sim}}$

indep. $\curvearrowright \{I_{\kappa\circledcirc}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma_2})\}_{\gamma_2}$

(iii) $A=0$ by cycl. rig $(\ell\Delta_{\Theta})(-1) \xrightarrow[\mathrm{Kum}]{\sim} \Pi_{\mu}(-1)$

splitting of $\mathcal{O}^{\times} \curvearrowright \underline{\underline{\Theta}}^{?}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma})/\mathcal{O}^{\mu}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma})$ $\{\underline{\underline{\Theta}}^{?}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma}/\mathcal{O}^{\mu}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma})\}$

$\mathcal{O}^{\times\mu}((\mathbb{M}^{\Theta}_{*\circledcirc})^{\gamma}) \times \{\infty\underline{\underline{\Theta}}_{\mathrm{env}}$

compat. w/ the splitting of Cor 1.12 , theta fcn

[IUTch II, Cor

(cycl. Rig (LGFT) $\mapsto$ {

uniradial.

$\Pi_{\underline{\mu}} := h(\Pi_{\mathbb{M}^{\Theta}_{*}}) \longrightarrow \Pi_{X}(\mathbb{M}^{\Theta}_{*})$

[IUTchII, Cor2.9] (bs-Gal thic Θ-eval)

(Cycl. Rig (CFT)) $\longrightarrow$ $\left\{ \begin{array}{l} \mu_{\ell}^{\wedge}(G_{\underline{v}}(\Pi_{\underline{v}})) \cong (\ell\Delta_{\Theta})(\Pi_{\underline{v}}) \\ \mu_{\ell}^{\wedge}(G_{\underline{v}}(\Pi^{\mathcal{D}}_{\underline{v}\circledast})) \cong (\ell\Delta_{\Theta})(\Pi^{\mathcal{D}}_{\underline{v}\circledast}) \end{array} \right.$

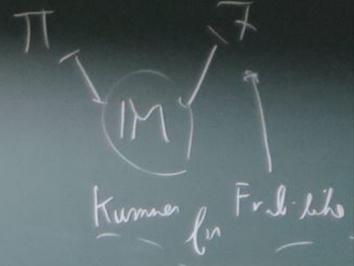
uniradi.

(a) $\underline{\Theta}_{\text{env}}$ ⇝ (a) $\underline{\Theta}_{\text{bs}}$

(b) $\underline{\Theta}^{A}_{\text{env}}$ ⇝ (b) $\underline{\Theta}^{A}_{\text{bs}}$

(c) $\underline{\Theta}^{|A|}_{\text{env}}$ ⇝ (c) $\underline{\Theta}^{|A|}_{\text{bs}}$

splitting.



theta fct

theta values

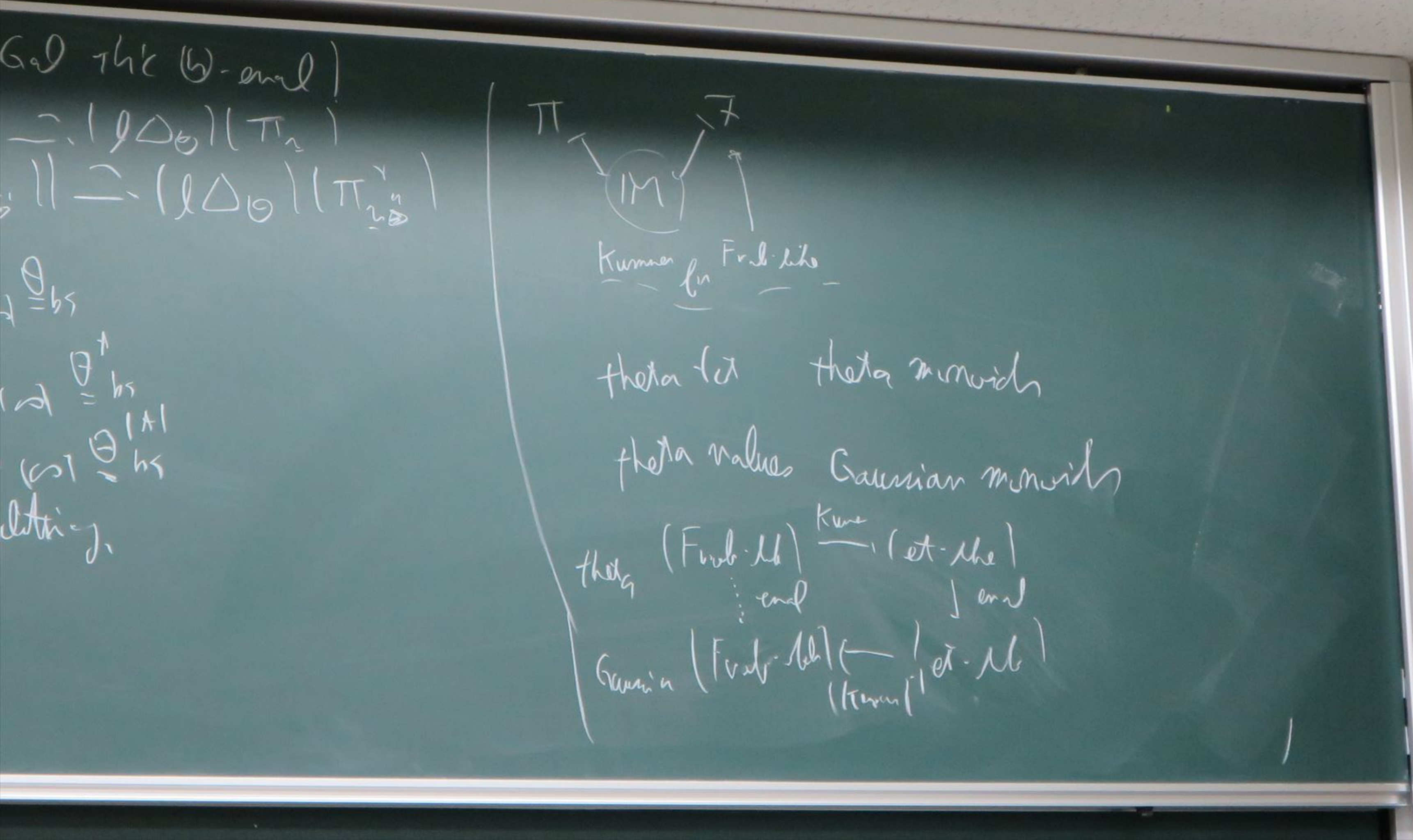
$\mathbb{M}^{\Theta}_{*\circledast}|^{\mathcal{D}} / \Theta^{\mathcal{N}}(\mathbb{M}^{\Theta}_{*\circledast}|^{\mathcal{D}}) \}$

A w/ the splitting of Cor 1.12 ↪ theta fct

$T_{\mathbb{M}^{\Theta}_{*}} := \varprojlim (\Pi_{\mathbb{M}^{\Theta}_{N}}) \twoheadrightarrow \Pi_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{v}}$

$\cup$

$\Theta^{?}((\mathbb{M}^{\Theta}_{*}|^{\mathcal{D}})$



Gal thic (θ)-end)

$\cong (l\Delta_\Theta)(\Pi_v)$

$|| \cong (l\Delta_\Theta)(\Pi^{Y}_{\underline{v}\otimes})$

$\underline{\Theta}_{bs}$

$\underline{\Theta}^{*}_{bs}$

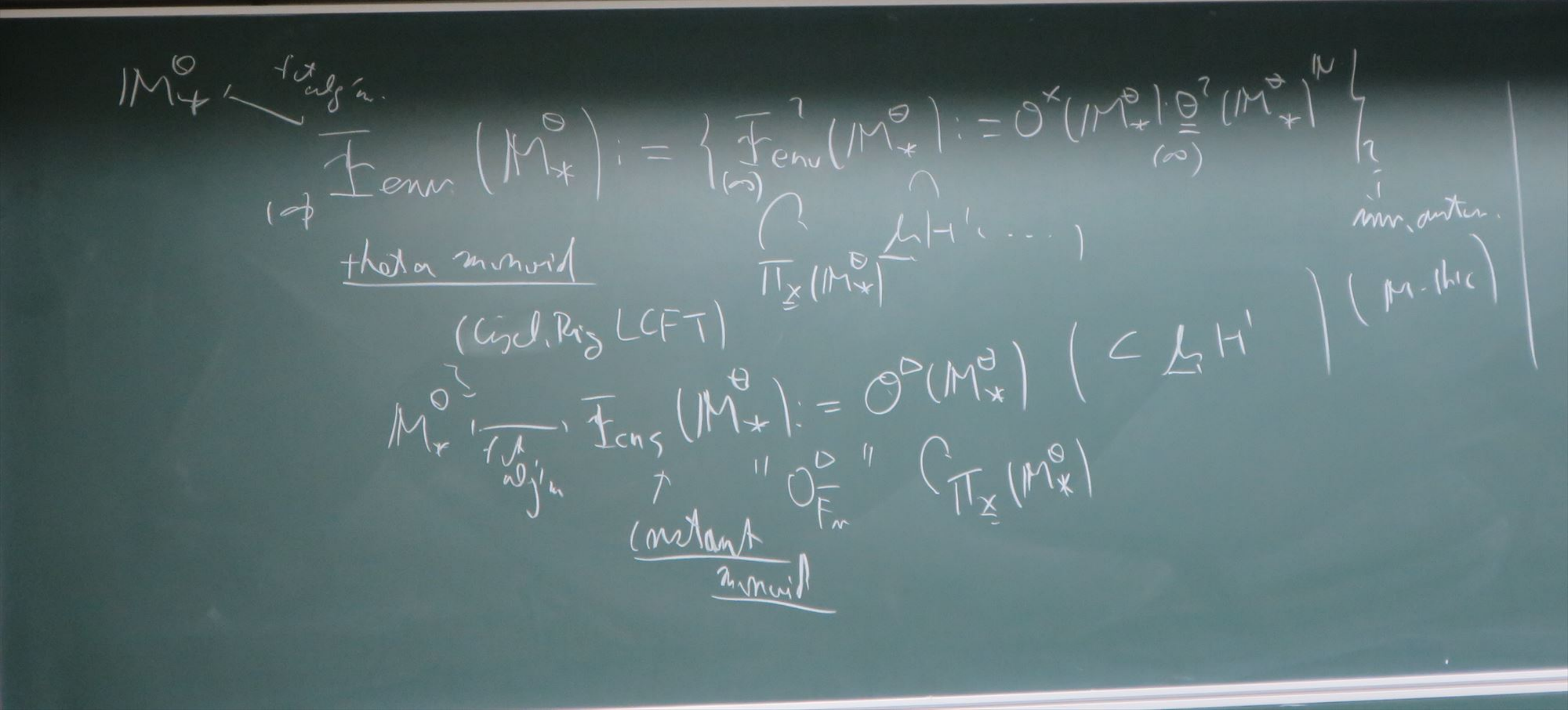
$\underline{\Theta}^{|*|}_{bs}$

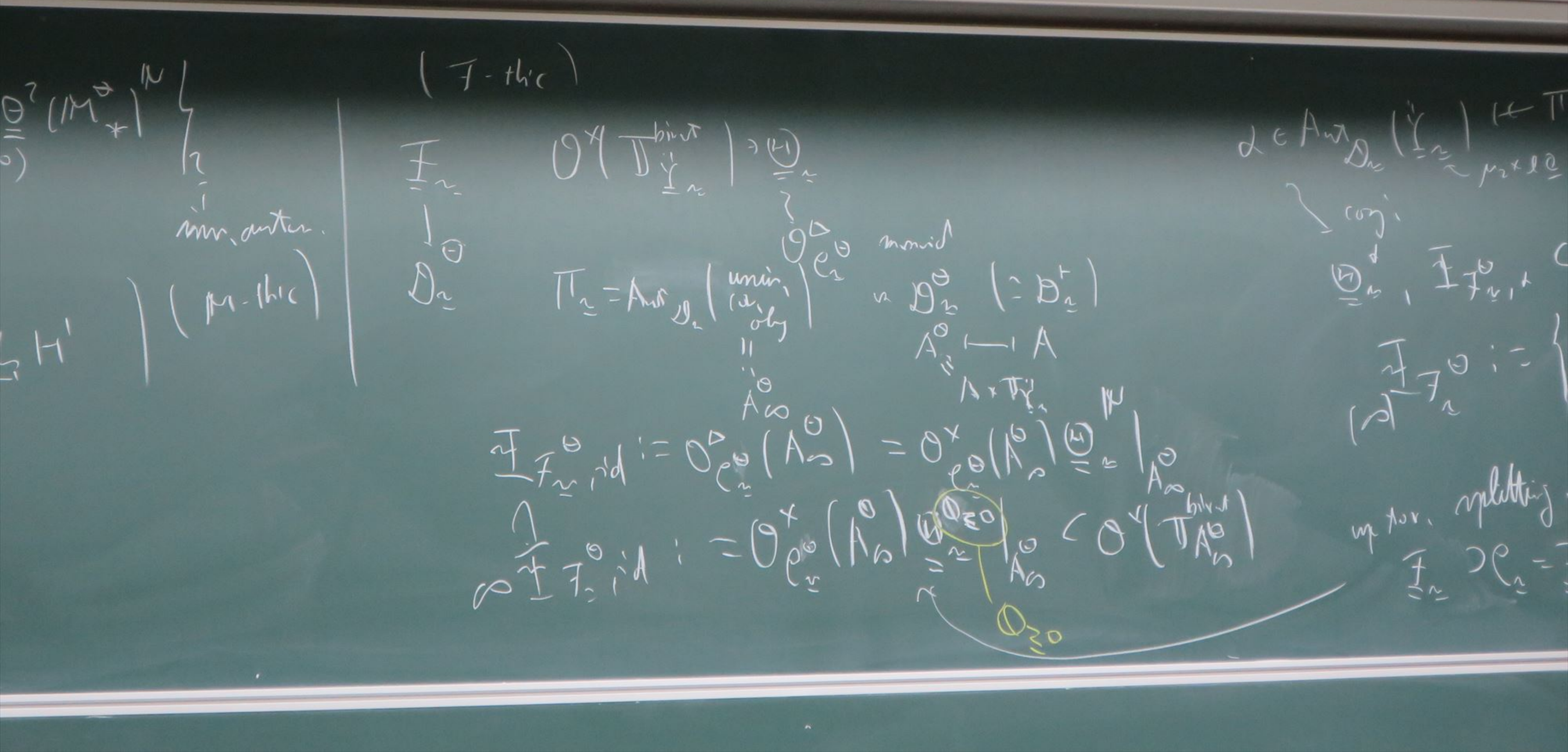
lting.

theta fct theta monoids

theta values Gaussian monoids

$$\begin{tikzcd} \text{theta} & (\text{Frob-like}) \arrow[r, "\text{Kum}"] \arrow[d, dotted, "\text{end}"] & (\text{et-like}) \arrow[d, "\text{end}"] \\ \text{Gaussian} & (\text{Frob-like}) & (\text{et-like}) \arrow[l, "(\text{Kummer})^{-1}"'] \end{tikzcd}$$





$\underline{\underline{\Theta}}^{?}(\mathbb{M}^{\Theta}_{*})^{\mathbb{N}}$ $\wr$?

o)

min. autom.

$\cong H^1$

(M-thic)

($\mathcal{F}$ - thic)

$\mathcal{F}_{\underline{v}}$ $\quad \mathcal{O}^{\times}(\Pi^{\text{birat}}_{\ddot{Y}_{\underline{v}}}) \supseteq \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$

$\wr$

$\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ monoid

$\Big|_{\Theta}$

$\mathcal{D}_{\underline{v}}$ $\quad \Pi_{\underline{v}} = \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\text{univ. cov. obj}) \rightsquigarrow \mathcal{D}^{\Theta}_{\underline{v}} \; (\cong \mathcal{D}^{\vdash}_{\underline{v}})$

$A^{\Theta} \longmapsto A$

$A^{\Theta} \cong A \times \Pi_{\ddot{Y}_{\underline{v}}}$

$\| \quad A^{\Theta}_{\infty}$

$$\mathcal{I}\mathcal{F}^{\Theta}_{\underline{v},\mathrm{id}} := \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) = \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}\Big|_{A^{\Theta}_{\infty}}$$

$$\cap$$

$$\rightsquigarrow \mathcal{I}\mathcal{F}^{\Theta}_{\underline{v},\mathrm{id}} := \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}^{\mathbb{Q}_{\geq 0}}_{\underline{v}}\Big|_{A^{\Theta}_{\infty}} \subset \mathcal{O}^{\times}(\Pi^{\text{birat}}_{A^{\Theta}_{\infty}})$$

$\mathbb{Q}_{\geq 0}$

α

$\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{Y}_{\underline{v}}) \; (\twoheadleftarrow \Pi$

$\cong \mu_2 \times l\underline{\mathbb{Z}}$

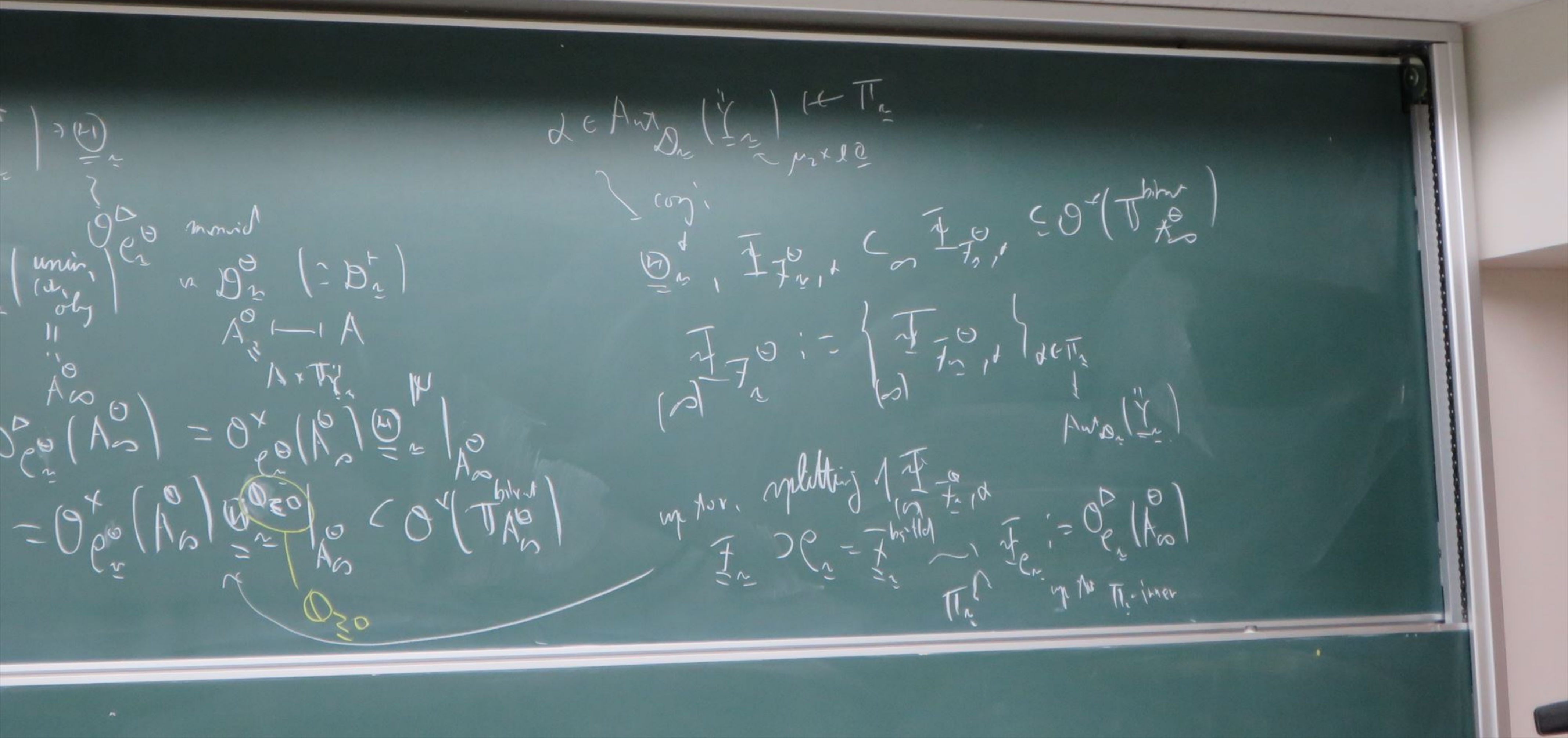
conj.

$\underline{\underline{\Theta}}^{\alpha}_{\underline{v}}, \; \mathcal{I}\mathcal{F}^{\Theta}_{\underline{v},\alpha}$

$(\rightsquigarrow) \; \mathcal{I}\mathcal{F}^{\Theta}_{\underline{v}} := \{$

up to tor. splitting

$\mathcal{F}_{\underline{v}} \supset \mathcal{C}_{\underline{v}} =$



$\Big| \to \underline{\underline{\Theta}}_{\underline{v}}$

$\wr$

$\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ monoid

$\left(\begin{matrix}\text{unin,}\\ \text{(rk,}\\ \text{obj}\end{matrix}\right) \curvearrowright \mathcal{D}^{\Theta}_{\underline{v}} \ (\cong \mathcal{D}^{\vdash}_{\underline{v}})$

$A^{\Theta}_{\underline{v}} \longmapsto A$

$A \times \Pi_{\ddot{\underline{Y}}_{\underline{v}}}$

$\|$

A^{Θ}_{∞}

$$\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) = \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}_{\underline{v}}\big|_{A^{\Theta}_{\infty}}$$

$$= \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}^{\mathbb{Q}_{\geq 0}}_{\underline{v}}\big|_{A^{\Theta}_{\infty}} \subset \mathcal{O}^{\times}(\Pi^{\text{birat}}_{A^{\Theta}_{\infty}})$$

$\mathbb{Q}_{\geq 0}$

$$\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\underline{Y}}_{\underline{v}}) \hookleftarrow \Pi_{\underline{v}}$$

$\cong \mu_2 \times l\underline{\mathbb{Q}}$

conj.

$$\underline{\underline{\Theta}}^{\alpha}_{\underline{v}},\ \underline{\underline{\mathcal{F}}}^{\Theta}_{\underline{v},\alpha} \subset_{\infty} \underline{\underline{\mathcal{F}}}^{\Theta}_{\underline{v},\alpha} \subseteq \mathcal{O}^{\times}(\Pi^{\text{birat}}_{A^{\Theta}_{\infty}})$$

$${}_{(\infty)}\underline{\underline{\mathcal{F}}}^{\Theta}_{\underline{v}} := \{{}_{(\infty)}\underline{\underline{\mathcal{F}}}^{\Theta}_{\underline{v},\alpha}\}_{\alpha \in \Pi_{\underline{v}}}$$

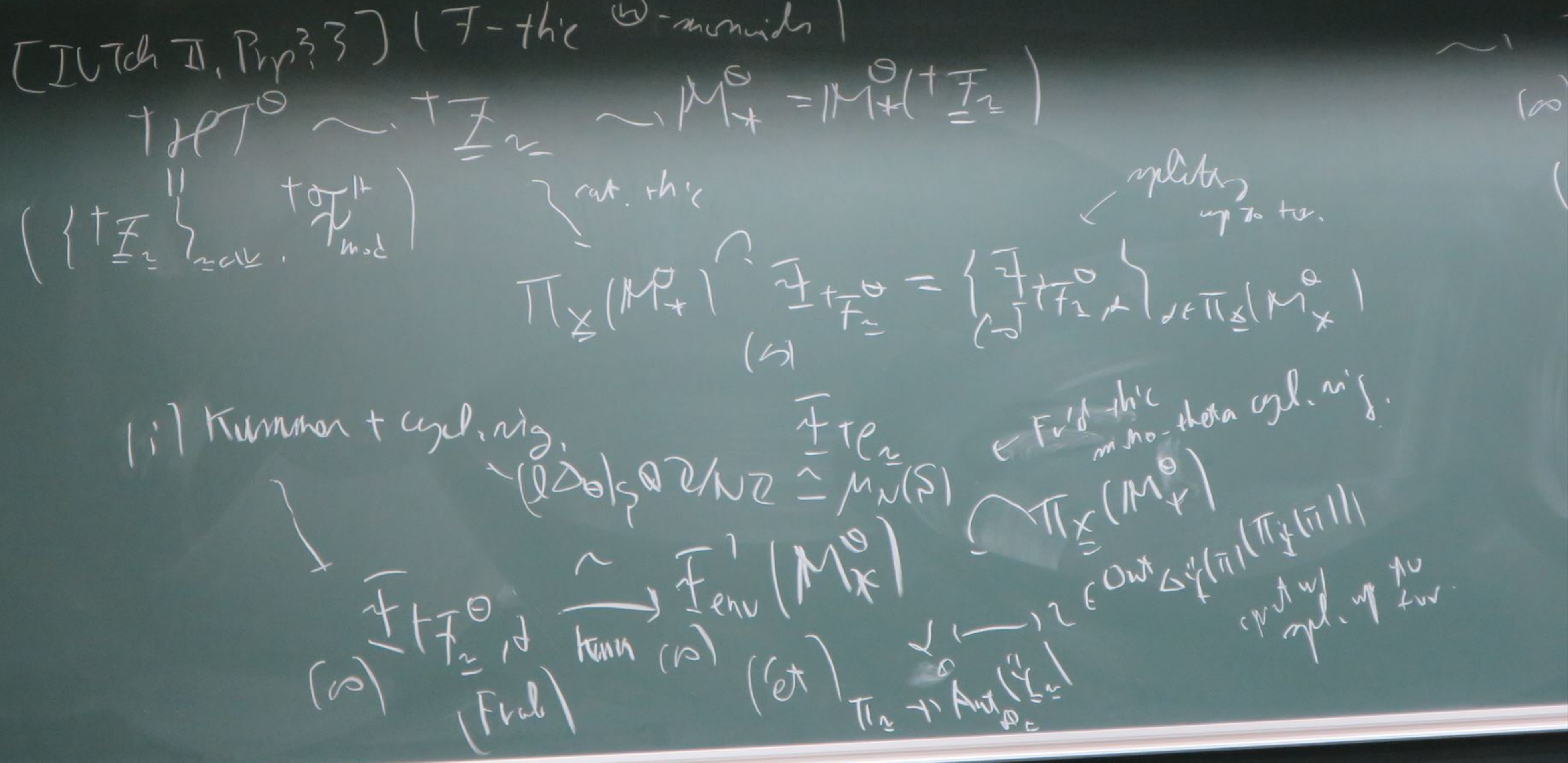
$\downarrow$

$\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\ddot{\underline{Y}}_{\underline{v}})$

up to r. splitting $1_{(\infty)}\underline{\underline{\mathcal{F}}}^{\Theta}_{\underline{v},\alpha}$

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \supset \mathcal{C}_{\underline{v}} = \underline{\underline{\mathcal{F}}}^{\text{birat}}_{\underline{v}} \rightsquigarrow \mathcal{F}_{\mathcal{C}_{\underline{v}}} := \mathcal{O}^{\triangleright}_{\mathcal{C}_{\underline{v}}}(A^{\Theta}_{\infty})$$

$\Pi_{\underline{v}} \curvearrowright$ up to $\Pi_{\underline{v}}$-inner



[IUTch II, Prp 3.3] ($\mathcal{F}$-thic $\circledast$-monoids)

$${}^{\dagger}\mathcal{HT}^{\Theta} \rightsquigarrow {}^{\dagger}\underline{\underline{\mathcal{F}}}_{\succ} \rightsquigarrow \mathcal{M}^{\Theta}_{*} = \mathcal{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\succ})$$

$$\Big(\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\succ}\}_{\succ\in\underline{\mathbb{V}}},\ {}^{\dagger}\mathcal{D}^{\Vdash}_{\mathrm{mod}}\Big)$$

$\rightsquigarrow$ cat. thic

$$\Pi_{\underline{\underline{X}}}(\mathcal{M}^{\Theta}_{*}) \curvearrowright \underset{(\infty)}{\overline{\underline{\underline{\mathcal{F}}}}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\succ}}} = \Big\{\underset{(\infty)}{\overline{\underline{\underline{\mathcal{F}}}}}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\succ},\alpha}\Big\}_{\alpha\in\Pi_{\underline{\underline{X}}}(\mathcal{M}^{\Theta}_{*})}$$

splitting up to tor.

(i) Kummer + cycl. rig.

$$\rightsquigarrow (l\Delta_{\Theta})_{\zeta}\otimes\mathbb{Z}/N\mathbb{Z} \cong \mu_N(\overline{\underline{\underline{\mathcal{F}}}}_{{}^{\dagger}\mathcal{C}_{\succ}})$$

← Fr'd-thic mono-theta cycl. rig.

$$\curvearrowright \Pi_{\underline{\underline{X}}}(\mathcal{M}^{\Theta}_{*})$$

$$\underset{(\infty)}{\overline{\underline{\underline{\mathcal{F}}}}}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\succ},\alpha} \xrightarrow[\text{Kumm}]{\sim} \underset{(\infty)}{\mathcal{F}^{1}_{\mathrm{env}}}(\mathcal{M}^{\Theta}_{*})$$

(Frob)

$$(\text{ét})\qquad \alpha \longmapsto \iota \in \mathrm{Out}_{\Delta_{\underline{Y}}(\Pi)}(\Pi_{\underline{Y}}(\Pi))$$

$$\Pi_{\succ} \to \mathrm{Aut}_{\mathcal{D}_{\succ}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\succ})$$

compat w/ spl. up to tor.

splitting
up to tor.

$\}_{\alpha \in \Pi_{\underline{v}}(\mathbb{M}^{\Theta}_{*})}$

thic
mono-theta cycl. rig.

$(\mathbb{M}^{\Theta}_{*})$

$\mathrm{Out}^{*}_{\Delta^{\ddagger}_{\underline{v}}(\Pi)}(\Pi_{\underline{v}}(\Pi))$

compat w/
spl. up to tor.

$$\sim\ {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\succ}} \xrightarrow[\text{Kmm}]{\sim} {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$$

classical cycl. rig.

(ii) Kummer + cycl. rig.

$$\Psi_{{}^{\dagger}\mathcal{F}_{\succ}} \xrightarrow[\text{Kmm}]{\sim} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$$

(const. monoids) $\curvearrowleft \Pi_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$

[IUTch II, Prop 3.4]

${}^{\dagger}\mathcal{F}_{\succ}$ full

(i) (split Θ-mono

${}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}$

$(\Pi_{\succ} \xrightarrow{\sim})$

$\mathcal{M}^{\Theta}_{*})$

✓ classical cycl. ris.

ris.

$\Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})$

$\curvearrowright \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*})$

[IUTchII, Prop 3.4] (Π thm for Θ-monoids)

$^{\dagger}\mathfrak{F}_{\underline{v}}$ full poly isom $\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathcal{M}^{\Theta}_{*}(^{\dagger}\mathfrak{F}_{\underline{v}})$

$\Pi \mapsto \mathcal{M}(\Pi) \overset{\sim}{\underset{\text{full poly}}{\to}} \mathcal{M}(\mathfrak{F}) \leftarrow \mathfrak{F}$

(i) (split Θ-monoids multirad.)

each isom ⊛ induces

$$\begin{tikzcd}
(ii) & \Psi_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[r, "\overset{⊛}{\sim}"] \arrow[d, phantom, "\cap"] & \Psi_{\mathrm{env}}(\mathcal{M}^{\Theta}(^{\dagger}\mathfrak{F}_{\underline{v}})) \arrow[r, "(\mathrm{Kum})^{-1}", "\sim"'] \arrow[d, phantom, "\cap"] & {}_{\infty}\Psi_{^{\dagger}\mathfrak{F}_{\underline{v}}}^{\Theta} \\
(\Pi_{\underline{v}} \curvearrowright) & \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[r, "\sim", "⊛"'] & \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*}(^{\dagger}\mathfrak{F}_{\underline{v}})) & \circlearrowright
\end{tikzcd}$$

compat w/ splittings up to tor.

$$\begin{tikzcd}
\Psi_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} \arrow[r, "\sim", "\circledast"'] & \Psi_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathfrak{F}}}_{\underline{v}}))^{\times} \arrow[r, "\sim", "(\kappa_{mm})^{-1}"'] & \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \\
G_{\underline{v}} \cong G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[u, phantom, "\curvearrowright"] \arrow[r, "\sim", "\circledast"'] & G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathfrak{F}}}_{\underline{v}})) \arrow[u, phantom, "\curvearrowright"] &
\end{tikzcd}$$

$\Pi_{\underline{v}} \longmapsto \Psi_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$

f.n. alg'm (Θ) compat w/ $\forall$ autom. of the pair

mult. rad.

$G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathfrak{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\mu}$

Ismet-multiple of

$G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathfrak{F}}}_{\underline{v}})) \curvearrowright \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}$

[IUTch II, Prop 3.3] ($\mathcal{F}$-thic Θ-monoids)

$\mathcal{M}^{\Theta} = \mathcal{M}^{\Theta}({}^{\dagger}\mathfrak{F}_{\underline{v}})$

$\Psi^{\times}_{{}^\dagger\mathcal{F}^{\vdash\Theta}_{\underline{v}}}$

(ii) (const monoid uniradiality)

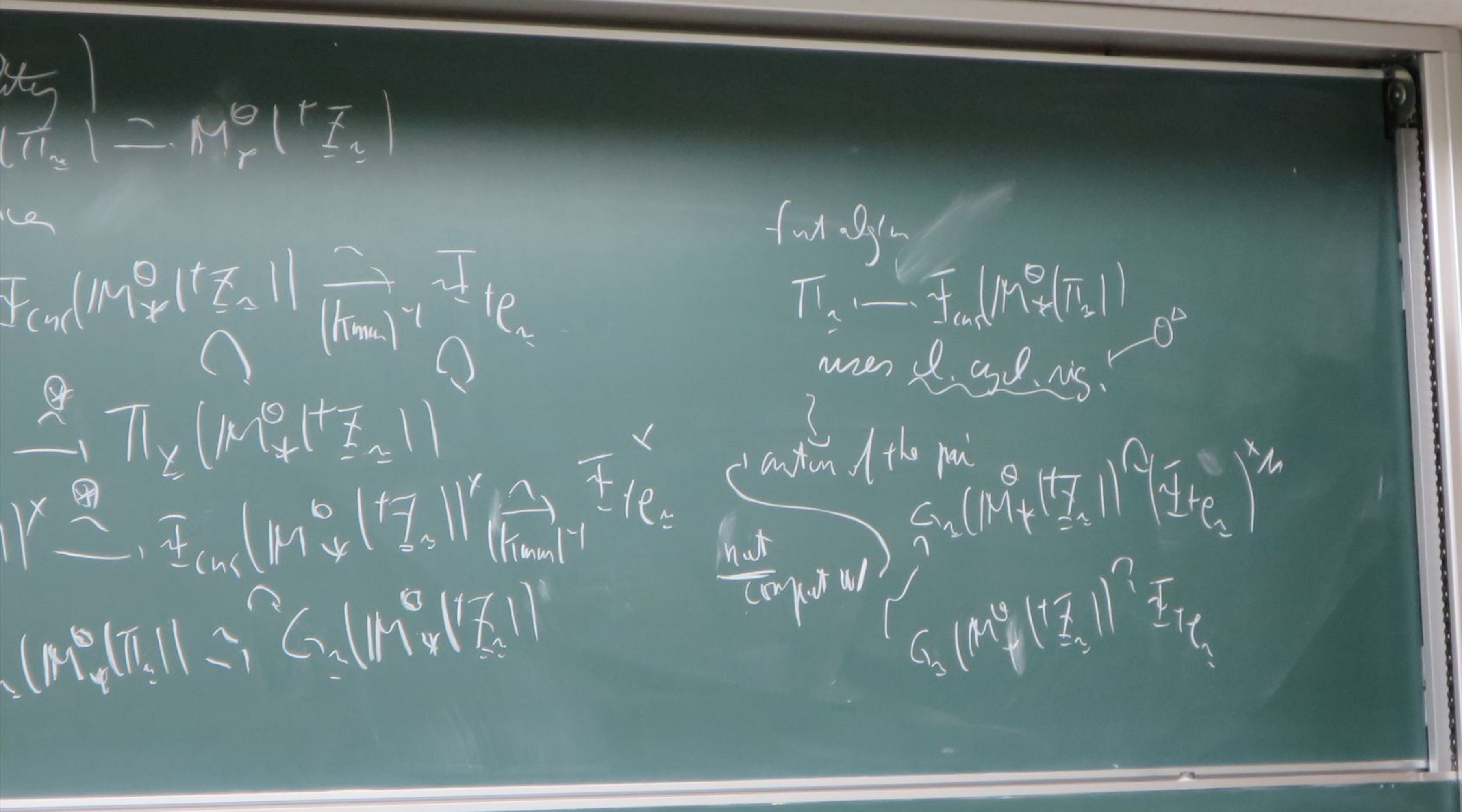
each isom $\circledast$ $\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{\underline{v}})$

induce

$$\begin{tikzcd}
\Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[r, "\circledast", "\sim"'] & \Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{\underline{v}})) \arrow[r, "\sim", "(\kappa_{\mathrm{mon}})^{-1}"'] \arrow[d, phantom, "\cap"] & \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \arrow[d, phantom, "\cap"] \\
\Pi_{\underline{v}} \cong \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[u, phantom, "\curvearrowright"] \arrow[r, "\circledast", "\sim"'] & \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{\underline{v}})) & {} \\
\Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} \arrow[r, "\circledast", "\sim"'] & \Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{\underline{v}}))^{\times} \arrow[r, "\sim", "(\kappa_{\mathrm{mon}})^{-1}"'] & \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \\
G_{\underline{v}} \cong G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \arrow[u, phantom, "\curvearrowright"] \arrow[r, "\sim"] & G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{\underline{v}})) \arrow[u, phantom, "\curvearrowright"] & {}
\end{tikzcd}$$

not

com



ity)

$(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}})$

ce

$$\begin{tikzcd}
\Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}})) \arrow[r, "\sim", "(\kappa_{\mathrm{Kmm}})^{-1}"'] & \Psi_{{}^{\dagger}\mathcal{E}_{\underline{v}}} \\
\cap & \cup
\end{tikzcd}$$

$\xrightarrow[\sim]{\circledast} \Pi_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}}))$

$))^{\times} \xrightarrow[\sim]{\circledast} \Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}}))^{\times} \xrightarrow[(\kappa_{\mathrm{Kmm}})^{-1}]{\sim} \Psi_{{}^{\dagger}\mathcal{E}_{\underline{v}}}^{\times}$

$(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \xrightarrow{\sim} \overset{\curvearrowright}{G}_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}}))$

funct alg'm

$\Pi_{\underline{v}} \mapsto \Psi_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$

uses ℓ. cycl. rig. $\leftarrow \Theta^{\Delta}$

action of the pair

$G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{E}_{\underline{v}}})^{\times\mu}$

not compat w/

$G_{\underline{v}}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathfrak{F}_{\underline{v}})) \curvearrowright \Psi_{{}^{\dagger}\mathcal{E}_{\underline{v}}}$

Gaussian monoids
$$\begin{tikzcd} \Pi \arrow[rd] & & \mathcal{F} \arrow[ld] \\ & \mathbb{M} & \end{tikzcd}$$

$\Delta =$

[IUTch II, Cor 3.5] ($\mathbb{M}$ the Gaussian monoids)

$\mathbb{M}^{\Theta}_{*}$ s.t. $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \cong \Pi_{\underline{X}}$

$A \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*}))$ $\quad$ ($I^{\ell}_{*} \subset \Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$)

(i) $A \rightsquigarrow I_{A} \subset \Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$

$\mathbb{F}_{\ell}^{\rtimes\pm} = \Delta_{C}(\mathbb{M}^{\Theta}_{*}) / \Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \curvearrowleft \Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$

$\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$-outer action

} induces isom's of pairs ($G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{*} \curvearrowright \mathcal{F}_{\mathrm{mod}}(\mathbb{M}^{\Theta}_{*})_{*}$)

$\mathbb{F}_{\ell}^{\rtimes\pm}$-symmetrising isom

$\Delta = \{0, \langle\ \rangle\}$

$\star, -\star$ $\xrightarrow{\text{symm.}}$ identify $|\star|$

write $\langle |\mathbb{F}_l| \rangle$

$\langle \mathbb{F}_l^{\circledast} \rangle$

(diag) $\subset \prod(\cdot)$

$|\star| \in |\mathbb{F}_l|$, $\mathbb{F}_l^{\circledast}$

can construct $\star$

(synchronizing indet's)

$\mathcal{F}_{\mathrm{cns}}(|\mathcal{M}^{\Theta}_{*}|)$

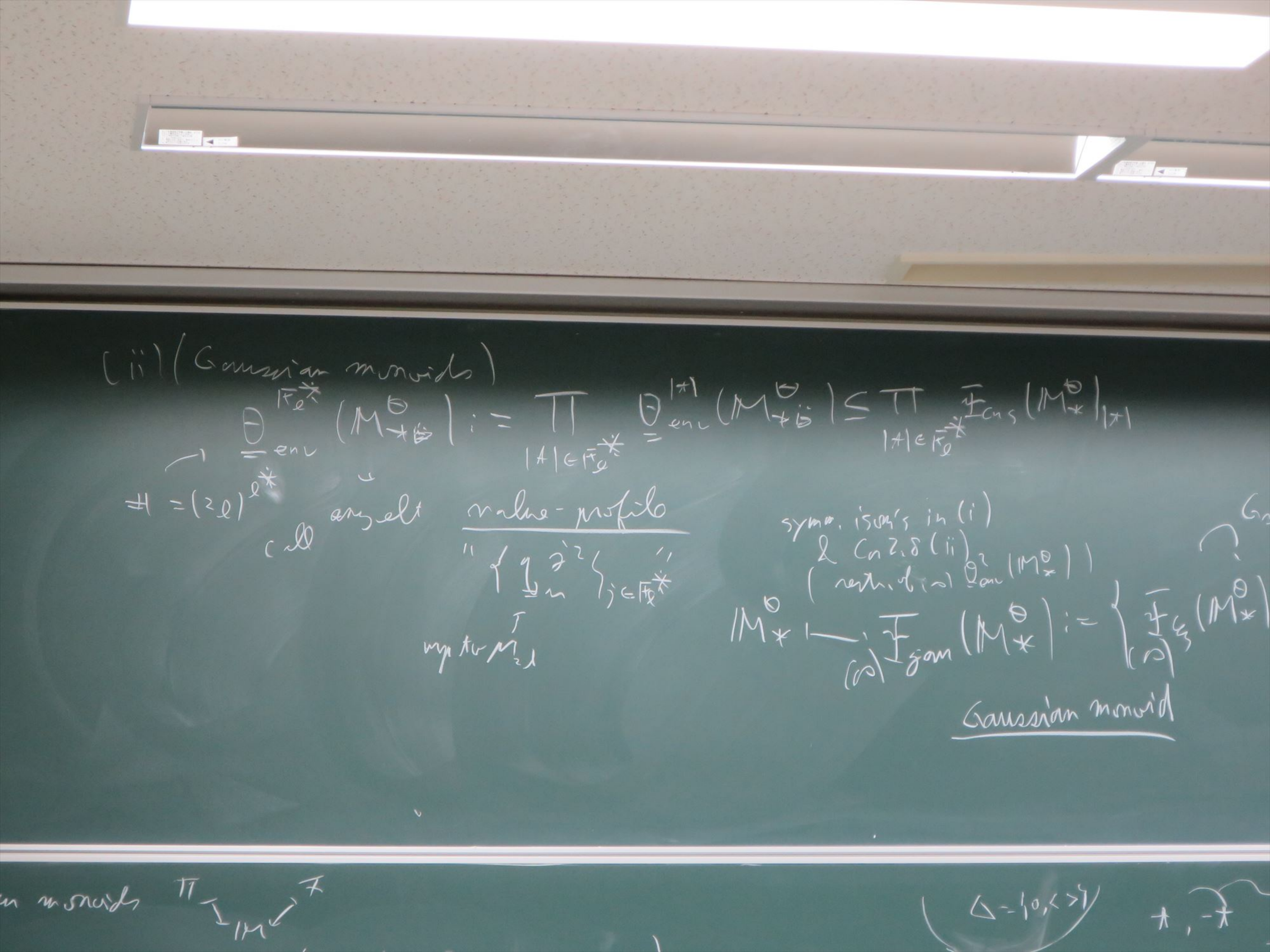
restr. to $\Pi_{\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}}}$, $D^{\vdash}_{\star}, \mu_-$

$$\begin{tikzcd}
\rightsquigarrow (\Pi_{\underline{\star}}(|\mathcal{M}^{\Theta}_{*}|) \supseteq) \Pi_{\underline{\star}\ddot{\circledcirc}}(|\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}}|) \arrow[r] \arrow[d, phantom, "\cap"] & G_{\underline{\star}}(|\mathcal{M}^{\Theta}_{*\ddot{\circledcirc}}|_{\langle|\mathbb{F}_l|\rangle}) \arrow[d, phantom, "\cap"] \\
\mathcal{F}_{\mathrm{cns}}(|\mathcal{M}^{\Theta}_{*}|) \arrow[r, "\sim"] & \mathcal{F}_{\mathrm{cns}}(|\mathcal{M}^{\Theta}_{*}|_{\langle|\mathbb{F}_l|\rangle})
\end{tikzcd}$$

compat w/ symm. isoms, well-def'd up to $\Pi_{\underline{\star}}(|\mathcal{M}^{\Theta}_{*}|)$-inner

indep. of $|\star| \in |\mathbb{F}_l|$ conj. synchro.

$|^{\Theta}_{*}|_{\star})$



(ii) (Gaussian monoids)

$$\underline{\underline{\Theta}}_{\mathrm{env}}^{\mathbb{F}_l^{\rtimes\pm}}(\mathbb{M}^{\Theta}_{*\mathfrak{D}}) := \prod_{|t| \in \mathbb{F}_l^{*}} \underline{\underline{\Theta}}_{\mathrm{env}}^{|t|}(\mathbb{M}^{\Theta}_{*\mathfrak{D}}) \subseteq \prod_{|t| \in \mathbb{F}_l^{*}} \mathcal{I}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$$

$\# = (2l)^{l^*}$

call any elt value-profile

" $\{\underline{\underline{q}}^{j^2}_{v}\}_{j \in \mathbb{F}_l^*}$ "

map to $\mathbb{M}_{\mathrm{mod}}$

symm. isom's in (i)

& Cor 2.8 (ii)

(restr. of (~) $\underline{\underline{\Theta}}^{?}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$)

$$\mathbb{M}^{\Theta}_{*} \longmapsto {}^{(\sim)}\mathcal{I}_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) := \{ {}^{(\sim)}\mathcal{I}_{\xi}(\mathbb{M}^{\Theta}_{*}) \ldots$$

Gaussian monoid

...n monoids $\prod \mathcal{I}_{\mathbb{M}}$

$\Delta = \{0, <>\}$

$\star, -\star$

$(\mathcal{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subseteq \prod_{|*|\in\mathbb{F}_l^{*}} \overline{\mathcal{F}}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{|*|}$

symm. isom's in (i)
& Cor 2.8 (ii)
(restn. of $\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*})$)

$\mathcal{G}_{\simeq}(\mathcal{M}^{\Theta}_{*\ddot{\blacktriangleright}}|_{\langle\mathbb{F}_l^{*}\rangle})$ ↷

$$\mathcal{M}^{\Theta}_{*} \longmapsto {}^{(\omega)}\overline{\mathcal{F}}_{\mathrm{gau}}(\mathcal{M}^{\Theta}_{*}) := \left\{ {}^{(\omega)}\overline{\mathcal{F}}_{\xi}(\mathcal{M}^{\Theta}_{*}) := \overline{\mathcal{F}}^{\times}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{\langle\mathbb{F}_l^{*}\rangle}^{\xi^{\overset{(\mathbb{Q}_{\geq 0})}{\mathbb{N}}}} \right\}_{\xi\text{: nml. profile}}$$

$$\overline{\mathcal{F}}^{\times}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{\langle\mathbb{F}_l^{*}\rangle} \cap \prod_{|*|\in\mathbb{F}_l^{*}} \overline{\mathcal{F}}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{|*|}$$

Gaussian monoid

$${}^{(\omega)}\overline{\mathcal{F}}_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}) \curvearrowleft (\Pi_{\underline{\underline{v}}}(\mathcal{M}^{\Theta}_{*})\supset)\,\Pi_{\underline{\underline{v}}\ddot{\blacktriangleright}}(\mathcal{M}^{\Theta}_{*\ddot{\blacktriangleright}}$$

$\Pi_{\underline{\underline{v}}\dot{\blacktriangleright}}(\mathcal{M}^{\Theta}$

$\Delta = \{0, \langle\ \rangle\}$

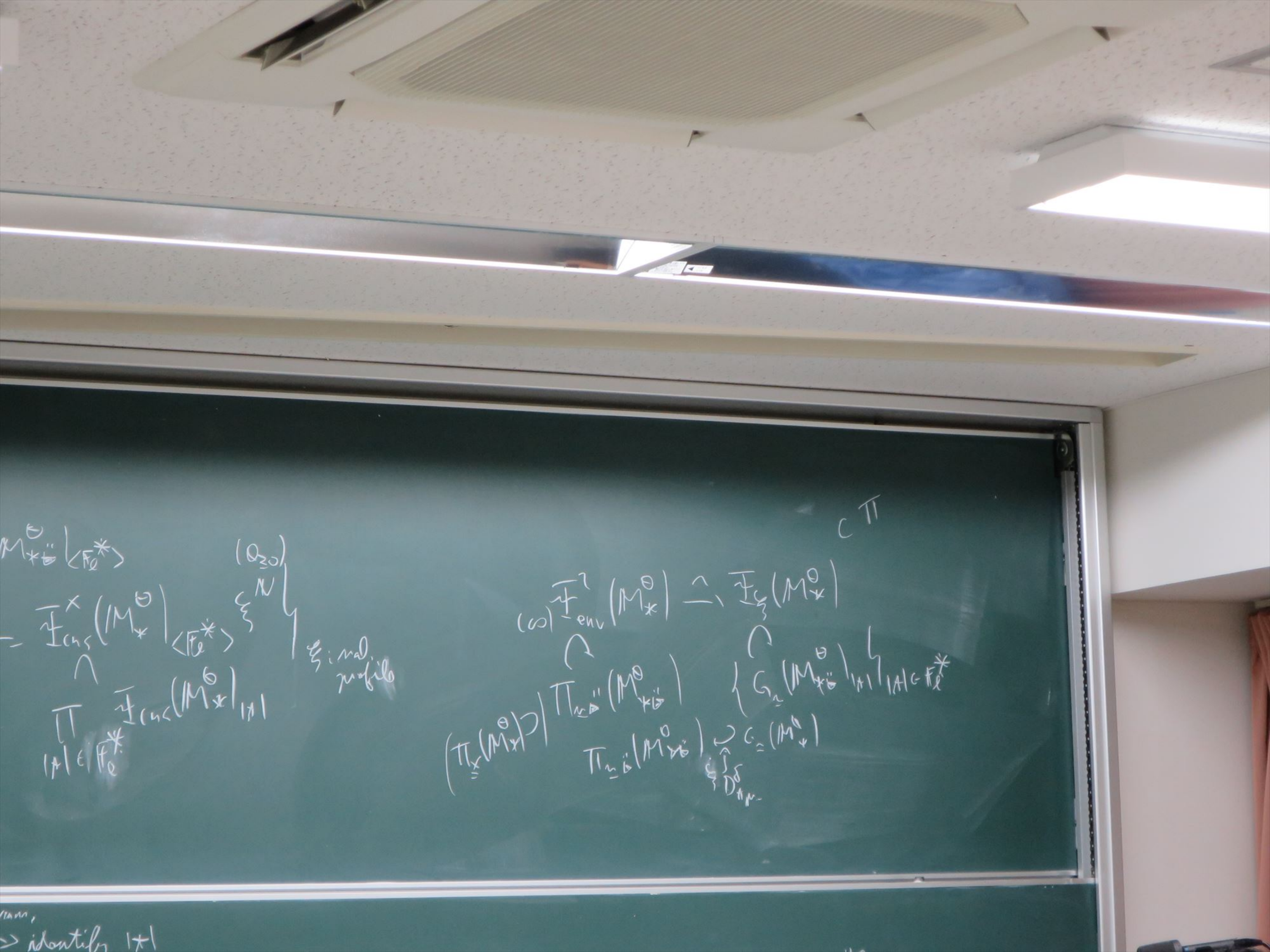
$*, -*$ ↷ symm. → identify $|*|$

write
$\langle|\mathbb{F}_l|\rangle$
$\langle\mathbb{F}_l^{*}\rangle$

$(\mathrm{diag}) \subset \prod_{|*|\in|\mathbb{F}_l|,\ \mathbb{F}_l^{*}} (\cdot)$

can construct

$\mathcal{M}^{\Theta}$



$$\mathcal{M}^{\Theta}_{*\ddot{\circ}}|_{\langle \mathbb{F}_l^{*} \rangle}$$

$$(\Omega_{\geq 0})$$

$$= \mathcal{F}^{\Vdash \times}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{\langle \mathbb{F}_l^{*} \rangle} \; \xi^{N} \Big\}$$

ξ: nml. profile

$$\cap$$

$$\prod_{|*| \in \mathbb{F}_l^{*}} \mathcal{F}^{\Vdash}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})|_{|*|}$$

$$c \; \Pi$$

$$(\omega) \; \mathcal{F}^{\Vdash ?}_{\mathrm{env}}(\mathcal{M}^{\Theta}_{*}) \xrightarrow{\sim} \mathcal{F}_{\xi}(\mathcal{M}^{\Theta}_{*})$$

$$\curvearrowright \qquad \curvearrowright$$

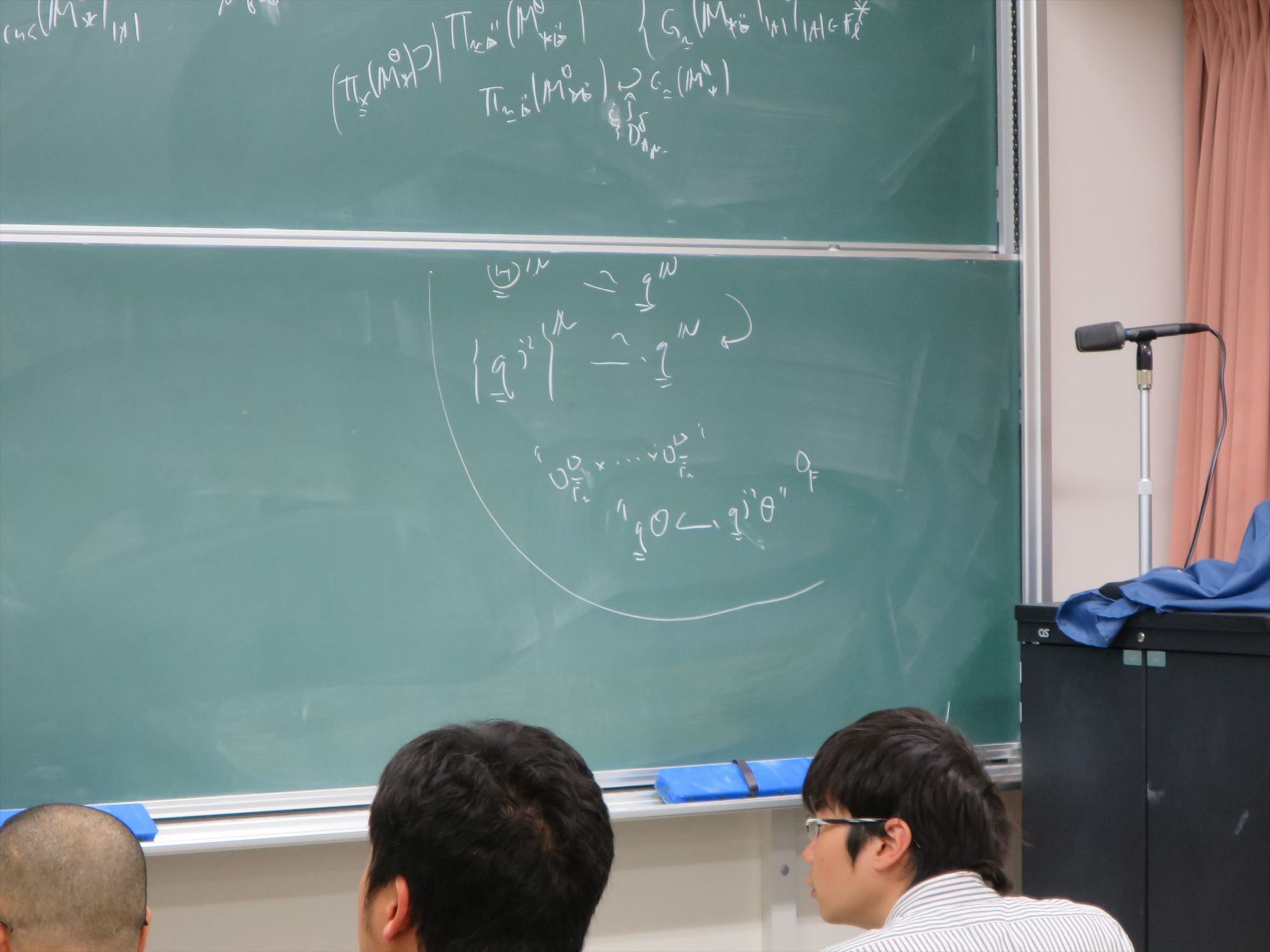
$$(\Pi_{\underline{\underline{x}}}(\mathcal{M}^{\Theta}_{*}) \supset) \; \Pi_{\underline{\underline{x}}\ddot{\circ}}(\mathcal{M}^{\Theta}_{*\ddot{\circ}}) \qquad \{ G_{\underline{\underline{v}}}(\mathcal{M}^{\Theta}_{*\ddot{\circ}})|_{|*|} \}_{|*| \in \mathbb{F}_l^{*}}$$

$$\Pi_{\underline{\underline{v}}\ddot{\circ}}(\mathcal{M}^{\Theta}_{*\ddot{\circ}}) \hookrightarrow G_{\underline{\underline{v}}}(\mathcal{M}^{\Theta}_{*})$$

$$\xi \; \hat{\int} \; D^{\delta}_{*\mu-}$$

ram,

=> identify |*|



$\ldots(M_{*})_{|*|}$

$\left(\Pi_{\underline{X}}(M^{\Theta}_{*})\supset\right)\ \Pi_{\underline{v}\ddot{\triangleright}}(M^{\Theta}_{*\ddot{\triangleright}})$ $\quad \{G_{\underline{v}}(M_{*\ddot{\triangleright}})_{|*|}\}_{|*|\in\mathbb{F}_l^{*}}$

$\Pi_{\underline{v}\ddot{\triangleright}}(M^{\Theta}_{*\ddot{\triangleright}}) \hookrightarrow G_{\underline{v}}(M^{\Theta}_{*})$

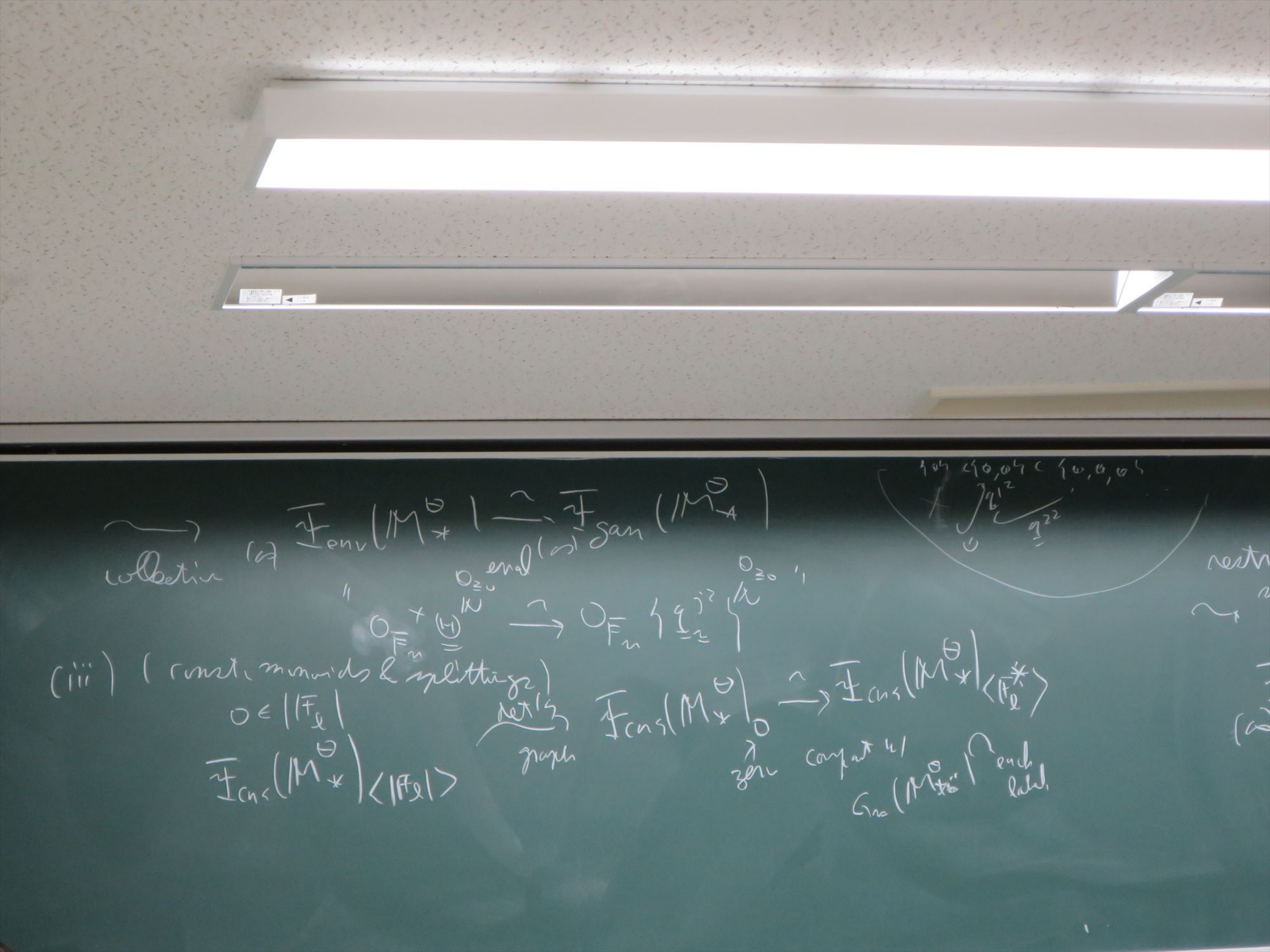
$\hat{\int}\ \xi\ D^{\delta}_{\triangleright\ldots}$

$\underline{\underline{(4)}}^{\mathbb{N}} \xrightarrow{\sim} \underline{q}^{\mathbb{N}}$

$\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \xrightarrow{\sim} \underline{\underline{q}}^{\mathbb{N}}$

"$\mathcal{O}^{\triangleright}_{\underline{F}_1}\times\cdots\times\mathcal{O}^{\triangleright}_{\underline{F}_l}$" $\quad \mathcal{O}_F$

"$\underline{\underline{q}}^{\mathcal{O}} \hookleftarrow \underline{\underline{q}}^{j^2\Theta}$"



$\underset{\text{collection}}{\xrightarrow{\sim}}$ (a) $\mathcal{F}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \underset{\text{eval}(\infty)}{\xrightarrow{\sim}} \mathcal{F}_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{\mathbb{A}})$

$\parallel$ $\mathcal{O}_{\geq 0}$ $\mathcal{O}_{\geq 0}$'s

$\mathcal{O}^{\times}_{F_n} \cdot \underline{\underline{\Theta}}^{\mathbb{N}} \xrightarrow{\sim} \mathcal{O}_{F_n}\{\underline{\underline{q}}^{j^2}\}_{j}$

$\{0\} \subset \{0, 0\} \subset \{0, 0, 0\}$

$\underline{\underline{q}}^{1^2}$ $\underline{\underline{q}}^{2^2}$ Θ

restr

$\rightsquigarrow$

(iii) (const. monoids & splittings)

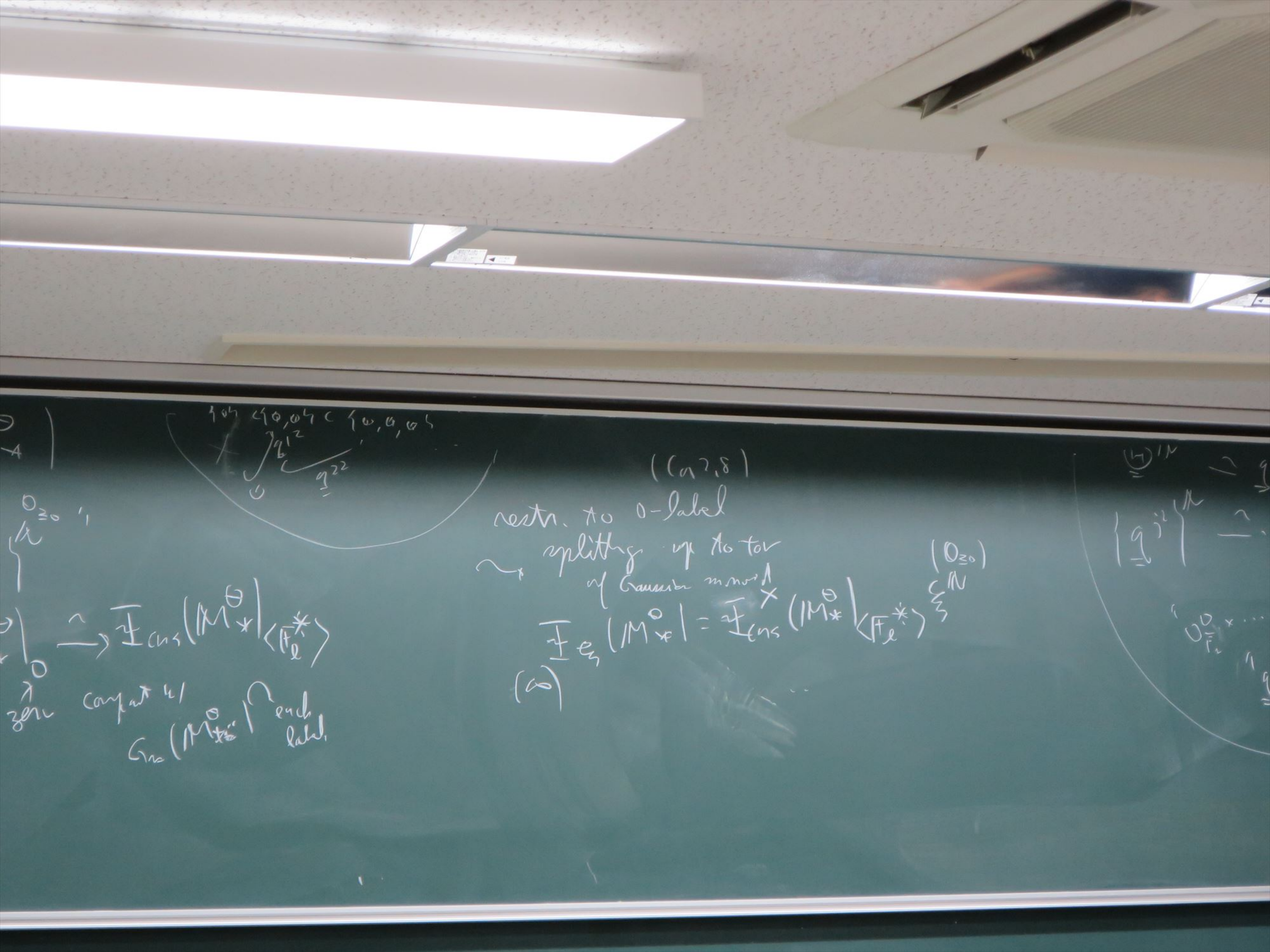
$0 \in |\mathbb{F}_l|$

$\mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})\langle|\mathbb{F}_l|\rangle \underset{\text{graph}}{\xrightarrow{\text{det'ic}}} \mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})|_{0} \xrightarrow{\sim} \mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})|_{\langle \mathbb{F}^{*}_{l} \rangle}$

$\uparrow$ zero

compat w/ $G_{v}(\mathbb{M}^{\Theta}_{*v}) \curvearrowright$ each label.

(∞



$\{0\} \subset \{0, \omega\} \subset \{\omega, 0, \omega\}$

q^{1^2} q^{2^2}

$\mathcal{O}_{\geq 0}$

$\Big|_{0} \xrightarrow{\sim} \Psi_{cns}(\mathcal{M}^{\Theta}_{*}|_{\langle \mathbb{F}_l^{*} \rangle})$

zero compat w/ $G_{\mathbb{Q}}(\mathcal{M}^{\Theta}_{*})$ ⌒ each label.

(Cor 7.8)

restr. to 0-label

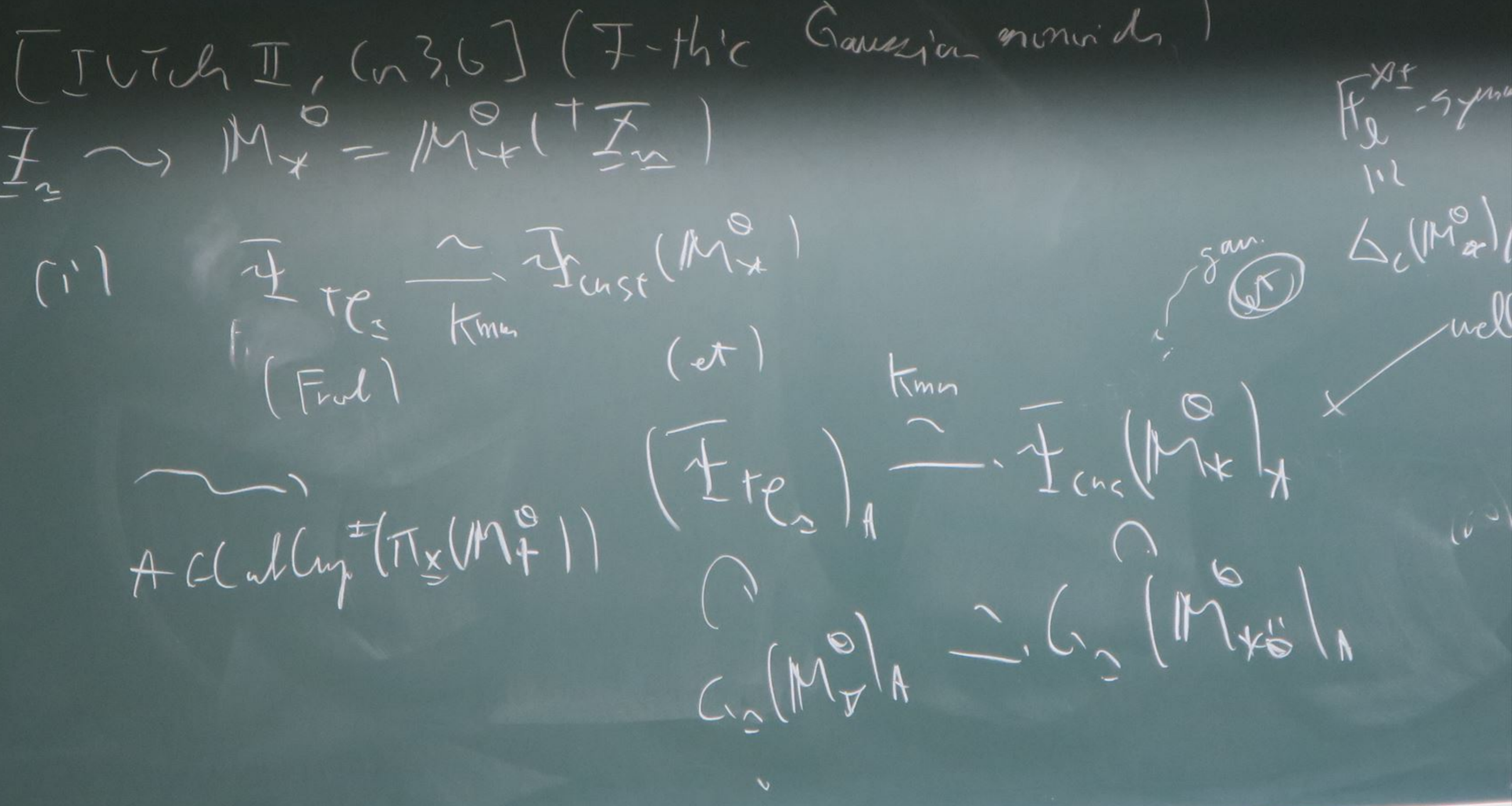
⇝ splitting up to tor of Gaussian monoid

$(\mathcal{O}_{\geq 0})$

$$\Psi_{\mathcal{E}_{\Im}}(\mathcal{M}^{\Theta}_{*}) = \Psi^{\times}_{cns}(\mathcal{M}^{\Theta}_{*}|_{\langle \mathbb{F}_l^{*} \rangle})^{\xi^{\mathbb{N}}}$$

(∞)

$\{q^{j^2}\}^{\mathbb{N}}$



[IUTch II, Cor 3.6] ($\mathbb{F}$-th'c Gaussian monoids)

$${}^{\dagger}\mathfrak{F}_{\succ} \rightsquigarrow \mathbb{M}_{*}^{\circledcirc} = \mathbb{M}_{*}^{\circledcirc}({}^{\dagger}\mathfrak{F}_{\succ})$$

(i) $\Psi_{{}^{\dagger}\mathcal{C}_{\succ}} \xrightarrow[\text{Kmm}]{\sim} \Psi_{\mathrm{cnst}}(\mathbb{M}_{*}^{\circledcirc})$

(Frd) (et)

$$\xrightarrow{\sim} A\,\mathcal{G}al\mathcal{G}y^{\pm}(\Pi_{\underline{x}}(\mathbb{M}_{*}^{\circledcirc}))$$

$$\begin{tikzcd}
(\Psi_{{}^{\dagger}\mathcal{C}_{\succ}})_A \arrow[r, "\sim", "\text{Kmm}"'] \arrow[d, phantom, "\cap"] & \Psi_{\mathrm{cns}}(\mathbb{M}_{*}^{\circledcirc})_A \arrow[d, phantom, "\cap"] \\
G_{\succ}(\mathbb{M}_{\nabla}^{\circledcirc})_A \arrow[r, "\sim"] & G_{\succ}(\mathbb{M}_{*\circledcirc}^{\circledcirc})_A
\end{tikzcd}$$

gau. $\circledast$

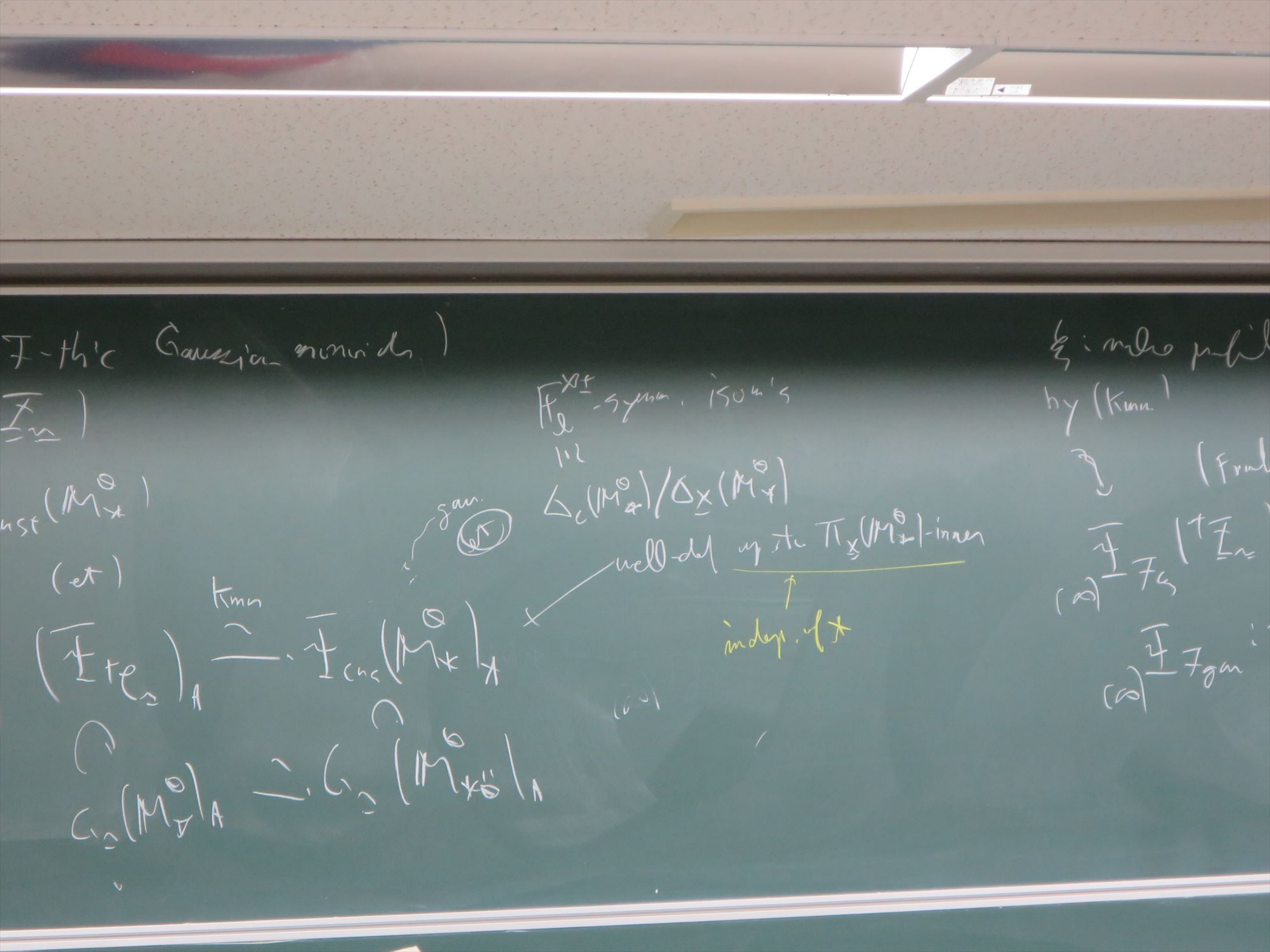
$\mathbb{F}_{l}^{\rtimes\pm}$-symm

$\wr\wr$

$\Delta_c(\mathbb{M}_{*}^{\circledcirc})$

well

(ii)



$\mathcal{F}$-th'c Gaussian monoids)

$\underline{\underline{\mathcal{F}}}_{\mathrm{mod}}$)

nst$(\mathcal{M}^{\Theta}_{*})$

(et)

$\mathbb{F}_l^{\rtimes\pm}$-symm. isom's

≀≀

$\Delta_c(\mathcal{M}^{\Theta}_{*})/\Delta_{\underline{X}}(\mathcal{M}^{\Theta}_{*})$

well-def up to $\Pi_{\underline{X}}(\mathcal{M}^{\Theta}_{*})$-inner

indep. of $*$

gau. $\circledast$

$$\begin{tikzcd}
(\overline{\Psi}_{\mathrm{env}})_A \arrow[r, "\sim", "\mathrm{Kmn}"'] & \overline{\Psi}_{\mathrm{cns}}(\mathcal{M}^{\Theta}_{*})_{*} \\
G_{\underline{v}}(\mathcal{M}^{\Theta}_{\nabla})_A \arrow[u, phantom, "\cap"] \arrow[r, "\sim"] & G_{\underline{v}}(\mathcal{M}^{\Theta}_{*\Theta})_A \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

(ii)

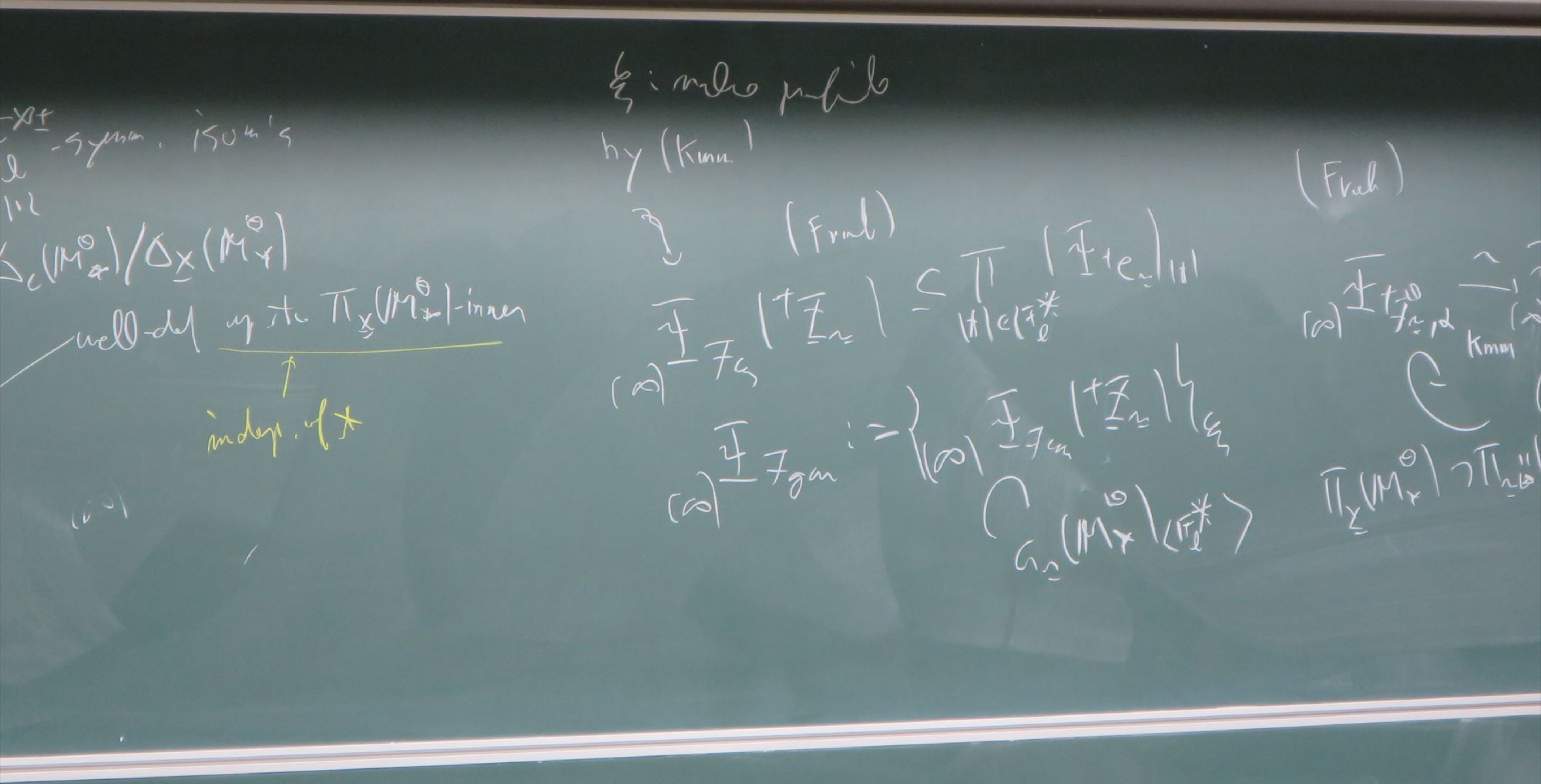
ξ: mult. prol

by (Kmn)

(Frob

$_{(\infty)}\overline{\Psi}_{\mathcal{F}_{\mathrm{LGP}}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\mathrm{mod}}$

$_{(\infty)}\overline{\Psi}_{\mathcal{F}_{\mathrm{gau}}}$:



-Xt± -symm. isom's

ℓ

1:2

$\Delta_c(\mathbb{M}^{\Theta}_{*})/\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$

well-def up to $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$-inner

indep. of *

(ii)

ξ : [illegible] profile

by (Kmm)

(Frml)

$$\Big\downarrow \wr \quad {}^{(\infty)}\overline{\Psi}_{\mathcal{F}_{\xi}}({}^{\dagger}\mathcal{D}_{\succ}) \subseteq \prod_{|t| \in \mathbb{F}_{\ell}^{*}} (\overline{\Psi}_{{}^{\dagger}e_{\succ}})_{|t|}$$

$${}^{(\infty)}\overline{\Psi}_{\mathcal{F}_{gau}} := \{ {}^{(\infty)}\overline{\Psi}_{\mathcal{F}_{\xi}}({}^{\dagger}\mathcal{D}_{\succ}) \}_{\xi}$$

$$\curvearrowleft G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})\langle \mathbb{F}_{\ell}^{*} \rangle$$

(Frob)

$${}^{(\infty)}\overline{\Psi}_{\mathcal{F}^{\vdash\Theta}_{\underline{v},\alpha}} \xrightarrow{\sim}$$

Kmm

$$\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \to \Pi_{\underline{v}}$$

ℓ)

$$) \subseteq \prod_{|t| \in |\mathbb{F}_\ell^{*}|} (\overline{\Psi}_{{}^\dagger e_{\underline{v}}})_{|t|}$$

$$= \{ {}^{(\infty)}\overline{\Psi}_{\mathcal{F}_{env}}({}^\dagger \mathfrak{D}_{\succ}) \}_{\xi}$$

$$\curvearrowright \; G_{\underline{v}}(M_*^{\Theta})_{\langle \mathbb{F}_\ell^{*} \rangle}$$

(Frob) (et) (et) (Frob)

$$\begin{tikzcd}
{}^{(\infty)}\overline{\Psi}_{\mathcal{F}^{\vdash \Theta}_{\underline{v}, \alpha}} \arrow[r, "\sim"', "Kmm"] & {}^{(\infty)}\overline{\Psi}^{\iota}_{env}(M_*^{\Theta}) \arrow[r, "\sim", "env"'] & {}^{(\infty)}\overline{\Psi}_{\xi}(M_*^{\Theta}) \arrow[r, "\sim", "(Kmm)"'] & \overline{\Psi}_{\mathcal{F}_{\xi}}^{\dagger(\infty)}({}^\dagger \mathfrak{D}_{\succ}) \\
\curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright \\
\Pi_{\underline{v}}(M_*^{\Theta}) \arrow[r, two heads] & \Pi_{\underline{v}}^{\ddot{}}(M_{\ddot{}}^{\Theta}) & \{G_{\underline{v}}(M_{\ddot{}}^{\Theta})_{|t|}\}_{t \in \mathbb{F}_\ell^{*}} \arrow[r, "\sim"] & \{G_{\underline{v}}(M_*^{\Theta})_{|t|}\}_{t \in \mathbb{F}_\ell^{*}}
\end{tikzcd}$$

compat as above

$$(\infty)\ \Psi_{\dagger\mathcal{F}^{\ominus}_{\triangle}} \xrightarrow[(\mathrm{Kmm})]{\sim} {}^{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\ominus}_{*}) \xrightarrow[\mathrm{end}]{\sim} \Psi_{\mathrm{gau}}(\mathbb{M}^{\ominus}_{*}) \xrightarrow[(\mathrm{Kmm})^{-1}]{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{F}_{\triangle})$$

$$(\mathrm{Frob}) \qquad (\mathrm{et}) \qquad (\mathrm{et}) \qquad (\mathrm{Frob})$$

(iii) (const monoids & splittings)

$$0 \in |\mathbb{F}_l| \rightsquigarrow (\Psi_{\dagger\mathcal{F}_{\triangle}})_0 \xrightarrow{\sim} (\Psi_{\dagger\mathcal{F}_{\triangle}})_{\langle \mathbb{F}_l^{*}\rangle} \cdot \mathrm{Im}(\xi)^{\overset{\mathcal{O}_{\geq 0}}{\cap}}$$

$$\text{splitting}\ {}_{(\infty)}\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\mathcal{F}_{\triangle}) = (\Psi_{\dagger\mathcal{F}_{\triangle}})_{\langle \mathbb{F}_l^{*}\rangle}$$

Π later use for next, cusic ${}^{\circ}\Pi_{\veebar} \hookleftarrow$ $\circlearrowleft$ j l_g-link $\mathbb{F}_l^{\veebar\pm}$-symm compatib w/

full poly

$$\mathcal{M}^{\Theta}_{*}(\Pi_{\veebar}) \simeq \mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathcal{F}_{\veebar})$$

/ each isom

$\circledast$

[IUTch II, Cor 3.7]

$$(-)\,\overline{\Psi}_{env}(\mathcal{M}^{\Theta}_{*}(\Pi_{\veebar})) \overset{\wedge}{\underset{env}{=}} (-)\,\overline{\Psi}_{\xi}(\mathcal{M}^{\Theta}_{*}(\Pi_{\veebar}))$$

$$\underset{\circledast}{\overset{\wedge}{=}} (-)\,\overline{\Psi}_{\xi}(\mathcal{M}^{\Theta}_{*}({}^{\dagger}\mathcal{F}_{\veebar})) \underset{(\kappa_{mon})^{-1}}{\xrightarrow{\sim}} (-)\,\overline{\Psi}_{\mathcal{F}_{\xi}}({}^{\dagger}\mathcal{F}_{\veebar})$$

Gaussian monoid mono-radiality