§1. Not ‘what’ but ‘why’

§2. The Membership Equation

§3. The Inter-universal Geometry of Categories

§4. Analogy with (Uniformizing) \( \mathcal{M}_x \)-objects

§1. Not ‘what’ but ‘why’:

What is anb. geom.?

connected scheme

\[ X \rightarrow \pi_1(X) \quad (\text{etale fund. gr.: } \text{SGA} 1) \]

... To what extent/for which \( X \), can one \( \pi_1(X) \rightarrow X \)?

Why does one care?

... not so clear, but influences fundamental direction of research

... i.e., without an answer to ‘why’? ... tends to degenerate into a

Original Apparent Motivation (Grothendieck? Deligne?)

meaningless contest of stoicism

(e.g., smaller portions of \( \pi_1 \), more general \( X \))

Diophantine Geometry: e.g., Mordell Conjecture

\[ X: \text{smooth, proper curve, genus } \geq 2 \]

\[ \text{number field } F; \quad G_F = \text{Gal}(F/F); \quad \varPi_X = \pi_1(X) \]

\[ X(F) = \{ x_1, \ldots, x_n, \ldots \} \quad \Rightarrow \quad \frac{\varPi_X}{G_F} \rightarrow S_{\infty}; \quad G_F \rightarrow \varPi_X \]

\( \Rightarrow \) if ‘Section Conjecture’ holds, then \( S_{\infty} \) arises from \( \exists x_\infty \)

\( \Rightarrow \) Contradiction ???

Speaker’s conclusion: This approach is of historical interest, but is not really the correct approach!
Here, we consider a new, different approach. cf. one mathematician's remark after work of Tomagao, Mochizuki on Grothendieck's Anabelian Conjecture (GC):

"In some sense, it's a shame that GC was proven"

... i.e., if GC was false, then \( \exists \) an interesting new non-scheme-theoretic geometry: \( \text{Aut}(\Gamma(x)) \) i.e., interesting new geometry (lying outside scheme theory)

... i.e., one should be interested in phenomena that lie on the BOUNDARY between 'anabelian' / 'non-anabelian'

... in a word, our answer is that =

§2. The Membership Equation:

motivation: ABC Conjecture \( \rightarrow \) Need geometry (e.g., derivative) \( \mathfrak{H}_1 \)

\[ \begin{align*}
  q_n & = 1, 2, 3, 5, 7, 11, 13, 17, \ldots \\
  \pi_1 & = 1, 2, 3, 5, 7, 11, 13, 17, \ldots
\end{align*} \]

\[ \begin{align*}
  & \text{form quotient by identifying} \\
  & \begin{align*}
    & \{ a_i \} \\
    & \begin{cases}
      a_i & \mapsto a_i \\
      b_i & \mapsto b_i
    \end{cases}
  \end{align*}
\]

\[ \begin{align*}
  & \text{Solve } (G \leq A) ! \\
  & \text{(contradicts 'axiom of foundation')}!
\end{align*} \]

relation to Hodge Theory; cf. 'indigenous bundles on a hyperbolic Riemann surface \( X \):

\[ \begin{align*}
  \mathbb{T} & \quad \text{Fiber} \\
  \mathbb{P} & \quad \text{moduli of fibers}
\end{align*} \]

Hodge section = identification of a fiber with \{ tall fibers \}
Pathology / distortion arising from this identification (= Hodge section = 'a= a').

Kodaira-Spencer

§3. The Inter-Universal Geometry of Categories:

Goal: to identify \( \bigcirc \) with \( \bigcirc \) (a loop) (cf. a o a)

... if 'math. theater' is built up with usual set-theoretic objects, then it typically has a very complicated \( \mathcal{C} \)-structure

\[ \downarrow \]

difficult to form a loop

\[ \downarrow \]

if take 'categories' as fund geom. objects, then 'simple \( \mathcal{C} \)-structure'

\[ \downarrow \]

IU Geom of Categories: 'categories up to equivalence'

Also, note:

- \( \text{M monoid} \Rightarrow \text{M}_{\text{obj}} \times \text{M}_{\text{mor}} \quad \text{End}(X) = M \) \( \Rightarrow \) 'gp./monoid' is a special case of 1-cats.

- \( \text{R ring} \Rightarrow \text{DR} \times \text{CR} \quad \text{R}^a \times \text{a functor} \quad \text{2-cats of 1-cats}. \)
...i.e., ring has '2 dims/levels': '+' and 'x'

ABC conj. concerns the relation betw. these two levels.

→ need to be able to consider them 'independently'

§ 4. Analogy with (uniformizing) $M_{3^P}$-objects (→ Filtered Frobenius crystals)

Observe:

\[ \mathbb{F}_p[x] \rightarrow \mathbb{F}_p[x] \rightarrow \ldots \]

alg. geom. over alg. geom. over

the relation betw. these two 'math. objects' is the model for § 3.

↓

'loop' of § 3 \[ \leftrightarrow \]

$M_{3^P}$-objects (i.e., come back to themselves w.r.t. Frob)

... cf. especially, uniformizing $M_{3^P}$-objects of 'p-adic Teichmüller theory'

'Intrinsic Hodge Theory'

(Cf., ICM '98)

→ 'give rise to p-adic unifs.'

of hyperbolic curves \[ \leftrightarrow \]

Koebe unif./\( \mathbb{C} \)

\[ \leftrightarrow \]

Bers unif./\( \mathbb{C} \)

... note: for $M_{3^P}$-objects, \( \text{ad.(Frob.)} = p \cdot \mathbb{C} \) \[ \rightarrow \]

(preview) in the case of ABC,

... note: perfection is a sort of 'limit'

\[ \leftrightarrow \]

IU geom. also a sort of 'limit' = 'analysis'

... rough:

Cf. 'Arakelov geom./scheme theory': take into account arch. primes (+ their analysis)

'IU geom./Arak.geom.' = generic primes (+ their analysis)
§1. Anabelian Geometry as a Special Case of Inter-universal Geometry

In a word, corresponds to the case of Galois categories

\[ \mathcal{B}(\Pi) : \text{obj: finite sets with cont. } \Pi \text{-action} \]
\[ \text{mor: } \Pi \text{-morphisms.} \]

\[ (\text{geom. via}) \quad \downarrow \quad (\text{geom. via}) \]
\[ \text{sets, up to equiv.} \longleftrightarrow \text{profinite gps (e.g., } \pi_1(\text{scheme}) \text{)} \]
\[ \text{up to automorphism} \]

then the discussion of \( I \) applies to anab geometry.

§2. Arithmetic Fields:

Number Fields:

\[ F(\mathbb{Q} < \infty) \quad G_F := G_{\text{gal}}(F/F) \]

Most basic result: \( \text{no. fields are anab.} \)

Theorem (Neukirch-Uchida): \( F_1, F_2 : \text{no. fields} \Rightarrow \) \[ \text{Isom}(F_1, F_2) \subseteq \text{Out Isom}(G_{F_1}, G_{F_2}) \]

Also, if \( G_F \to G_F^{\text{sol}} \) denotes the ‘maximal solvable quotient’, then:

Theorem (Uchida): \( F_1, F_2 : \text{no. fields} \Rightarrow \) \[ \text{Isom}(F_1, F_2) \subseteq \text{Out Isom}(G_{F_1}^{\text{sol}}, G_{F_2}^{\text{sol}}) \]
\( p \)-adic local fields: \( K/\mathbb{Q}_p < \infty \), \( G_K := \text{Gal}(K/k) \)

\[ \Rightarrow \text{N/TM is false for } G_K ! \]  \Rightarrow \text{possibility of interesting new geometry!} 

(cf. I, §4; III)

But if consider higher ramification filtration

Theorem (M): \( K_1, K_2 \): \( p \)-adic local fields \( \Rightarrow \)
\[ \text{Isom}(K_1, K_2) \sim \text{OutIsom}^\text{ram.}(G_{K_1}, G_{K_2}) \]

\[ \text{Preserves higher ram. filt.} \]

\[ \text{not quite absolute!} \]

§3. Hyperbolic Curves:

\( K: \text{field}, \)
\( X: \text{hyperbolic curve } /K: \) (smooth, proper, geom. conn. curve over \( F \)) \( \setminus \) (reduced divisor of \( \text{deg. } r \)) \( \leq 2g - 2 + r > 0 \)

\[ \downarrow \]

\[ 1 \rightarrow \Delta_X \rightarrow \Pi_X \rightarrow G_K \rightarrow 1 \]

\( \Delta_X := \pi_2(K_{\overline{\kappa}}) \quad \parallel \quad \text{Gr}_K := \text{Gal}(K/k) \)
\( \Pi_X := \pi_1^+(X) \) \( \text{same ramification at cusp} \) (vacuous if \( \text{char}(K) = 0 \))

note: if \( \text{char}(K) = 0 \), then \( \Delta_X \) det'd up to isom. by \( (g, r) \), center-free.
if \( \text{char}(K) = p \), then \( \Delta_X \) is not center-free (cf. below), but is center-free.

note: if \( K \) sub-p-adic (i.e., \( \exists \kappa \) (\( \text{fin. gen. extn. of } \mathbb{Q}_p \)) : e.g., no. pts of p-adic local fields)
then \( \text{Gr}_K \) center-free (false for, e.g., finite fields.)
characteristic $p > 0$:

**Theorem (Tamagawa)** For $i = 1, 2$, $\mathbb{P}^1 / \mathbb{F}_p \cong \infty$, $X_i$: affine hyperbolic curve / $\mathbb{F}_p$

$X_i \rightarrow X_i$: profinite universal covering

$\text{Isom}(\tilde{X}_i / X_i, \tilde{X}_2 / X_1) \Rightarrow \text{Isom}(\tilde{X}_1, \tilde{X}_2)$

Remark: An unpublished extension to proper case (M)

**Theorem (Tamagawa)** $X_i$: hyperbolic curve / $\mathbb{F}_p$ (for $i = 1, 2$) $\Rightarrow$

(i) There are finitely many $X_2$ s.t. $\tilde{\Pi X}_1 \cong \tilde{\Pi X}_2$

(ii) $X_i$: genus 0 $\Rightarrow$

$\left\{ (\tilde{\Pi X}_1, \tilde{\Pi X}_2) \right\} \Rightarrow \left( X_1 \cong X_2 \right)$

characteristic 0:

$S$: Name\,(K) $\ni S = \lim_{\rightarrow} S_i$, where $\rightarrow S_i \Rightarrow S_j$ $\Leftarrow$

if $\emptyset \neq \Sigma$ is a set of primes, write:

$\Delta^\Sigma_S$: maximal pro-$\Sigma$ quotient of $\Delta_S$

$\Pi^\Sigma_S: = \Pi_S / \text{ker}(\Delta_S \rightarrow \Delta^\Sigma_S)$

**Theorem (M)** $K$: sub-p-adic; $S$: smooth pro-variety / $K$; $X$: hyperbolic curve / $K$; $p \in \Sigma$ $\Rightarrow$

$\text{Hom}^\text{dominant}_K (S, X) \sim \text{Out Hom}^\text{open}_K (\Pi^\Sigma_S, \Pi^\Sigma_X)$
\[ K \text{ generalized sub-padic: } K \overset{\exists}{\rightarrow} \text{ (is an extension of } \mathbb{Q}(W(\overline{F}_p))) \]

(\Rightarrow G_{Tk} \text{ center-free})

\text{Theorem (M): } K \text{ generalized sub-padic; for } i=1,2, X_i: \text{ hyperbolic curve } / K, \forall \Sigma \Rightarrow

\text{Isom}_K(X_1, X_2) \cong \text{OutIsom}_{G_K}(\prod_i^\Sigma X_1, \prod_i^\Sigma X_2)

\text{Remark: cf. Tamagawa } \overline{F}_p

\text{Absolute Results / Number Fields:}

\text{Corollary: } F_i/\mathbb{Q} < \infty; X_i: \text{ hyperbolic curve } / F_i; \forall \Sigma \text{ arbitrary } \Rightarrow

\text{Isom}(X_1, X_2) \cong \text{OutIsom}(\prod_i^\Sigma X_1, \prod_i^\Sigma X_2)

\text{If } |\Sigma| = 1, \text{ and, for } p \in \Sigma, \text{ p-torsion points of Jacobian of } X \text{ (hyponormal over } k) \text{ are defined over } K, \text{ then}

G_{Tk} \rightarrow \text{Out}(\Delta_x^\Sigma)

\text{maximal Solvable Quotient}

\text{Corollary: } F_i/\mathbb{Q} < \infty; X_i: \text{ hyperbolic curve } / F_i; \Sigma \text{-solvable } \Rightarrow

\text{Isom}(X_1, X_2) \cong \text{OutIsom}(\prod_i^\Sigma X_1, \prod_i^\Sigma X_2)

\text{Remark: Since 'NU for p-adic local fields' is false, unclear (UNKNOWN)}

\text{if abs. result holds for } p\text{-adic local fields, = 'abs-p'}

\text{... my expectation: } \underline{\text{false}} \text{ (Corollary of work on ABC ???)
§1. p-adic Absolute Anabelian Geometry

Expectation: \text{absGrC} \text{ false!}

Motivation: 'ANALOGY'

<table>
<thead>
<tr>
<th>\text{p-adic local fields}</th>
<th>\mathbb{F}_p(\mathbb{C})</th>
</tr>
</thead>
<tbody>
<tr>
<td>treating ( T \times X ) absolutely</td>
<td>treating ( X ) not as an object over ( \mathbb{F}_p(\mathbb{C}) ), but rather as an object over ( \mathbb{F}_p )</td>
</tr>
<tr>
<td>( \text{cf. differentiation over } \mathbb{F}_p^1 )</td>
<td>( \downarrow )</td>
</tr>
<tr>
<td>Should only be able to recover properties preserved by coordinate transformations ( t \mapsto t + (?)^2 + \ldots )</td>
<td></td>
</tr>
</tbody>
</table>

\( \downarrow \)

one expects that one should be able to recover

(i) the (log) special fiber
(ii) the generic fiber if \( X \) is a constant curve

§2. The Logarithmic Special Fiber

\text{Theorem (M via Itohara):} For \( i=1,2 \), \( K_i/\mathbb{Q}_p < \infty \), \( X_i \text{ hyperelliptic curve}/K_i \) with log special fiber \( X^\log_{k_i} \)

\( \forall \Pi x_1 \Rightarrow \Pi x_2 \text{ functionally} \) \( x_1^\log_{k_1} \cong x_2^\log_{k_2} \)
§3. Canonical Curves

<table>
<thead>
<tr>
<th>Hyperbolic Curves</th>
<th>Abelian Varieties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bers embedding</td>
<td>Hyperbolic geometry of Siegel upper half plane</td>
</tr>
<tr>
<td>( p )-adic Teichmüller theory</td>
<td>Sernesi-Tate theory</td>
</tr>
<tr>
<td>Canonical liftings</td>
<td>Canonical liftings</td>
</tr>
</tbody>
</table>

\[ \Pi_{X_1} \cong \Pi_{X_2} \implies (i) \text{ } X_1 \text{ can. lift } \iff \text{ } X_2 \text{ can. lift} \]

(ii) If can. lift, then log sp. fiber isom. lifts uniquely to \( \tilde{X}_1 \cong \tilde{X}_2 \)

**Theorem (1):** For \( i=1,2 \), \( K_i/\mathbb{Q}_p \) abs. unram., \( p > 5 \); \( X_i \) hyperbolic curve/\( K_i \),

\[ \Pi_{X_1} \cong \Pi_{X_2} \implies (i) \ X_1 \text{ can. lift } \iff \text{ } X_2 \text{ can. lift} \]

(ii) If can. lift, then log sp. fiber isom. lifts uniquely to \( \tilde{X}_1 \cong \tilde{X}_2 \)

**Definition:** \( X \) absolute if \( \forall X \) s.t. \( \Pi_{X} \cong \Pi_{X'} \), \( \exists X \cong X' \)

**Corollary:** The points determined by absolute curves are Zariski dense in \( \text{Mg,r}(\mathbb{Q}_p), p > 5 \)

**Remarks:**
(i) So far, these are the only known absolute curves.
(ii) Thus, 'can. lifts are like constant curves' (cf. abel. vars)
(iii) 'Can' is first application of \( p \)-adic Teichmüller theory (1)
(iv) cf. 'can. lifts of abel. vars. are CM': \( \overline{\mathbb{Q}_p} \subseteq \overline{\mathbb{Q}} \text{ Aut} \)

\( \text{cf.} \ '\text{abs. } \Pi_{X}' \)
84. Absolute Relative Anabelian Geometry


$\text{Loc}_K(X)$: obj: $Y$ s.t. $\exists$ fin. et. $Y \to X$

mor: $Y_1 \to Y_2$ fin. et., $/K$ (not nec. $/X$ !)

$\text{DLoc}_K(X)$: obj: $Y \subseteq \mathbb{Z}$ s.t. $\exists$ fin. et. $Y \to X$; 2 hyperbolic curves,

$Y \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}$ usual compactification embedding

mor: $Y_1 \to Y_2$ dominant, $/K$.

Thus, $\exists$ natural faithful $\text{Loc}_K(X) \to \text{DLoc}_K(X)$

$\Rightarrow$ rel $\text{pGp}$ implies the following absolute results:

**Theorem (M)** For $i = 1, 2$, $K_i / \mathbb{Q} < \infty$; $X_i$: hyp. curve $/ K_i$, $\alpha: \Pi X_1 \cong \Pi X_2$.

$\Rightarrow$

$\exists$

$\text{Loc}_{K_i}(X_1) \cong \text{Loc}_{K_2}(X_2)$; $\text{DLoc}_{K_i}(X_1) \cong \text{DLoc}_{K_2}(X_2)$

... functorial in $\alpha$

**Corollary:** notation as in Theorem.

(i) Suppose $(q_i, r_i) = (1, 1)$. Then $\alpha$ preserves decomp. gps. of torsion points.

(ii) Suppose $X_i$ isogenous (i.e., admits a common fin. et. covering) to a genus $0$ hyp. curve defined over a number field. Then $\alpha$ preserves decomp. gps. of all closed points.
Remark: "Cor" may be interpreted as a sort of trace section conjecture. cf. [1], §4, the importance of the why: $\prod_{x \in X} \mathbf{1}_x$, i.e., when do not obviously occur? $\mathcal{O}_{X,K}$

Suppose $x \in \overline{X} \setminus X$; "cusp" $\Rightarrow$

\[ 1 \to I_x \to D_x \to G_K \to 1 \]
\[ \mathbb{Z}(1) \]

Torsor of splittings, over $H^1(G_K, I_x) \cong (K^\times)^\wedge$

For arb. (resp. stable reduction) $X$, local (resp. local integral) coord.

$t \in I_x$, \{t \mathcal{N}\} \rightsquigarrow$ reduction of structure gp \( (K^\times)^\wedge \to K^\times \) (resp. \((K^\times)^\wedge \to \mathcal{O}_X^\times \))

Corollary: notation as in Theorem. Suppose $X$ isogenous to genus 0 hyp. curve.

(i) $\alpha$ preserves red. of str. gp. \( (K^\times)^\wedge \to K^\times \)

(ii) In stable red. case, $\alpha$ preserves red. of str. gp. \( (K^\times)^\wedge \to \mathcal{O}_K^\times \)

Remarks: (i) These results extend immediately to André's tempered fundamental gp.

(ii) "Cor" is related via "étale theta function" (and Riemann) \( (= \text{sort of p-adic analytic version of theta fn.}) \)

...to Hodge-Tate theory, ABC (c.f. functional equation of classical theta function)