COMMENTS ON THE MANUSCRIPT (2018-08 VERSION) BY SCHOLZE-STIX CONCERNING INTER-UNIVERSAL TEICHMÜLLER THEORY (IUTCH)

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In the following, we make various additional *Comments* concerning the August 2018 version of the manuscript [SS2018-08] by Scholze-Stix (SS), to *supplement* the comments made in [Cmt2018-05] concerning the May 2018 version [SS2018-05] of this manuscript. Most of the Comments of [Cmt2018-05] were *not addressed* in [SS2018-08] and hence, in particular, *continue to remain valid* concering [SS2018-08]. In addition, we would like to make the following supplementary Comments:

(C1) : Remark 5, "For fixed ... $h(P) \leq b$.": I can only say that it is a *very challenging task to document the depth of my astonishment* when I first read this Remark! This Remark may be described as a **breath-takingly (melo?)dramatic self-declaration**, on the part of SS, of their *profound ignorance* of the *elementary theory of heights*, at the advanced undergraduate/beginning graduate level. Indeed,

> the *finiteness statement* at the beginning of the paragraph follows immediately, by considering the *j-invariant* (say, multiplied by a suitable positive integer N, which depends only on d and b) of the elliptic curve under consideration, from the *finiteness* of the set of complex numbers that satisfy a monic polynomial equation of degree d with coefficients $\in \mathbb{Z}$ of absolute value $\leq C$, for some fixed real number C that depends only on d and b.

To repeat, this sort of argument lies well within the framework of *advanced undergraduate/beginning graduate-level mathematics*. It is *entirely inconceivable* that any researcher with substantial experience working with heights of rational points would attempt to prove this sort of finiteness statement by invoking such a *nontrivial result* as Faltings' theorem. Anyone familiar with the proof of Faltings' theorem will also recognize immediately that the proof of Faltings' theorem ultimately reduces to the elementary observation reviewed above, i.e., that the finiteness of the set of rational points (of, say, a proper variety) of bounded height over number fields of bounded degree follows immediately from elementary considerations, namely, from the finiteness of the set of solutions of monic polynomial equations of bounded degree with bounded coefficients $\in \mathbb{Z}$. (Another problem with the argument in Remark 5 is that it is never mentioned *why the discriminant of $k/\mathbb{Q}$ is bounded.*

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Such a bound is necessary in order to conclude that the abelian variety A has *good reduction outside a fixed finite set of primes* that depends only on d and b.)

(C2) : §2.1, (3): By comparison to the corresponding passage in [SS2018-05], "explain" was replaced by "convince". As discussed in [Cmt2018-05], (C3), the *fundamental problems* that arise when one attempts to *"identify identical objects along the identity"* were discussed at length in the March 2018 discussions and are discussed in detail in [Rpt2018] (cf., especially [Rpt2018], (T3); [Rpt2018], §10; [Rpt2018], (SSId), (SSIdFs), (SSIdEx), (ModEll)).

(C3) §2.1.2, second sentence of the final paragraph, "equivalence of categories": This is not such a central issue, but the *"equivalence of categories"* asserted here is **false** as stated since it does not take into account the *choices* of the *prime number* "l" and the *set of valuations* "$\underline{\mathbb{V}}$" (both of which are required to satisfy certain conditions).

(C4) Footnote 7, "K is algebraically closed and thus the image of log is divisible rather than contained in the maximal ideal": This is not such a central issue, but this statement is a bit *misleading* in the following sense: *Unlike divisibility*, the property that the image of the p-adic logarithm is not contained in the maximal ideal already holds in the case of *finite extensions of* $\mathbb{Q}_p$ that are *sufficiently ramified* over $\mathbb{Q}_p$.

(C5) §2.1.4, the latter portion of the final paragraph: The discussion here was reworded in way that appropriately addresses [Cmt2018-05], (C8).

(C6) §2.1.5, the discussion following the first display: The discussion here was reworded in way that appropriately addresses [Cmt2018-05], (C9). There is, however, a *misprint*: "$\overline{k}_V$" should be replaced by "$\overline{k}_v$".

(C7) Footnote 8, "convincingly in our opinion": The phrase "convincingly in our opinion" was added. This topic was discussed extensively in [Cmt2018-05], (C7); [Rpt2018] (cf., especially, the portions of [Rpt2018] referred to in [Cmt2018-05], (C7)).

(C8) §2.1.6, the discussion following the first display: The discussion here was reworded in way that appropriately addresses [Cmt2018-05], (C10).

(C9) §2.1.8, second sentence of the first paragraph, "$\pi_1(X)$", "tempered coverings of X": The modifications here (of the corresponding passage in [SS2018-05]) — i.e., which amount to replacing *local objects* at bad primes by *global objects* over number fields — seem *somewhat strange*. That is to say, although both descriptions are rather rough and sketchy, the corresponding passage in [SS2018-05] is *much more accurate* than the [SS2018-08] version of this passage. Indeed, the essential portion of the theory of theta values takes place locally at the bad primes and is then formally extended to global data. This theory makes use, in an essential way, of the tempered fundamental group at bad primes. Moreover, the phrase *"tempered coverings of X"* is *meaningless*, since tempered coverings are *only defined locally.*

(C10) §2.1.8, second paragraph, "(j-th) concrete Θ-pilot object": This phrase is inappropriate since there is *no* "j-th Θ-pilot object" in IUTch. There is a "j-th component" of (what SS refer to as) the "concrete Θ-pilot object", but there is *only one* "concrete Θ-pilot object".

(C11) §2.2, second paragraph: Unlike the case with [SS2018-05], the terms *"multiradial algorithm"* and *"processions of tensor packets of log-shells"* are mentioned. On the other hand, it is clear from the discussion of §2.2 that

> *SS still completely misunderstand the way in which the mathematical objects referred to by these terms are used, in an essential way, in IUTch* (cf. (C12), (C13), (C14), below).

(C12) Footnote 10, "Mochizuki does not properly distinguish them, which is part of our main concern"; §2.2, third sentence of the second paragraph, "As ... work" (cf. also [Cmt2018-05], (C16)): This assertion of [SS2018-08] is *central* to the arguments of [SS2018-08] and reflects a *fundamental misunderstanding* of SS. The issue of distinguishing the *abstract category-theoretic versions of pilot objects* determined by the *intrinsic structure* of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips from their *concrete (multiradial!) representations* on *tensor-packets of log-shells* is one of the **most central aspects** of IUTch (cf., e.g., [IUTchIII], Theorem 3.11; the proof of [IUTchIII], Corollary 3.12)! That is to say, in IUTch,

> the *q- and Θ-concrete realizations* (in the terminology of [SS2018-08]) — i.e., the complicated *"intertwining"*, or *relationship*, between the *value group portion* and the *unit group portion* of the data that constitutes an *"abstract category-theoretic"* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip — correspond precisely to the ***q*- and Θ-arithmetic holomorphic structures** (i.e., roughly speaking, to the *distinct ring structures*) *in the domain and codomain of the Θ-link*

— cf. the discussion of [Rpt2018], §12, especially, [Rpt2018], (LbLV). Thus, in summary,

> the issue of *"not properly distinguishing..."* arises in [SS2018-08] precisely as a consequence of the fact that in [SS2018-08], the **arithmetic holomorphic structures** *in the domain and codomain of the Θ-link* are **not distinguished**

— i.e., *not as a consequence of any logical flaw in IUTch.*

(C13) Footnote 12 (cf. also [Cmt2018-05], (C15); the discussion of (T9) at the end of [Rpt2018], §4): The *"simplifications"* discussed here correspond precisely to the *"*id-*version"* discussed in detail in [Rpt2018], §10, especially, [Rpt2018], (SSId). As discussed in [Rpt2018], (SSIdFs), once one makes these simplifications, one can *no longer apply the multiradial algorithms* of [IUTchIII], Theorem 3.11. Further explanations of [Rpt2018], (SSIdFs), in somewhat more elementary terms — involving *real and complex vector spaces* — may be found in [Rpt2018], (SSIdEx), (ModEll), (HstMod).

(C14) §2.2, third paragraph, "spell out all identifications of copies of real numbers"; §2.2, fourth paragraph, "consistently identify all of these"; §2.2, displayed diagram; §2.2, fourth paragraph, "wanted to introduce scalars of j^2 somewhere" (cf. [Cmt2018-05], (C17), (N1), (N2), (N3), (N4), (N5)): In some sense, the *main assertion* of SS underlying this argument in §2.2 concerning identifications of copies of $\mathbb{R}$ is the following:

(Lin) the relationship between any two of these copies of $\mathbb{R}$ is a simple, straightforward **linear relationship**, given by *multiplication by some scalar*, i.e., *multiplication by some positive real number*.

Here, it should be stated clearly that this assertion (Lin), which underlies the argument of §2.2, is **completely false**. That is to say, *such simple linear relationships do indeed exist* between the copies of $\mathbb{R}$ arising (via the Θ-link) from the various $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips involved. On the other hand, whenever *indeterminacies* are involved, as in the case of the *multiradial representation of the* Θ*-pilot*, the relationship between log-volumes of regions *subject to* and *not subject to* such indeterminacies is *much more complicated* and *depends nontrivially on the geometry of the particular region* under consideration. In particular, this relationship is **highly non-linear**. We refer to [Rpt2018], (MlLV), (LVEx), (DsInd), for a discussion of this phenomenon, which includes an *elementary example* (namely, [Rpt2018], (LVEx)) of this phenomenon, involving real vector spaces.

Bibliography

[IUTchI] S. Mochizuki, *Inter-universal Teichmüller Theory I: Construction of Hodge Theaters*, RIMS Preprint **1756** (August 2012).

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, RIMS Preprint **1757** (August 2012).

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, RIMS Preprint **1758** (August 2012).

[IUTchIV] S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, RIMS Preprint **1759** (August 2012).

[Alien] S. Mochizuki, *The Mathematics of Mutually Alien Copies: from Gaussian Integrals to Inter-universal Teichmüller Theory*, RIMS Preprint **1854** (July 2016).

[Rpt2018] S. Mochizuki, *Report on Discussions, Held During the Period March 15 – 20, 2018, Concerning Inter-universal Teichmüller Theory (IUTch)*, September 2018, available at:

http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html

[Cmt2018-05] S. Mochizuki, *Comments on the manuscript by Scholze-Stix concerning inter-universal Teichmüller theory (IUTch)*, July 2018, available at:
http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html

[SS2018-05] P. Scholze and J. Stix, *Why abc is still a conjecture*, manuscript, May 2018, available at:
http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html

[SS2018-08] P. Scholze and J. Stix, *Why abc is still a conjecture*, manuscript, August 2018, available at:
http://www.kurims.kyoto-u.ac.jp/~motizuki/IUTch-discussions-2018-03.html

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