Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia Groups and Profinite Dehn Twists

Yuichiro Hoshi and Shinichi Mochizuki

Received March 31, 2011.
Revised December 28, 2011.
2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.
Key words and phrases. anabelian geometry, combinatorial anabelian geometry, profinite Dehn twist, semi-graph of anabelioids, inertia group, hyperbolic curve, configurationa space.
The first author was supported by Grant-in-Aid for Young Scientists (B), No. 22740012, Japan Society for the Promotion of Science.

Abstract.

Let Σ be a nonempty set of prime numbers. In the present paper, we continue our study of the pro-Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields of characteristic zero. Our first main result asserts, roughly speaking, that if an F-admissible automorphism [i.e., an automorphism that preserves the *fiber subgroups* that arise as kernels associated to the various natural projections of the configuration space under consideration to configuration spaces of lower dimension] of a configuration space group arises from an *F-admissible* automorphism of a configuration space group [arising from a configuration space] of *strictly higher dimension*, then it is *necessarily FC-admissible*, i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups. After discussing various abstract profinite combinatorial technical tools involving semi-graphs of anabelioids of PSC-type that are motivated by the well-known classical theory of *topological surfaces*, we proceed to develop a theory of *profinite Dehn twists*, i.e., an abstract profinite combinatorial analogue of classical Dehn twists associated to cycles on topological surfaces. This theory of profinite Dehn twists leads naturally to *comparison results* between the abstract combinatorial machinery developed in the present paper and more classical scheme-theoretic constructions. In particular, we obtain a *purely combinatorial description* of the *Galois action* associated to a [*scheme-theoretic*!] *degenerating family of hyperbolic curves* over a complete equicharacteristic discrete valuation ring of characteristic zero. Finally, we apply the theory of *profinite Dehn twists* to prove a *"geometric version of the Grothendieck Conjecture"* for — i.e., put another way, we compute the *centralizer* of the *geometric monodromy* associated to — the tautological curve over the moduli stack of pointed smooth curves.

Contents

§ Introduction

Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset of the set of prime numbers $\mathfrak{Primes}$. In the present paper, we continue our study [cf. [SemiAn], [CmbGC], [CmbCsp], [MT], [NodNon]] of the *anabelian geometry* of *semi-graphs of anabelioids of [pro-*Σ*] PSC-type*, i.e., semi-graphs of anabelioids that arise from a *pointed stable curve* over an algebraically closed field of characteristic zero. Roughly speaking, such a "semi-graph of anabelioids" may be thought of as a slightly modified, Galois category-theoretic formulation of the *"graph of profinite groups"* associated to such a pointed stable curve that takes into account the cusps [i.e., marked points] of the pointed stable curve, and in which the profinite groups that appear are regarded as being defined only *up to inner automorphism*. At a more conceptual level, the notion of a semi-graph of anabelioids of PSC-type may be thought of as a sort of **abstract profinite combinatorial analogue** of the notion of a **hyperbolic topological surface of finite type**, i.e., the underlying topological surface of a hyperbolic Riemann surface of finite type. One central object of study in this context is the notion of an *outer representation of IPSC-type* [cf. [NodNon], Definition 2.4, (i)], which may be thought of as an abstract profinite combinatorial analogue of the scheme-theoretic notion of a *degenerating family of hyperbolic curves* over a complete discrete valuation ring. In [NodNon], we studied a *purely combinatorial* generalization of this notion, namely, the notion of an *outer representation of NN-type* [cf. [NodNon], Definition 2.4, (iii)], which may be thought of as an abstract profinite combinatorial analogue of the topological notion of a **family of hyperbolic topological surfaces of finite type over a circle**. Here, we recall that such families are a central object of study in the theory of hyperbolic threefolds.

Another central object of study in the combinatorial anabelian geometry of hyperbolic curves [cf. [CmbCsp], [MT], [NodNon]] is the notion of a *configuration space group* [cf. [MT], Definition 2.3, (i)], i.e., the pro-Σ fundamental group of the configuration space associated to a hyperbolic curve over an algebraically closed field of characteristic zero, where Σ is either equal to $\mathfrak{Primes}$ or of cardinality one. In [MT], it was shown [cf. [MT], Corollary 6.3] that, if one excludes the case of hyperbolic curves of type $(g, r) \in \{(0, 3),\ (1, 1)\}$, then, up to a permutation of the factors of the configuration space under consideration, any automorphism of a configuration space group is necessarily *F-admissible* [cf. [CmbCsp], Definition 1.1, (ii)], i.e., preserves the *fiber subgroups* that arise as kernels associated to the various natural projections of the configuration space under consideration to configuration spaces of lower dimension.

In §1, we prove our *first main result* [cf. Corollary 1.9], by means of techniques that extend the techniques of [MT], §4, i.e., techniques that center around applying the fact that the *first Chern class* associated to the diagonal divisor in a product of two copies of a proper hyperbolic curve consists, in essence, of the *identity matrix* [cf. Lemma 1.3, (iii)]. This result asserts, roughly speaking, that if an F-admissible automorphism of a configuration space group arises from an **F-admissible** automorphism of a configuration space group [arising from a configuration space] of **strictly higher dimension**, then it is **necessarily FC-admissible** [cf. [CmbCsp], Definition 1.1, (ii)], i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups.

Theorem A (**F-admissibility and FC-admissibility).** *Let* Σ *be a set of prime numbers which is either of* **cardinality one** *or* **equal to the set of all prime numbers***;* n *a positive integer;* (g, r) *a pair of nonnegative integers such that* $2g - 2 + r > 0$*;* X *a hyperbolic curve of type* (g, r) *over an algebraically closed field* k *of characteristic* $\notin \Sigma$*;* X_n *the* n*-th configuration space of* X*;* Π_n *the maximal pro-*Σ *quotient of the fundamental group of* X_n*; "*$\mathrm{Out}^{\mathrm{FC}}(-)$*", "*$\mathrm{Out}^{\mathrm{F}}(-)$*"* $\subseteq$ *"*$\mathrm{Out}(-)$*" the subgroups of FC- and F-admissible [cf.* [CmbCsp]*, Definition 1.1, (ii)] outomorphisms [cf. the discussion entitled "Topological groups" in §0] of "*$(-)$*". Then the following hold:*

(i) *Let* $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})$*. Then* α *induces the* **same** *outomorphism of* Π_n *relative to the various quotients* $\Pi_{n+1} \twoheadrightarrow \Pi_n$ *by fiber subgroups of length* 1 *[cf.* [MT]*, Definition 2.3, (iii)]. In particular, we obtain a natural homomorphism*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)\,.$$

(ii) *The image of the homomorphism*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

of (i) is **contained** *in*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)\,.$$

For the convenience of the reader, we remark that our treatment of Theorem A in §1 does not require any knowledge of the theory of semi-graphs of anabelioids. On the other hand, in a sequel to the present paper, we intend to prove a substantial stengthening of Theorem A, whose proof makes quite essential use of the theory of [CmbGC], [CmbCsp], and [NodNon] [i.e., in particular, of the theory of semi-graphs of anabelioids of PSC-type].

In §2 and §3, we develop various *technical tools* that will play a crucial role in the subsequent development of the theory of the present paper. In §2, we study various **fundamental operations** on semi-graphs of anabelioids of PSC-type. A more detailed description of these operations may be found in the discussion at the beginning of §2, as well as in the various *illustrations* referred to in this discussion. Roughly speaking, these operations may be thought of as *abstract profinite combinatorial analogues* of various well-known operations that occur in the theory of **"surgery" on topological surfaces** — i.e.,

- **restriction** *to a subsurface* arising from a decomposition, such as a "pants decomposition", of the surface or *to a* [suitably positioned] *cycle*;
- **partially compactifying** the surface by adding "missing points";
- **cutting a surface** along a [suitably positioned] cycle;
- **gluing together two surfaces** along [suitably positioned] cycles.

Most of §2 is devoted to the abstract combinatorial formulation of these operations, as well as to the verification of various basic properties involving these operations.

In §3, we develop the local theory of the second cohomology group with compact supports associated to various sub-semi-graphs and components of a semi-graph of anabelioids of PSC-type. Roughly speaking, this theory may be thought of as a sort of *abstract profinite combinatorial analogue* of the **local theory of orientations** *on a topological surface* S, i.e., the theory of the locally defined cohomology modules

$$(U, x) \;\mapsto\; H^2(U, U \setminus \{x\}; \mathbb{Z}) \quad (\cong \mathbb{Z})$$

— where $U \subseteq S$ is an open subset, $x \in U$. In the abstract profinite combinatorial context of the present paper, the various locally defined second cohomology groups with compact supports give rise to *cyclotomes*, i.e., copies of quotients of the once-Tate-twisted Galois module $\widehat{\mathbb{Z}}(1)$. The main result that we obtain in §3 concerns various **canonical synchronizations of cyclotomes** [cf. Corollary 3.9], i.e., canonical isomorphisms between these cyclotomes associated to various local portions of the given semi-graph of anabelioids of PSC-type which are *compatible* with the various fundamental operations studied in §2.

In §4, we apply the technical tools developed in §2, §3 to define and study the notion of a **profinite Dehn [multi-]twist** [cf. Definition 4.4; Theorem 4.8, (iv)]. This notion is, needless to say, a natural abstract profinite combinatorial analogue of the usual notion of a Dehn twist in the theory of topological surfaces. On the other hand, it is defined, in keeping with the spirit of the present paper, in a fashion that is *purely combinatorial*, i.e., without resorting to the "crutch" of considering, for instance, profinite closures of Dehn twists associated to cycles on topological surfaces. Our *main results* in §4 [cf. Theorem 4.8, (i), (iv); Proposition 4.10, (ii)] assert, roughly speaking, that profinite Dehn twists satisfy a *structure theory* of the sort that one would expect from the analogy with the topological case, and that this structure theory is *compatible*, in a suitable sense, with the various fundamental operations studied in §2.

Theorem B (**Properties of profinite Dehn multi-twists).** *Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Write*

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$$

for the group of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$ and

$$\mathrm{Dehn}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \{\, \alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \mid \alpha_{\mathcal{G}|_v} = \mathrm{id}_{\mathcal{G}|_v} \textit{ for any } v \in \mathrm{Vert}(\mathcal{G})\,\}$$

— where we write $\alpha_{\mathcal{G}|_v}$ for the restriction of α to the semi-graph of anabelioids $\mathcal{G}|_v$ of pro-Σ PSC-type determined by $v \in \mathrm{Vert}(\mathcal{G})$ [cf. Definitions 2.1, (iii); 2.14, (ii); Remark 2.5.1, (ii)]; we shall refer to an element of $\mathrm{Dehn}(\mathcal{G})$ *as a* **profinite Dehn multi-twist** *of $\mathcal{G}$. Then the following hold:*

(i) **(Normality)** $\mathrm{Dehn}(\mathcal{G})$ *is* **normal** *in* $\mathrm{Aut}(\mathcal{G})$.

(ii) **(Structure of the group of profinite Dehn multi-twists)** *Write*

$$\Lambda_{\mathcal{G}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(H^2_c(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma}), \widehat{\mathbb{Z}}^{\Sigma})$$

for the **cyclotome** *associated to $\mathcal{G}$ [cf. Definitions 3.1, (ii), (iv); 3.8, (i)]. Then there exists a natural isomorphism*

$$\mathfrak{D}_{\mathcal{G}} \colon \mathrm{Dehn}(\mathcal{G}) \xrightarrow{\sim} \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$$

that is **functorial**, *in* $\mathcal{G}$, *with respect to isomorphisms of semi-graphs of anabelioids. In particular,* $\mathrm{Dehn}(\mathcal{G})$ *is a* **finitely generated free** $\widehat{\mathbb{Z}}^{\Sigma}$**-module of rank** $\mathrm{Node}(\mathcal{G})^{\sharp}$. *We shall refer to a nontrivial profinite Dehn multi-twist whose image* $\in \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$ *lies in a direct summand [i.e., in a single "*$\Lambda_{\mathcal{G}}$*"] as a* **profinite Dehn twist**.

(iii) **(Exact sequence relating profinite Dehn multi-twists and glueable outomorphisms)** *Write*

$$\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

*for the [closed] subgroup of "***glueable***" collections of outomorphisms of the direct product* $\prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$ *consisting of elements* $(\alpha_v)_{v \in \mathrm{Vert}(\mathcal{G})}$ *such that* $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ *for any* $v, w \in \mathrm{Vert}(\mathcal{G})$ *— where we write* $\mathcal{G}|_v$ *for the semi-graph of anabelioids of pro-*Σ *PSC-type determined by* $v \in \mathrm{Vert}(\mathcal{G})$ *[cf. Definition 2.1, (iii)] and* $\chi_v \colon \mathrm{Aut}(\mathcal{G}|_v) \to (\widehat{\mathbb{Z}}^{\Sigma})^*$ *for the* **pro-**$\boldsymbol{\Sigma}$ **cyclotomic character** *of* $v \in \mathrm{Vert}(\mathcal{G})$ *[cf. Definition 3.8, (ii)]. Then the natural homomorphism*

$$\begin{array}{ccc} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) & \longrightarrow & \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \\ \alpha & \mapsto & (\alpha_{\mathcal{G}|_v})_{v \in \mathrm{Vert}(\mathcal{G})} \end{array}$$

factors through $\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$, *and, moreover, the resulting homomorphism* $\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$ *[cf. (i)] fits into an* **exact sequence** *of profinite groups*

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \xrightarrow{\rho^{\mathrm{Vert}}_{\mathcal{G}}} \mathrm{Glu}(\mathcal{G}) \longrightarrow 1\,.$$

The approach of §2, §3, §4 is *purely combinatorial* in nature. On the other hand, in §5, we return briefly to the world of *[log] schemes* in order to **compare** the **purely combinatorial** constructions of §2, §3, §4 to analogous constructions from **scheme theory**. The *main techinical result* [cf. Theorem 5.7] of §5 asserts that the **purely combinatorial synchronizations of cyclotomes** constructed in §3, §4 for the profinite Dehn twists associated to the various nodes of the semi-graph of anabelioids of PSC-type under consideration **coincide** with certain natural **scheme-theoretic synchronizations of cyclotomes**. This technical result is obtained, roughly speaking, by applying the various fundamental operations of §2 so as to reduce to the case where the semi-graph of anabelioids of PSC-type under consideration admits a **symmetry** that **permutes the nodes** [cf. Fig. 6]; the desired coincidence of synchronizations is then obtained by observing that both the combinatorial and the scheme-theoretic synchronizations are *compatible with this symmetry*. One way to understand this fundamental coincidence of synchronizations is as a sort of *abstract combinatorial analogue* of the *cyclotomic synchronization* given in [GalSct], Theorem 4.3; [AbsHyp], Lemma 2.5, (ii) [cf. Remark 5.9.1, (i)]. Another way to understand this fundamental coincidence of synchronizations is as a statement to the effect that

> the **Galois action** associated to a [*scheme-theoretic*!] *degenerating family of hyperbolic curves* over a complete equicharacteristic discrete valuation ring of characteristic zero — i.e., "an outer representation of IPSC-type" — admits a **purely combinatorial description** [cf. Corollary 5.9, (iii)].

That is to say, one central problem in the theory of outer Galois representations associated to hyperbolic curves over arithmetic fields is precisely the problem of giving such a *"purely combinatorial description"* of the outer Galois representation. Indeed, this point of view plays a central role in the theory of the *Grothendieck-Teichmüller group*. Thus, although an explicit solution to this problem is well out of reach at the present time in the case of *number fields* or *mixed-characteristic local fields*, the theory of §5 yields a *solution to this problem* in the case of *complete equicharacteristic discrete valuation fields of characteristic zero*. One consequence of this solution is the following *criterion* for an outer representation to be of *IPSC-type* [cf. Corollary 5.10].

Theorem C (**Combinatorial/group-theoretic nature of scheme-theoreticity**). *Let* (g, r) *be a pair of nonnegative integers such that* $2g - 2 + r > 0$*;* Σ *a nonempty set of prime numbers;* R *a complete discrete valuation ring whose residue field* k *is separably closed of characteristic* $\notin \Sigma$*;* $S^{\log}$ *the log scheme obtained by equipping* $S \stackrel{\mathrm{def}}{=} \operatorname{Spec} R$ *with the log structure determined by the maximal ideal of* R*;* $(\overline{\mathcal{M}}_{g,r})_S$ *the* **moduli stack** *of* r*-pointed stable curves of genus* g *over* S *whose* r *marked points are equipped with an ordering;* $(\mathcal{M}_{g,r})_S \subseteq (\overline{\mathcal{M}}_{g,r})_S$ *the open substack of* $(\overline{\mathcal{M}}_{g,r})_S$ *parametrizing smooth curves;* $(\overline{\mathcal{M}}^{\log}_{g,r})_S$ *the log stack obtained by equipping* $(\overline{\mathcal{M}}_{g,r})_S$ *with the log structure associated to the divisor with normal crossings* $(\overline{\mathcal{M}}_{g,r})_S \setminus (\mathcal{M}_{g,r})_S \subseteq (\overline{\mathcal{M}}_{g,r})_S$*;* $x \in (\overline{\mathcal{M}}_{g,r})_S(k)$ *a* k*-valued point of* $(\overline{\mathcal{M}}_{g,r})_S$*;* $\widehat{\mathcal{O}}$ *the completion of the local ring of* $(\overline{\mathcal{M}}_{g,r})_S$ *at the image of* x*;* $T^{\log}$ *the log scheme obtained by* *equipping* $T \overset{\mathrm{def}}{=} \mathrm{Spec}\,\widehat{\mathcal{O}}$ *with the log structure induced by the log structure of* $(\overline{\mathcal{M}}^{\log}_{g,r})_S$*;* $t^{\log}$ *the log scheme obtained by equipping the closed point of* T *with the log structure induced by the log structure of* $T^{\log}$*;* $X^{\log}_t$ *the stable log curve over* $t^{\log}$ *corresponding to the natural strict (*1*-)morphism* $t^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_S$*;* $I_{T^{\log}}$ *the maximal pro-*Σ *quotient of the log fundamental group* $\pi_1(T^{\log})$ *of* $T^{\log}$*;* $I_{S^{\log}}$ *the maximal pro-*Σ *quotient of the log fundamental group* $\pi_1(S^{\log})$ *of* $S^{\log}$*;* $\mathcal{G}_{X^{\log}}$ *the semi-graph of anabelioids of pro-*Σ *PSC-type determined by the stable log curve* $X^{\log}_t$ *[cf.* [CmbGC]*, Example 2.5];* $\rho^{\mathrm{univ}}_{X^{\log}_t} \colon I_{T^{\log}} \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ *the natural outer representation associated to* $X^{\log}_t$ *[cf. Definition 5.5];* I *a profinite group;* $\rho \colon I \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ *an outer representation of pro-*Σ *PSC-type [cf.* [NodNon]*, Definition 2.1, (i)]. Then the following conditions are equivalent:*

(i) ρ *is of* **IPSC-type** *[cf.* [NodNon]*, Definition 2.4, (i)].*

(ii) *There exist a morphism of log schemes* $\phi^{\log} \colon S^{\log} \to T^{\log}$ *over* S *and an* **isomorphism of outer representations** *of pro-*Σ *PSC-type* $\rho \overset{\sim}{\to} \rho^{\mathrm{univ}}_{X^{\log}_t} \circ I_{\phi^{\log}}$ *[cf.* [NodNon]*, Definition 2.1, (i)] — where we write* $I_{\phi^{\log}} \colon I_{S^{\log}} \to I_{T^{\log}}$ *for the homomorphism induced by* $\phi^{\log}$ *— i.e., there exist an* **automorphism** β *of* $\mathcal{G}_{X^{\log}}$ *and an isomorphism* $\alpha \colon I \overset{\sim}{\to} I^{\log}_S$ *such that the diagram*

$$\begin{array}{ccc} I & \xrightarrow{\ \rho\ } & \mathrm{Aut}(\mathcal{G}_{X^{\log}}) \\ {\scriptstyle\alpha}\downarrow\wr & & \downarrow\wr \\ I_{S^{\log}} & \xrightarrow{\rho_{X^{\log}_t} \circ I_{\phi^{\log}}} & \mathrm{Aut}(\mathcal{G}_{X^{\log}}) \end{array}$$

— where the right-hand vertical arrow is the automorphism of $\mathrm{Aut}(\mathcal{G}_{X^{\log}})$ *induced by* β *— commutes.*

(iii) *There exist a morphism of log schemes* $\phi^{\log} \colon S^{\log} \to T^{\log}$ *over* S *and an* **isomorphism** $\alpha \colon I \overset{\sim}{\to} I^{\log}_S$ *such that* $\rho = \rho^{\mathrm{univ}}_{X^{\log}_t} \circ I_{\phi^{\log}} \circ \alpha$ *— where we write* $I_{\phi^{\log}} \colon I_{S^{\log}} \to I_{T^{\log}}$ *for the homomorphism induced by* $\phi^{\log}$ *— i.e., the automorphism "*β*" of (ii) may be taken to be the* **identity***.*

Before proceeding, in this context we observe that one fundamental intrinsic difference between outer representations of IPSC-type and more general outer representations of NN-type is that, unlike the case with outer representations of IPSC-type, the *period matrices* associated to outer representations of NN-type may, in general, *fail to be nondegenerate* — cf. the discussion of Remark 5.9.2.

Here, we remark in passing that in a sequel to the present paper, the theory of §5 will play an important role in the proofs of certain applications to the theory of *tempered fundamental groups* developed in [André].

Finally, in §6, we apply the theory of *profinite Dehn twists* developed in §4 to prove a **"geometric version of the Grothendieck Conjecture"** for — i.e., put another way, we compute the *centralizer* of the *geometric monodromy* associated to — the tautological curve over the moduli stack of pointed smooth curves [cf. Theorems 6.13; 6.14].

Theorem D (**Centralizers of geometric monodromy groups arising from moduli stacks of pointed curves).** *Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$; Σ a nonempty set of prime numbers; k an algebraically closed field of characteristic zero. Write $(\mathcal{M}_{g,r})_k$ for the* **moduli stack** *of r-pointed smooth curves of genus g over k whose r marked points are equipped with an ordering; $\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}$ for the* **tautological curve** *over $\mathcal{M}_{g,r}$ [cf. the discussion entitled "Curves" in §0]; $\Pi_{\mathcal{M}_{g,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{M}_{g,r})_k)$ for the étale fundamental group of the moduli stack $(\mathcal{M}_{g,r})_k$; $\Pi_{g,r}$ for the maximal pro-Σ quotient of the kernel $N_{g,r}$ of the natural surjection $\pi_1((\mathcal{C}_{g,r})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{g,r})_k) = \Pi_{\mathcal{M}_{g,r}}$; $\Pi_{\mathcal{C}_{g,r}}$ for the quotient of the étale fundamental group $\pi_1((\mathcal{C}_{g,r})_k)$ of $(\mathcal{C}_{g,r})_k$ by the kernel of the natural surjection $N_{g,r} \twoheadrightarrow \Pi_{g,r}$; $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ for the group of outomorphisms [cf. the discussion entitled "Topological groups" in §0] of $\Pi_{g,r}$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi_{g,r}$. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{g,r} \longrightarrow \Pi_{\mathcal{C}_{g,r}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow 1\,,$$

which determines an outer representation

$$\rho_{g,r} \colon \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\Pi_{g,r})\,.$$

Then the following hold:

(i) *Let $H \subseteq \Pi_{\mathcal{M}_{g,r}}$ be an open subgroup of $\Pi_{\mathcal{M}_{g,r}}$. Suppose that one of the following two conditions is satisfied:*

  (a) *$2g-2+r > 1$, i.e., $(g, r) \notin \{(0, 3), (1, 1)\}$.*

  (b) *$(g, r) = (1, 1)$, $2 \in \Sigma$, and $H = \Pi_{\mathcal{M}_{g,r}}$.*

Then the composite of natural homomorphisms

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{g,r}}}(\Pi_{\mathcal{C}_{g,r}})/\mathrm{Inn}(\Pi_{g,r})$$

$$\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{g,r})}(\mathrm{Im}(\rho_{g,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,r})}(\rho_{g,r}(H))$$

[cf. the discussion entitled "Topological groups" in §0] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H))\,.$$

Here, we recall that $\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k)$ *is isomorphic to*

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \textit{if } (g,r) = (0,4); \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } (g,r) \in \{(1,1),(1,2),(2,0)\}; \\ \{1\} & \textit{if } (g,r) \notin \{(0,4),(1,1),(1,2),(2,0)\}\,. \end{cases}$$

(ii) *Let* $H \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ *be a closed subgroup of* $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ *that contains an open subgroup of* $\mathrm{Im}(\rho_{g,r}) \subseteq \mathrm{Out}(\Pi_{g,r})$. *Suppose that*

$$2g-2+r > 1, \ \textit{i.e.,}\ (g,r) \notin \{(0,3),(1,1)\}.$$

Then H *is* **almost slim** *[cf. the discussion entitled "Topological groups" in §0]. If, moreover,*

$$2g-2+r > 2, \ \textit{i.e.,}\ (g,r) \notin \{(0,3),(0,4),(1,1),(1,2),(2,0)\},$$

then H *is* **slim** *[cf. the discussion entitled "Topological groups" in §0].*

§0. Notations and Conventions

Sets: If S is a set, then we shall denote by 2^S the *power set* of S and by $S^\sharp$ the *cardinality* of S.

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of all prime numbers. The notation $\mathbb{N}$ will be used to denote the set or [additive] monoid of nonnegative rational integers. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. If $p \in \mathfrak{Primes}$, then the notation $\mathbb{Z}_p$ (respectively, $\mathbb{Q}_p$) will be used to denote the p-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$). If $\Sigma \subseteq \mathfrak{Primes}$, then the notation $\widehat{\mathbb{Z}}^\Sigma$ will be used to denote the pro-Σ completion of $\mathbb{Z}$.

Monoids: We shall write M^{gp} for the *groupification* of a monoid M.

Topological groups: Let G be a topological group and $\mathbf{P}$ a property of topological groups [e.g., "abelian" or "pro-Σ" for some $\Sigma \subseteq \mathfrak{Primes}$]. Then we shall say that G is *almost* $\mathbf{P}$ if there exists an open subgroup of G that is $\mathbf{P}$.

Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$; $C_G(H)$) the *centralizer* (respectively, *normalizer*; *commensurator*) of H in G, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \,|\, ghg^{-1} = h \text{ for any } h \in H \,\},$$

$$(\text{respectively, } N_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \,|\, g \cdot H \cdot g^{-1} = H \,\};$$

$$C_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \mid H \cap g{\cdot}H{\cdot}g^{-1} \text{ is of finite index in } H \text{ and } g{\cdot}H{\cdot}g^{-1} \,\});$$

we shall refer to $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ as the *center* of G. It is immediate from the definitions that

$$Z_G(H) \subseteq N_G(H) \subseteq C_G(H) \; ; \;\; H \subseteq N_G(H) \, .$$

We shall say that the closed subgroup H is *centrally terminal* (respectively, *normally terminal*; *commensurably terminal*) in G if $H = Z_G(H)$ (respectively, $H = N_G(H)$; $H = C_G(H)$). We shall say that G is *slim* if $Z_G(U) = \{1\}$ for any open subgroup U of G.

Let G be a topological group. Then we shall write G^{ab} for the *abelianization* of G, i.e., the quotient of G by the closure of the commutator subgroup of G.

Let G be a topological group. Then we shall write $\mathrm{Aut}(G)$ for the group of [continuous] automorphisms of G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. Now suppose that G is *center-free* [i.e., $Z(G) = \{1\}$]. Then we have an exact sequence of groups

$$1 \longrightarrow G \, (\stackrel{\sim}{\to} \mathrm{Inn}(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1 \, .$$

If J is a group and $\rho \colon J \to \mathrm{Out}(G)$ is a homomorphism, then we shall denote by

$$G \stackrel{\mathrm{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of profinite groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \stackrel{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1 \, .$$

Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ with respect to which the above exact sequence relating $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ determines an exact sequence of *profinite groups*. In particular, one verifies easily that if, moreover, J is *profinite* and $\rho\colon J \to \mathrm{Out}(G)$ is *continuous*, then the above exact sequence involving $G \overset{\mathrm{out}}{\rtimes} J$ determines an exact sequence of *profinite groups*.

Let G, J be *profinite* groups. Suppose that G is *center-free* and *topologically finitely generated*. Let $\rho\colon J \to \mathrm{Out}(G)$ be a *continuous* homomorphism. Write $\mathrm{Aut}_J(G \overset{\mathrm{out}}{\rtimes} J)$ for the group of [continuous] automorphisms of $G \overset{\mathrm{out}}{\rtimes} J$ that preserve and induce the identity automorphism on the quotient J. Then one verifies easily that the operation of restricting to G determines an *isomorphism* of profinite groups

$$\mathrm{Aut}_J(G \overset{\mathrm{out}}{\rtimes} J)/\mathrm{Inn}(G) \overset{\sim}{\longrightarrow} Z_{\mathrm{Out}(G)}(\mathrm{Im}(\rho))\,.$$

Let G and H be topological groups. Then we shall refer to a homomorphism of topological groups $\phi\colon G \to H$ as a *split injection* (respectively, *split surjection*) if there exists a homomorphism of topological groups $\psi\colon H \to G$ such that $\psi \circ \phi$ (respectively, $\phi \circ \psi$) is the identity automorphism of G (respectively, H).

Log schemes: When a *scheme* appears in a diagram of log schemes, the scheme is to be understood as the log scheme obtained by equipping the scheme with the *trivial log structure*. If $X^{\log}$ is a log scheme, then we shall refer to the largest open subscheme of the underlying scheme of $X^{\log}$ over which the log structure is trivial as the *interior* of $X^{\log}$. Fiber products of fs log schemes are to be understood as fiber products taken in the category of fs log schemes.

Curves: We shall use the terms "*hyperbolic curve*", "*cusp*", "*stable log curve*", "*smooth log curve*", and "*tripod*" as they are defined in [CmbGC], §0; [Hsh], §0. If (g, r) is a pair of nonnegative integers such that $2g - 2 + r > 0$, then we shall denote by $\overline{\mathcal{M}}_{g,r}$ the *moduli stack of* r*-pointed stable curves of genus* g over $\mathbb{Z}$ whose r marked points are *equipped with an ordering*, by $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ *parametrizing smooth curves*, by $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus$

$\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$, by $\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ the *tautological curve* over $\overline{\mathcal{M}}_{g,r}$, and by $\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$ the corresponding *tautological divisor of marked points* of $\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$. Then the divisor given by the union of $\overline{\mathcal{D}}_{g,r}$ with the inverse image in $\overline{\mathcal{C}}_{g,r}$ of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ determines a log structure on $\overline{\mathcal{C}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{C}}^{\log}_{g,r}$. Thus, we obtain a (1-)morphism of log stacks $\overline{\mathcal{C}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,r}$. We shall denote by $\mathcal{C}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$ the interior of $\overline{\mathcal{C}}^{\log}_{g,r}$. Thus, we obtain a (1-)morphism of stacks $\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}$. Let S be a scheme. Then we shall write $(\overline{\mathcal{M}}_{g,r})_S \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$, $(\mathcal{M}_{g,r})_S \stackrel{\mathrm{def}}{=} \mathcal{M}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$, $(\overline{\mathcal{M}}^{\log}_{g,r})_S \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}^{\log}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$, $(\overline{\mathcal{C}}_{g,r})_S \stackrel{\mathrm{def}}{=} \overline{\mathcal{C}}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$, $(\mathcal{C}_{g,r})_S \stackrel{\mathrm{def}}{=} \mathcal{C}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$, and $(\overline{\mathcal{C}}^{\log}_{g,r})_S \stackrel{\mathrm{def}}{=} \overline{\mathcal{C}}^{\log}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} S$.

Let n be a positive integer and $X^{\log}$ a stable log curve of type (g, r) over a log scheme $S^{\log}$. Then we shall refer to the log scheme obtained by pulling back the (1-)morphism $\overline{\mathcal{M}}^{\log}_{g,r+n} \to \overline{\mathcal{M}}^{\log}_{g,r}$ given by forgetting the last n points via the classifying (1-)morphism $S^{\log} \to \overline{\mathcal{M}}^{\log}_{g,r}$ of $X^{\log}$ as the *n-th log configuration space* of $X^{\log}$.

§1. F-admissibility and FC-admissibility

In the present §, we consider the *FC-admissibility* [cf. [CmbCsp], Definition 1.1, (ii)] of F-admissible automorphisms [cf. [CmbCsp], Definition 1.1, (ii)] of configuration space groups [cf. [MT], Definition 2.3, (i)]. Roughly speaking, we prove that if an F-admissible automorphism of a configuration space group arises from an F-admissible automorphism of a configuration space group [arising from a configuration space] of *strictly higher dimension*, then it is necessarily *FC-admissible*, i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups [cf. Theorem 1.8, Corollary 1.9 below].

Lemma 1.1 (Representations arising from certain families of hyperbolic curves). *Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; l a prime number; k an algebraically closed field of characteristic $\neq l$; B and C hyperbolic curves over k of type (g, r); n a positive integer. Suppose that $(r, n) \neq (0, 1)$. For $i = 1, \cdots, n$, let $f_i \colon B \xrightarrow{\sim} C$ be an isomorphism over k; s_i the section of $B \times_k C \xrightarrow{\mathrm{pr}_1} B$ determined by the isomorphism f_i. Suppose that, for any $i \neq j$,*

$\mathrm{Im}(s_i) \cap \mathrm{Im}(s_j) = \emptyset$. *Write*

$$Z \stackrel{\mathrm{def}}{=} B \times_k C \ \setminus \bigcup_{i=1,\cdots,n} \mathrm{Im}(s_i) \ \subseteq \ B \times_k C$$

for the complement of the images of the s_i*'s, where* i *ranges over the integers such that* $1 \leq i \leq n$*;* pr *for the composite* $Z \hookrightarrow B \times_k C \stackrel{\mathrm{pr}_1}{\to} B$ *[thus,* $\mathrm{pr}\colon Z \to B$ *is a* **family of hyperbolic curves** *of type* $(g, r+n)$*];* Π_B *(respectively,* Π_C*;* Π_Z*) the maximal pro-*l *quotient of the étale fundamental group* $\pi_1(B)$ *(respectively,* $\pi_1(C)$*;* $\pi_1(Z)$*) of* B *(respectively,* C*;* Z*);* $\underline{\mathrm{pr}}\colon \Pi_Z \twoheadrightarrow \Pi_B$ *for the surjection induced by* pr*;* $\Pi_{Z/B}$ *for the kernel of* $\underline{\mathrm{pr}}$*;* $\rho_{Z/B}\colon \Pi_B \to \mathrm{Out}(\Pi_{Z/B})$ *for the outer representation of* Π_B *on* $\Pi_{Z/B}$ *determined by the exact sequence*

$$1 \longrightarrow \Pi_{Z/B} \longrightarrow \Pi_Z \stackrel{\underline{\mathrm{pr}}}{\longrightarrow} \Pi_B \longrightarrow 1\,.$$

Let $\overline{b}$ *be a geometric point of* B *and* $Z_{\overline{b}}$ *the geometric fiber of* $\mathrm{pr}\colon Z \to B$ *at* $\overline{b}$*. For* $i = 1, \cdots, n$*, fix an inertia subgroup [among its various conjugates] of the étale fundamental group* $\pi_1(Z_{\overline{b}})$ *of* $Z_{\overline{b}}$ *associated to the cusp of* $Z_{\overline{b}}$ *determined by the section* s_i *and denote by*

$$I_{s_i} \subseteq \Pi_{Z/B}$$

the image in $\Pi_{Z/B}$ *of this inertia subgroup of* $\pi_1(Z_{\overline{b}})$*. Then the following hold:*

(i) **(Fundamental groups of fibers)** *The quotient* $\Pi_{Z/B}$ *of the étale fundamental group* $\pi_1(Z_{\overline{b}})$ *of the geometric fiber* $Z_{\overline{b}}$ **coincides with the maximal pro-*l* quotient** *of* $\pi_1(Z_{\overline{b}})$*.*

(ii) **(Abelianizations of the fundamental groups of fibers)** *For* $i = 1, \cdots, n$*, write* $J_{s_i} \subseteq \Pi^{\mathrm{ab}}_{Z/B}$ *for the image of* $I_{s_i} \subseteq \Pi_{Z/B}$ *in* $\Pi^{\mathrm{ab}}_{Z/B}$*. Then the composite* $I_{s_i} \hookrightarrow \Pi_{Z/B} \twoheadrightarrow \Pi^{\mathrm{ab}}_{Z/B}$ *determines an* **isomorphism** $I_{s_i} \stackrel{\sim}{\to} J_{s_i}$*; moreover, the inclusions* $J_{s_i} \hookrightarrow \Pi^{\mathrm{ab}}_{Z/B}$ *determine an* **exact sequence**

$$1 \longrightarrow (\bigoplus_{i=1}^{n} J_{s_i})/J_r \longrightarrow \Pi^{\mathrm{ab}}_{Z/B} \longrightarrow \Pi^{\mathrm{ab}}_C \longrightarrow 1$$

— *where*

$$J_r \subseteq \bigoplus_{i=1}^{n} J_{s_i}$$

is a $\mathbb{Z}_l$*-submodule such that*

$$J_r \simeq \begin{cases} \mathbb{Z}_l & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases}$$

and, moreover, if $r = 0$ *and* $i = 1, \cdots, n$*, then the composite*

$$J_r \hookrightarrow \bigoplus_{i=1}^{n} J_{s_i} \overset{\mathrm{pr}_{s_i}}{\twoheadrightarrow} J_{s_i}$$

is an **isomorphism***.*

(iii) **(Unipotency of a certain natural representation)** *The action of* Π_B *on* $\Pi_{Z/B}^{\mathrm{ab}}$ *determined by* $\rho_{Z/B}$ **preserves** *the exact sequence*

$$1 \longrightarrow (\bigoplus_{i=1}^{n} J_{s_i})/J_r \longrightarrow \Pi_{Z/B}^{\mathrm{ab}} \longrightarrow \Pi_C^{\mathrm{ab}} \longrightarrow 1$$

[cf. (ii)] and **induces the identity automorphisms** *on the subquotients* $(\bigoplus_{i=1}^{n} J_{s_i})/J_r$ *and* Π_C^{ab}*; in particular, the natural homomorphism* $\Pi_B \to \mathrm{Aut}_{\mathbb{Z}_l}(\Pi_{Z/B}^{\mathrm{ab}})$ *factors through a* **uniquely determined** *homomorphism*

$$\Pi_B \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C^{\mathrm{ab}}, (\bigoplus_{i=1}^{n} J_{s_i})/J_r\Big).$$

Proof. Assertion (i) follows immediately from the [easily verified] fact that the natural action of $\pi_1(B)$ on $\pi_1(Z_{\overline{b}})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l$ is *unipotent* — cf., e.g., [Hsh], Proposition 1.4, (i), for more details. [Note that although [Hsh], Proposition 1.4, (i), is only stated in the case where the hyperbolic curves corresponding to B and C are *proper*, the same proof may be applied to the case where these hyperbolic curves are *affine*.] Assertion (ii) follows immediately, in light of our assumption that $(r, n) \neq (0, 1)$, from assertion (i), together with the well-known structure of the maximal pro-l quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\neq l$. Finally, we verify assertion (iii). The fact that the action of Π_B on $\Pi_{Z/B}$ *preserves* the exact sequence appearing in the statement of assertion (iii) follows immediately from the fact that the surjection $\Pi_{Z/B}^{\mathrm{ab}} \twoheadrightarrow \Pi_C^{\mathrm{ab}}$ is *induced by* the open immersion $Z \hookrightarrow B \times_k C$ over B. The fact that the action in question *induces the identity automorphism* on $(\bigoplus_{i=1}^{n} J_{s_i})/J_r$

(respectively, Π_C^{ab}) follows immediately from the fact that the f_i's are *isomorphisms* (respectively, the fact that the surjection $\Pi_{Z/B}^{\mathrm{ab}} \twoheadrightarrow \Pi_C^{\mathrm{ab}}$ is *induced by* the open immersion $Z \hookrightarrow B \times_k C$ over B). Q.E.D.

Lemma 1.2 (**Maximal cuspidally central quotients of certain fundamental groups**). *In the notation of Lemma 1.1, for $i = 1, \cdots, n$, write*

$$\Pi_{Z/B} \twoheadrightarrow \Pi_{(Z/B)[i]} \ (\twoheadrightarrow \Pi_C)$$

for the quotient of $\Pi_{Z/B}$ by the normal closed subgroup topologically normally generated by the I_{s_j}'s, where j ranges over the integers such that $1 \leq j \leq n$ and $j \neq i$;

$$\Pi_{(Z/B)[i]} \twoheadrightarrow \mathbb{E}_{(Z/B)[i]}$$

for the **maximal cuspidally central quotient** *[cf.* [AbsCsp]*, Definition 1.1, (i)] relative to the surjection $\Pi_{(Z/B)[i]} \twoheadrightarrow \Pi_C$ determined by the natural open immersion $Z \hookrightarrow B \times_k C$;*

$$I_{s_i}^{\mathbb{E}} \subseteq \mathbb{E}_{(Z/B)[i]}$$

for the kernel of the natural surjection $\mathbb{E}_{(Z/B)[i]} \twoheadrightarrow \Pi_C$; and

$$\mathbb{E}_{Z/B} \stackrel{\mathrm{def}}{=} \mathbb{E}_{(Z/B)[1]} \times_{\Pi_C} \cdots \times_{\Pi_C} \mathbb{E}_{(Z/B)[n]}.$$

Then the following hold:

(i) **(Cuspidal inertia subgroups)** *Let $1 \leq i, j \leq n$ be integers. Then the homomorphism $I_{s_i} \to I_{s_j}^{\mathbb{E}}$ determined by the composite $I_{s_i} \hookrightarrow \Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{(Z/B)[j]}$ is an* **isomorphism** *(respectively,* **trivial***) if $i = j$ (respectively, $i \neq j$).*

(ii) **(Surjectivity)** *The homomorphism $\Pi_{Z/B} \to \mathbb{E}_{Z/B}$ determined by the natural surjections $\Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{(Z/B)[i]}$ — where i ranges over the integers such that $1 \leq i \leq n$ — is* **surjective***.*

(iii) **(Maximal cuspidally central quotients and abelianizations)** *The quotient $\Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{Z/B}$ of $\Pi_{Z/B}$ [cf. (ii)] coincides with the* **maximal cuspidally central quotient** *[cf.* [AbsCsp]*, Definition 1.1, (i)] relative to the surjection $\Pi_{Z/B} \twoheadrightarrow \Pi_C$ determined by the natural open immersion $Z \hookrightarrow B \times_k C$. In particular, the natural surjection $\Pi_{Z/B} \twoheadrightarrow \Pi_{Z/B}^{\mathrm{ab}}$* **factors**

through *the surjection* $\Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{Z/B}$*, and the resulting surjection* $\mathbb{E}_{Z/B} \twoheadrightarrow \Pi^{\mathrm{ab}}_{Z/B}$ *fits into a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bigoplus_{i=1}^{n} I^{\mathbb{E}}_{s_i} & \longrightarrow & \mathbb{E}_{Z/B} & \longrightarrow & \Pi_C & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\bigoplus_{i=1}^{n} J_{s_i})/J_r & \longrightarrow & \Pi^{\mathrm{ab}}_{Z/B} & \longrightarrow & \Pi^{\mathrm{ab}}_C & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are **exact***, and the vertical arrows are* **surjective***. Moreover, the left-hand vertical arrow coincides with the surjection induced by the natural isomorphisms* $I_{s_i} \overset{\sim}{\to} J_{s_i}$ *[cf. Lemma 1.1, (ii)] and* $I_{s_i} \overset{\sim}{\to} I^{\mathbb{E}}_{s_i}$ *[cf. (i)]. Finally, if* $r \neq 0$*, then the right-hand square is* **cartesian***.*

Proof. Assertion (i) follows immediately from the definition of the quotient $\mathbb{E}_{(Z/B)[j]}$ of $\Pi_{Z/B}$, together with the well-known structure of the maximal pro-l quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\neq l$ [cf., e.g., [MT], Lemma 4.2, (iv), (v)]. Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from assertions (i), (ii) [cf. [AbsCsp], Proposition 1.6, (iii)]. Q.E.D.

Lemma 1.3 (**The kernels of representations arising from certain families of hyperbolic curves).** *In the notation of Lemmas 1.1, 1.2, suppose that* $r \neq 0$*. Then the following hold:*

(i) **(Unipotency of a certain natural outer representation)** *Consider the action of* Π_B *on* $\mathbb{E}_{Z/B}$ *determined by the natural isomorphism*

$$\mathbb{E}_{Z/B} \overset{\sim}{\longrightarrow} \Pi^{\mathrm{ab}}_{Z/B} \times_{\Pi^{\mathrm{ab}}_C} \Pi_C$$

[cf. Lemma 1.2, (iii)], together with the natural action of Π_B *on* $\Pi^{\mathrm{ab}}_{Z/B}$ *induced by* $\rho_{Z/B}$ *and the trivial action of* Π_B *on* Π_C*. Then the* **outer** *action of* Π_B *on* $\mathbb{E}_{Z/B}$ *induced by this action* **coincides** *with the natural* **outer** *action of* Π_B *on* $\mathbb{E}_{Z/B}$ *induced by* $\rho_{Z/B}$*. In particular, relative to the natural identification* $I_{s_i} \overset{\sim}{\to} I^{\mathbb{E}}_{s_i}$ *[cf. Lemma 1.2, (i)], the above action of* Π_B *on* $\mathbb{E}_{Z/B}$ **factors through** *the homomorphism*

$$\Pi_B \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C, \bigoplus_{i=1}^{n} I_{s_i}\Big) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi^{\mathrm{ab}}_C, \bigoplus_{i=1}^{n} I_{s_i}\Big)$$

obtained in Lemma 1.1, (iii).

(ii) **(Homomorphisms arising from a certain extension)** *For* $i = 1, \cdots, n$*, write* ϕ_i *for the composite*

$$\Pi_B \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C^{\mathrm{ab}}, \bigoplus_{j=1}^{n} I_{s_j}\Big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C^{\mathrm{ab}},\ I_{s_i}\Big)$$

— where the first arrow is the homomorphism of (i), and the second arrow is the homomorphism determined by the projection $\mathrm{pr}_i\colon \bigoplus_{j=1}^{n} I_{s_j} \twoheadrightarrow I_{s_i}$*. Then the homomorphism* ϕ_i **coincides** *with the image of the element of* $H^2(\Pi_B \times \Pi_C, I_{s_i})$ *determined by the extension*

$$1 \longrightarrow I_{s_i} \longrightarrow \Pi^{\mathbb{E}}_{Z[i]} \longrightarrow \Pi_B \times \Pi_C \longrightarrow 1$$

— where we write $\Pi^{\mathbb{E}}_{Z[i]} \overset{\mathrm{def}}{=} \Pi_Z/\mathrm{Ker}(\Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{Z/B[i]})$ *— of* $\Pi_B \times \Pi_C$ *by* $I_{s_i} \overset{\sim}{\to} I^{\mathbb{E}}_{s_i}$ *[cf. Lemma 1.2, (i)] via the composite*

$$H^2(\Pi_B \times \Pi_C, I_{s_i}) \overset{\sim}{\to} H^1(\Pi_B, H^1(\Pi_C, I_{s_i})) \overset{\sim}{\to} \mathrm{Hom}\Big(\Pi_B, \mathrm{Hom}(\Pi_C, I_{s_i})\Big)$$

— where the first arrow is the isomorphism determined by the Hochschild-Serre spectral sequence relative to the surjection $\Pi_B \times \Pi_C \overset{\mathrm{pr}_1}{\twoheadrightarrow} \Pi_B$*.*

(iii) **(Factorization)** *Write* $\overline{B}$ *(respectively,* $\overline{C}$*) for the compactification of* C *(respectively,* B*) and* $\Pi_{\overline{B}}$ *(respectively,* $\Pi_{\overline{C}}$*) for the maximal pro-l quotient of the étale fundamental group* $\pi_1(\overline{B})$ *(respectively,* $\pi_1(\overline{C})$*) of* $\overline{B}$ *(respectively,* $\overline{C}$*). Then the homomorphism* ϕ_i *of (ii)* **factors** *as the composite*

$$\Pi_B \twoheadrightarrow \Pi_{\overline{B}}^{\mathrm{ab}} \overset{\sim}{\to} \Pi_{\overline{C}}^{\mathrm{ab}} \overset{\sim}{\to} \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_{\overline{C}}^{\mathrm{ab}}, I_{s_i}\Big) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C^{\mathrm{ab}}, I_{s_i}\Big)$$

— where the first (respectively, second; fourth) arrow is the homomorphism induced by $B \hookrightarrow \overline{B}$ *(respectively,* $f_i\colon B \overset{\sim}{\to} C$*;* $C \hookrightarrow \overline{C}$*), and the third arrow is the isomorphism determined by the Poincaré duality isomorphism in étale cohomology, relative to the natural isomorphism* $I_{s_i} \overset{\sim}{\to} \mathbb{Z}_l(1)$*. [Here, the "(1)" denotes a "Tate twist".]*

(iv) **(Kernel of a certain natural representation)** *The kernel of the homomorphism* $\Pi_B \to \mathrm{Aut}_{\mathbb{Z}_l}(\Pi_{Z/B}^{\mathrm{ab}})$ *determined by* $\rho_{Z/B}$ **coincides** *with the kernel of the natural surjection* $\Pi_B \twoheadrightarrow \Pi_{\overline{B}}^{\mathrm{ab}}$*.*

Proof. Assertions (i), (ii) follow immediately from the various definitions involved. Next, we verify assertion (iii). It follows from assertion (ii), together with [MT], Lemma 4.2, (ii), (v) [cf. also the discussion surrounding [MT], Lemma 4.2], that, relative to the natural isomorphism $I_{s_i} \xrightarrow{\sim} \mathbb{Z}_l(1)$, the image of $\phi_i \in \mathrm{Hom}(\Pi_B, \mathrm{Hom}_{\mathbb{Z}_l}(\Pi_C^{\mathrm{ab}}, I_{s_i}))$ via the isomorphisms

$$\mathrm{Hom}(\Pi_B, \mathrm{Hom}_{\mathbb{Z}_l}(\Pi_C^{\mathrm{ab}}, I_{s_i})) \xrightarrow{\sim} \mathrm{Hom}(\Pi_B, \mathrm{Hom}_{\mathbb{Z}_l}(\Pi_C^{\mathrm{ab}}, \mathbb{Z}_l(1)))$$
$$\xleftarrow{\sim} H^2(\Pi_B \times \Pi_C, \mathbb{Z}_l(1)) \xrightarrow{\sim} H^2(B \times_k C, \mathbb{Z}_l(1))$$

— where the first (respectively, second) isomorphism is the isomorphism induced by the above isomorphism $I_{s_i} \xrightarrow{\sim} \mathbb{Z}_l(1)$ (respectively, the Hochschild-Serre spectral sequence relative to the surjection $\Pi_B \times \Pi_C \overset{\mathrm{pr}_1}{\twoheadrightarrow} \Pi_B$) — is the *first Chern class* of the invertible sheaf associated to the divisor determined by the scheme-theoretic image of $s_i \colon B_i \hookrightarrow B \times_k C$. Thus, since the section s_i *extends uniquely* to a section $\overline{s}_i \colon \overline{B} \hookrightarrow \overline{B} \times_k \overline{C}$, whose scheme-theoretic image we denote by $\mathrm{Im}(\overline{s}_i)$, it follows that the homomorphism $\phi_i \in \mathrm{Hom}(\Pi_B, \mathrm{Hom}_{\mathbb{Z}_l}(\Pi_C^{\mathrm{ab}}, I_{s_i}))$ *coincides* with the image of the *first Chern class* of the invertible sheaf on $\overline{B} \times_k \overline{C}$ associated to the divisor $\mathrm{Im}(\overline{s}_i)$ via the composite

$$H^2(\overline{B} \times_k \overline{C}, \mathbb{Z}_l(1)) \xleftarrow{\sim} H^2(\overline{B} \times_k \overline{C}, I_{s_i}) \to H^2(B \times_k C, I_{s_i})$$
$$\xleftarrow{\sim} H^2(\Pi_B \times \Pi_C, I_{s_i}) \xrightarrow{\sim} \mathrm{Hom}\Big(\Pi_B, \mathrm{Hom}_{\mathbb{Z}_l}(\Pi_C, I_{s_i})\Big)$$

— where the first arrow is the isomorphism induced by the above isomorphism $I_{s_i} \xrightarrow{\sim} \mathbb{Z}_l(1)$, and the second arrow is the homomorphism induced by the natural open immersion $B \times_k C \hookrightarrow \overline{B} \times_k \overline{C}$. In particular, assertion (iii) follows immediately from [Mln], Chapter VI, Lemma 12.2 [cf. also the argument used in the proof of [MT], Lemma 4.4]. Finally, we verify assertion (iv). To this end, we recall that by Lemma 1.1, (iii), the homomorphism $\Pi_B \to \mathrm{Aut}_{\mathbb{Z}_l}(\Pi_{Z/B}^{\mathrm{ab}})$ *factors through* the homomorphism $\Pi_B \to \mathrm{Hom}_{\mathbb{Z}_l}\Big(\Pi_C^{\mathrm{ab}}, \bigoplus_{i=1}^n J_{s_i}\Big)$ of assertion (i). Thus, assertion (iv) follows immediately from assertion (iii). This completes the proof of assertion (iv). Q.E.D.

Definition 1.4. For $\square \in \{\circ, \bullet\}$, let $\Sigma^\square$ be a set of prime numbers which is either *of cardinality one* or *equal to the set of all prime numbers*; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$; $X^\square$ a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of characteristic $\notin \Sigma^\square$; $d^\square$ a positive integer; $X^\square_{d^\square}$ the $d^\square$-th configuration space of $X^\square$ [cf. [MT], Definition 2.1, (i)]; $\Pi^\square_{d^\square}$ the pro-$\Sigma^\square$ configuration space group [cf. [MT], Definition 2.3, (i)] obtained by forming the maximal pro-$\Sigma^\square$ quotient of the étale fundamental group $\pi_1(X^\square_{d^\square})$ of $X^\square_{d^\square}$.

(i) We shall say that an isomorphism of profinite groups $\alpha\colon \Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PF-admissible* [i.e., "permutation-fiber-admissible"] if α induces a bijection between the set of fiber subgroups [cf. [MT], Definition 2.3, (iii)] of $\Pi^\circ_{d^\circ}$ and the set of fiber subgroups of $\Pi^\bullet_{d^\bullet}$. We shall say that an outer isomorphism $\Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PF-admissible* if it is determined by a PF-admissible isomorphism.

(ii) We shall say that an isomorphism of profinite groups $\alpha\colon \Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PC-admissible* [i.e., "permutation-cusp-admissible"] if the following condition is satisfied: Let

$$\{1\} = K_{d^\circ} \subseteq K_{d^\circ-1} \subseteq \cdots \subseteq K_m \subseteq \cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 = \Pi^\circ_{d^\circ}$$

be the standard fiber filtration of $\Pi^\circ_{d^\circ}$ [cf. [CmbCsp], Definition 1.1, (i)]; then for any integer $1 \leq a \leq d^\circ$, the image $\alpha(K_a) \subseteq \Pi^\bullet_{d^\bullet}$ is a fiber subgroup of $\Pi^\bullet_{d^\bullet}$ of length $d^\circ - a$ [cf. [MT], Definition 2.3, (iii)], and, moreover, the isomorphism $K_{a-1}/K_a \xrightarrow{\sim} \alpha(K_{a-1})/\alpha(K_a)$ determined by α induces a bijection between the set of cuspidal inertia subgroups of K_{a-1}/K_a and the set of cuspidal inertia subgroups of $\alpha(K_{a-1})/\alpha(K_a)$. [Note that it follows immediately from the various definitions involved that the profinite group K_{a-1}/K_a (respectively, $\alpha(K_{a-1})/\alpha(K_a)$) is equipped with a natural structure of pro-Σ° (respectively, pro-$\Sigma^\bullet$) surface group [cf. [MT], Definition 1.2].] We shall say that an outer isomorphism $\Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PC-admissible* if it is determined by a PC-admissible isomorphism.

(iii) We shall say that an isomorphism of profinite groups $\alpha\colon \Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PFC-admissible* [i.e., "permutation-fiber-cusp-admissible"] if α is PF-admissible and PC-admissible. We shall say that an outer isomorphism $\Pi^\circ_{d^\circ} \xrightarrow{\sim} \Pi^\bullet_{d^\bullet}$ is *PFC-admissible* if it is determined by a PFC-admissible isomorphism.

(iv) We shall say that an isomorphism of profinite groups $\alpha\colon \Pi^{\circ}_{d^{\circ}} \overset{\sim}{\to} \Pi^{\bullet}_{d^{\bullet}}$ is *PF-cuspidalizable* if there exists a commutative diagram

$$\begin{array}{ccc} \Pi^{\circ}_{d^{\circ}+1} & \overset{\sim}{\longrightarrow} & \Pi^{\bullet}_{d^{\bullet}+1} \\ \downarrow & & \downarrow \\ \Pi^{\circ}_{d^{\circ}} & \underset{\alpha}{\overset{\sim}{\longrightarrow}} & \Pi^{\bullet}_{d^{\bullet}} \end{array}$$

— where the upper horizontal arrow is a PF-admissible isomorphism, and the left-hand (respectively, right-hand) vertical arrow is the surjection obtained by forming the quotient by a fiber subgroup of length 1 [cf. [MT], Definition 2.3, (iii)] of $\Pi^{\circ}_{d^{\circ}+1}$ (respectively, $\Pi^{\bullet}_{d^{\bullet}+1}$). We shall say that an outer isomorphism $\Pi^{\circ}_{d^{\circ}} \overset{\sim}{\to} \Pi^{\bullet}_{d^{\bullet}}$ is *PF-cuspidalizable* if it is determined by a PF-cuspidalizable isomorphism.

Remark 1.4.1. It follows immediately from the various definitions involved that, in the notation of Definition 1.4, an automorphism α of $\Pi^{\circ}_{d^{\circ}}$ is *PF-admissible* (respectively, *PC-admissible*; *PFC-admissible*) if and only if there exists an automorphism σ of $\Pi^{\circ}_{d^{\circ}}$ that lifts the outomorphism [cf. the discussion entitled "*Topological groups*" in §0] of $\Pi^{\circ}_{d^{\circ}}$ naturally determined by a permutation of the d° factors of the configuration space involved such that the composite $\alpha \circ \sigma$ is *F-admissible* (respectively, *C-admissible*; *FC-admissible*) [cf. [CmbCsp], Definition 1.1, (ii)]. In particular, a(n) *F-admissible* (respectively, *C-admissible*; *FC-admissible*) automorphism of $\Pi^{\circ}_{d^{\circ}}$ is *PF-admissible* (respectively, *PC-admissible*; *PFC-admissible*):

$$\begin{array}{ccccc} \text{F-admissible} & \Longleftarrow & \text{FC-admissible} & \Longrightarrow & \text{C-admissible} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{PF-admissible} & \Longleftarrow & \text{PFC-admissible} & \Longrightarrow & \text{PC-admissible}\,. \end{array}$$

Proposition 1.5 (Properties of PF-admissible isomorphisms). *In the notation of Definition 1.4, let* $\alpha\colon \Pi^{\circ}_{d^{\circ}} \overset{\sim}{\to} \Pi^{\bullet}_{d^{\bullet}}$ *be an isomorphism. Then the following hold:*

(i) $\Sigma^{\circ} = \Sigma^{\bullet}$.

(ii) *Suppose that the isomorphism* α *is* **PF-admissible**. *Let* $1 \leq n \leq d^\circ$ *be an integer and* $H \subseteq \Pi^\circ_{d^\circ}$ *a fiber subgroup of* **length** $\boldsymbol{n}$ *of* $\Pi^\circ_{d^\circ}$. *Then the subgroup* $\alpha(H) \subseteq \Pi^\bullet_{d^\bullet}$ *is a fiber subgroup of* **length** $\boldsymbol{n}$ *of* $\Pi^\bullet_{d^\bullet}$. *In particular, it holds that* $d^\circ = d^\bullet$.

(iii) *Write* $\Xi^\circ \subseteq \Pi^\circ_{d^\circ}$ *(respectively,* $\Xi^\bullet \subseteq \Pi^\bullet_{d^\bullet}$*) for the normal closed subgroup of* $\Pi^\circ_{d^\circ}$ *(respectively,* $\Pi^\bullet_{d^\bullet}$*) obtained by taking the intersection of the various fiber subgroups of length* $d^\circ - 1$ *(respectively,* $d^\bullet - 1$*). Then the isomorphism* α *is* **PF-admissible** *if and only if* α *induces an isomorphism* $\Xi^\circ \xrightarrow{\sim} \Xi^\bullet$.

Proof. Assertion (i) follows immediately from the [easily verified] fact that $\Sigma^\Box$ may be characterized as the smallest set of primes Σ^* for which $\Pi^\Box_{d^\Box}$ is pro-Σ^*. Assertion (ii) follows immediately from the various definitions involved. Finally, we verify assertion (iii). The *necessity* of the condition follows immediately from assertion (ii). The *sufficiency* of the condition follows immediately from a similar argument to the argument used in the proof of [CmbCsp], Proposition 1.2, (i). This completes the proof of assertion (iii). Q.E.D.

Lemma 1.6 (C-admissibility of certain isomorphisms). *In the notation of Definition 1.4, let* $\alpha_2 \colon \Pi^\circ_2 \xrightarrow{\sim} \Pi^\bullet_2$, $\alpha^1_1 \colon \Pi^\circ_1 \xrightarrow{\sim} \Pi^\bullet_1$, $\alpha^2_1 \colon \Pi^\circ_1 \xrightarrow{\sim} \Pi^\bullet_1$ *be isomorphisms of profinite groups which, for* $i = 1, 2$, *fit into a commutative diagram*

$$\begin{array}{ccc} \Pi^\circ_2 & \xrightarrow{\alpha_2} & \Pi^\bullet_2 \\ \mathrm{pr}^\circ_{\{i\}} \big\downarrow & & \big\downarrow \mathrm{pr}^\bullet_{\{i\}} \\ \Pi^\circ_1 & \xrightarrow{\alpha^i_1} & \Pi^\bullet_1 \end{array}$$

— where the vertical arrow "$\mathrm{pr}^\Box_{\{i\}}$*" is the surjection induced by the projection "*$X^\Box_2 \to X^\Box_1$*" obtained by projecting to the* i*-th factor. Then the isomorphism* α^1_1 *is* **C-admissible**. *In particular,* $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.

Proof. Write $\Sigma \stackrel{\text{def}}{=} \Sigma^\circ = \Sigma^\bullet$ [cf. Proposition 1.5, (i)]. Now it follows from the well-known structure of the maximal pro-Σ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\notin \Sigma$ that $\Pi^\Box_1$ is a *free pro-*Σ group if and only if $r^\Box \neq 0$ [cf. [CmbGC], Remark 1.1.3]. Thus, if $r^\circ = r^\bullet = 0$, then it is immediate that α^1_1 is *C-admissible*; moreover, it follows, by considering the rank of the abelianization of $\Pi^\Box_1$ [cf. [CmbGC], Remark 1.1.3], that $g^\circ = g^\bullet$. In particular, to verify Lemma 1.6, we may assume without loss of generality that r°, $r^\bullet \neq 0$. Then it follows from [CmbGC], Theorem 1.6, (i), that, to verify Lemma 1.6, it suffices to show that α_1^1 is *numerically cuspidal* [cf. [CmbGC], Definition 1.4, (ii)], i.e., to show that the following *assertion* holds:

> Let $\Pi_Y^\circ \subseteq \Pi_1^\circ$ be an open subgroup of Π_1°. Write $\Pi_Y^\bullet \stackrel{\mathrm{def}}{=} \alpha_1^1(\Pi_Y^\circ) \subseteq \Pi_1^\bullet$, $Y^\circ \to X^\circ$ (respectively, $Y^\bullet \to X^\bullet$) for the connected finite étale covering of X° (respectively, $X^\bullet$) corresponding to the open subgroup $\Pi_Y^\circ \subseteq \Pi_1^\circ$ (respectively, $\Pi_Y^\bullet \subseteq \Pi_1^\bullet$), and (g_Y°, r_Y°) (respectively, $(g_Y^\bullet, r_Y^\bullet)$) for the type of Y° (respectively, $Y^\bullet$). Then it holds that $r_Y^\circ = r_Y^\bullet$.

On the other hand, in the notation of the above *assertion*, one verifies easily that for any $l \in \Sigma$ and $\Box \in \{\circ, \bullet\}$, if $\Pi_{Y'}^\Box \subseteq \Pi_1^\Box$ is an open subgroup of $\Pi_1^\Box$ contained in $\Pi_Y^\Box$, then the natural inclusion $\Pi_{Y'}^\Box \hookrightarrow \Pi_Y^\Box$ induces a *surjection*

$$\mathrm{Ker}((\Pi_{Y'}^\Box)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y'}}^\Box)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l \twoheadrightarrow \mathrm{Ker}((\Pi_Y^\Box)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y}}^\Box)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l$$

— where we write $(\Pi_{\overline{Y}}^\Box)$, $(\Pi_{\overline{Y'}}^\Box)$ for the maximal pro-Σ quotients of the étale fundamental groups of the compactifications $\overline{Y}$, $\overline{Y}'$ of Y, Y', respectively. Thus, since any open subgroup of Π_1° contains a *characteristic* open subgroup of Π_1°, it follows immediately from the well-known fact that for $\Box \in \{\circ, \bullet\}$, $(\Pi_Y^\Box)^{\mathrm{ab}}$ (respectively, $(\Pi_{\overline{Y}}^\Box)^{\mathrm{ab}}$) is a *free* $\widehat{\mathbb{Z}}^\Sigma$-*module of rank* $2g_Y^\Box + r_Y^\Box - 1$ (respectively, $2g_Y^\Box$) [cf., e.g., [CmbGC], Remark 1.1.3] that to verify the above *assertion*, it suffices to verify that if $\Pi_Y^\circ \subseteq \Pi_1^\circ$ in the above *assertion* is *characteristic*, then the isomorphism $\Pi_Y^\circ \xrightarrow{\sim} \Pi_Y^\bullet$ determined by α_1^1 induces an isomorphism of $\mathrm{Ker}((\Pi_Y^\circ)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y}}^\circ)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l$ with $\mathrm{Ker}((\Pi_Y^\bullet)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y}}^\bullet)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l$ for some $l \in \Sigma$.

To this end, for $\Box \in \{\circ, \bullet\}$, write $\Pi_Z^\Box \subseteq \Pi_2^\Box$ for the normal open subgroup of $\Pi_2^\Box$ obtained by forming the inverse image via the surjection

$$(\mathrm{pr}_{\{1\}}^\Box, \mathrm{pr}_{\{2\}}^\Box) \colon \Pi_2^\Box \twoheadrightarrow \Pi_1^\Box \times \Pi_1^\Box$$

of the image of the natural inclusion $\Pi_Y^\Box \times \Pi_Y^\Box \hookrightarrow \Pi_1^\Box \times \Pi_1^\Box$; $Z^\Box \to X_2^\Box$ for the connected finite étale covering corresponding to this normal open subgroup $\Pi_Z^\Box \subseteq \Pi_2^\Box$; $\Pi_{Z/Y}^\Box$ for the kernel of the natural surjection $\Pi_Z^\Box \twoheadrightarrow \Pi_Y^\Box$ induced by the composite $Z^\Box \to X_2^\Box \hookrightarrow X^\Box \times_k X^\Box \stackrel{\mathrm{pr}_1}{\to} X^\Box$. Then the natural surjection $\Pi_Z^\Box \twoheadrightarrow \Pi_Y^\Box$ determines a representation

$$\Pi_Y^\Box \longrightarrow \mathrm{Aut}((\Pi_{Z/Y}^\Box)^{\mathrm{ab}});$$

moreover, the isomorphisms α_2, α_1^1, and α_1^2 determine a commutative diagram

$$\begin{array}{ccc} \Pi_Y^\circ & \longrightarrow & \mathrm{Aut}((\Pi_{Z/Y}^\circ)^{\mathrm{ab}}) \\ \downarrow & & \downarrow \\ \Pi_Y^\bullet & \longrightarrow & \mathrm{Aut}((\Pi_{Z/Y}^\bullet)^{\mathrm{ab}}) \end{array}$$

— where the vertical arrows are *isomorphisms*. [Here, we note that since $\Pi_Y^\bullet$ is a *characteristic* subgroup of $\Pi_1^\bullet$, and the composite $\alpha_1^2 \circ (\alpha_1^1)^{-1}$ is an *automorphism* of $\Pi_1^\bullet$, it follows that $\Pi_Y^\bullet = \alpha_1^2(\Pi_Y^\circ)$, hence that α_2 induces an isomorphism $\Pi_Z^\circ \xrightarrow{\sim} \Pi_Z^\bullet$.] On the other hand, it follows from the definition of $Z^\square$ that $Z^\square$ is isomorphic to the open subscheme of $Y^\square \times_k Y^\square$ obtained by forming the complement of the graphs of the various elements of $\mathrm{Aut}(Y^\square/X^\square)$. Thus, it follows from Lemma 1.3, (iv) — by replacing the various profinite groups involved by their maximal pro-l quotients for some $l \in \Sigma$ — that the isomorphism $\Pi_Y^\circ \xrightarrow{\sim} \Pi_Y^\bullet$ determined by α_1^1 induces an isomorphism of $\mathrm{Ker}((\Pi_Y^\circ)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y}}^\circ)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l$ with $\mathrm{Ker}((\Pi_Y^\bullet)^{\mathrm{ab}} \twoheadrightarrow (\Pi_{\overline{Y}}^\bullet)^{\mathrm{ab}}) \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l$ for some $l \in \Sigma$. This completes the proof of Lemma 1.6. Q.E.D.

Lemma 1.7 (**PFC-admissibility of certain PF-admissible isomorphisms**)**.** *In the notation of Definition 1.4, let* $\alpha\colon \Pi_{d^\circ}^\circ \xrightarrow{\sim} \Pi_{d^\bullet}^\bullet$ *be a* **PF-admissible** *isomorphism. Then the following condition implies that the isomorphism* α *is* **PFC-admissible***:*

> *Let* $H^\circ \subseteq \Pi_{d^\circ}^\circ$ *be a* **fiber subgroup** *of length 1 [cf.* [MT]*, Definition 2.3, (iii)]. Write* $H^\bullet \stackrel{\mathrm{def}}{=} \alpha(H^\circ) \subseteq \Pi_{d^\bullet}^\bullet$ *for the* **fiber subgroup** *of length 1 obtained as the image of* H° *via* α *[cf. Proposition 1.5, (ii)]. [Thus, it follows immediately from the various definitions involved that* H° *(respectively,* $H^\bullet$*) is equipped with a natural structure of pro-*Σ° *(respectively, pro-*$\Sigma^\bullet$*)* **surface group***.] Then the isomorphism* $H^\circ \xrightarrow{\sim} H^\bullet$ *induced by* α *is* **C-admissible***.*

Proof. Let $\square \in \{\circ, \bullet\}$. Then one may verify easily that the following *fact* holds:

> Let $1 \leq a \leq d^\square$ be an integer and $F' \subseteq F \subseteq \Pi_{d^\square}^\square$ fiber subgroups of $\Pi_{d^\square}^\square$ such that F is of length a, and F' is of length $a-1$. Then there exists a *fiber*

subgroup $H \subseteq F \subseteq \Pi^{\Box}_{d^{\Box}}$ *of* $\Pi^{\Box}_{d^{\Box}}$ *of length* 1 such that the composite

$$H \hookrightarrow F \twoheadrightarrow F/F'$$

arises from a natural open immersion of a hyperbolic curve of type $(g^{\Box}, r^{\Box}+d^{\Box}-1)$ *into a hyperbolic curve of type* $(g^{\Box}, r^{\Box}+d^{\Box}-a)$. [Note that it follows immediately from the various definitions involved that H (respectively, F/F') is equipped with a natural structure of pro-$\Sigma^{\Box}$ surface group.] In particular, the composite is a *surjection* whose kernel is topologically normally generated by suitable cuspidal inertia subgroups of H; moreover, any cuspidal inertia subgroup of F/F' may be obtained as the image of a cuspidal inertia subgroup of H.

On the other hand, one may verify easily that Lemma 1.7 follows immediately from the above *fact*. This completes the proof of Lemma 1.7. Q.E.D.

Theorem 1.8 (PFC-admissibility of certain isomorphisms). *For* $\Box \in \{\circ, \bullet\}$, *let* $\Sigma^{\Box}$ *be a set of prime numbers which is either* **of cardinality one** *or* **equal to the set of all prime numbers**; $(g^{\Box}, r^{\Box})$ *a pair of nonnegative integers such that* $2g^{\Box}-2+r^{\Box} > 0$; $X^{\Box}$ *a hyperbolic curve of type* $(g^{\Box}, r^{\Box})$ *over an algebraically closed field of characteristic* $\notin \Sigma^{\Box}$; $d^{\Box}$ *a positive integer;* $\Pi^{\Box}_{d^{\Box}}$ *the pro-*$\Sigma^{\Box}$ *configuration space group [cf.* [MT], *Definition 2.3, (i)] obtained by forming the maximal pro-*$\Sigma^{\Box}$ *quotient of the étale fundamental group of the* $d^{\Box}$*-th configuration space of* $X^{\Box}$;

$$\alpha \colon \Pi^{\circ}_{d^{\circ}} \xrightarrow{\sim} \Pi^{\bullet}_{d^{\bullet}}$$

an **isomorphism of [abstract] groups**. *If*

$$\{(g^{\circ}, r^{\circ}), (g^{\bullet}, r^{\bullet})\} \cap \{(0,3), (1,1)\} \neq \emptyset ,$$

then we suppose further that the isomorphism α *is* **PF-admissible** *[cf. Definition 1.4, (i)]. Then the following hold:*

(i) $\Sigma^{\circ} = \Sigma^{\bullet}$.

(ii) *The isomorphism* α *is an* **isomorphism of profinite groups**.

(iii) *The isomorphism* α *is* **PF-admissible**. *In particular,* $d^{\circ} = d^{\bullet}$.

(iv) *If* α *is* **PF-cuspidalizable** *[cf. Definition 1.4, (iv)], then* α *is* **PFC-admissible** *[cf. Definition 1.4, (iii)]. In particular,* $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.

Proof. Assertion (ii) follows from [NS], Theorem 1.1. In light of assertion (ii), assertion (i) follows from Proposition 1.5, (i). Assertion (iii) follows from Proposition 1.5, (ii); [MT], Corollary 6.3, together with the assumption appearing in the statement of Theorem 1.8. Assertion (iv) follows immediately from Lemmas 1.6, 1.7. Q.E.D.

Corollary 1.9 (**F-admissibility and FC-admissibility**). *Let* Σ *be a set of prime numbers which is either of* **cardinality one** *or* **equal to the set of all prime numbers***;* n *a positive integer;* (g, r) *a pair of nonnegative integers such that* $2g - 2 + r > 0$*;* X *a hyperbolic curve of type* (g, r) *over an algebraically closed field* k *of characteristic* $\notin \Sigma$*;* X_n *the* n*-th configuration space of* X*;* Π_n *the maximal pro-*Σ *quotient of the fundamental group of* X_n*; "*$\mathrm{Out}^{\mathrm{FC}}(-)$*", "*$\mathrm{Out}^{\mathrm{F}}(-)$*"* $\subseteq$ *"*$\mathrm{Out}(-)$*" the subgroups of FC- and F-admissible [cf.* [CmbCsp]*, Definition 1.1, (ii)] outomorphisms [cf. the discussion entitled "Topological groups" in §0] of "*$(-)$*". Then the following hold:*

(i) *Let* $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})$*. Then* α *induces the* **same** *outomorphism of* Π_n *relative to the various quotients* $\Pi_{n+1} \twoheadrightarrow \Pi_n$ *by fiber subgroups of length* 1 *[cf.* [MT]*, Definition 2.3, (iii)]. In particular, we obtain a natural homomorphism*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n) .$$

(ii) *The image of the homomorphism*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

of (i) is **contained** *in*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n) .$$

Proof. First, we verify assertion (i). Let H^1, $H^2 \subseteq \Pi_{n+1}$ be two distinct fiber subgroups of Π_{n+1} of length 1. Observe that the normal closed subgroup $H \subseteq \Pi_{n+1}$ of Π_{n+1} topologically generated by H^1 and H^2 is a *fiber subgroup of* Π_{n+1} *of length* 2 [cf. [MT], Proposition 2.4, (iv)], hence is equipped with a natural structure of pro-Σ configuration space group, with respect to which $H^i \subseteq H$ may be regarded as a *fiber* *subgroup of length* 1 [cf. [MT], Proposition 2.4, (ii)]. Moreover, it follows immediately from the scheme-theoretic definition of the various configuration space groups involved that one has natural outer isomorphisms $\Pi_{n+1}/H^i \xrightarrow{\sim} \Pi_n$ and $H/H^1 \xrightarrow{\sim} H/H^2$. Thus, since for $i \in \{1,2\}$, we have natural outer isomorphisms

$$\Pi_n \xleftarrow{\sim} \Pi_{n+1}/H^i \xrightarrow{\sim} (H/H^i) \overset{\text{out}}{\rtimes} \Pi_{n+1}/H$$

[cf. the discussion entitled "*Topological groups*" in §0] which are compatible with the various natural outer isomorphisms discussed above, one verifies easily [cf. the argument given in the first paragraph of the proof of [CmbCsp], Theorem 4.1] that to complete the proof of assertion (i), by replacing Π_{n+1} by H, it suffices to verify assertion (i) in the case where $n = 1$. The rest of the proof of assertion (i) is devoted to verifying assertion (i) in the case where $n = 1$.

Let $\widetilde{\alpha} \in \mathrm{Aut}^{\mathrm{F}}(\Pi_2)$ be an *F-admissible* automorphism of Π_2; $\widetilde{\alpha}^1$, $\widetilde{\alpha}^2 \in \mathrm{Aut}(\Pi_1)$ the automorphisms of Π_1 induced by $\widetilde{\alpha}$ relative to the quotients $\Pi_2 \twoheadrightarrow \Pi_2/H^1 \xrightarrow{\sim} \Pi_1$, $\Pi_2 \twoheadrightarrow \Pi_2/H^2 \xrightarrow{\sim} \Pi_1$, respectively. Now it is immediate that to complete the proof of assertion (i), it suffices to verify that the difference $\widetilde{\alpha}^1 \circ (\widetilde{\alpha}^2)^{-1} \in \mathrm{Aut}(\Pi_1)$ is Π_1-*inner*. Therefore, it follows immediately from [JR], Theorem B, that to complete the proof of assertion (i), it suffices to verify that

$(*_1)$: for any normal open subgroup $N \subseteq \Pi_1$ of Π_1, it holds that $\widetilde{\alpha}^1(N) = \widetilde{\alpha}^2(N)$.

To this end, let $N \subseteq \Pi_1$ be a normal open subgroup of Π_1. Write $\Pi_N \overset{\text{def}}{=} \Pi_2 \times_{\Pi_1} N$ for the fiber product of $\Pi_2 \twoheadrightarrow \Pi_2/H^1 \xrightarrow{\sim} \Pi_1$ and $N \hookrightarrow \Pi_1$ and F_N for the kernel of the composite $\Pi_N = \Pi_2 \times_{\Pi_1} N \overset{\mathrm{pr}_1}{\hookrightarrow} \Pi_2 \twoheadrightarrow \Pi_2/H^2 \xrightarrow{\sim} \Pi_1$. Then the surjection $\Pi_N \twoheadrightarrow N \times \Pi_1$ determined by the natural surjection $\Pi_N \twoheadrightarrow \Pi_N/F_N \xrightarrow{\sim} \Pi_1$ and the second projection $\Pi_N = \Pi_2 \times_{\Pi_1} N \overset{\mathrm{pr}_2}{\twoheadrightarrow} N$ fits into a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_N & \longrightarrow & \Pi_N & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & N \times \Pi_1 & \xrightarrow{\mathrm{pr}_2} & \Pi_1 & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. Write $\rho_N \colon \Pi_1 \to \mathrm{Aut}(F_N^{\mathrm{ab}})$ for the natural action determined by the upper horizontal sequence and $V_N \subseteq F_N^{\mathrm{ab}}$ for the kernel of the natural surjection $F_N^{\mathrm{ab}} \twoheadrightarrow N^{\mathrm{ab}}$ induced by the left-hand vertical arrow.

Now we *claim* that

> $(*_2)$: the action ρ_N of Π_1 on F_N^{ab} preserves $V_N \subseteq F_N^{\mathrm{ab}}$, and, moreover, the resulting action $\rho_N^V \colon \Pi_1 \to \mathrm{Aut}(V_N)$ factors as the composite
>
> $$\Pi_1 \twoheadrightarrow \Pi_1/N \hookrightarrow \mathrm{Aut}(V_N)$$
>
> — where the second arrow is *injective*.

Indeed, the fact that the action ρ_N of Π_1 on F_N^{ab} preserves $V_N \subseteq F_N^{\mathrm{ab}}$ follows immediately from the definition of ρ_N [cf. also the above commutative diagram]. Next, let us observe that it follows immediately from the various definitions involved that if we write $f \colon Y \to X$ for the connected finite étale Galois covering of X corresponding to $N \subseteq \Pi_1$, then the right-hand square of the above diagram arises from a commutative diagram of schemes

$$\begin{array}{ccc} (Y \times_k X) \setminus \Gamma_f & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow & & \| \\ Y \times_k X & \xrightarrow{\mathrm{pr}_2} & X \end{array}$$

— where we write $\Gamma_f \subseteq Y \times_k X$ for the *graph* of f, and the left-hand vertical arrow is the natural open immersion. Thus, it follows immediately from a similar argument to the argument used in the proof of Lemma 1.1, (i) [cf. also [Hsh], Proposition 1.4, (i)], that F_N, N are naturally isomorphic to the maximal pro-Σ quotients of the étale fundamental groups of geometric fibers of the families of hyperbolic curves $Y \times_k X \setminus \Gamma_f$, $Y \times_k X \overset{\mathrm{pr}_2}{\to} X$ over X, respectively. Therefore, by the well-known structure of the maximal pro-Σ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\notin \Sigma$, we conclude — by considering the natural action of Π_1 on the set of cusps of the family of hyperbolic curves $Y \times_k X \setminus \Gamma_f \overset{\mathrm{pr}_2}{\to} X$ — that the resulting action $\rho_N^V \colon \Pi_1 \to \mathrm{Aut}(V_N)$ factors as the composite $\Pi_1 \twoheadrightarrow \Pi_1/N \to \mathrm{Aut}(V_N)$, and that if X is *affine* (respectively, *proper*), then for any $l \in \Sigma$, the resulting representation $\Pi_1/N \to \mathrm{Aut}(V_N \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l)$ is isomorphic to

> the *regular representation of* Π_1/N *over* $\mathbb{Q}_l$ (respectively, the *quotient of the regular representation of* Π_1/N *over* $\mathbb{Q}_l$ *by the trivial subrepresentation [of dimension* 1*]*).

In particular, as is well-known, the homomorphism $\Pi_1/N \to \mathrm{Aut}(V_N \otimes_{\widehat{\mathbb{Z}}^\Sigma} \mathbb{Q}_l)$, hence also the homomorphism $\Pi_1/N \to \mathrm{Aut}(V_N)$, is *injective*. This completes the proof of the *claim* $(*_2)$.

Next, let us observe that since $\widetilde{\alpha}$ is *F-admissible*, it follows immediately from the definition of "ρ_N^V" that the automorphism $\widetilde{\alpha}$ induces a commutative diagram

$$\begin{array}{ccc} \Pi_1 & \xrightarrow{\rho_N^V} & \mathrm{Aut}(V_N) \\ {\scriptstyle \widetilde{\alpha}^2}\downarrow\wr & & \downarrow\wr \\ \Pi_1 & \xrightarrow{\rho^V_{\widetilde{\alpha}^1(N)}} & \mathrm{Aut}(V_{\widetilde{\alpha}^1(N)}) \end{array}$$

— where the vertical arrows are *isomorphisms* that are induced by $\widetilde{\alpha}$. Thus, by considering the kernels of ρ_N^V, $\rho^V_{\widetilde{\alpha}^1(N)}$, one concludes from the *claim* $(*_2)$ that $\widetilde{\alpha}^1(N) = \widetilde{\alpha}^2(N)$. This completes the proof of $(*_1)$, hence also of assertion (i).

Assertion (ii) follows immediately from Theorem 1.8, (iv) [cf. also Remark 1.4.1]. This completes the proof of Corollary 1.9. Q.E.D.

Remark 1.9.1. The *discrete versions* of Theorem 1.8, Corollary 1.9 will be discussed in a sequel to the present paper.

§2. Various operations on semi-graphs of anabelioids of PSC-type

In the present §, we study various operations on semi-graphs of anabelioids of PSC-type. These operations include the following:

(Op1) the operation of *restriction* to a sub-semi-graph [satisfying certain conditions] of the underlying semi-graph [cf. Definition 2.2, (ii); Fig. 2 below],

(Op2) the operation of *partial compactification* [cf. Definition 2.4, (ii); Fig. 3 below],

(Op3) the operation of *resolution* of a given set [satisfying certain conditions] of nodes [cf. Definition 2.5, (ii); Fig. 4 below], and

(Op4) the operation of *generization* [cf. Definition 2.8; Fig. 5 below].

A basic reference for the theory of *semi-graphs of anabelioids of PSC-type* is [CmbGC]. We shall use the terms "*semi-graph of anabelioids of PSC-type*", "*PSC-fundamental group of a semi-graph of anabelioids of*

PSC-type", "*finite étale covering of semi-graphs of anabelioids of PSC-type*", "*vertex*", "*edge*", "*cusp*", "*node*", "*verticial subgroup*", "*edge-like subgroup*", "*nodal subgroup*", "*cuspidal subgroup*", and "*sturdy*" as they are defined in [CmbGC], Definition 1.1. Also, we shall apply the various notational conventions established in [NodNon], Definition 1.1, and refer to the "PSC-fundamental group of a semi-graph of anabelioids of PSC-type" simply as the "fundamental group" [of the semi-graph of anabelioids of PSC-type]. That is to say, we shall refer to the maximal pro-Σ quotient of the fundamental group of a semi-graph of anabelioids of pro-Σ PSC-type [as a semi-graph of anabelioids!] as the "fundamental group of the semi-graph of anabelioids of PSC-type".

Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_{\mathcal{G}}$ for the [pro-Σ] fundamental group of $\mathcal{G}$, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$. Then since the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ is *topologically finitely generated*, the profinite topology of $\Pi_{\mathcal{G}}$ induces [profinite] topologies on $\mathrm{Aut}(\Pi_{\mathcal{G}})$ and $\mathrm{Out}(\Pi_{\mathcal{G}})$ [cf. the discussion entitled "*Topological groups*" in §0]. If, moreover, we write

$$\mathrm{Aut}(\mathcal{G})$$

for the automorphism group of $\mathcal{G}$, then by the discussion preceding [CmbGC], Lemma 2.1, the natural homomorphism

$$\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$$

is an *injection with closed image*. [Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph — cf. [SemiAn], Definition 2.1; [SemiAn], Remark 2.4.2.] Thus, by equipping $\mathrm{Aut}(\mathcal{G})$ with the topology induced via this homomorphism by the topology of $\mathrm{Out}(\Pi_{\mathcal{G}})$, we may regard $\mathrm{Aut}(\mathcal{G})$ as being equipped with the structure of a *profinite group*.

Definition 2.1.

(i) For $z \in \mathrm{VCN}(\mathcal{G})$ such that $z \in \mathrm{Vert}(\mathcal{G})$ (respectively, $z \in \mathrm{Edge}(\mathcal{G})$), we shall say that a closed subgroup of $\Pi_{\mathcal{G}}$ is a *VCN-subgroup* of $\Pi_{\mathcal{G}}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$ if the closed subgroup is a verticial (respectively, an edge-like) subgroup of

$\Pi_{\mathcal{G}}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$. For $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ such that $\widetilde{z} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (respectively, $\widetilde{z} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$), we shall say that a closed subgroup of $\Pi_{\mathcal{G}}$ is the *VCN-subgroup* of $\Pi_{\mathcal{G}}$ associated to $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ if the closed subgroup is the verticial (respectively, edge-like) subgroup of $\Pi_{\mathcal{G}}$ associated to $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ [cf. [NodNon], Definition 1.1, (vi)].

(ii) For $z \in \mathrm{VCN}(\mathcal{G})$, we shall write

$$\mathcal{G}_z$$

for the anabelioid corresponding to $z \in \mathrm{VCN}(\mathcal{G})$.

(iii) For $v \in \mathrm{Vert}(\mathcal{G})$, we shall write

$$\mathcal{G}|_v$$

for the semi-graph of anabelioids of pro-Σ PSC-type defined as follows [cf. Fig. 1 below]: We take $\mathrm{Vert}(\mathcal{G}|_v)$ to consist of the single element "v", $\mathrm{Cusp}(\mathcal{G}|_v)$ to be the set of branches of $\mathcal{G}$ which abut to v, and $\mathrm{Node}(\mathcal{G}|_v)$ to be the empty set. We take the anabelioid of $\mathcal{G}|_v$ corresponding to the unique vertex "v" to be $\mathcal{G}_v$ [cf. (ii)]. For each edge $e \in \mathcal{E}(v)$ of $\mathcal{G}$ and each branch b of e that abuts to the vertex v, we take the anabelioid of $\mathcal{G}|_v$ corresponding to the branch b to be a copy of the anabelioid $\mathcal{G}_e$ [cf. (ii)]. For each edge $e \in \mathcal{E}(v)$ of $\mathcal{G}$ and each branch b of e that abuts, relative to $\mathcal{G}$, to the vertex v, we take the morphism of anabelioids $(\mathcal{G}|_v)_{e_b} \to (\mathcal{G}|_v)_v$ of $\mathcal{G}|_v$ — where we write e_b for the cusp of $\mathcal{G}|_v$ corresponding to b — to be the morphism of anabelioids $\mathcal{G}_e \to \mathcal{G}_v$ associated, relative to $\mathcal{G}$, to the branch b. Thus, one has a *natural morphism*

$$\mathcal{G}|_v \longrightarrow \mathcal{G}$$

of semi-graphs of anabelioids.

Remark 2.1.1. Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex of $\mathcal{G}$ and $\Pi_v \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup of $\Pi_{\mathcal{G}}$ associated to $v \in \mathrm{Vert}(\mathcal{G})$. Then it follows immediately from the various definitions involved that the fundamental group of $\mathcal{G}|_v$ is *naturally isomorphic to* Π_v, and that we have a natural identification

$$\mathrm{Aut}(\mathcal{G}_v) \simeq \mathrm{Out}(\Pi_v)$$

and a natural injection

$$\mathrm{Aut}(\mathcal{G}|_v) \hookrightarrow \mathrm{Aut}(\mathcal{G}_v)\,.$$

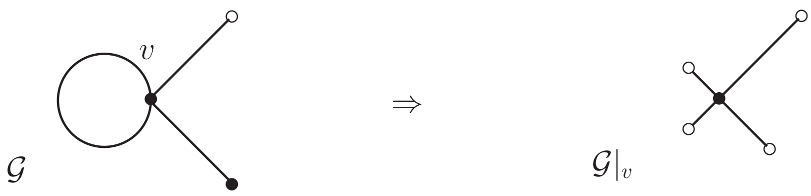

Figure 1: $\mathcal{G}|_v$

Definition 2.2 (cf. the operation (Op1) discussed at the beginning the present §2)**.**

(i) Let $\mathbb{K}$ be a [not necessarily finite] semi-graph and $\mathbb{H}$ a sub-semi-graph of $\mathbb{K}$ [cf. [SemiAn], the discussion following the figure entitled "A Typical Semi-graph"]. Then we shall say that $\mathbb{H}$ is of *PSC-type* if the following three conditions are satisfied:

(1) $\mathbb{H}$ is *finite* [i.e., the set consisting of vertices and edges of $\mathbb{H}$ is finite] and *connected*.

(2) $\mathbb{H}$ has *at least one vertex*.

(3) If v is a vertex of $\mathbb{H}$, and e is an edge of $\mathbb{K}$ that abuts to v, then e is an edge of $\mathbb{H}$. [Thus, if e abuts both to a vertex lying in $\mathbb{H}$ and to a vertex not lying in $\mathbb{H}$, then the resulting edge of $\mathbb{H}$ is a "cusp", i.e., an open edge.]

Thus, a sub-semi-graph of *PSC-type* $\mathbb{H}$ is completely determined by the set of vertices that lie in $\mathbb{H}$.

(ii) Let $\mathbb{H}$ be a sub-semi-graph of *PSC-type* [cf. (i)] of $\mathbb{G}$. Then one may verify easily that the semi-graph of anabelioids obtained by restricting $\mathcal{G}$ to $\mathbb{H}$ [cf. the discussion preceding [SemiAn], Definition 2.2] is of *pro-Σ PSC-type*. Here, we recall that the semi-graph of anabelioids obtained by restricting $\mathcal{G}$ to $\mathbb{H}$ is the semi-graph of anabelioids such that the underlying semi-graph is $\mathbb{H}$; for each vertex v (respectively, edge e) of $\mathbb{H}$, the anabelioid corresponding to v (respectively, e) is $\mathcal{G}_v$ (respectively,

$\mathcal{G}_e$) [cf. Definition 2.1, (ii)]; for each branch b of an edge e of $\mathbb{H}$ that abuts to a vertex v of $\mathbb{H}$, the morphism associated to b is the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $\mathbb{G}$ corresponding to b. We shall write

$$\mathcal{G}|_{\mathbb{H}}$$

for this semi-graph of anabelioids of pro-Σ PSC-type and refer to $\mathcal{G}|_{\mathbb{H}}$ as the *semi-graph of anabelioids of pro-Σ PSC-type obtained by restricting $\mathcal{G}$ to* $\mathbb{H}$ [cf. Fig. 2 below]. Thus, one has a *natural morphism*

$$\mathcal{G}|_{\mathbb{H}} \longrightarrow \mathcal{G}$$

of semi-graphs of anabelioids.

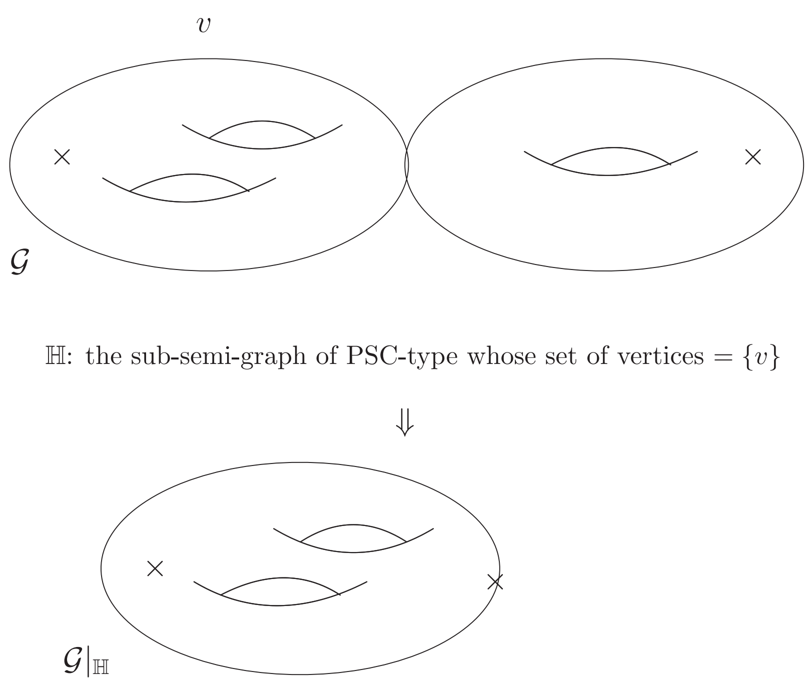

Figure 2: Restriction

Definition 2.3. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$.

(i) We shall say that $\mathcal{G}$ is of *type* (g, r) if $\mathcal{G}$ arises from a stable log curve of type (g, r) over an algebraically closed field of characteristic $\notin \Sigma$, i.e., $\mathrm{Cusp}(\mathcal{G})$ is of cardinality r, and, moreover,

$$\mathrm{rank}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}}^{\mathrm{ab}}) = 2g + \mathrm{Cusp}(\mathcal{G})^{\sharp} - c_{\mathcal{G}}$$

— where

$$c_{\mathcal{G}} \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if } \mathrm{Cusp}(\mathcal{G}) = \emptyset, \\ 1 & \text{if } \mathrm{Cusp}(\mathcal{G}) \neq \emptyset. \end{cases}$$

[Here, we recall that it follows from the discussion of [CmbGC], Remark 1.1.3, that $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ is a *free* $\widehat{\mathbb{Z}}^{\Sigma}$*-module of finite rank*.]

(ii) Let $\mathbb{H}$ be a sub-semi-graph of *PSC-type* [cf. Definition 2.2, (i)]. Then we shall say that $\mathbb{H}$ is of *type* (g, r) if the semi-graph of anabelioids $\mathcal{G}|_{\mathbb{H}}$, which is of pro-$\Sigma$ PSC-type [cf. Definition 2.2, (ii)], is of type (g, r) [cf. (i)].

(iii) Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex. Then we shall say that v is of *type* (g, r) if the semi-graph of anabelioids $\mathcal{G}|_v$, which is of pro-Σ PSC-type [cf. Definition 2.1, (iii)], is of type (g, r) [cf. (i)].

(iv) We shall say that $\mathcal{G}$ is *totally degenerate* if each vertex of $\mathcal{G}$ is of type $(0, 3)$ [cf. (iii)].

(v) One may verify easily that there exists a *unique*, up to isomorphism, semi-graph of anabelioids of pro-Σ PSC-type that is of *type* (g, r) [cf. (i)] and *has no node*. We shall write

$$\mathcal{G}_{g,r}^{\mathrm{model}}$$

for this semi-graph of anabelioids of pro-Σ PSC-type.

Remark 2.3.1. It follows immediately from the various definitions involved that there exists a *unique* pair (g, r) of nonnegative integers such that $\mathcal{G}$ is of *type* (g, r) [cf. Definition 2.3, (i)].

Definition 2.4 (cf. the operation (Op2) discussed at the beginning the present §2)**.**

(i) We shall say that a subset $S \subseteq \mathrm{Cusp}(\mathcal{G})$ of $\mathrm{Cusp}(\mathcal{G})$ is *omittable* if the following condition is satisfied: For each vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, if v is of type (g, r) [cf. Definition 2.3, (iii); Remark 2.3.1], then it holds that $2g - 2 + r - (\mathcal{E}(v) \cap S)^\sharp > 0$.

(ii) Let $S \subseteq \mathrm{Cusp}(\mathcal{G})$ be a subset of $\mathrm{Cusp}(\mathcal{G})$ which is *omittable* [cf. (i)]. Then by eliminating the cusps [i.e., the open edges] contained in S, and, for each vertex v of $\mathcal{G}$, replacing the anabelioid $\mathcal{G}_v$ corresponding to v by the anabelioid of finite étale coverings of $\mathcal{G}_v$ that restrict to a trivial covering over the cusps contained in S that abut to v, we obtain a semi-graph of anabelioids

$$\mathcal{G}_{\bullet S}$$

of *pro-Σ PSC-type*. We shall refer to $\mathcal{G}_{\bullet S}$ as the *partial compactification of $\mathcal{G}$ with respect to S* [cf. Fig. 3 below]. Thus, for each $v \in \mathrm{Vert}(\mathcal{G}) = \mathrm{Vert}(\mathcal{G}_{\bullet S})$, the pro-$\Sigma$ fundamental group of the anabelioid $(\mathcal{G}_{\bullet S})_v$ corresponding to $v \in \mathrm{Vert}(\mathcal{G}) = \mathrm{Vert}(\mathcal{G}_{\bullet S})$ may be naturally identified, up to inner automorphism, with the quotient of a verticial subgroup $\Pi_v \subseteq \Pi_\mathcal{G}$ of $\Pi_\mathcal{G}$ associated to $v \in \mathrm{Vert}(\mathcal{G}) = \mathrm{Vert}(\mathcal{G}_{\bullet S})$ by the subgroup of Π_v topologically normally generated by the $\Pi_e \subseteq \Pi_v$ for $e \in \mathcal{E}(v) \cap S$. If, moreover, we write $\Pi_{\mathcal{G}_{\bullet S}}$ for the [pro-Σ] fundamental group of $\mathcal{G}_{\bullet S}$ and $N_S \subseteq \Pi_\mathcal{G}$ for the normal closed subgroup of $\Pi_\mathcal{G}$ topologically normally generated by the cuspidal subgroups of $\Pi_\mathcal{G}$ associated to elements of S, then we have a *natural outer isomorphism*

$$\Pi_\mathcal{G}/N_S \xrightarrow{\sim} \Pi_{\mathcal{G}_{\bullet S}}.$$

Remark 2.4.1.

(i) Let $S_1 \subseteq S_2 \subseteq \mathrm{Cusp}(\mathcal{G})$ be subsets of $\mathrm{Cusp}(\mathcal{G})$. Then it follows immediately from the various definitions involved that the *omittability* of S_2 [cf. Definition 2.4, (i)] implies the *omittability* of S_1.

(ii) If $\mathcal{G}$ is *sturdy*, then it follows from the various definitions involved that $\mathrm{Cusp}(\mathcal{G})$, hence also any subset of $\mathrm{Cusp}(\mathcal{G})$ [cf. (i)], is *omittable*. Moreover, the partial compactification of $\mathcal{G}$ with respect to $\mathrm{Cusp}(\mathcal{G})$ coincides with the *compactification of* $\mathcal{G}$ [cf. [CmbGC], Remark 1.1.6; [NodNon], Definition 1.11].

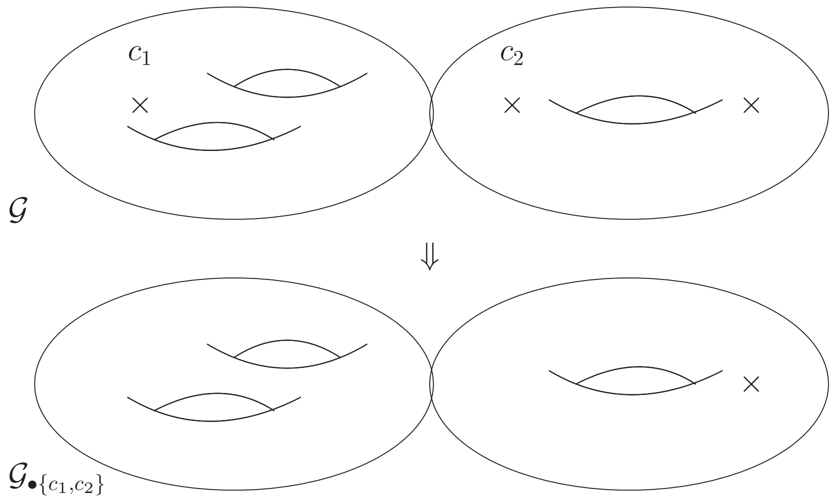

Figure 3: Partial compactification

Definition 2.5 (cf. the operation (Op3) discussed at the beginning the present §2)**.** Let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$.

(i) We shall say that S is of *separating type* if the semi-graph obtained by removing the closed edges corresponding to the elements of S from $\mathbb{G}$ is disconnected. Moreover, for each node $e \in \mathrm{Node}(\mathcal{G})$, we shall say that e is of *separating type* if $\{e\} \subseteq \mathrm{Node}(\mathcal{G})$ is of separating type.

(ii) Suppose that S is *not of separating type* [cf. (i)]. Then one may define a semi-graph of anabelioids of pro-Σ PSC-type as follows: We take the underlying semi-graph $\mathbb{G}_{\succ S}$ to be the semi-graph obtained by replacing each node e of $\mathbb{G}$ contained in S such that $\mathcal{V}(e) = \{v_1, v_2\} \subseteq \mathrm{Vert}(\mathcal{G})$ — where v_1, v_2 are *not necessarily distinct* — by two cusps that abut to v_1, $v_2 \in \mathrm{Vert}(\mathcal{G})$, respectively. We take the anabelioid corresponding to a vertex v (respectively, node e) of $\mathbb{G}_{\succ S}$ to be $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$). [Note that the set of vertices (respectively, nodes) of $\mathbb{G}_{\succ S}$ may be naturally identified with $\mathrm{Vert}(\mathcal{G})$ (respectively, $\mathrm{Node}(\mathcal{G}) \setminus S$).] We take the anabelioid corresponding to a cusp of $\mathbb{G}_{\succ S}$ arising from a *cusp* e *of* $\mathcal{G}$ to be $\mathcal{G}_e$. We take the anabelioid corresponding to a cusp of $\mathbb{G}_{\succ S}$ *arising from a node* e *of* $\mathcal{G}$ to be $\mathcal{G}_e$. For each branch b of $\mathbb{G}_{\succ S}$ that abuts to a vertex v of a node e (respectively, of a cusp e that does *not arise from a node of* $\mathbb{G}$), we take the morphism associated to b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $\mathbb{G}$ corresponding to b. For each branch b of $\mathbb{G}_{\succ S}$ that abuts to a vertex v of a cusp of $\mathbb{G}_{\succ S}$ that *arises from a node* e *of* $\mathbb{G}$, we take the morphism associated to b to be the morphism $\mathcal{G}_e \to \mathcal{G}_v$ associated to the branch of $\mathbb{G}$ corresponding to b. We shall denote the resulting semi-graph of anabelioids of pro-Σ PSC-type by

$$\mathcal{G}_{\succ S}$$

and refer to $\mathcal{G}_{\succ S}$ as the *semi-graph of anabelioids of pro-*Σ *PSC-type obtained from* $\mathcal{G}$ *by resolving* S [cf. Fig. 4 below]. Thus, one has a *natural morphism*

$$\mathcal{G}_{\succ S} \longrightarrow \mathcal{G}$$

of semi-graphs of anabelioids.

Remark 2.5.1.

(i) Let $S_1 \subseteq S_2 \subseteq \mathrm{Node}(\mathcal{G})$ be subsets of $\mathrm{Node}(\mathcal{G})$. Then it follows immediately from the various definitions involved that if S_2 is *not of separating type* [cf. Definition 2.5, (i)], then S_1 is *not of separating type*.

(ii) Let $v \in \mathrm{Vert}(\mathcal{G})$ be a vertex of $\mathcal{G}$. Then one may verify easily that there exists a *unique* sub-semi-graph of *PSC-type* [cf. Definition 2.2, (i)] $\mathbb{G}_v$ of $\mathbb{G}$ such that the set of vertices of $\mathbb{G}_v$ is equal to $\{v\}$. Moreover, one may also verify easily that $\mathrm{Node}(\mathcal{G}|_{\mathbb{G}_v})$ [cf. Definition 2.2, (ii)] is *not of separating type* [cf. Definition 2.5, (i)], relative to $\mathcal{G}|_{\mathbb{G}_v}$, and that the semi-graph of anabelioids of pro-Σ PSC-type

$$(\mathcal{G}|_{\mathbb{G}_v})_{\succ \mathrm{Node}(\mathcal{G}|_{\mathbb{G}_v})}$$

[cf. Definition 2.5, (ii)] is *naturally isomorphic to* $\mathcal{G}|_v$ [cf. Definition 2.1, (iii)].

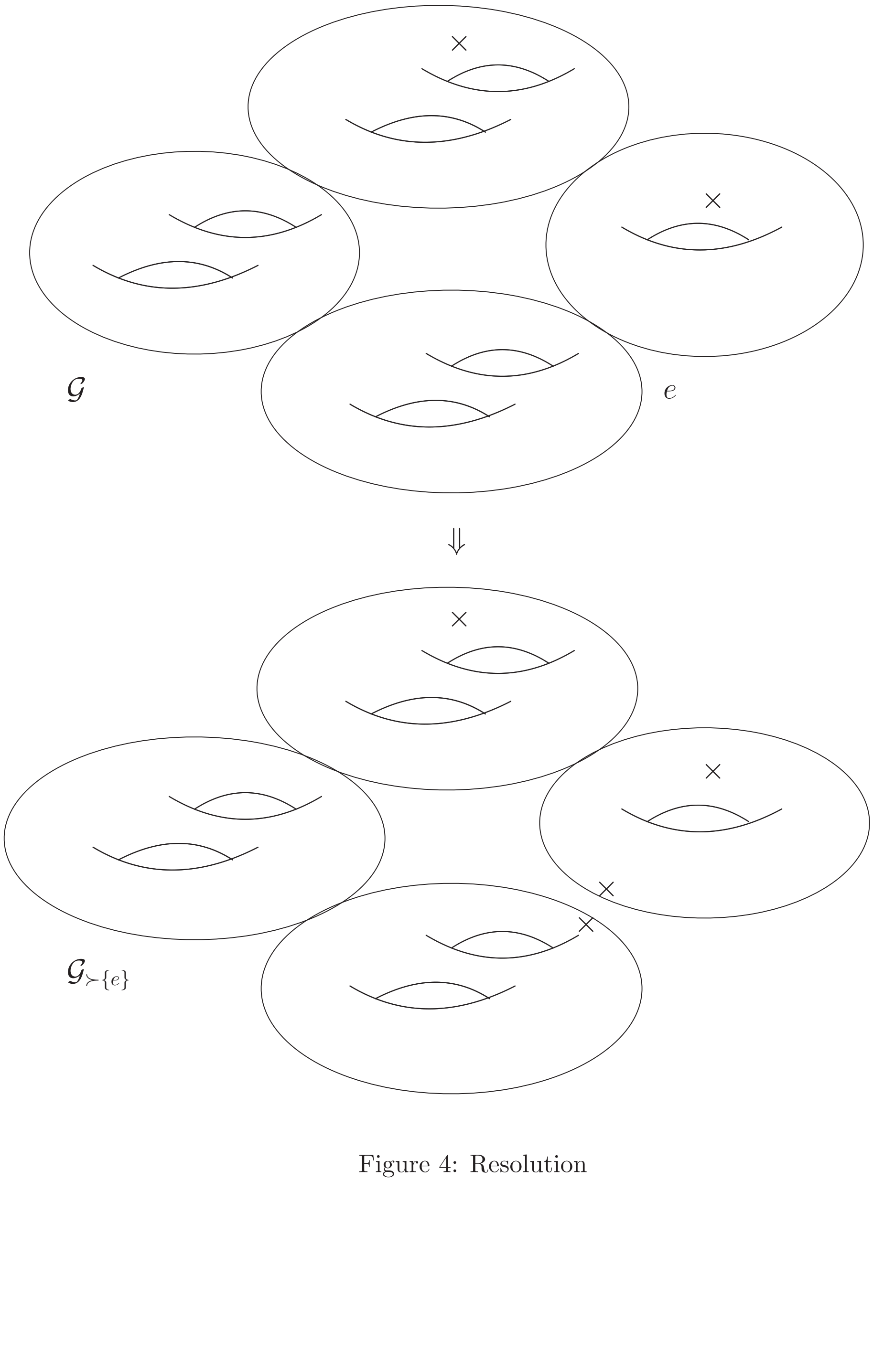

Figure 4: Resolution

Definition 2.6.

(i) Let $S \subseteq \mathrm{VCN}(\mathcal{G})$ be a subset of $\mathrm{VCN}(\mathcal{G})$. Then we shall denote by

$$\mathrm{Aut}^{S}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$$

the [closed] subgroup of $\mathrm{Aut}(\mathcal{G})$ consisting of automorphisms α of $\mathcal{G}$ such that the automorphism of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ induced by α *preserves* S and by

$$\mathrm{Aut}^{|S|}(\mathcal{G}) \subseteq \mathrm{Aut}^{S}(\mathcal{G})$$

the [closed] subgroup of $\mathrm{Aut}(\mathcal{G})$ consisting of automorphisms α of $\mathcal{G}$ such that the automorphism of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ induced by α *preserves* and *induces the identity automorphism of* S. Moreover, we shall write

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{|\mathrm{VCN}(\mathcal{G})|}(\mathcal{G}).$$

(ii) Let $H \subseteq \Pi_{\mathcal{G}}$ be a closed subgroup of $\Pi_{\mathcal{G}}$. Then we shall denote by

$$\mathrm{Out}^{H}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$$

the [closed] subgroup of $\mathrm{Out}(\Pi_{\mathcal{G}})$ consisting of outomorphisms [cf. the discussion entitled "*Topological groups*" in §0] of $\Pi_{\mathcal{G}}$ which preserve the $\Pi_{\mathcal{G}}$-conjugacy class of $H \subseteq \Pi_{\mathcal{G}}$. Moreover, we shall denote by

$$\mathrm{Aut}^{H}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\mathcal{G}) \cap \mathrm{Out}^{H}(\Pi_{\mathcal{G}}).$$

(iii) Let $\mathbb{H}$ be a sub-semi-graph of *PSC-type* [cf. Definition 2.2, (i)] of $\mathbb{G}$. Then $\mathrm{VCN}(\mathcal{G}|_{\mathbb{H}})$ [cf. Definition 2.2, (ii)] may be regarded as a subset of $\mathrm{VCN}(\mathcal{G})$. We shall write

$$\begin{aligned} \mathrm{Aut}^{|\mathbb{H}|}(\mathcal{G}) &\stackrel{\mathrm{def}}{=} \mathrm{Aut}^{|\mathrm{VCN}(\mathcal{G}|_{\mathbb{H}})|}(\mathcal{G}) \\ &\subseteq \mathrm{Aut}^{\mathbb{H}}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{VCN}(\mathcal{G}|_{\mathbb{H}})}(\mathcal{G}) \\ &\qquad = \mathrm{Aut}^{\mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})}(\mathcal{G}). \end{aligned}$$

Proposition 2.7 (Subgroups determined by sets of components). *Let* $S \subseteq \mathrm{VCN}(\mathcal{G})$ *be a nonempty subset of* $\mathrm{VCN}(\mathcal{G})$. *Then:*

(i) *It holds that*

$$\mathrm{Aut}^{|S|}(\mathcal{G}) = \bigcap_{z \in S} \mathrm{Aut}^{\Pi_z}(\mathcal{G})$$

— where we use the notation Π_z *to denote a VCN-subgroup [cf. Definition 2.1, (i)] of* $\Pi_{\mathcal{G}}$ *associated to* $z \in \mathrm{VCN}(\mathcal{G})$.

(ii) *It holds that*

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) = \bigcap_{z \in \mathrm{VCN}(\mathcal{G})} \mathrm{Out}^{\Pi_z}(\Pi_{\mathcal{G}})$$

— where we use the notation Π_z *to denote a VCN-subgroup of* $\Pi_{\mathcal{G}}$ *associated to* $z \in \mathrm{VCN}(\mathcal{G})$.

(iii) *The closed subgroups* $\mathrm{Aut}^{|S|}(\mathcal{G})$, $\mathrm{Aut}^{S}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$ *are* **open** *in* $\mathrm{Aut}(\mathcal{G})$. *Moreover, the closed subgroup* $\mathrm{Aut}^{|S|}(\mathcal{G}) \subseteq \mathrm{Aut}^{S}(\mathcal{G})$ *is* **normal** *in* $\mathrm{Aut}^{S}(\mathcal{G})$. *In particular,* $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$ *is* **normal** *in* $\mathrm{Aut}(\mathcal{G})$.

Proof. Assertion (i) follows immediately from [CmbGC], Proposition 1.2, (i). Next, we verify assertion (ii). It follows immediately from [CmbGC], Proposition 1.5, (ii), that the right-hand side of the equality in the statement of assertion (ii) is *contained in* $\mathrm{Aut}(\mathcal{G})$. Thus, assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the *finiteness* of the semi-graph $\mathbb{G}$, together with the various definitions involved. Q.E.D.

Definition 2.8 (cf. the operation (Op4) discussed at the beginning the present §2)**.** Let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$. Then we define the semi-graph of anabelioids of pro-Σ PSC-type

$$\mathcal{G}_{\rightsquigarrow S}$$

as follows:

(i) We take $\mathrm{Cusp}(\mathcal{G}_{\rightsquigarrow S}) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G})$.

(ii) We take $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow S}) \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}) \setminus S$.

(iii) We take $\mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ to be the set of connected components of the semi-graph obtained from $\mathbb{G}$ by omitting the edges $e \in \mathrm{Edge}(\mathcal{G}) \setminus S$. Alternatively, one may take $\mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ to be the set of equivalence classes of elements of $\mathrm{Vert}(\mathcal{G})$ with respect to the equivalence relation "$\sim$" defined as follows: for v, $w \in \mathrm{Vert}(\mathcal{G})$, $v \sim w$ if either $v = w$ or there exist n elements $e_1, \cdots, e_n \in S$ of S and $n+1$ vertices $v_0, v_1, \cdots, v_n \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$ such that $v_0 \stackrel{\mathrm{def}}{=} v$, $v_n \stackrel{\mathrm{def}}{=} w$, and, for $1 \leq i \leq n$, it holds that $\mathcal{V}(e_i) = \{v_{i-1}, v_i\}$.

(iv) For each branch b of an edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ $(= \mathrm{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) and each vertex $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ of $\mathcal{G}_{\rightsquigarrow S}$, *$b$ abuts, relative to $\mathcal{G}_{\rightsquigarrow S}$, to v* if b abuts, relative to $\mathcal{G}$, to an element of the equivalence class v [cf. (iii)].

(v) For each edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ $(= \mathrm{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) of $\mathcal{G}_{\rightsquigarrow S}$, we take the anabelioid of $\mathcal{G}_{\rightsquigarrow S}$ corresponding to $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ to be $\mathcal{G}_e$ [cf. Definition 2.1, (ii)].

(vi) Let $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ be a vertex of $\mathcal{G}_{\rightsquigarrow S}$. Then one verifies easily that there exists a *unique* sub-semi-graph of *PSC-type* [cf. Definition 2.2, (i)] $\mathbb{H}_v$ of $\mathbb{G}$ such that the set of vertices of $\mathbb{H}_v$ consists of the elements of the equivalence class v [cf. (iii)]. Write

$$T_v \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v}) \setminus (S \cap \mathrm{Node}(\mathcal{G}|_{\mathbb{H}_v}))$$

[cf. Definition 2.2, (ii)]. Then we take the anabelioid of $\mathcal{G}_{\rightsquigarrow S}$ corresponding to $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ to be the anabelioid determined by the finite étale coverings of

$$(\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v}$$

[cf. Definition 2.5, (ii)] of degree a product of primes $\in \Sigma$.

(vii) Let b be a branch of an edge $e \in \mathrm{Edge}(\mathcal{G}_{\rightsquigarrow S})$ $(= \mathrm{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) that abuts to a vertex $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$. Then since b abuts to v, one verifies easily that there exists a *unique* vertex w of $\mathcal{G}$ which belongs to the equivalent class v [cf. (iii)] such that b abuts to w relative to $\mathcal{G}$. We take the morphism of anabelioids associated to b, relative to $\mathcal{G}_{\rightsquigarrow S}$, to be the morphism naturally determined by the morphism of anabelioids

$$\mathcal{G}_e \to \mathcal{G}_w$$

corresponding to the branch b relative to $\mathcal{G}$ and the morphism of semi-graphs of anabelioids of pro-Σ PSC-type

$$\mathcal{G}|_w \to (\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v}$$

[cf. (vi); Definition 2.1, (iii)]. Here, we recall that the anabelioid obtained by considering the connected finite étale coverings of $\mathcal{G}|_w$ may be naturally identified with $\mathcal{G}_w$ [cf. Remark 2.1.1].

We shall refer to this semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}_{\rightsquigarrow S}$ as the *generization of* $\mathcal{G}$ *with respect to* S [cf. Fig. 5 below].

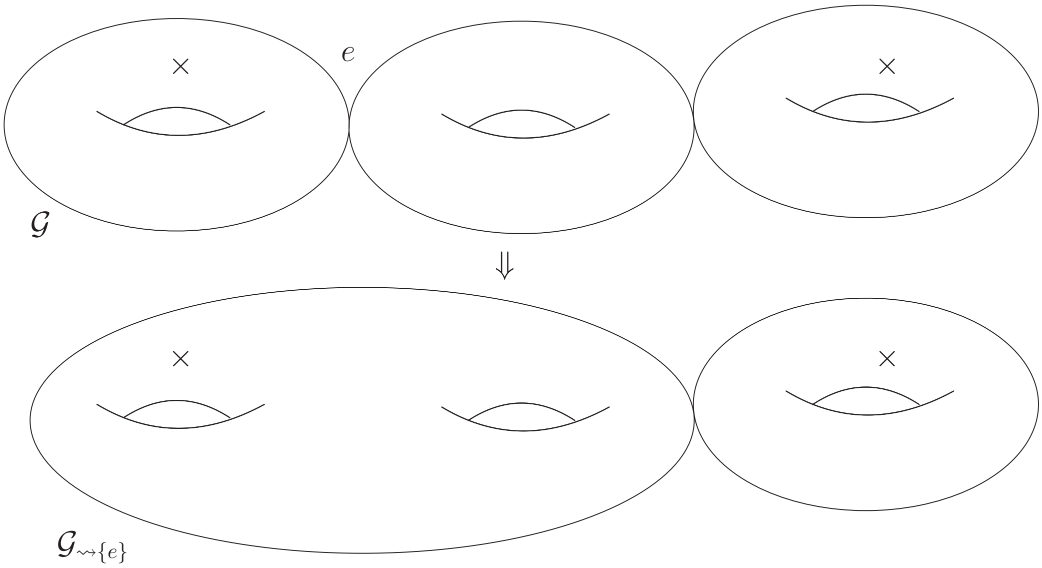

Figure 5: Generization

Remark 2.8.1. It follows immediately from the various definitions involved that if $\mathcal{G}$ is of *type* (g, r) [cf. Definition 2.3, (i)], then the generization $\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G})}$ of $\mathcal{G}$ with respect to $\mathrm{Node}(\mathcal{G})$ is isomorphic to $\mathcal{G}^{\mathrm{model}}_{g,r}$ [cf. Definition 2.3, (v)].

Proposition 2.9 (**Specialization outer isomorphisms**). *Let* $S \subseteq \mathrm{Node}(\mathcal{G})$ *be a subset of* $\mathrm{Node}(\mathcal{G})$. *Write* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *for the [pro-*Σ*] fundamental group of the generization* $\mathcal{G}_{\rightsquigarrow S}$ *of* $\mathcal{G}$ *with respect to* S *[cf. Definition 2.8]. Then the following hold:*

(i) *There exists a* **natural outer isomorphism** *of profinite groups*

$$\Phi_{\mathcal{G}_{\rightsquigarrow S}} : \Pi_{\mathcal{G}_{\rightsquigarrow S}} \stackrel{\sim}{\longrightarrow} \Pi_{\mathcal{G}}$$

which satisfies the following three conditions:

(1) $\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ *induces a bijection between the set of cuspidal subgroups of* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *and the set of cuspidal subgroups of* $\Pi_{\mathcal{G}}$*.*

(2) $\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ *induces a bijection between the set of nodal subgroups of* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *and the set of nodal subgroups of* $\Pi_{\mathcal{G}}$ *associated to the elements of* $\mathrm{Node}(\mathcal{G}) \setminus S$*.*

(3) *Let* $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ *be a vertex of* $\mathcal{G}_{\rightsquigarrow S}$*;* $\mathbb{H}_v$*,* T_v *as in Definition 2.8, (vi). Then* $\Phi_{\mathcal{G}_{\rightsquigarrow S}}$ *induces a bijection between the* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$*-conjugacy class of any verticial subgroup* $\Pi_v \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *of* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *associated to* $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow S})$ *and the* $\Pi_{\mathcal{G}}$*-conjugacy class of subgroups determined by the image of the outer homomorphism*

$$\Pi_{(\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v}} \longrightarrow \Pi_{\mathcal{G}}$$

induced by the natural morphism $(\mathcal{G}|_{\mathbb{H}_v})_{\succ T_v} \to \mathcal{G}$ *[cf. Definitions 2.2, (ii); 2.5, (ii)] of semi-graphs of anabelioids of pro-*Σ *PSC-type.*

Moreover, any two outer isomorphisms $\Pi_{\mathcal{G}_{\rightsquigarrow S}} \stackrel{\sim}{\to} \Pi_{\mathcal{G}}$ *that satisfy the above three conditions differ by composition with a* **graphic** *[cf.* [CmbGC]*, Definition 1.4, (i)] outomorphism [cf. the discussion entitled "Topological groups" in §0] of* $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$*.*

(ii) *The isomorphism*

$$\mathrm{Out}(\Pi_{\mathcal{G}}) \stackrel{\sim}{\longrightarrow} \mathrm{Out}(\Pi_{\mathcal{G}_{\rightsquigarrow S}})$$

induced by the natural outer isomorphism of (i) determines an injection

$$\mathrm{Aut}^S(\mathcal{G}) \hookrightarrow \mathrm{Aut}(\mathcal{G}_{\rightsquigarrow S})$$

[cf. Definition 2.6, (i)].

Proof. First, we verify assertion (i). An outer isomorphism that satisfies the three conditions of assertion (i) may be obtained by observing that, after sorting through the various definitions involved, a finite étale covering of $\mathcal{G}_{\rightsquigarrow S}$ amounts to the same data as a finite étale covering of $\mathcal{G}$. The final portion of assertion (i) follows immediately, in light of the three conditions in the statement of assertion (i), from

[CmbGC], Proposition 1.5, (ii). This completes the proof of assertion (i). Assertion (ii) follows immediately from [CmbGC], Proposition 1.5, (ii), together with the three conditions in the statement of assertion (i). This completes the proof of Proposition 2.9. Q.E.D.

Definition 2.10. Let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$. Write $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ for the [pro-Σ] fundamental group of the generization $\mathcal{G}_{\rightsquigarrow S}$ of $\mathcal{G}$ with respect to S [cf. Definition 2.8]. Then we shall refer to the natural outer isomorphism

$$\Phi_{\mathcal{G}_{\rightsquigarrow S}} : \Pi_{\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$$

obtained in Proposition 2.9, (i), as the *specialization outer isomorphism with respect to* S.

Proposition 2.11 (**Commensurable terminality of closed subgroups determined by certain semi-graphs).** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of* $\mathbb{G}$ *and* $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)]. Then the natural morphism* $(\mathcal{G}|_{\mathbb{H}})_{\succ S} \to \mathcal{G}$ *[cf. Definitions 2.2, (ii); 2.5, (ii)] of semi-graphs of anabelioids of pro-*Σ *PSC-type determines an outer* **injection** *of profinite groups*

$$\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}} \hookrightarrow \Pi_{\mathcal{G}} .$$

Moreover, the image of this outer injection is **commensurably terminal** *in* $\Pi_{\mathcal{G}}$ *[cf. the discussion entitled "Topological groups" in §0].*

Proof. Write $\mathcal{H} \stackrel{\mathrm{def}}{=} (\mathcal{G}|_{\mathbb{H}})_{\succ S}$ and $T \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}|_{\mathbb{H}}) \setminus S$. Note that it follows from the definition of $\mathcal{G}|_{\mathbb{H}}$ that T may be regarded as the subset of $\mathrm{Node}(\mathcal{G})$ determined by $\mathrm{Node}(\mathcal{H})$; for simplicity, we shall identify T with $\mathrm{Node}(\mathcal{H})$. Now it follows immediately from the definition of "$\mathcal{G}_{\rightsquigarrow T}$" that the composite

$$\Pi_{\mathcal{H}} \xrightarrow{\Phi_{\mathbb{H},S}} \Pi_{\mathcal{G}} \xrightarrow[\sim]{\Phi^{-1}_{\mathcal{G}_{\rightsquigarrow T}}} \Pi_{\mathcal{G}_{\rightsquigarrow T}}$$

factors through a verticial subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow T}}$ of $\Pi_{\mathcal{G}_{\rightsquigarrow T}}$ associated to a vertex $v \in \mathrm{Vert}(\mathcal{G}_{\rightsquigarrow T})$, and that the composite

$$\Pi_{\mathcal{H}} \longrightarrow \Pi_{(\mathcal{G}_{\rightsquigarrow T})|_v}$$

of the resulting outer homomorphism $\Pi_{\mathcal{H}} \to \Pi_v$ [which is well-defined in light of the *commensurable terminality* of Π_v in $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ — cf. [CmbGC], Proposition 1.2, (ii)] and the natural outer isomorphism $\Pi_v \simeq \Pi_{(\mathcal{G}_{\rightsquigarrow T})|_v}$ [cf. Remark 2.1.1] may be identified with "$\Phi^{-1}_{\mathcal{H}_{\rightsquigarrow T}}$" [cf. Definition 2.10]. Thus, Proposition 2.11 follows immediately from the fact that $\Phi_{\mathcal{H}_{\rightsquigarrow T}}$ is an outer *isomorphism*, together with the fact that $\Pi_v \subseteq \Pi_{\mathcal{G}_{\rightsquigarrow S}}$ is *commensurably terminal* in $\Pi_{\mathcal{G}_{\rightsquigarrow S}}$ [cf. [CmbGC], Proposition 1.2, (ii)]. This completes the proof of Proposition 2.11. Q.E.D.

Lemma 2.12 (**Restrictions of outomorphisms**). *Let $H \subseteq \Pi_\mathcal{G}$ be a closed subgroup of $\Pi_\mathcal{G}$ which is* **normally terminal** *[cf. the discussion entitled "Topological groups" in §0] and $\alpha \in \mathrm{Out}^H(\Pi_\mathcal{G})$ [cf. Definition 2.6, (ii)]. Then the following hold:*

(i) *There exists a lifting $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_\mathcal{G})$ of α such that $\widetilde{\alpha}$* **preserves** *the closed subgroup $H \subseteq \Pi_\mathcal{G}$. Moreover, such a lifting $\widetilde{\alpha}$ is* **uniquely determined** *up to composition with an H-inner automorphism of $\Pi_\mathcal{G}$.*

(ii) *Write α_H for the outomorphism [cf. the discussion entitled "Topological groups" in §0] of H determined by the restriction of a lifting $\widetilde{\alpha}$ as obtained in (i) to the closed subgroup $H \subseteq \Pi_\mathcal{G}$. Then the map*

$$\mathrm{Out}^H(\Pi_\mathcal{G}) \longrightarrow \mathrm{Out}(H)$$

given by assigning $\alpha \mapsto \alpha_H$ is a **homomorphism**.

(iii) *The homomorphism*

$$\mathrm{Out}^H(\Pi_\mathcal{G}) \longrightarrow \mathrm{Out}(H)$$

obtained in (ii) **depends only** *on the conjugacy class of the closed subgroup $H \subseteq \Pi_\mathcal{G}$, i.e., if we write $H^\gamma \stackrel{\mathrm{def}}{=} \gamma \cdot H \cdot \gamma^{-1}$ for $\gamma \in \Pi_\mathcal{G}$, then the diagram*

$$\begin{array}{ccc} \mathrm{Out}^H(\Pi_\mathcal{G}) & \longrightarrow & \mathrm{Out}(H) \\ \| & & \downarrow \\ \mathrm{Out}^{H^\gamma}(\Pi_\mathcal{G}) & \longrightarrow & \mathrm{Out}(H^\gamma) \end{array}$$

— where the upper (respectively, lower) horizontal arrow is the homomorphism given by mapping $\alpha \mapsto \alpha_H$ (respectively, $\alpha \mapsto \alpha_{H^\gamma}$), and the right-hand vertical arrow is the isomorphism

obtained by mapping $\phi \in \mathrm{Out}(H)$ *to*

$$H^\gamma \xrightarrow[\sim]{\mathrm{Inn}(\gamma^{-1})} H \xrightarrow[\sim]{\phi} H \xrightarrow[\sim]{\mathrm{Inn}(\gamma)} H^\gamma$$

— **commutes**.

Proof. Assertion (i) follows immediately from the *normal terminality* of H in $\Pi_{\mathcal{G}}$. Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the various definitions involved. Q.E.D.

Definition 2.13. Let $H \subseteq \Pi_{\mathcal{G}}$ be a [closed] subgroup of $\Pi_{\mathcal{G}}$ which is *normally terminal* [cf. the discussion entitled "*Topological groups*" in §0]. Then we shall write

$$\mathrm{Out}^{|H|}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Out}^{H}(\Pi_{\mathcal{G}})$$

for the closed subgroup of $\mathrm{Out}^{H}(\Pi_{\mathcal{G}})$ consisting of outomorphisms [cf. the discussion entitled "*Topological groups*" in §0] α of $\Pi_{\mathcal{G}}$ such that the image α_H of α via the homomorphism $\mathrm{Out}^{H}(\mathcal{G}) \to \mathrm{Out}(H)$ obtained in Lemma 2.12, (ii), is *trivial*. Also, we shall write

$$\mathrm{Aut}^{|H|}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{|H|}(\Pi_{\mathcal{G}}) \cap \mathrm{Aut}(\mathcal{G}) .$$

Definition 2.14.

(i) Let $T \subseteq \mathrm{Cusp}(\mathcal{G})$ be an *omittable* [cf. Definition 2.4, (i)] subset of $\mathrm{Cusp}(\mathcal{G})$. Write $\Pi_{\mathcal{G}_{\bullet T}}$ for the [pro-Σ] fundamental group of $\mathcal{G}_{\bullet T}$ [cf. Definition 2.4, (ii)] and $N_T \subseteq \Pi_{\mathcal{G}}$ for the normal closed subgroup of $\Pi_{\mathcal{G}}$ topologically normally generated by the cuspidal subgroups of $\Pi_{\mathcal{G}}$ associated to elements of T. Then one verifies easily that the natural outer isomorphism $\Pi_{\mathcal{G}}/N_T \xrightarrow{\sim} \Pi_{\mathcal{G}_{\bullet T}}$ [cf. Definition 2.4, (ii)] induces a homomorphism $\mathrm{Out}^{N_T}(\Pi_{\mathcal{G}}) \to \mathrm{Out}(\Pi_{\mathcal{G}_{\bullet T}})$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}^{T}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}(\mathcal{G}_{\bullet T}) \\ \downarrow & & \downarrow \\ \mathrm{Out}^{N_T}(\Pi_{\mathcal{G}}) & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}_{\bullet T}}) \end{array}$$

— where the vertical arrows are the *natural injections*. For $\alpha \in \mathrm{Out}^{N_T}(\Pi_{\mathcal{G}})$, we shall write

$$\alpha_{\mathcal{G}_{\bullet T}} \in \mathrm{Out}(\Pi_{\mathcal{G}_{\bullet T}})$$

for the image of α via the lower horizontal arrow in the above commutative diagram. If, moreover, $\alpha \in \mathrm{Aut}^{T}(\mathcal{G})$, then, in light of the *injectivity* of the right-hand vertical arrow in the above diagram, we shall write [by abuse of notation]

$$\alpha_{\mathcal{G}_{\bullet T}} \in \mathrm{Aut}(\mathcal{G}_{\bullet T})$$

for the image of α via the upper horizontal arrow in the above commutative diagram.

(ii) Let $\mathbb{H}$ be a sub-semi-graph of *PSC-type* [cf. Definition 2.2, (i)] of $\mathbb{G}$ and $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ [cf. Definition 2.2, (ii)] a subset of $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ that is *not of separating type* [cf. Definition 2.5, (i)]. Write $\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}$ for the [pro-Σ] fundamental group of $(\mathcal{G}|_{\mathbb{H}})_{\succ S}$ [cf. Definition 2.5, (ii)]. Then the natural outer homomorphism $\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}} \to \Pi_{\mathcal{G}}$ is an outer *injection* whose image is *commensurably terminal* [cf. Proposition 2.11]. Thus, it follows from Lemma 2.12, (iii), that we have a homomorphism $\mathrm{Out}^{\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}}(\Pi_{\mathcal{G}}) \to \mathrm{Out}(\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}})$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}^{\mathbb{H}\succ S}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Aut}^{\mathbb{H}}(\mathcal{G}) \cap \mathrm{Aut}^{S}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}((\mathcal{G}|_{\mathbb{H}})_{\succ S}) \\ \downarrow & & \downarrow \\ \mathrm{Out}^{\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}}(\Pi_{\mathcal{G}}) & \longrightarrow & \mathrm{Out}(\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}) \end{array}$$

— where the vertical arrows are the *natural injections*. For $\alpha \in \mathrm{Out}^{\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}}(\Pi_{\mathcal{G}})$, we shall write

$$\alpha_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}} \in \mathrm{Out}(\Pi_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}})$$

for the image of α via the lower horizontal arrow in the above commutative diagram. If, moreover, $\alpha \in \mathrm{Aut}^{\mathbb{H}\succ S}(\mathcal{G})$, then, in light of the *injectivity* of the right-hand vertical arrow in the above diagram, we shall write [by abuse of notation]

$$\alpha_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}} \in \mathrm{Aut}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$$

for the image of α via the upper horizontal arrow in the above commutative diagram. Finally, if $T \subseteq \mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ is an *omittable* subset of $\mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$, then we shall write

$$\mathrm{Aut}^{\mathbb{H}\succ S\bullet T}(\mathcal{G}) \subseteq \mathrm{Aut}^{\mathbb{H}\succ S}(\mathcal{G})$$

for the inverse image of the closed subgroup $\mathrm{Aut}^{T}((\mathcal{G}|_{\mathbb{H}})_{\succ S}) \subseteq \mathrm{Aut}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ of $\mathrm{Aut}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ in $\mathrm{Aut}^{\mathbb{H}\succ S}(\mathcal{G})$ via the upper horizontal arrow $\mathrm{Aut}^{\mathbb{H}\succ S}(\mathcal{G}) \to \mathrm{Aut}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ of the above commutative diagram; thus, we have a natural homomorphism [cf. (i)]

$$\begin{array}{ccc} \mathrm{Aut}^{\mathbb{H}\succ S\bullet T}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}) \\ \alpha & \mapsto & \alpha_{((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}}\,. \end{array}$$

(iii) Let $z \in \mathrm{VCN}(\mathcal{G})$ be an element of $\mathrm{VCN}(\mathcal{G})$ and $\Pi_z \subseteq \Pi_{\mathcal{G}}$ a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$. Then it follows from [CmbGC], Proposition 1.2, (ii), that the closed subgroup $\Pi_z \subseteq \Pi_{\mathcal{G}}$ is *commensurably terminal*. Thus, it follows from Lemma 2.12, (iii), that we obtain a homomorphism $\mathrm{Out}^{\Pi_z}(\Pi_{\mathcal{G}}) \to \mathrm{Out}(\Pi_z)$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}^{\{z\}}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}(\mathcal{G}_z) \\ \downarrow & & \downarrow \wr \\ \mathrm{Out}^{\Pi_z}(\Pi_{\mathcal{G}}) & \longrightarrow & \mathrm{Out}(\Pi_z) \end{array}$$

— where the left-hand vertical arrow is *injective*, and the right-hand vertical arrow is an *isomorphism*. For $\alpha \in \mathrm{Out}^{\Pi_z}(\Pi_{\mathcal{G}})$, we shall write

$$\alpha_z \in \mathrm{Out}(\Pi_z)$$

for the image of α via the lower horizontal arrow in the above commutative diagram.

§3. Synchronization of cyclotomes

In the present §, we introduce and study the notion of the *second cohomology group with compact supports* of a semi-graph of anabelioids of PSC-type [cf. Definition 3.1, (ii), (iii) below]. In particular, we show that such cohomology groups are compatible with *graph-theoretic localization* [cf. Definition 3.4, Lemma 3.5 below]. This leads naturally to a discussion of the phenomenon of *synchronization* among *the various cyclotomes* [cf. Definition 3.8 below] arising from a semi-graph of anabelioids of PSC-type [cf. Corollary 3.9 below].

Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_{\mathcal{G}}$ for the [pro-Σ] fundamental group of $\mathcal{G}$, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$.

Definition 3.1. Let M be a finitely generated $\widehat{\mathbb{Z}}^{\Sigma}$-module and $v \in \mathrm{Vert}(\mathcal{G})$ a vertex of $\mathcal{G}$.

(i) We shall write

$$H^2(\mathcal{G}, M) \stackrel{\mathrm{def}}{=} H^2(\Pi_{\mathcal{G}}, M)$$

— where we regard M as being equipped with the trivial action of $\Pi_{\mathcal{G}}$ — and refer to $H^2(\mathcal{G}, M)$ as the *second cohomology group* of $\mathcal{G}$.

(ii) Let s be a section of the natural surjection $\mathrm{Cusp}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{G})$. Given a central extension of profinite groups

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1\,,$$

and a cusp $e \in \mathrm{Cusp}(\mathcal{G})$, we shall refer to a section of this extension over the edge-like subgroup $\Pi_{s(e)} \subseteq \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$ determined by $s(e) \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$ as a *trivialization of this extension at the cusp* e. We shall write

$$H^2_c(\mathcal{G}, M)$$

for the set of equivalence classes

$$[E, (\iota_e \colon \Pi_{s(e)} \to E)_{e \in \mathrm{Cusp}(\mathcal{G})}]$$

of collections of data $(E, (\iota_e \colon \Pi_{s(e)} \to E)_{e \in \mathrm{Cusp}(\mathcal{G})})$ as follows: (a) E is a central extension of profinite groups

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1\,;$$

(b) for each $e \in \mathrm{Cusp}(\mathcal{G})$, ι_e is a trivialization of this extension at the cusp e. The equivalence relation "$\sim$" is then defined as follows: for two collections of data $(E, (\iota_e))$ and $(E', (\iota'_e))$, we shall write $(E, (\iota_e)) \sim (E', (\iota'_e))$ if there exists an isomorphism of profinite groups $\alpha\colon E \xrightarrow{\sim} E'$ over $\Pi_{\mathcal{G}}$ which induces the identity automorphism of M, and, moreover, for each $e \in \mathrm{Cusp}(\mathcal{G})$, maps ι_e to ι'_e. We shall refer to $H^2_c(\mathcal{G}, M)$ as the *second cohomology group with compact supports* of $\mathcal{G}$.

(iii) We shall write

$$H^2_c(v, M) \stackrel{\mathrm{def}}{=} H^2_c(\mathcal{G}|_v, M)$$

[cf. (ii); Definition 2.1, (iii)] and refer to $H^2_c(v, M)$ as the *second cohomology group with compact supports* of v.

(iv) The set $H^2_c(\mathcal{G}, M)$ is equipped with a *natural structure of* $\widehat{\mathbb{Z}}^{\Sigma}$-*module* defined as follows:

- Let $[E, (\iota_e)]$, $[E', (\iota'_e)] \in H^2_c(\mathcal{G}, M)$. Then the fiber product $E \times_{\Pi_{\mathcal{G}}} E'$ of the surjections $E \twoheadrightarrow \Pi_{\mathcal{G}}$, $E' \twoheadrightarrow \Pi_{\mathcal{G}}$ is an extension of $\Pi_{\mathcal{G}}$ by $M \times M$. Thus, the quotient S of $E \times_{\Pi_{\mathcal{G}}} E'$ by the image of the composite

$$\begin{array}{ccccc} M & \hookrightarrow & M \times M & \hookrightarrow & E \times_{\Pi_{\mathcal{G}}} E' \\ m & \mapsto & (m, -m) & & \end{array}$$

  is an extension of $\Pi_{\mathcal{G}}$ by M. On the other hand, it follows from the definition of S that for each $e \in \mathrm{Cusp}(\mathcal{G})$, the sections ι_e and ι'_e naturally determine a section $\iota^S_e\colon \Pi_{s(e)} \to S$ over $\Pi_{s(e)}$. Thus, we define

$$[E, (\iota_e)] + [E', (\iota'_e)] \stackrel{\mathrm{def}}{=} [S, (\iota^S_e)]\,.$$

  Here, one may verify easily that the equivalence class $[S, (\iota^S_e)]$ depends only on the equivalence classes $[E, (\iota_e)]$, $[E', (\iota'_e)]$, and that this definition of "+" determines a module structure on $H^2_c(\mathcal{G}, M)$.

- Let $[E, (\iota_e)] \in H^2_c(\mathcal{G}, M)$ be an element of $H^2_c(\mathcal{G}, M)$ and $a \in \widehat{\mathbb{Z}}^{\Sigma}$. Now the composite $E \times M \stackrel{\mathrm{pr}_1}{\twoheadrightarrow} E \twoheadrightarrow \Pi_{\mathcal{G}}$ determines an extension of $\Pi_{\mathcal{G}}$ by $M \times M$. Thus, the quotient P of $E \times M$ by the image of the composite

$$\begin{array}{ccccc} M & \hookrightarrow & M \times M & \hookrightarrow & E \times M \\ m & \mapsto & (m, -am) & & \end{array}$$

  is an extension of $\Pi_{\mathcal{G}}$ by M. On the other hand, it follows from the definition of P that for each $e \in \mathrm{Cusp}(\mathcal{G})$, the section ι_e and the zero homomorphism $\Pi_{s(e)} \to M$ naturally determine a section $\iota_e^P \colon \Pi_{s(e)} \to P$ over $\Pi_{s(e)}$. Thus, we define

$$a \cdot [E, (\iota_e)] \stackrel{\text{def}}{=} [P, (\iota_e^P)] \, .$$

Here, one may verify easily that the equivalence class $[P, (\iota_e^P)]$ depends only on the equivalence class $[E, (\iota_e)]$ and $a \in \widehat{\mathbb{Z}}^{\Sigma}$, and that this definition of "$\cdot$" determines a $\widehat{\mathbb{Z}}^{\Sigma}$-module structure on $H^2_c(\mathcal{G}, M)$.

Finally, we note that it follows from Lemma 3.2 below that the $\widehat{\mathbb{Z}}^{\Sigma}$-module "$H^2_c(\mathcal{G}, M)$" does *not depend* on the choice of the section s. More precisely, the $\widehat{\mathbb{Z}}^{\Sigma}$-module "$H^2_c(\mathcal{G}, M)$" is *uniquely determined by* $\mathcal{G}$ *and* M *up to the natural isomorphism* obtained in Lemma 3.2.

Lemma 3.2 (**Independence of the choice of section**). *Let* M *be a finitely generated* $\widehat{\mathbb{Z}}^{\Sigma}$*-module and* s, s' *sections of the natural surjection* $\mathrm{Cusp}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{G})$. *Write* $H^2_c(\mathcal{G}, M, s)$, $H^2_c(\mathcal{G}, M, s')$ *for the* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules "*$H^2_c(\mathcal{G}, M)$*" defined in Definition 3.1 by means of the sections* s, s', *respectively. Then there exists a* **natural isomorphism** *of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*

$$H^2_c(\mathcal{G}, M, s) \xrightarrow{\sim} H^2_c(\mathcal{G}, M, s') \, .$$

Proof. Let $[E, (\iota_e)] \in H^2_c(\mathcal{G}, M, s)$ be an element of $H^2_c(\mathcal{G}, M, s)$. Now it follows from the various definitions involved that, for each $e \in \mathrm{Cusp}(\mathcal{G})$, there exists an element $\gamma_e \in \Pi_{\mathcal{G}}$ such that $\Pi_{s'(e)} = \gamma_e \cdot \Pi_{s(e)} \cdot \gamma_e^{-1}$. For each $e \in \mathrm{Cusp}(\mathcal{G})$, fix a lifting $\widetilde{\gamma}_e \in E$ of $\gamma_e \in \Pi_{\mathcal{G}}$ and write $\iota'_e \colon \Pi_{s'(e)} \to E$ for the section given by

$$\begin{array}{ccc} \Pi_{s'(e)} = \gamma_e \cdot \Pi_{s(e)} \cdot \gamma_e^{-1} & \longrightarrow & E \\ \gamma_e a \gamma_e^{-1} & \mapsto & \widetilde{\gamma}_e \iota_e(a) \widetilde{\gamma}_e^{-1} \, . \end{array}$$

Then it follows immediately from the fact that $M \subseteq E$ is *contained* in the center $Z(E)$ of E that this section ι'_e does *not depend* on the choice of the lifting $\widetilde{\gamma}_e \in E$ of $\gamma_e \in \Pi_{\mathcal{G}}$. Moreover, it follows immediately from the various definitions involved that the assignment "$[E, (\iota_e)] \mapsto [E, (\iota'_e)]$" determines an *isomorphism of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*

$$H^2_c(\mathcal{G}, M, s) \xrightarrow{\sim} H^2_c(\mathcal{G}, M, s') \, .$$

This completes the proof of Lemma 3.2. Q.E.D.

Lemma 3.3 (Exactness of certain sequences). *Let M be a finitely generated $\widehat{\mathbb{Z}}^{\Sigma}$-module. Suppose that* $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. *Then the natural inclusions $\Pi_e \hookrightarrow \Pi_{\mathcal{G}}$ — where e ranges over the cusps of $\mathcal{G}$, and, for each cusp $e \in \mathrm{Cusp}(\mathcal{G})$, we use the notation Π_e to denote an edge-like subgroup of $\Pi_{\mathcal{G}}$ associated to the cusp e — determine an exact sequence of $\widehat{\mathbb{Z}}^{\Sigma}$-modules*

$$\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}}^{\mathrm{ab}}, M) \longrightarrow \bigoplus_{e \in \mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_e, M) \longrightarrow H_c^2(\mathcal{G}, M) \longrightarrow 0\,.$$

Proof. Let s be a section of the natural surjection $\mathrm{Cusp}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{G})$. Then given an element

$$(\phi_e \colon \Pi_e \to M)_{e \in \mathrm{Node}(\mathcal{G})} \in \bigoplus_{e \in \mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_e, M)\,,$$

one may construct an element

$$[M \times \Pi_{\mathcal{G}} \ (\overset{\mathrm{pr}_2}{\twoheadrightarrow} \Pi_{\mathcal{G}}), (\iota_e \colon \Pi_{s(e)} \to M \times \Pi_{\mathcal{G}})_{e \in \mathrm{Node}(\mathcal{G})}]$$

— where we write $\iota_e \colon \Pi_{s(e)} \to M \times \Pi_{\mathcal{G}}$ for the section determined by $\phi_e \colon \Pi_{s(e)} \to M$ and the natural inclusion $\Pi_{s(e)} \hookrightarrow \Pi_{\mathcal{G}}$ — of $H_c^2(\mathcal{G}, M)$. In particular, we obtain a map $\bigoplus_{e \in \mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_e, M) \to H_c^2(\mathcal{G}, M)$, which, as is easily verified, is a *homomorphism of $\widehat{\mathbb{Z}}^{\Sigma}$-modules*. Now the exactness of the sequence in question follows immediately from the fact that $\Pi_{\mathcal{G}}$ is *free pro-Σ* [cf. [CmbGC], Remark 1.1.3]. This completes the proof of Lemma 3.3. Q.E.D.

Definition 3.4. Let M be a finitely generated $\widehat{\mathbb{Z}}^{\Sigma}$-module.

(i) Let $\mathcal{E}$ be a semi-graph of anabelioids. Denote by $\mathrm{VCN}(\mathcal{E})$ the set of components of $\mathcal{E}$ [i.e., the set of vertices and edges of $\mathcal{E}$] and, for each $z \in \mathrm{VCN}(\mathcal{E})$, by $\Pi_{\mathcal{E}_z}$ the fundamental group of the anabelioid $\mathcal{E}_z$ of $\mathcal{E}$ corresponding to $z \in \mathrm{VCN}(\mathcal{E})$. Then we define a *central extension of $\mathcal{G}$ by M* to be a collection of data

$$(\mathcal{E}, \alpha = (\alpha_z \colon M \hookrightarrow \Pi_{\mathcal{E}_z})_{z \in \mathrm{VCN}(\mathcal{E})}, \beta \colon \mathcal{E}/\alpha \overset{\sim}{\to} \mathcal{G})$$

as follows:

(a) For each $z \in \mathrm{VCN}(\mathcal{E})$, $\alpha_z \colon M \hookrightarrow \Pi_{\mathcal{E}_z}$ is an injective homomorphism of profinite groups whose image is contained in the center $Z(\Pi_{\mathcal{E}_z})$ of $\Pi_{\mathcal{E}_z}$. [Thus, the image of α_z is a *normal* closed subgroup of $\Pi_{\mathcal{E}_z}$.]

(b) For each branch b of an edge e that abuts to a vertex v of $\mathcal{E}$, we assume that the outer homomorphism $\Pi_{\mathcal{E}_e} \to \Pi_{\mathcal{E}_v}$ associated to b is *injective* and fits into a *commutative diagram* of [outer] homomorphisms of profinite groups

$$\begin{array}{ccc} M & = & M \\ \alpha_e \downarrow & & \downarrow \alpha_v \\ \Pi_{\mathcal{E}_e} & \longrightarrow & \Pi_{\mathcal{E}_v} \end{array}$$

— i.e., where the lower horizontal arrow is the outer injection associated to b.

(c) Write $\mathcal{E}/\alpha$ for the semi-graph of anabelioids defined as follows: We take the underlying semi-graph of $\mathcal{E}/\alpha$ to be the underlying semi-graph of $\mathcal{E}$; for each $z \in \mathrm{VCN}(\mathcal{E})$, we take the anabelioid $(\mathcal{E}/\alpha)_z$ of $\mathcal{E}/\alpha$ corresponding to $z \in \mathrm{VCN}(\mathcal{E})$ to be the anabelioid determined by the profinite group $\Pi_{\mathcal{E}_z}/\mathrm{Im}(\alpha_z)$ [cf. condition (a)]; for each branch b of an edge e that abuts to a vertex v of $\mathcal{E}$, we take the associated morphism of anabelioids $(\mathcal{E}/\alpha)_e \to (\mathcal{E}/\alpha)_v$ to be the morphism of anabelioids naturally determined by the morphism $\mathcal{E}_e \to \mathcal{E}_v$ associated, relative to $\mathcal{E}$, to b [cf. condition (b)].

(d) $\beta \colon \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}$ is an isomorphism of semi-graphs of anabelioids.

There is an evident notion of *isomorphisms of central extensions of* $\mathcal{G}$ *by* M. Also, given a central extension of $\mathcal{G}$ by M, and a section s of the natural surjection $\mathrm{Cusp}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{G})$, there is an evident notion of *trivialization of the given central extension of* $\mathcal{G}$ *by* M *at a cusp of* $\mathcal{G}$ [cf. the discussion of Definition 3.1, (ii), (iv)].

(ii) Let

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1$$

be a *central extension of* $\Pi_{\mathcal{G}}$ *by* M. Then we shall define a semi-graph of anabelioids

$$\mathcal{G}_E$$

— which we shall refer to as the *semi-graph of anabelioids associated to the central extension* E — as follows: We take the underlying semi-graph of $\mathcal{G}_E$ to be the underlying semi-graph of $\mathcal{G}$. We take the anabelioid of $\mathcal{G}_E$ corresponding to $z \in \mathrm{VCN}(\mathcal{G})$ to be the anabelioid determined by the fiber product $E \times_{\Pi_\mathcal{G}} \Pi_z$ of the surjection $E \to \Pi_\mathcal{G}$ and a natural inclusion $\Pi_z \hookrightarrow \Pi_\mathcal{G}$ — where we use the notation $\Pi_z \subseteq \Pi_\mathcal{G}$ to denote a VCN-subgroup [cf. Definition 2.1, (i)] of $\Pi_\mathcal{G}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$; for each branch b of an edge e that abuts to a vertex v of $\mathcal{G}$, if we write $(\mathcal{G}_E)_v$, $(\mathcal{G}_E)_e$ for the anabelioids of $\mathcal{G}_E$ corresponding to v, e, respectively, then we take the morphism of anabelioids $(\mathcal{G}_E)_e \to (\mathcal{G}_E)_v$ associated to the branch b to be the morphism naturally determined by the morphism of anabelioids $\mathcal{G}_e \to \mathcal{G}_v$ associated, relative to $\mathcal{G}$, to b.

(iii) In the notation of (ii), one may verify easily that the semi-graph of anabelioids $\mathcal{G}_E$ associated to the central extension E is equipped with a *natural structure of central extension of* $\mathcal{G}$ *by* M. More precisely, for each $z \in \mathrm{VCN}(\mathcal{G})$, if we denote by $\alpha_z \colon M \hookrightarrow \Pi_{(\mathcal{G}_E)_z} = E \times_{\Pi_\mathcal{G}} \Pi_z$ the homomorphism determined by the natural inclusion $M \hookrightarrow E$ and the trivial homomorphism $M \to \Pi_z$, then there exists a natural isomorphism $\beta \colon \mathcal{G}_E/(\alpha_z)_{z\in\mathrm{VCN}(\mathcal{G})} \xrightarrow{\sim} \mathcal{G}$ such that the collection of data

$$(\mathcal{G}_E, (\alpha_z)_{z\in\mathrm{VCN}(\mathcal{G})}, \beta)$$

forms a central extension of $\mathcal{G}$ by M, which we shall refer to as the *central extension of* $\mathcal{G}$ *by* M *associated to the central extension* E.

Lemma 3.5 (Graph-theoretic localizability of central extensions of fundamental groups). *Let* M *be a finitely generated* $\widehat{\mathbb{Z}}^\Sigma$*-module. Then the following hold:*

(i) **(Exactness and centrality)** *Let*

$$(\mathcal{E}, \alpha = (\alpha_z \colon M \hookrightarrow \Pi_{\mathcal{E}_z})_{z\in\mathrm{VCN}(\mathcal{E})}, \beta \colon \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}) \qquad (\ddagger_1)$$

be a **central extension of $\mathcal{G}$ by M** *[cf. Definition 3.4, (i)]. Write* $\Pi_{\mathcal{E}}$ *for the pro-*Σ *fundamental group of* $\mathcal{E}$*, i.e., the maximal pro-*Σ *quotient of the fundamental group of* $\mathcal{E}$ *[cf. the discussion preceding* [SemiAn]*, Definition 2.2]. Then the composite* $\mathcal{E} \to \mathcal{E}/\alpha \overset{\beta}{\xrightarrow{\sim}} \mathcal{G}$ *determines an* **exact sequence** *of profinite groups*

$$1 \longrightarrow M \longrightarrow \Pi_{\mathcal{E}} \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1 \qquad (\ddagger_2)$$

which is **central**.

(ii) **(Natural isomorphism I)** *In the notation of (i), the central extension of* $\mathcal{G}$ *by* M *associated to the central extension* $(\ddagger_2)$ *[cf. Definition 3.4, (iii)] is* **naturally isomorphic***, as a central extension of* $\mathcal{G}$ *by* M*, to* $(\ddagger_1)$*.*

(iii) **(Natural isomorphism II)** *Let*

$$1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1$$

be a **central extension** *of* $\Pi_{\mathcal{G}}$ *by* M*. Then the pro-*Σ *fundamental group of the semi-graph of anabelioids* $\mathcal{G}_E$ *associated to the central extension* E *[cf. Definition 3.4, (ii)] — i.e., the maximal pro-*Σ *quotient of the fundamental group of* $\mathcal{G}_E$ *— is* **naturally isomorphic***, over* $\Pi_{\mathcal{G}}$*, to* E*.*

(iv) **(Equivalence of categories)** *The correspondences of (i), (ii), (iii) determine a* **natural equivalence of categories** *between the category of central extensions of* $\mathcal{G}$ *by* M *and the category of central extensions of* $\Pi_{\mathcal{G}}$ *by* M*. [Here, we take the morphisms in both categories to be the* **isomorphisms** *of central extensions of the sort under consideration.] Moreover, this equivalence extends to a similar* **natural equivalence of categories** *between categories of central extensions equipped with* **trivializations at the cusps of $\mathcal{G}$** *[cf. Definitions 3.1, (ii); 3.4, (i)].*

Proof. First, we verify assertion (i). If $\mathrm{Node}(\mathcal{G}) = \emptyset$, then assertion (i) is immediate; thus, suppose that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. For each connected finite étale covering $\mathcal{E}' \to \mathcal{E}$ of $\mathcal{E}$, denote by $\Pi_{\mathcal{E}'}$ the pro-Σ fundamental group of $\mathcal{E}'$, by $\mathrm{VCN}(\mathcal{E}')$ the set of components of $\mathcal{E}'$ [i.e., the set of vertices and edges of $\mathcal{E}'$], and by $\mathrm{Vert}(\mathcal{E}')$ the set of vertices of $\mathcal{E}'$; for each $z \in \mathrm{VCN}(\mathcal{E}')$, denote by $\mathcal{E}'_z$ the anabelioid of $\mathcal{E}'$ corresponding to $z \in \mathrm{VCN}(\mathcal{E}')$ and by $\Pi_{\mathcal{E}'_z}$ the fundamental group of $\mathcal{E}'_z$. Now we *claim* that

> $(*_1)$: the composite in question $\mathcal{E} \to \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}$ induces an *isomorphism* between the underlying semi-graphs, as well as an outer *surjection* $\Pi_{\mathcal{E}} \twoheadrightarrow \Pi_{\mathcal{G}}$.

Indeed, the fact that the composite in question determines an *isomorphism* between the underlying semi-graphs follows from conditions (c), (d) of Definition 3.4, (i). In particular, we obtain a *bijection* $\mathrm{VCN}(\mathcal{E}) \xrightarrow{\sim} \mathrm{VCN}(\mathcal{G})$. Now for each $z \in \mathrm{VCN}(\mathcal{E}) \xrightarrow{\sim} \mathrm{VCN}(\mathcal{G})$, again by conditions (c), (d) of Definition 3.4, (i), the composite $\mathcal{E} \to \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}$ induces an outer *surjection* $\Pi_{\mathcal{E}_z} \twoheadrightarrow \Pi_z$, where we use the notation $\Pi_z \subseteq \Pi_{\mathcal{G}}$ to denote a VCN-subgroup [cf. Definition 2.1, (i)] of $\Pi_{\mathcal{G}}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$. Therefore, in light of the *isomorphism* verified above between the semi-graphs of $\mathcal{E}$ and $\mathcal{G}$, one may verify easily that the natural outer homomorphism $\Pi_{\mathcal{E}} \to \Pi_{\mathcal{G}}$ is *surjective*. This completes the proof of the *claim* $(*_1)$.

For each vertex $v \in \mathrm{Vert}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G})$ [cf. *claim* $(*_1)$], it follows from the assumption that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$ that any verticial subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$ associated to a vertex $v \in \mathrm{Vert}(\mathcal{G})$ is a *free pro*-Σ group [cf. [CmbGC], Remark 1.1.3]; thus, there exists a *section* of the natural surjection $\Pi_{\mathcal{E}_v} \twoheadrightarrow \Pi_v$. Now for each vertex $v \in \mathrm{Vert}(\mathcal{G})$, let us fix such a section of the natural surjection $\Pi_{\mathcal{E}_v} \twoheadrightarrow \Pi_v$, hence also — since the extension $\Pi_{\mathcal{E}_v}$ of Π_v by M is *central* [cf. condition (a) of Definition 3.4, (i)] — an isomorphism $t_v \colon M \times \Pi_v \xrightarrow{\sim} \Pi_{\mathcal{E}_v}$. Let $\mathcal{G}_1 \to \mathcal{G}$ be a connected finite étale Galois covering of $\mathcal{G}$ and write $\mathcal{E}_1 \stackrel{\mathrm{def}}{=} \mathcal{E} \times_{\mathcal{G}} \mathcal{G}_1$. Then it follows from the *claim* $(*_1)$ that $\mathcal{E}_1$ is *connected*; moreover, one may verify easily that the structure of central extension of $\mathcal{G}$ by M on $\mathcal{E}$ naturally determines a structure of central extension of $\mathcal{G}_1$ by M on $\mathcal{E}_1$, and that for each vertex $v \in \mathrm{Vert}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G})$ and each vertex $w \in \mathrm{Vert}(\mathcal{E}_1) \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G}_1)$ that lies over v, the normal closed subgroup $\Pi_{(\mathcal{E}_1)_w} \subseteq \Pi_{\mathcal{E}_v}$ *corresponds to* $M \times \Pi_w \subseteq M \times \Pi_v$ relative to the isomorphism $t_v \colon M \times \Pi_v \xrightarrow{\sim} \Pi_{\mathcal{E}_v}$ fixed above, i.e., we obtain an isomorphism $t_w \colon M \times \Pi_w \xrightarrow{\sim} \Pi_{(\mathcal{E}_1)_w}$.

Now for a finite quotient $M \twoheadrightarrow Q$ of M and a connected finite étale Galois covering $\mathcal{G}_1 \to \mathcal{G}$ of $\mathcal{G}$, we shall say that a connected finite étale covering $\mathcal{E}_2 \to \mathcal{E}$ of $\mathcal{E}$ satisfies the *condition* $(\dagger_{Q,\mathcal{G}_1})$ if the following two conditions are satisfied:

$(\dagger^1_{Q,\mathcal{G}_1})$ $\mathcal{E}_2 \to \mathcal{E}$ *factors through* $\mathcal{E}_1 \stackrel{\mathrm{def}}{=} \mathcal{E} \times_{\mathcal{G}} \mathcal{G}_1 \to \mathcal{E}$, the resulting covering $\mathcal{E}_2 \to \mathcal{E}_1$ is *Galois*, and for each vertex $v \in \mathrm{VCN}(\mathcal{E}_1)$, the composite

$$M \hookrightarrow \Pi_{(\mathcal{E}_1)_v} \to \Pi_{\mathcal{E}_1} \twoheadrightarrow \Pi_{\mathcal{E}_1}/\Pi_{\mathcal{E}_2}$$

is *surjective*, with kernel equal to the kernel of $M \twoheadrightarrow Q$.

$(\dagger^2_{Q,\mathcal{G}_1})$ $\mathcal{E}_2 \to \mathcal{E}$ is *Galois*.

Then we *claim* that

> $(*_2)$: for any finite quotient $M \twoheadrightarrow Q$ of M and any connected finite étale Galois covering $\mathcal{G}_1 \to \mathcal{G}$, there exists — after possibly replacing $\mathcal{G}_1 \to \mathcal{G}$ by a connected finite étale Galois covering of $\mathcal{G}$ that factors through $\mathcal{G}_1 \to \mathcal{G}$ — a connected finite étale covering of $\mathcal{E}$ which satisfies the condition $(\dagger_{Q,\mathcal{G}_1})$.

Indeed, let $M \twoheadrightarrow Q$ be a finite quotient of M, $\mathcal{G}_1 \to \mathcal{G}$ a connected finite étale Galois covering of $\mathcal{G}$, and $\mathcal{E}_1 \stackrel{\mathrm{def}}{=} \mathcal{E} \times_{\mathcal{G}} \mathcal{G}_1$. For each vertex $v \in \mathrm{Vert}(\mathcal{E}_1) \stackrel{\sim}{\to} \mathrm{Vert}(\mathcal{G}_1)$ [cf. the above discussion], denote by $\Pi_{(\mathcal{E}_1)_v} \twoheadrightarrow Q_v$ the quotient of $\Pi_{(\mathcal{E}_1)_v}$ obtained by forming the composite

$$\Pi_{(\mathcal{E}_1)_v} \overset{t_v}{\stackrel{\sim}{\leftarrow}} M \times \Pi_v \overset{\mathrm{pr}_1}{\twoheadrightarrow} M \twoheadrightarrow Q\,.$$

Thus, we have a natural isomorphism $Q \stackrel{\sim}{\to} Q_v$. Next, let e be a node of $\mathcal{E}_1$; b, b' the two *distinct* branches of e; v, v' the [*not necessarily distinct*] vertices of $\mathcal{E}_1$ to which b, b' abut. Then since the quotient Q [$\simeq Q_v \simeq Q_{v'}$] is *finite*, one may verify easily that — after possibly replacing $\mathcal{G}_1 \to \mathcal{G}$ by a connected finite étale Galois covering of $\mathcal{G}$ that factors through $\mathcal{G}_1 \to \mathcal{G}$ — the kernels of the two composites $\Pi_{(\mathcal{E}_1)_e} \hookrightarrow \Pi_{(\mathcal{E}_1)_v} \twoheadrightarrow Q_v$, $\Pi_{(\mathcal{E}_1)_e} \hookrightarrow \Pi_{(\mathcal{E}_1)_{v'}} \twoheadrightarrow Q_{v'}$ — where $\Pi_{(\mathcal{E}_1)_e} \hookrightarrow \Pi_{(\mathcal{E}_1)_v}$, $\Pi_{(\mathcal{E}_1)_e} \hookrightarrow \Pi_{(\mathcal{E}_1)_{v'}}$ are the natural outer injections corresponding to b, b', respectively — *coincide*. Moreover, if we write $N_e \subseteq \Pi_{(\mathcal{E}_1)_e}$ for this kernel, then it follows immediately from condition (b) of Definition 3.4, (i), that the actions of Q induced by the natural isomorphisms $Q \stackrel{\sim}{\to} Q_v \stackrel{\sim}{\leftarrow} \Pi_{(\mathcal{E}_1)_e}/N_e$, $Q \stackrel{\sim}{\to} Q_{v'} \stackrel{\sim}{\leftarrow} \Pi_{(\mathcal{E}_1)_e}/N_e$ on the connected finite étale Galois covering of $(\mathcal{E}_1)_e$ corresponding to $N_e \subseteq \Pi_{(\mathcal{E}_1)_e}$ *coincide*. Therefore, since the underlying semi-graph of $\mathcal{E}_1$ is *finite*, by applying this argument to the various nodes of $\mathcal{E}_1$ and then gluing the connected finite étale Galois coverings of the various $(\mathcal{E}_1)_v$'s corresponding to the quotients $\Pi_{(\mathcal{E}_1)_v} \twoheadrightarrow Q_v$ to one another by means of *Q-equivariant isomorphisms*, we obtain a connected finite étale Galois covering $\mathcal{E}_2 \to \mathcal{E}_1$ which satisfies the condition $(\dagger^1_{Q,\mathcal{G}_1})$.

Write $\mathcal{E}_2^0 \to \mathcal{E}$ for the Galois closure of the connected finite étale covering $\mathcal{E}_2 \to \mathcal{E}$; thus, since $\mathcal{E}_1$ is *Galois* over $\mathcal{E}$, we have connected finite étale *Galois* coverings $\mathcal{E}_2^0 \to \mathcal{E}_2 \to \mathcal{E}_1$ of $\mathcal{E}_1$. Now it follows immediately from the condition ($\dagger^1_{Q,\mathcal{G}_1}$) that $\mathcal{E}_2 \to \mathcal{E}_1$ induces an *isomorphism* between the underlying semi-graphs. In particular, it follows from Lemma 3.6 below, in light of the *claim* ($*_1$), that the natural outer homomorphisms $\Pi_{\mathcal{E}_2} \hookrightarrow \Pi_{\mathcal{E}_1} \twoheadrightarrow \Pi_{\mathcal{G}_1}$ induce outer *isomorphisms* $\Pi_{\mathcal{E}_2}/\Pi^{\mathrm{vert}}_{\mathcal{E}_2} \overset{\sim}{\to} \Pi_{\mathcal{E}_1}/\Pi^{\mathrm{vert}}_{\mathcal{E}_1} \overset{\sim}{\to} \Pi_{\mathcal{G}_1}/\Pi^{\mathrm{vert}}_{\mathcal{G}_1} \simeq \pi_1^{\mathrm{top}}(\mathbb{G}_1)^{\Sigma}$, where we write "$\Pi^{\mathrm{vert}}_{(-)} \subseteq \Pi_{(-)}$" for the normal closed subgroup of "$\Pi_{(-)}$" topologically normally generated by the verticial subgroups and $\pi_1^{\mathrm{top}}(\mathbb{G}_1)^{\Sigma}$ for the pro-Σ completion of the [discrete] topological fundamental group of the underlying semi-graph $\mathbb{G}_1$ of $\mathcal{G}_1$. On the other hand, since for each vertex $v \in \mathrm{Vert}(\mathcal{E}) \overset{\sim}{\to} \mathrm{Vert}(\mathcal{G})$ and each vertex $w \in \mathrm{Vert}(\mathcal{E}_1) \overset{\sim}{\to} \mathrm{Vert}(\mathcal{G}_1)$ that lies over v, the isomorphism $t_w \colon M \times \Pi_w \overset{\sim}{\to} \Pi_{(\mathcal{E}_1)_w}$ *arises from the isomorphism* $t_v \colon M \times \Pi_v \overset{\sim}{\to} \Pi_{\mathcal{E}_v}$, one may verify easily that the closed subgroup $\Pi_{(\mathcal{E}_2)_w} \subseteq \Pi_{\mathcal{E}_v}$ is *normal*. [Here, we regard $w \in \mathrm{Vert}(\mathcal{E}_1)$ as an element of $\mathrm{Vert}(\mathcal{E}_2)$ by the bijection $\mathrm{Vert}(\mathcal{E}_2) \overset{\sim}{\to} \mathrm{Vert}(\mathcal{E}_1)$ induced by $\mathcal{E}_2 \to \mathcal{E}_1$.] In particular, it follows immediately that the connected finite étale Galois covering $\mathcal{E}_2^0 \to \mathcal{E}_2$ *arises from a normal open subgroup of the quotient* $\Pi_{\mathcal{E}_2} \twoheadrightarrow \Pi_{\mathcal{E}_2}/\Pi^{\mathrm{vert}}_{\mathcal{E}_2} \overset{\sim}{\to} \pi_1(\mathbb{G}_1)^{\Sigma}$. Therefore, there exists a connected finite étale Galois covering $\mathcal{G}'_1 \to \mathcal{G}$ that factors through $\mathcal{G}_1 \to \mathcal{G}$ [and arises from a normal open subgroup of the quotient $\Pi_{\mathcal{G}_1} \twoheadrightarrow \pi_1^{\mathrm{top}}(\mathbb{G}_1)^{\Sigma}$] such that the connected finite étale covering $\mathcal{E}_2 \times_{\mathcal{G}_1} \mathcal{G}'_1$ of $\mathcal{E}$ is *Galois*. Now it follows immediately from the fact that $\mathcal{E}_2 \to \mathcal{E}$ satisfies the condition ($\dagger^1_{Q,\mathcal{G}_1}$) that $\mathcal{E}_2 \times_{\mathcal{G}_1} \mathcal{G}'_1 \to \mathcal{E}$ satisfies both conditions ($\dagger^1_{Q,\mathcal{G}'_1}$) and ($\dagger^2_{Q,\mathcal{G}'_1}$), as desired. This completes the proof of the *claim* ($*_2$).

Next, we *claim* that

> ($*_3$): the composite $\mathcal{E} \to \mathcal{E}/\alpha \overset{\sim}{\to} \mathcal{G}$, together with the composites
> $$M \hookrightarrow \Pi_{\mathcal{E}_v} \to \Pi_{\mathcal{E}}$$
> for $v \in \mathrm{Vert}(\mathcal{E})$, determine an exact sequence of profinite groups
> $$1 \longrightarrow M \longrightarrow \Pi_{\mathcal{E}} \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1\,.$$

Indeed, it follows immediately from the *claim* ($*_2$) — by arguing as in the final portion of the proof of ($*_2$) — that any connected finite étale Galois covering of $\mathcal{E}$ is a subcovering of a covering of $\mathcal{E}$ which satisfies the condition ($\dagger_{Q,\mathcal{G}_1}$) for some finite quotient $M \twoheadrightarrow Q$ of M and some connected finite étale Galois covering $\mathcal{G}_1$ of $\mathcal{G}$. Therefore, the *exactness* of the sequence in question follows immediately from the various definitions involved, together with the *claim* ($*_1$). This completes the proof of the *claim* ($*_3$).

Finally, we *claim* that

> $(*_4)$: the exact sequence of profinite groups
>
> $$1 \longrightarrow M \longrightarrow \Pi_{\mathcal{E}} \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1$$
>
> of $(*_3)$ is *central*, i.e., if we write $\rho\colon \Pi_{\mathcal{G}} \to \mathrm{Aut}(M)$ for the representation of $\Pi_{\mathcal{G}}$ on M determined by this extension $\Pi_{\mathcal{E}}$, then ρ is *trivial*.

Indeed, it follows immediately from condition (a) of Definition 3.4, (i), that $\Pi_{\mathcal{G}}^{\mathrm{vert}} \subseteq \mathrm{Ker}(\rho)$, where we write $\Pi_{\mathcal{G}}^{\mathrm{vert}} \subseteq \Pi_{\mathcal{G}}$ for the normal closed subgroup of $\Pi_{\mathcal{G}}$ topologically normally generated by the verticial subgroups of $\Pi_{\mathcal{G}}$. On the other hand, it follows immediately from condition (b) of Definition 3.4, (i), by *"parallel transporting"* along loops on $\mathbb{G}$, that the restriction to $\pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \pi_1^{\mathrm{top}}(\mathbb{G})^{\Sigma}$ of the representation $[\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}/\Pi_{\mathcal{G}}^{\mathrm{vert}} \xrightarrow{\sim}]\ \pi_1^{\mathrm{top}}(\mathbb{G})^{\Sigma} \to \mathrm{Aut}(M)$ [cf. Lemma 3.6 below] induced by ρ — where we write $\pi_1^{\mathrm{top}}(\mathbb{G})$ for the [discrete] topological fundamental group of the semi-graph $\mathbb{G}$ and $\pi_1^{\mathrm{top}}(\mathbb{G})^{\Sigma}$ for the pro-Σ completion of $\pi_1^{\mathrm{top}}(\mathbb{G})$ — is *trivial*. In particular, since the subgroup $\pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \pi_1^{\mathrm{top}}(\mathbb{G})^{\Sigma}$ is *dense*, the representation ρ is *trivial*, as desired. This completes the proof of the *claim* $(*_4)$, hence also the proof of assertion (i).

Assertion (ii) follows immediately from the various definitions involved. Next, we verify assertion (iii). It follows immediately from assertion (i), together with Definition 3.4, (iii), that if we write $\Pi_{\mathcal{G}_E}$ for the pro-Σ fundamental group of $\mathcal{G}_E$, then we have a natural exact sequence of profinite groups

$$1 \longrightarrow M \longrightarrow \Pi_{\mathcal{G}_E} \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1\,.$$

On the other hand, it follows immediately from the definition of $\mathcal{G}_E$ that one may construct a *tautological profinite covering* of $\mathcal{G}_E$ [i.e., a pro-object of the category $\mathcal{B}(\mathcal{G}_E)$ that appears in the discussion following [SemiAn], Definition 2.1] equipped with a *tautological action* by E. In particular, one obtains an outer surjection $\Pi_{\mathcal{G}_E} \twoheadrightarrow E$ that is compatible with the respective outer surjections to $\Pi_{\mathcal{G}}$. Thus, one concludes from the "Five Lemma" that this outer surjection $\Pi_{\mathcal{G}_E} \twoheadrightarrow E$ is an outer isomorphism, as desired. This completes the proof of assertion (iii). Assertion (iv) follows immediately, in light of assertions (i), (ii), (iii), from the various definitions involved. This completes the proof of Lemma 3.5. Q.E.D.

Lemma 3.6 (**Quotients by verticial subgroups).** *Let $\mathcal{H}$ be a semi-graph of anabelioids. Write $\Pi_{\mathcal{H}}$ for the pro-Σ fundamental group of $\mathcal{H}$ [i.e., the pro-Σ quotient of the fundamental group of $\mathcal{H}$] and $\Pi_{\mathcal{H}}^{\text{vert}} \subseteq \Pi_{\mathcal{H}}$ for the normal closed subgroup of $\Pi_{\mathcal{H}}$ topologically normally generated by the verticial subgroups of $\Pi_{\mathcal{H}}$. Then the natural injection $\Pi_{\mathcal{H}}^{\text{vert}} \hookrightarrow \Pi_{\mathcal{H}}$ determines an* **exact** *sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathcal{H}}^{\text{vert}} \longrightarrow \Pi_{\mathcal{H}} \longrightarrow \pi_1^{\text{top}}(\mathbb{H})^{\Sigma} \longrightarrow 1$$

— where we write $\pi_1^{\text{top}}(\mathbb{H})^{\Sigma}$ for the pro-Σ completion of the [discrete] topological fundamental group $\pi_1^{\text{top}}(\mathbb{H})$ of the underlying semi-graph $\mathbb{H}$ of $\mathcal{H}$.

Proof. This follows immediately from the various definitions involved. Q.E.D.

Theorem 3.7 (**Properties of the second cohomology group with compact supports).** *Let Σ be a nonempty set of prime numbers, $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type, and M a finitely generated $\widehat{\mathbb{Z}}^{\Sigma}$-module. Then the following hold:*

(i) **(Change of coefficients)** *There exists a* **natural isomorphism** *of $\widehat{\mathbb{Z}}^{\Sigma}$-modules*

$$H_c^2(\mathcal{G}, M) \xrightarrow{\sim} H_c^2(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma}) \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} M$$

that is **functorial** *with respect to isomorphisms of the pair $(\mathcal{G}, M)$. If, moreover, $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, then there exists a* **natural isomorphism** *of $\widehat{\mathbb{Z}}^{\Sigma}$-modules*

$$H_c^2(\mathcal{G}, M) \xrightarrow{\sim} H^2(\mathcal{G}, M)$$

that is **functorial** *with respect to isomorphisms of the pair $(\mathcal{G}, M)$.*

(ii) **(Structure as an abstract profinite group)** *The second cohomology group with compact supports $H_c^2(\mathcal{G}, M)$ of $\mathcal{G}$ is [noncanonically]* **isomorphic to** *M.*

(iii) **(Synchronization with respect to generization)** *Let $S \subseteq \mathrm{Node}(\mathcal{G})$ be a subset of $\mathrm{Node}(\mathcal{G})$. Then the* **specialization** *outer isomorphism $\Phi_{\mathcal{G}_{\rightsquigarrow S}} : \Pi_{\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ with respect to S [cf. Definition 2.10] determines a* **natural isomorphism**

$$H_c^2(\mathcal{G}, M) \xrightarrow{\sim} H_c^2(\mathcal{G}_{\rightsquigarrow S}, M)$$

that is **functorial** *with respect to isomorphisms of the triple* $(\mathcal{G}, S, M)$*.*

(iv) **(Synchronization with respect to "surgery")** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of* $\mathbb{G}$*,* $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)], and* $T \subseteq \mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ *[cf. Definition 2.5, (ii)] an* **omittable** *[cf. Definition 2.4, (i)] subset of* $\mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$*. Then there exists a* **natural isomorphism** *— given by* **"extension by zero"** *—*

$$H^2_c(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}, M) \xrightarrow{\sim} H^2_c(\mathcal{G}, M)$$

[cf. Definition 2.4, (ii)] that is **functorial** *with respect to isomorphisms of the quintuple* $(\mathcal{G}, \mathbb{H}, S, T, M)$*. In particular, for each vertex* $v \in \mathrm{Vert}(\mathcal{G})$ *of* $\mathcal{G}$*, there exists a* **natural isomorphism** *of* $\widehat{\mathbb{Z}}^\Sigma$*-modules*

$$H^2_c(v, M) \xrightarrow{\sim} H^2_c(\mathcal{G}, M)$$

[cf. Remark 2.5.1, (ii)] that is **functorial** *with respect to isomorphisms of the triple* $(\mathcal{G}, v, M)$*.*

(v) **(Homomorphisms induced by finite étale coverings)** *Let* $\mathcal{H} \to \mathcal{G}$ *be a connected finite étale covering of* $\mathcal{G}$*. Then the* **image** *of the natural homomorphism*

$$H^2_c(\mathcal{G}, M) \longrightarrow H^2_c(\mathcal{H}, M)$$

is given by

$$[\Pi_{\mathcal{G}} : \Pi_{\mathcal{H}}] \cdot H^2_c(\mathcal{H}, M)\,.$$

Proof. Assertion (iii) follows immediately from condition (1) of Proposition 2.9, (i).

Next, we verify assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. The existence of a *natural isomorphism* $H^2_c(\mathcal{G}, M) \xrightarrow{\sim} H^2_c(\mathcal{G}, \widehat{\mathbb{Z}}^\Sigma) \otimes_{\widehat{\mathbb{Z}}^\Sigma} M$ follows immediately from Lemma 3.3. On the other hand, the fact that $H^2_c(\mathcal{G}, M)$ is [noncanonically] *isomorphic to* M follows immediately from Lemma 3.3, together with the following well-known facts [cf. [CmbGC], Remark 1.1.3]:

(A) $\Pi_{\mathcal{G}}$ is a *free pro-*Σ group.

(B) For any cusp $e_0 \in \mathrm{Cusp}(\mathcal{G})$ of $\mathcal{G}$, the natural homomorphism of $\widehat{\mathbb{Z}}^{\Sigma}$-modules

$$\bigoplus_{e \in \mathrm{Cusp}(\mathcal{G}) \setminus \{e_0\}} \Pi_e \longrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$$

is a *split injection of free* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules* [cf. the discussion entitled "*Topological groups*" in §0], and its image contains the image of Π_{e_0} in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$.

This completes the proof of assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$.

Next, we verify assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$. The existence of a *natural isomorphism* $H_c^2(\mathcal{G}, M) \overset{\sim}{\to} H^2(\mathcal{G}, M)$ is well-known [cf., e.g., [NSW], Theorem 2.7.7]. Now it follows from assertion (iii) that to verify assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, we may assume without loss of generality — by replacing $\mathcal{G}$ by $\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G})}$ — that $\mathrm{Node}(\mathcal{G}) = \emptyset$. Then the existence of a *natural isomorphism* $H_c^2(\mathcal{G}, M) \overset{\sim}{\to} H_c^2(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma}) \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} M$ and the fact that $H_c^2(\mathcal{G}, M)$ is [non-canonically] *isomorphic* to M follow immediately from the existence of a *natural isomorphism* $H_c^2(\mathcal{G}, M) \overset{\sim}{\to} H^2(\mathcal{G}, M)$ and the fact that any compact Riemann surface of genus $\neq 0$ is a "$K(\pi, 1)$" space [i.e., its universal covering is contractible], together with the well-known structure of the second cohomology group of a compact Riemann surface. This completes the proof of assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$.

Next, we verify assertion (iv) in the case where $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$, i.e., $((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T} = \mathcal{G}_{\bullet T}$. Thus, suppose that $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$. Now define a homomorphism of $\widehat{\mathbb{Z}}^{\Sigma}$-modules

$$H_c^2(\mathcal{G}_{\bullet T}, M) \longrightarrow H_c^2(\mathcal{G}, M)$$

as follows: Let $\widetilde{\mathcal{G}}_{\bullet T} \to \mathcal{G}_{\bullet T}$ be a universal covering of $\mathcal{G}_{\bullet T}$ which is compatible [in the evident sense] with the universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ of $\mathcal{G}$, $s^{\bullet}$ a section of the natural surjection $\mathrm{Cusp}(\widetilde{\mathcal{G}}_{\bullet T}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{G}_{\bullet T})$, and $[E^{\bullet}, (\iota_e^{\bullet} \colon \Pi_{s^{\bullet}(e)} \to E^{\bullet})_{e \in \mathrm{Cusp}(\mathcal{G}_{\bullet T})}] \in H_c^2(\mathcal{G}_{\bullet T}, M)$ an element of $H_c^2(\mathcal{G}_{\bullet T}, M)$. Write E for the fiber product of the surjection $E^{\bullet} \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}}$ and the natural surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}}$ [arising from the compatibility of the respective universal coverings]. Next, we introduce notation as follows:

- for $e \in \mathrm{Cusp}(\mathcal{G}_{\bullet T})$ $(= \mathrm{Cusp}(\mathcal{G}) \setminus T \subseteq \mathrm{Cusp}(\mathcal{G}))$, denote by $\iota_e \colon \Pi_e \to E$ — where we use the notation $\Pi_e \subseteq \Pi_{\mathcal{G}}$ to denote an edge-like subgroup of $\Pi_{\mathcal{G}}$ associated to e such that the composite $\Pi_e \hookrightarrow \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}}$ determines an isomorphism of Π_e with $\Pi_{s^\bullet(e)} \subseteq \Pi_{\mathcal{G}_{\bullet T}}$ — the section over Π_e naturally determined by the composite

$$\Pi_e \xrightarrow{\sim} \Pi_{s^\bullet(e)} \xrightarrow{\iota_e^\bullet} E^\bullet ,$$

and

- for $e \in \mathrm{Cusp}(\mathcal{G}) \setminus \mathrm{Cusp}(\mathcal{G}_{\bullet T})$ $(= T \subseteq \mathrm{Cusp}(\mathcal{G}))$, denote by $\iota_e \colon \Pi_e \to E$ — where we use the notation $\Pi_e \subseteq \Pi_{\mathcal{G}}$ to denote an edge-like subgroup of $\Pi_{\mathcal{G}}$ associated to e — the section over Π_e naturally determined by the *trivial* homomorphism $\Pi_e \to E^\bullet$.

Then it follows immediately from the various definitions involved that the assignment "$[E^\bullet, (\iota_e^\bullet)_{e \in \mathrm{Cusp}(\mathcal{G}_{\bullet T})}] \mapsto (E, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})})$" determines a *homomorphism of* $\widehat{\mathbb{Z}}^\Sigma$*-modules*

$$H^2_c(\mathcal{G}_{\bullet T}, M) \longrightarrow H^2_c(\mathcal{G}, M) ,$$

as desired.

Next, we verify that this homomorphism $H^2_c(\mathcal{G}_{\bullet T}, M) \to H^2_c(\mathcal{G}, M)$ is an *isomorphism*. First, let us observe that it follows from assertion (ii) that, to verify that the homomorphism in question is an *isomorphism*, it suffices to verify that it is *surjective*. The rest of the proof of assertion (iv) in the case where $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$ is devoted to verifying this *surjectivity*. To verify the desired *surjectivity*, by induction on the cardinality $T^\sharp$ of the *finite* set T, we may assume without loss of generality that $T^\sharp = 1$, i.e., $T = \{e_0\}$ for some $e_0 \in \mathrm{Cusp}(\mathcal{G})$.

To verify the desired *surjectivity*, let $[E, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})}] \in H^2_c(\mathcal{G}, M)$ be an element of $H^2_c(\mathcal{G}, M)$. Then since $\Pi_{\mathcal{G}}$ is a *free pro-*Σ group, there exists a continuous section $\Pi_{\mathcal{G}} \to E$ of the surjection $E \twoheadrightarrow \Pi_{\mathcal{G}}$, hence also — since the extension E of $\Pi_{\mathcal{G}}$ is *central* — an isomorphism $M \times \Pi_{\mathcal{G}} \xrightarrow{\sim} E$. Write $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi$ for the *maximal cuspidally central quotient* [cf. [AbsCsp], Definition 1.1, (i)] relative to the surjection $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}}$, E_Π for the quotient of E by the normal closed subgroup of E corresponding to $\{1\} \times \mathrm{Ker}(\Pi_{\mathcal{G}} \twoheadrightarrow \Pi) \subseteq M \times \Pi_{\mathcal{G}}$ [thus, $E_\Pi \xleftarrow{\sim} M \times \Pi$], and $N \subseteq E_\Pi$ for the image of the composite

$$\Pi_{s(e_0)} \overset{\iota_{e_0}}{\hookrightarrow} E \twoheadrightarrow E_\Pi .$$

Now we *claim* that $N \subseteq E_\Pi$ is *contained in the center* $Z(E_\Pi)$ of E_Π, hence also *normal* in E_Π. Indeed, since the composite

$$\Pi_{s(e_0)} \hookrightarrow \Pi_{\mathcal{G}} \twoheadrightarrow \Pi$$

is *injective*, and its image coincides with the kernel of the natural surjection $\Pi \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}}$, it holds that the image of the composite

$$\Pi_{s(e_0)} \overset{\iota_{e_0}}{\hookrightarrow} E \twoheadrightarrow E_\Pi \overset{\sim}{\leftarrow} M \times \Pi$$

is *contained in* $M \times \mathrm{Ker}(\Pi \to \Pi_{\mathcal{G}_{\bullet T}})$. On the other hand, since the extension E of $\Pi_\mathcal{G}$ is *central*, it follows from the definition of the quotient Π of $\Pi_\mathcal{G}$ that the image of $M \times \mathrm{Ker}(\Pi \twoheadrightarrow \Pi_{\mathcal{G}_{\bullet T}})$ in E_Π via $M \times \Pi \overset{\sim}{\to} E_\Pi$ is *contained in the center* $Z(E_\Pi)$ of E_Π. This completes the proof of the above *claim*.

Now it follows from the definition of $N \subseteq E_\Pi$, together with the above *claim*, that we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & \Pi_\mathcal{G} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & M & \longrightarrow & E_\Pi/N & \longrightarrow & \Pi_{\mathcal{G}_{\bullet T}} & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. In particular, we obtain an extension E_Π/N of $\Pi_{\mathcal{G}_{\bullet T}}$ by M, which is *central* since the extension E is central. For $e \in \mathrm{Cusp}(\mathcal{G}_{\bullet T}) = \mathrm{Cusp}(\mathcal{G}) \setminus \{e_0\}$, write $\Pi_e^\bullet \subseteq \Pi_{\mathcal{G}_{\bullet T}}$ for the edge-like subgroup of $\Pi_{\mathcal{G}_{\bullet T}}$ [associated to $e \in \mathrm{Cusp}(\mathcal{G}_{\bullet T})$] determined by the image of $\Pi_{s(e)} \subseteq \Pi_\mathcal{G}$ and $\iota_e^\bullet$ for the section $\Pi_e^\bullet \to E_\Pi/N$ over $\Pi_e^\bullet$ determined by $\iota_e \colon \Pi_{s(e)} \to E$. Then it follows immediately from the various definitions involved that the image of

$$[E_\Pi/N, (\iota_e^\bullet)_{e' \in \mathrm{Cusp}(\mathcal{G}_{\bullet T})}] \in H_c^2(\mathcal{G}_{\bullet T}, M)$$

in $H_c^2(\mathcal{G}, M)$ is $[E, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})}] \in H_c^2(\mathcal{G}, M)$. This completes the proof of the desired *surjectivity* and hence of assertion (iv) in the case where $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$.

Next, to complete the proof of assertion (iv) in the *general case*, one verifies immediately that it suffices to verify assertion (iv) in the case where $T = \emptyset$, i.e., $((\mathcal{G}|_\mathbb{H})_{\succ S})_{\bullet T} = (\mathcal{G}|_\mathbb{H})_{\succ S}$. Thus, suppose that $T = \emptyset$. Write $\mathcal{H} \overset{\mathrm{def}}{=} (\mathcal{G}|_\mathbb{H})_{\succ S}$. To define a *natural homomorphism of* $\widehat{\mathbb{Z}}^\Sigma$*-modules* $H_c^2(\mathcal{H}, M) \to H_c^2(\mathcal{G}, M)$, let $\widetilde{\mathcal{H}} \to \mathcal{H}$ be a universal covering of $\mathcal{H}$ which is compatible [in the evident sense] with the universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ of $\mathcal{G}$, $s_\mathcal{H}$ a section of the natural surjection $\mathrm{Cusp}(\widetilde{\mathcal{H}}) \twoheadrightarrow \mathrm{Cusp}(\mathcal{H})$, and $[E^\mathcal{H}, (\iota_e^\mathcal{H} \colon \Pi_{s_\mathcal{H}(e)} \to E^\mathcal{H})_{e \in \mathrm{Cusp}(\mathcal{H})}] \in H_c^2(\mathcal{H}, M)$ an element of $H_c^2(\mathcal{H}, M)$. Since the extension $E^\mathcal{H}$ of $\Pi_\mathcal{H}$ by M is *central*, the section $\iota_e^\mathcal{H} \colon \Pi_{s_\mathcal{H}(e)} \to E^\mathcal{H}$ naturally determines an *isomorphism*

$$M \times \Pi_{s_\mathcal{H}(e)} \xrightarrow{\sim} E^\mathcal{H} \times_{\Pi_\mathcal{H}} \Pi_{s_\mathcal{H}(e)}$$

of the direct product $M \times \Pi_{s_{\mathcal{H}}(e)}$ with the fiber product $E^{\mathcal{H}} \times_{\Pi_{\mathcal{H}}} \Pi_{s_{\mathcal{H}}(e)}$ of the surjection $E^{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}$ and the natural inclusion $\Pi_{s_{\mathcal{H}}(e)} \hookrightarrow \Pi_{\mathcal{H}}$. Write $\mathcal{G}_{E^{\mathcal{H}}}$ for the semi-graph of anabelioids associated to the central extension $E^{\mathcal{H}}$ [cf. Definition 3.4, (ii)]. Then one may define a central extension of $\mathcal{G}$ by M

$$(\mathcal{E}, \alpha, \beta \colon \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G})$$

[cf. Definition 3.4, (i)] whose restriction to $\mathcal{H}$, relative to the isomorphism $\beta \colon \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}$, is isomorphic to the semi-graph of anabelioids $\mathcal{G}_{E^{\mathcal{H}}}$ as follows: We take the underlying semi-graph of $\mathcal{E}$ to be the underlying semi-graph of $\mathcal{G}$; for each vertex $v \in \mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})$, we take the anabelioid $\mathcal{E}_v$ of $\mathcal{E}$ corresponding to the vertex $v \in \mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})$ to be the anabelioid $(\mathcal{G}_{E^{\mathcal{H}}})_v$ of $\mathcal{G}_{E^{\mathcal{H}}}$ corresponding to the vertex v; for each vertex $v \in \mathrm{Vert}(\mathcal{G}) \setminus \mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})$, we take the anabelioid $\mathcal{E}_v$ of $\mathcal{E}$ corresponding to $v \in \mathrm{Vert}(\mathcal{G}) \setminus \mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})$ to be the anabelioid associated to the profinite group $M \times \Pi_v$. Then the above *isomorphisms* $M \times \Pi_{s_{\mathcal{H}}(e)} \xrightarrow{\sim} E^{\mathcal{H}} \times_{\Pi_{\mathcal{H}}} \Pi_{s_{\mathcal{H}}(e)}$ induced by the various $\iota_e^{\mathcal{H}}$'s naturally determine the remaining data [i.e., consisting of anabelioids associated to edges and morphisms of anabelioids associated to branches] necessary to define a *semi-graph of anabelioids* $\mathcal{E}$ which is naturally equipped with a structure of *central extension of* $\mathcal{G}$ *by* M whose restriction to $\mathcal{H}$ is naturally isomorphic to the semi-graph of anabelioids $\mathcal{G}_{E^{\mathcal{H}}}$, as desired.

Now it follows from Lemma 3.5, (i), that if we denote by $\Pi_{\mathcal{E}}$ the pro-Σ fundamental group of $\mathcal{E}$ — i.e., the maximal pro-Σ quotient of the fundamental group of $\mathcal{E}$ — then $\Pi_{\mathcal{E}}$ is a *central* extension of $\Pi_{\mathcal{G}}$ by M. Thus, it follows from the *equivalences of categories* of Lemma 3.5, (iv), that the *sections* $\iota_e^{\mathcal{H}}$ — where e ranges over the cusps of $\mathcal{G}$ that abut to a vertex of $\mathcal{G}|_{\mathbb{H}}$ — and the *tautological sections* $\Pi_{e'} \hookrightarrow M \times \Pi_{e'} = \Pi_{\mathcal{E}_{e'}}$ — where e' ranges over the cusps of $\mathcal{G}$ that do *not* abut to a vertex of $\mathcal{G}|_{\mathbb{H}}$ — naturally determine an equivalence class $[\Pi_{\mathcal{E}}, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})}] \in H_c^2(\mathcal{G}, M)$. In particular, we obtain a map

$$H_c^2(\mathcal{H}, M) \longrightarrow H_c^2(\mathcal{G}, M)$$

by assigning $[E^{\mathcal{H}}, (\iota_e^{\mathcal{H}})_{e \in \mathrm{Cusp}(\mathcal{H})}] \mapsto [\Pi_{\mathcal{E}}, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})}]$. Moreover, it follows immediately from the various definitions involved that this map is a *homomorphism of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*, as desired.

Next, we verify that this homomorphism $H_c^2(\mathcal{H}, M) \to H_c^2(\mathcal{G}, M)$ is an *isomorphism*. Since, for any vertex $v \in \mathrm{Vert}(\mathcal{G}|_{\mathbb{H}})$, the natural morphism $\mathcal{G}|_v \to \mathcal{G}$ *factors through* $(\mathcal{G}|_{\mathbb{H}})_{\succ S} = \mathcal{H} \to \mathcal{G}$, by replacing $\mathcal{H}$ by $\mathcal{G}|_v$ [cf. Remark 2.5.1, (ii)], we may assume without loss of generality that $\mathcal{H} = \mathcal{G}|_v$. Moreover, if $\mathrm{Node}(\mathcal{G}) = \emptyset$, then assertion (iv) in the case where $T = \emptyset$ is immediate; thus, we may assume without loss of generality that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$. On the other hand, it follows from assertion (ii) that to verify that the homomorphism in question is an *isomorphism*, it suffices to verify that it is *surjective*. The rest of the proof of assertion (iv) in the case where $T = \emptyset$ is devoted to verifying the *surjectivity* of the homomorphism $H^2_c(v, M) \to H^2_c(\mathcal{G}, M)$.

Let $\mathcal{J}$ be a semi-graph of anabelioids of pro-Σ PSC-type such that there exist a vertex $w \in \mathrm{Vert}(\mathcal{J})$ and an "omittable" cusp $e \in \mathcal{C}(w)$ [i.e., a cusp that abuts to w such that $\{e\}$ is *omittable*] such that $\mathcal{J}_{\bullet\{e\}}$ is isomorphic to $\mathcal{G}$, and, moreover, the isomorphism $\mathcal{J}_{\bullet\{e\}} \xrightarrow{\sim} \mathcal{G}$ induces an *isomorphism* of $(\mathcal{J}|_w)_{\bullet\{e\}} \xrightarrow{\sim} \mathcal{G}|_v$. [Note that one may verify easily that such a semi-graph of anabelioids of pro-Σ PSC type always exists.] Then it follows immediately from assertion (iv) in the case where $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$, together with the various definitions involved, that we have a commutative diagram

$$\begin{array}{ccccc} H^2_c(v, M) & \xrightarrow{\sim} & H^2_c((\mathcal{J}|_w)_{\bullet\{e\}}, M) & \xrightarrow{\sim} & H^2_c(w, M) \\ \downarrow & & & & \downarrow \\ H^2_c(\mathcal{G}, M) & \xrightarrow{\sim} & H^2_c(\mathcal{J}_{\bullet\{e\}}, M) & \xrightarrow{\sim} & H^2_c(\mathcal{J}, M) \end{array}$$

— where the left-hand horizontal arrows are *isomorphisms* induced by the isomorphisms $(\mathcal{J}|_w)_{\bullet\{e\}} \xrightarrow{\sim} \mathcal{G}|_v$, $\mathcal{J}_{\bullet\{e\}} \xrightarrow{\sim} \mathcal{G}$, respectively, and the right-hand horizontal arrows are *isomorphisms* obtained by applying assertion (iv) in the case where $\mathbb{H} = \mathbb{G}$ and $S = \emptyset$. In particular, to verify the desired *surjectivity* of the homomorphism $H^2_c(v, M) \to H^2_c(\mathcal{G}, M)$, by replacing $\mathcal{G}$ (respectively, v) by $\mathcal{J}$ (respectively, w), we may assume without loss of generality that $\mathcal{C}(v) \neq \emptyset$.

To verify the desired *surjectivity* of the homomorphism $H^2_c(v, M) \to H^2_c(\mathcal{G}, M)$ in the case where $\mathcal{C}(v) \neq \emptyset$, let $[E, (\iota_e)_{e \in \mathrm{Cusp}(\mathcal{G})}] \in H^2_c(\mathcal{G}, M)$ be an element of $H^2_c(\mathcal{G}, M)$. Now it follows from Lemma 3.3, together with the assumption that $\mathcal{C}(v) \neq \emptyset$, that we have two exact sequences of $\widehat{\mathbb{Z}}^\Sigma$-modules

$$\mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi^{\mathrm{ab}}_{\mathcal{G}}, M) \longrightarrow \bigoplus_{e \in \mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi_e, M) \longrightarrow H^2_c(\mathcal{G}, M) \longrightarrow 0\,;$$

$$\mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi^{\mathrm{ab}}_{\mathcal{G}|_v}, M) \longrightarrow \bigoplus_{e \in \mathrm{Cusp}(\mathcal{G}|_v)} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi_e, M) \longrightarrow H^2_c(v, M) \longrightarrow 0\,.$$

Let $e_0 \in \mathcal{C}(v)$ be a cusp of $\mathcal{G}$ that abuts to v. Here, note that it follows immediately from the definition of $\mathcal{G}|_v$ that e_0 may be regarded as a cusp of $\mathcal{G}|_v$. Then it follows immediately from the facts (A), (B) used in the proof of assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$

that there exists a lifting $(\phi_e)_{e\in\mathrm{Cusp}(\mathcal{G})} \in \bigoplus_{e\in\mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi_e, M)$ of $[E,(\iota_e)_{e\in\mathrm{Cusp}(\mathcal{G})}] \in H^2_c(\mathcal{G}, M)$ [with respect to the first exact sequence of the above display] such that if $e \neq e_0$, then $\phi_e = 0$. Write $(\psi_e)_{e\in\mathrm{Cusp}(\mathcal{G}|_v)} \in \bigoplus_{e\in\mathrm{Cusp}(\mathcal{G}|_v)} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi_e, M)$ for the element such that $\psi_{e_0} = \phi_{e_0}$, $\psi_e = 0$ for $e \neq e_0$. Then it follows immediately from the definitions of the above exact sequences and the homomorphism $H^2_c(v, M) \to H^2_c(\mathcal{G}, M)$ in question that the image of $(\psi_e)_{e\in\mathrm{Cusp}(\mathcal{G}|_v)} \in \bigoplus_{e\in\mathrm{Cusp}(\mathcal{G}|_v)} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(\Pi_e, M)$ in $H^2_c(v, M)$ is mapped to $[E,(\iota_e)_{e\in\mathrm{Cusp}(\mathcal{G})}] \in H^2_c(\mathcal{G}, M)$ via the homomorphism $H^2_c(v, M) \to H^2_c(\mathcal{G}, M)$ in question. This completes the proof of assertion (iv) in the case where $T = \emptyset$, hence also of assertion (iv) in the general case.

Finally, we verify assertion (v). If $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, then it follows immediately from a similar argument to the argument used in the proof of assertions (i), (ii) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, together with the well-known structure of the second cohomology group of a compact Riemann surface, that assertion (v) holds. Next, suppose that $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$. Write $\mathcal{G}^{\circledcirc}$ for the *double* of $\mathcal{G}$ [cf. [CmbGC], Proposition 2.2, (i)] — i.e., the analogue in the theory of semi-graphs of anabelioids of pro-Σ PSC-type to the well-known "double" of a Riemann surface with boundary. Write $\mathcal{H}^{\circledcirc}$ for the *double* of $\mathcal{H}$. Then it follows from the various definitions involved that the connected finite étale covering $\mathcal{H} \to \mathcal{G}$ determines a connected finite étale covering $\mathcal{H}^{\circledcirc} \to \mathcal{G}^{\circledcirc}$ of degree $[\Pi_\mathcal{G} : \Pi_\mathcal{H}]$. Next, let us observe $\mathcal{G}$ (respectively, $\mathcal{H}$) may be naturally identified with the restriction [cf. Definition 2.2, (ii)] of $\mathcal{G}^{\circledcirc}$ (respectively, $\mathcal{H}^{\circledcirc}$) to a suitable sub-semi-graph of *PSC-type* of the underlying semi-graph of $\mathcal{G}^{\circledcirc}$ (respectively, $\mathcal{H}^{\circledcirc}$). Thus, it follows from assertion (iv) that we have a commutative diagram of $\widehat{\mathbb{Z}}^\Sigma$-modules

$$\begin{array}{ccc} H^2_c(\mathcal{G}, M) & \xrightarrow{\sim} & H^2_c(\mathcal{G}^{\circledcirc}, M) \\ \downarrow & & \downarrow \\ H^2_c(\mathcal{H}, M) & \xrightarrow{\sim} & H^2_c(\mathcal{H}^{\circledcirc}, M) \end{array}$$

— where the horizontal arrows are the isomorphisms of assertion (iv), and the vertical arrows are the homomorphisms induced by the connected finite étale coverings $\mathcal{H} \to \mathcal{G}$, $\mathcal{H}^{\circledcirc} \to \mathcal{G}^{\circledcirc}$, respectively — and hence that assertion (v) in the case where $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$ follows immediately from assertion (v) in the case where $\mathrm{Cusp}(\mathcal{G}) = \emptyset$. This completes the proof of assertion (v). Q.E.D.

Definition 3.8.

(i) We shall write

$$\Lambda_{\mathcal{G}} \stackrel{\text{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(H^2_c(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma}), \widehat{\mathbb{Z}}^{\Sigma})$$

and refer to $\Lambda_{\mathcal{G}}$ as the *cyclotome associated to* $\mathcal{G}$. For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\Lambda_v \stackrel{\text{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(H^2_c(v, \widehat{\mathbb{Z}}^{\Sigma}), \widehat{\mathbb{Z}}^{\Sigma})$$

and refer to Λ_v as the *cyclotome associated to* $v \in \mathrm{Vert}(\mathcal{G})$. Note that it follows from Theorem 3.7, (ii), that the cyclotomes $\Lambda_{\mathcal{G}}$ and Λ_v are *free* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules of rank 1*.

(ii) We shall write

$$\chi_{\mathcal{G}} \colon \mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\Lambda_{\mathcal{G}}) \simeq (\widehat{\mathbb{Z}}^{\Sigma})^*$$

for the natural homomorphism induced by the natural action of $\mathrm{Aut}(\mathcal{G})$ on $H^2_c(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma})$ and refer to $\chi_{\mathcal{G}}$ as the *pro-*Σ *cyclotomic character* of $\mathcal{G}$. For a vertex $v \in \mathrm{Vert}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\chi_v \stackrel{\text{def}}{=} \chi_{\mathcal{G}|_v} \colon \mathrm{Aut}(\mathcal{G}|_v) \longrightarrow \mathrm{Aut}(\Lambda_v) \simeq (\widehat{\mathbb{Z}}^{\Sigma})^*$$

and refer to χ_v as the *pro-*Σ *cyclotomic character* of v.

Remark 3.8.1. One verifies easily that if $l \in \Sigma$, then the composite

$$\mathrm{Aut}(\mathcal{G}) \stackrel{\chi_{\mathcal{G}}}{\to} (\widehat{\mathbb{Z}}^{\Sigma})^* \twoheadrightarrow \mathbb{Z}_l^*$$

coincides with the pro-l cyclotomic character of $\mathrm{Aut}(\mathcal{G})$ defined in the statement of [CmbGC], Lemma 2.1.

Corollary 3.9 (**Synchronization of cyclotomes**)**.** *Let* Σ *be a nonempty set of prime numbers and* $\mathcal{G}$ *a semi-graph of anabelioids of pro-*Σ *PSC-type. Then the following hold:*

(i) **(Synchronization with respect to generization)** *Let* $S \subseteq \mathrm{Node}(\mathcal{G})$ *be a subset of* $\mathrm{Node}(\mathcal{G})$*. Then the specialization outer* *isomorphism* $\Phi_{\mathcal{G}\rightsquigarrow S} : \Pi_{\mathcal{G}\rightsquigarrow S} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ *with respect to* S *[cf. Definition 2.10] determines a* **natural isomorphism**

$$\Lambda_{\mathcal{G}\rightsquigarrow S} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

that is **functorial** *with respect to isomorphisms of the pair* $(\mathcal{G}, S)$.

(ii) **(Synchronization with respect to "surgery")** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of* $\mathbb{G}$, $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)], and* $T \subseteq \mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ *[cf. Definition 2.5, (ii)] an* **omittable** *[cf. Definition 2.4, (i)] subset of* $\mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$. *Then there exists a* **natural isomorphism** *— given by* **"extension by zero"** —

$$\Lambda_{\mathcal{G}} \xrightarrow{\sim} \Lambda_{((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}}$$

[cf. Definition 2.4, (ii)] that is **functorial** *with respect to isomorphisms of the quadruple* $(\mathcal{G}, \mathbb{H}, S, T)$. *In particular, [by taking the inverse of this isomorphism] we obtain, for each vertex* $v \in \mathrm{Vert}(\mathcal{G})$ *of* $\mathcal{G}$, *a* **natural isomorphism** *of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*

$$\mathfrak{syn}_v : \Lambda_v \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

that is **functorial** *with respect to isomorphisms of the pair* $(\mathcal{G}, v)$.

(iii) **(Synchronization with respect to finite étale coverings)** *Let* $\mathcal{H} \to \mathcal{G}$ *be a connected finite étale covering of* $\mathcal{G}$. *Then there exists a* **natural isomorphism**

$$\Lambda_{\mathcal{H}} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

that is **functorial** *with respect to isomorphisms of the pair* $(\mathcal{G}, \mathcal{H})$.

(iv) **(Synchronization of cyclotomic characters)** *Let* $v \in \mathrm{Vert}(\mathcal{G})$ *be a vertex of* $\mathcal{G}$ *and* $\alpha \in \mathrm{Aut}^{\{v\}}(\mathcal{G})$ *[cf. Definition 2.6, (i)]. Then it holds that*

$$\chi_{\mathcal{G}}(\alpha) = \chi_v(\alpha_{\mathcal{G}|_v})$$

[cf. Definitions 2.14, (ii); 3.8, (ii); Remark 2.5.1, (ii)].

(v) **(Synchronization associated to branches)** *Let* $e \in \mathrm{Edge}(\mathcal{G})$ *be an edge of* $\mathcal{G}$*,* b *a branch of* e *that abuts to a vertex* $v \in \mathcal{V}(e)$*, and* $\Pi_e \subseteq \Pi_{\mathcal{G}}$ *an edge-like subgroup of* $\Pi_{\mathcal{G}}$ *associated to* $e \in \mathrm{Edge}(\mathcal{G})$*. Then there exists a* **natural isomorphism**

$$\mathfrak{syn}_b \colon \Pi_e \xrightarrow{\sim} \Lambda_v$$

that is **functorial** *with respect to isomorphisms of the quadruple* $(\mathcal{G}, b, e, v)$*.*

(vi) **(Difference between two synchronizations associated to the two branches of a node)** *Let* $e \in \mathrm{Node}(\mathcal{G})$ *be a node of* $\mathcal{G}$ *with branches* $b_1 \neq b_2$ *that abut to vertices* v_1*,* $v_2 \in \mathrm{Vert}(\mathcal{G})$*, respectively. Then the two composites*

$$\Pi_e \overset{\mathfrak{syn}_{b_1}}{\xrightarrow{\sim}} \Lambda_{v_1} \overset{\mathfrak{syn}_{v_1}}{\xrightarrow{\sim}} \Lambda_{\mathcal{G}} \ ; \ \Pi_e \overset{\mathfrak{syn}_{b_2}}{\xrightarrow{\sim}} \Lambda_{v_2} \overset{\mathfrak{syn}_{v_2}}{\xrightarrow{\sim}} \Lambda_{\mathcal{G}}$$

differ by the automorphism of $\Lambda_{\mathcal{G}}$ *given by multiplication by* $-1 \in \widehat{\mathbb{Z}}^{\Sigma}$*.*

Proof. Assertion (i) (respectively, (ii)) follows immediately from Theorem 3.7, (iii) (respectively, Theorem 3.7, (iv)). Assertion (iv) follows immediately from assertion (ii).

Next, we verify assertion (iii). It follows immediately from Theorem 3.7, (v), that the homomorphism of $\widehat{\mathbb{Z}}^{\Sigma}$-modules $\Lambda_{\mathcal{H}} \to \Lambda_{\mathcal{G}}$ obtained by applying the functor "$\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(-, \widehat{\mathbb{Z}}^{\Sigma})$" to the induced homomorphism $H^2_c(\mathcal{G}, \widehat{\mathbb{Z}}^{\Sigma}) \to H^2_c(\mathcal{H}, \widehat{\mathbb{Z}}^{\Sigma})$ and dividing by the index $[\Pi_{\mathcal{G}} : \Pi_{\mathcal{H}}]$ is an *isomorphism.* This completes the proof of assertion (iii).

Next, we verify assertion (v). First, we observe that to verify assertion (v), by replacing $\mathcal{G}$ by $\mathcal{G}|_v$ and $e \in \mathrm{Edge}(\mathcal{G})$ by the cusp of $\mathcal{G}|_v$ corresponding to b, we may assume without loss of generality that $e \in \mathrm{Cusp}(\mathcal{G})$. Then we have homomorphisms of $\widehat{\mathbb{Z}}^{\Sigma}$-modules

$$\begin{aligned} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_e, \Pi_e) \ &\hookrightarrow \ \textstyle\bigoplus_{e' \in \mathrm{Cusp}(\mathcal{G})} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{e'}, \Pi_e) \\ &\twoheadrightarrow \ H^2_c(\mathcal{G}, \Pi_e) \qquad \overset{\sim}{\to} \qquad \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{G}}, \Pi_e) \end{aligned}$$

— where the first arrow is the natural inclusion into the component indexed by e, and the second arrow is the surjection appearing in the exact sequence of Lemma 3.3 in the case where $M = \Pi_e$. Here, we note that it follows immediately from the facts (A), (B) used in the proof of Theorem 3.7, (i), (ii), that the composite of these homomorphisms is an *isomorphism.* Therefore, we obtain a natural isomorphism

$$\mathfrak{syn}_b \colon \Pi_e \xrightarrow{\sim} \Lambda_{\mathcal{G}}$$

by forming the inverse of the image of the identity automorphism of Π_e via the composite of the homomorphisms of the above display. This completes the proof of assertion (v).

Finally, we verify assertion (vi). First, we observe that one may verify easily that there exist

- a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}^\dagger$,
- a sub-semi-graph of PSC-type $\mathbb{K}^\dagger$ of the underlying semi-graph of $\mathcal{H}^\dagger$,
- an omittable subset $S^\dagger \subseteq \mathrm{Cusp}((\mathcal{H}^\dagger)|_{\mathbb{K}^\dagger})$, and
- an isomorphism

$$((\mathcal{H}^\dagger)|_{\mathbb{K}^\dagger})_{\bullet S^\dagger} \xrightarrow{\sim} \mathcal{G}$$

such that the node $e_{\mathcal{H}^\dagger} \in \mathrm{Node}(\mathcal{H}^\dagger)$ of $\mathcal{H}^\dagger$ corresponding, relative to the isomorphism $((\mathcal{H}^\dagger)|_{\mathbb{K}^\dagger})_{\bullet S^\dagger} \xrightarrow{\sim} \mathcal{G}$, to the node $e \in \mathrm{Node}(\mathcal{G})$ is *not of separating type*. [Note that it follows immediately from the various definitions involved that $\mathrm{Node}(\mathcal{G}) \xleftarrow{\sim} \mathrm{Node}(((\mathcal{H}^\dagger)|_{\mathbb{K}^\dagger})_{\bullet S^\dagger})$ may be regarded as a subset of $\mathrm{Node}(\mathcal{H}^\dagger)$.] Thus, it follows immediately from assertions (i), (ii) — by replacing $\mathcal{G}$ (respectively, e) by $(\mathcal{H}^\dagger)_{\rightsquigarrow \mathrm{Node}(\mathcal{H}^\dagger)\setminus\{e_{\mathcal{H}^\dagger}\}}$ (respectively, $e_{\mathcal{H}^\dagger}$) — that to verify assertion (vi), we may assume without loss of generality that $\mathrm{Node}(\mathcal{G}) = \{e\}$, and that e is *not of separating type*.

Next, we observe that one may verify easily that there exists a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}^\ddagger$ such that

- $\mathrm{Node}(\mathcal{H}^\ddagger)$ consists of *precisely two* elements $e_{\mathcal{H}^\ddagger}, e'_{\mathcal{H}^\ddagger}$;
- $\mathcal{V}(e_{\mathcal{H}^\ddagger})$ consists of *precisely one* element $v_{\mathcal{H}^\ddagger}$ of *type* $(0,3)$ [cf. Definition 2.3, (iii)].
- $e'_{\mathcal{H}^\ddagger}$ is of *separating type*;
- $(\mathcal{H}^\ddagger)_{\rightsquigarrow \{e'_{\mathcal{H}^\ddagger}\}}$ is isomorphic to $\mathcal{G}$.

Thus, if we write $\mathbb{K}^\ddagger$ for the *unique* sub-semi-graph of *PSC-type* of the underlying semi-graph of $\mathcal{H}^\ddagger$ whose set of vertices $= \{v_{\mathcal{H}^\ddagger}\}$, then it follows immediately from assertions (i), (ii) — by replacing $\mathcal{G}$ (respectively, e) by $\mathcal{H}^\ddagger|_{\mathbb{K}^\ddagger}$ (respectively, $e_{\mathcal{H}^\ddagger}$) — that to verify assertion (vi), we may assume without loss of generality that $\mathrm{Node}(\mathcal{G}) = \{e\}$, that e is *not of* *separating type* [so $\mathrm{Vert}(\mathcal{G})$ consists of *precisely one* element], and that $\mathcal{G}$ is of *type* $(1, 1)$.

Write $v \in \mathrm{Vert}(\mathcal{G})$ for the *unique* vertex of $\mathcal{G}$. Note that it follows immediately from the various assumptions on $\mathcal{G}$ that $\mathcal{G}|_v$ is of *type* $(0, 3)$. Write e_1, $e_2 \in \mathrm{Cusp}(\mathcal{G}|_v)$ for the cusps of $\mathcal{G}|_v$ corresponding, respectively, to the two branches b_1, b_2 of the node e; write $e_3 \in \mathrm{Cusp}(\mathcal{G}|_v)$ for the *unique* element of $\mathrm{Cusp}(\mathcal{G}|_v) \setminus \{e_1, e_2\}$. Then since $\mathcal{G}|_v$ is of *type* $(0, 3)$, there exists a *graphic* isomorphism of $\mathcal{G}|_v$ with the semi-graph of anabelioids of pro-Σ PSC-type [*without nodes*] determined by the tripod [cf. the discussion entitled "*Curves*" in §0] $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over an algebraically closed field k of characteristic $\notin \Sigma$ such that the cusps e_1, e_2 of $\mathcal{G}|_v$ correspond to the cusps 0, ∞ of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$, respectively, relative to the graphic isomorphism. Thus, by considering the automorphism of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over k given by "$t \mapsto 1/t$", we obtain an automorphism $\tau_v \in \mathrm{Aut}(\mathcal{G}|_v)$ of $\mathcal{G}|_v$ that maps $e_1 \mapsto e_2$, $e_2 \mapsto e_1$. Moreover, since this automorphism of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ induces an automorphism of the stable log curve of type $(1, 1)$ obtained by identifying the cusps 0 and ∞ of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$, we also obtain an automorphism $\tau_{\mathcal{G}} \in \mathrm{Aut}(\mathcal{G})$ of $\mathcal{G}$. Note that it follows immediately from the definition of τ_v, together with the well-known structure of the étale fundamental group of the tripod $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$, that the automorphism τ_v induces the *identity automorphism* of the anabeloid $(\mathcal{G}|_v)_{e_3}$ corresponding to e_3.

Next, let us observe that it follows immediately from the definition of $\mathcal{G}|_v$, together with the proof of assertion (v), that for $i = 1, 2$, there exists a natural isomorphism $\Pi_e \xrightarrow{\sim} \Pi_{e_i}$ — where we use the notations Π_e, Π_{e_i} to denote edge-like subgroups of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}|_v}$ associated to e, e_i, respectively — such that the composite

$$\Pi_e \xrightarrow{\sim} \Pi_{e_i} \xrightarrow[\sim]{\mathfrak{syn}_{b'_i}} \Lambda_v \ [= \Lambda_{\mathcal{G}|_v}] \xrightarrow[\sim]{\mathfrak{syn}_v} \Lambda_{\mathcal{G}}$$

— where we write b'_i for the [*unique*] branch of e_i — coincides with the composite in question

$$\Pi_e \xrightarrow[\sim]{\mathfrak{syn}_{b_i}} \Lambda_v \xrightarrow[\sim]{\mathfrak{syn}_v} \Lambda_{\mathcal{G}} .$$

Next, let us observe that it follows immediately from the *functoriality* portion of assertion (v) that the automorphisms τ_v, $\tau_{\mathcal{G}}$ induce a commutative diagram of $\widehat{\mathbb{Z}}^{\Sigma}$-modules

$$\begin{array}{ccccccc}
\Pi_e & \xrightarrow{\sim} & \Pi_{e_1} & \xrightarrow{\mathfrak{syn}_{b'_1}} & \Lambda_v \ [= \Lambda_{\mathcal{G}|_v}] & \xrightarrow{\mathfrak{syn}_v} & \Lambda_{\mathcal{G}} \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr & & \downarrow\wr \\
\Pi_e & \xrightarrow{\sim} & \Pi_{e_2} & \xrightarrow{\mathfrak{syn}_{b'_2}} & \Lambda_v \ [= \Lambda_{\mathcal{G}|_v}] & \xrightarrow{\mathfrak{syn}_v} & \Lambda_{\mathcal{G}}
\end{array}$$

— where the vertical arrows are the isomorphisms induced by the automorphisms τ_v, $\tau_{\mathcal{G}}$. Now by considering the well-known *local structure of a stable log curve in a neighborhood of a node*, one may verify easily that the left-hand vertical arrow in the above diagram is the automorphism of Π_e given by *multiplication by* $-1 \in \widehat{\mathbb{Z}}^{\Sigma}$. Thus, to complete the verification of assertion (vi), it suffices, in light of the commutativity of the above diagram, to verify that $\tau_v \in \mathrm{Aut}(\mathcal{G}|_v)$ induces the *identity automorphism* of $\Lambda_{\mathcal{G}|_v} = \Lambda_v$. On the other hand, this follows immediately from assertion (v), applied to the cusp e_3, together with the fact that the automorphism τ_v induces the *identity automorphism* of $(\mathcal{G}|_v)_{e_3}$. This completes the proof of assertion (vi). Q.E.D.

§4. Profinite Dehn multi-twists

In the present §, we introduce and discuss the notion of a *profinite Dehn multi-twist*. Although our definition of this notion [cf. Definition 4.4 below] is entirely *group-theoretic* in nature, our main result concerning this notion [cf. Theorem 4.8 below] asserts, in effect, that this group-theoretic notion coincides with the usual *geometric* notion of a "Dehn multi-twist".

Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_{\mathcal{G}}$ for the [pro-Σ] fundamental group of $\mathcal{G}$, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$.

Definition 4.1. We shall say that $\mathcal{G}$ is *cyclically primitive* (respectively, *noncyclically primitive*) if $\mathrm{Node}(\mathcal{G})^{\sharp} = 1$, and the unique node of $\mathcal{G}$ is not of separating type (respectively, is of separating type) [cf. Definition 2.5, (i)].

Remark 4.1.1. If $\mathcal{G}$ is cyclically primitive (respectively, noncyclically primitive), then $\mathrm{Vert}(\mathcal{G})^\sharp = 1$ (respectively, 2), and the [discrete] topological fundamental group $\pi_1^{\mathrm{top}}(\mathbb{G})$ of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ is noncanonically isomorphic to $\mathbb{Z}$ (respectively, is trivial).

Lemma 4.2 (**Structure of the fundamental group of a noncyclically primitive semi-graph of anabelioids of PSC-type).** *Suppose that* $\mathcal{G}$ *is* **noncyclically primitive** *[cf. Definition 4.1]. Let* $v, w \in \mathrm{Vert}(\mathcal{G})$ *be the* **two distinct** *vertices of* $\mathcal{G}$ *[cf. Remark 4.1.1];* $\widetilde{e}, \widetilde{v}, \widetilde{w} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ *elements of* $\mathrm{VCN}(\widetilde{\mathcal{G}})$ *such that* $\widetilde{v}(\mathcal{G}) = v$, $\widetilde{w}(\mathcal{G}) = w$, *and, moreover,* $\widetilde{e} \in \mathcal{N}(\widetilde{v}) \cap \mathcal{N}(\widetilde{w})$. *Then the natural inclusions* $\Pi_{\widetilde{e}}$, $\Pi_{\widetilde{v}}$, $\Pi_{\widetilde{w}} \hookrightarrow \Pi_{\mathcal{G}}$ *determine an* **isomorphism** *of pro-*Σ *groups*

$$\varinjlim(\Pi_{\widetilde{v}} \hookleftarrow \Pi_{\widetilde{e}} \hookrightarrow \Pi_{\widetilde{w}}) \xrightarrow{\sim} \Pi_{\mathcal{G}}$$

— where the inductive limit is taken in the category of pro-Σ *groups.*

Proof. This may be thought of as a consequence of the "*van Kampen Theorem*" in elementary algebraic topology. At a more *combinatorial* level, one may reason as follows: It follows immediately from the *simple structure of the underlying semi-graph* $\mathbb{G}$ that there is a natural *equivalence of categories* between

- the category of finite sets with continuous $\Pi_{\mathcal{G}}$-action [and $\Pi_{\mathcal{G}}$-equivariant morphisms] and
- the category of finite sets with continuous actions of $\Pi_{\widetilde{v}}$, $\Pi_{\widetilde{w}}$ which restrict to the *same* action on $\Pi_{\widetilde{e}}$ [and $\Pi_{\widetilde{v}}$-, $\Pi_{\widetilde{w}}$-equivariant morphisms].

The isomorphism between $\Pi_{\mathcal{G}}$ and the inductive limit appearing in the statement of Lemma 4.2 now follows *formally* from this equivalence of categories. Q.E.D.

Lemma 4.3 (**Infinite cyclic tempered covering of a cyclically primitive semi-graph of anabelioids of PSC-type).** *Suppose that* $\mathcal{G}$ *is* **cyclically primitive** *[cf. Definition 4.1]. Denote by* $\pi_1^{\mathrm{temp}}(\mathcal{G})$ *the tempered fundamental group of* $\mathcal{G}$ *[cf. the discussion preceding* [SemiAn], *Proposition 3.6], by* $\pi_1^{\mathrm{top}}(\mathbb{G})$ *[*$\simeq \mathbb{Z}$ *— cf. Remark 4.1.1] the [discrete] topological fundamental group of the underlying semi-graph* $\mathbb{G}$ *of* $\mathcal{G}$*, and by* $\mathcal{G}_\infty \to \mathcal{G}$ *the connected tempered covering of* $\mathcal{G}$ *corresponding to the natural surjection* $\pi_1^{\mathrm{temp}}(\mathcal{G}) \twoheadrightarrow \pi_1^{\mathrm{top}}(\mathbb{G})$ *[where we refer to* [SemiAn], *§3,* *concerning tempered coverings of a semi-graph of anabelioids]. Then the following hold:*

(i) **(Exact sequence)** *The natural morphism* $\mathcal{G}_\infty \to \mathcal{G}$ *induces an exact sequence*

$$1 \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}) \longrightarrow \pi_1^{\mathrm{top}}(\mathbb{G}) \longrightarrow 1\,.$$

Moreover, the subgroup $\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \subseteq \pi_1^{\mathrm{temp}}(\mathcal{G})$ *of* $\pi_1^{\mathrm{temp}}(\mathcal{G})$ *is* **characteristic***.*

(ii) **(Automorphism groups)** *There exist natural* **injective** *homomorphisms*

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)\ ,\ \ \pi_1^{\mathrm{top}}(\mathbb{G}) \hookrightarrow \mathrm{Aut}(\mathcal{G}_\infty)$$

— where we write $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ *for the group of automorphisms of* $\mathcal{G}_\infty$ *that induce the identity automorphism of the underlying semi-graph of* $\mathcal{G}_\infty$*. Moreover, the centralizer of* $\pi_1^{\mathrm{top}}(\mathbb{G})$ *in* $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ *satisfies the equality*

$$Z_{\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)}(\pi_1^{\mathrm{top}}(\mathbb{G})) = \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})\,.$$

(iii) **(Action of the fundamental group of the underlying semi-graph)** *Let* $\gamma_\infty \in \pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \mathrm{Aut}(\mathcal{G}_\infty)$ *[cf. (ii)] be a generator of* $\pi_1^{\mathrm{top}}(\mathbb{G}) \simeq \mathbb{Z}$*. Write* $\mathrm{Vert}(\mathcal{G}_\infty)$*,* $\mathrm{Node}(\mathcal{G}_\infty)$*, and* $\mathrm{Cusp}(\mathcal{G}_\infty)$ *for the sets of vertices, nodes [i.e., closed edges], and cusps [i.e., open edges] of* $\mathcal{G}_\infty$*, respectively. Then there exist bijections*

$$V : \mathbb{Z} \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G}_\infty)\ ,\ \ N : \mathbb{Z} \xrightarrow{\sim} \mathrm{Node}(\mathcal{G}_\infty)\,,$$

$$C : \mathbb{Z} \times \mathrm{Cusp}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_\infty)$$

such that, for each $a \in \mathbb{Z}$*,*

- *the set of edges that abut to the vertex* $V(a)$ *is equal to the disjoint union of* $\{N(a), N(a+1)\}$ *and* $\{\, C(a,z) \,|\, z \in \mathrm{Cusp}(\mathcal{G})\}$*;*
- *the automorphism of* $\mathrm{Vert}(\mathcal{G}_\infty)$ *(respectively,* $\mathrm{Node}(\mathcal{G}_\infty)$*;* $\mathrm{Cusp}(\mathcal{G}_\infty)$*) induced by* $\gamma_\infty \in \mathrm{Aut}(\mathcal{G}_\infty)$ *maps* $V(a)$ *(respectively,* $N(a)$*;* $C(a,z)$*) to* $V(a+1)$ *(respectively,* $N(a+1)$*;* $C(a+1,z)$*).*

(iv) **(Restriction to a finite sub-semi-graph)** *Let* $a \leq b \in \mathbb{Z}$ *be integers. Denote by* $\mathbb{G}_{[a,b]}$ *the [***uniquely determined***] sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of the underlying semi-graph of* $\mathcal{G}_\infty$ *such that the set of vertices of* $\mathbb{G}_{[a,b]}$ *is equal to* $\{V(a), V(a+1), \cdots, V(b)\}$ *[cf. (iii)]; denote by* $\mathcal{G}_{[a,b]}$ *the semi-graph of anabelioids obtained by restricting* $\mathcal{G}_\infty$ *to* $\mathbb{G}_{[a,b]}$ *[cf. the discussion preceding* [SemiAn]*, Definition 2.2]. Then* $\mathcal{G}_{[a,b]}$ *is a* **semi-graph of anabelioids of pro-Σ PSC-type**. *Moreover,* $\mathcal{G}_{[a,a+1]}$ *is* **noncyclically primitive**.

(v) **(Restriction to a sub-semi-graph having precisely one vertex)** *Let* $a \leq c \leq b \in \mathbb{Z}$ *be integers. Then the natural morphism of semi-graphs of anabelioids* $\mathcal{G}_{[c,c]} \to \mathcal{G}_{[a,b]}$ *[cf. (iv)] determines an isomorphism* $\mathcal{G}_{[c,c]} \xrightarrow{\sim} \mathcal{G}_{[a,b]}|_{V(c)}$ *— where we regard* $V(c) \in \mathrm{Vert}(\mathcal{G}_\infty)$ *as a vertex of* $\mathcal{G}_{[a,b]}$. *Moreover, if we write* $v \in \mathrm{Vert}(\mathcal{G})$ *for the* **unique** *vertex of* $\mathcal{G}$ *[cf. Remark 4.1.1], then the composite of natural morphisms of semi-graphs of anabelioids* $\mathcal{G}_{[c,c]} \to \mathcal{G}_\infty \to \mathcal{G}$ *determines an isomorphism of* $\mathcal{G}_{[c,c]}$ *with* $\mathcal{G}|_v$.

(vi) **(Natural isomorphisms between restrictions to finite sub-semi-graphs)** *Let* $a \leq b \in \mathbb{Z}$ *be integers and* $\gamma_\infty \in \pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \mathrm{Aut}(\mathcal{G}_\infty)$ *the automorphism of* $\mathcal{G}_\infty$ *appearing in (iii). Then* γ_∞ *determines an isomorphism* $\mathcal{G}_{[a,b]} \xrightarrow{\sim} \mathcal{G}_{[a+1,b+1]}$.

Proof. First, we verify assertion (i). To show that the natural morphism $\mathcal{G}_\infty \to \mathcal{G}$ induces an exact sequence

$$1 \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}) \longrightarrow \pi_1^{\mathrm{top}}(\mathbb{G}) \longrightarrow 1\,,$$

it suffices to verify that every tempered covering of $\mathcal{G}_\infty$ determines, via the morphism $\mathcal{G}_\infty \to \mathcal{G}$, a tempered covering of $\mathcal{G}$. But this follows immediately, in light of the definition of a tempered covering, from the *finiteness* of the underlying semi-graph $\mathbb{G}$ and the *topologically finitely generated* nature of the verticial subgroups of the tempered fundamental group $\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty)$ of $\mathcal{G}_\infty$. On the other hand, the fact that the subgroup $\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \subseteq \pi_1^{\mathrm{temp}}(\mathcal{G})$ is characteristic follows immediately from the observation that the quotient $\pi_1^{\mathrm{temp}}(\mathcal{G}) \twoheadrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G})/\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty)$ may be characterized as the *maximal discrete free quotient* of $\pi_1^{\mathrm{temp}}(\mathcal{G})$ [cf. the argument of [André], Lemma 6.1.1]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The existence of a natural injection $\pi_1^{\mathrm{top}}(\mathbb{G}) \hookrightarrow \mathrm{Aut}(\mathcal{G}_\infty)$ follows immediately from the definition of the connected tempered covering $\mathcal{G}_\infty \to \mathcal{G}$, together with the fact that $\pi_1^{\mathrm{top}}(\mathbb{G})$ is *abelian*. On the other hand, it follows immediately from assertion (i), together with the various definitions involved, that any element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ determines — up to composition with an element of $\pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \mathrm{Aut}(\mathcal{G}_\infty)$ — an automorphism of $\mathcal{G}_\infty$. Therefore, by composing with a suitable element of $\pi_1^{\mathrm{top}}(\mathbb{G}) \subseteq \mathrm{Aut}(\mathcal{G}_\infty)$, one obtains a *uniquely determined* element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$, hence also a natural injective homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$. Next, to verify the equality $Z_{\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)}(\pi_1^{\mathrm{top}}(\mathbb{G})) = \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$, observe that $\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty)$ is *center-free* [cf. [SemiAn], Example 2.10; [SemiAn], Proposition 3.6, (iv)]; this implies that we have a natural isomorphism $\pi_1^{\mathrm{temp}}(\mathcal{G}) \simeq \pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \overset{\mathrm{out}}{\rtimes} \pi_1^{\mathrm{top}}(\mathbb{G})$ [cf. the discussion entitled "*Topological groups*" in §0]. Thus, in light of the [easily verified] inclusion $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq Z_{\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)}(\pi_1^{\mathrm{top}}(\mathbb{G}))$, the desired equality follows immediately from [CmbGC], Proposition 1.5, (ii). This completes the proof of assertion (ii).

Assertions (iii), (iv), (v), and (vi) follow immediately from the definition of the connected tempered covering $\mathcal{G}_\infty \to \mathcal{G}$. Q.E.D.

Definition 4.4. We shall write

$$\mathrm{Dehn}(\mathcal{G}) \overset{\mathrm{def}}{=} \{\, \alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \mid \alpha_{\mathcal{G}|_v} = \mathrm{id}_{\mathcal{G}|_v} \text{ for any } v \in \mathrm{Vert}(\mathcal{G}) \,\}$$

— where we refer to Definitions 2.1, (iii); 2.14, (ii); Remark 2.5.1, (ii), concerning "$\alpha_{\mathcal{G}|_v}$". We shall refer to an element of $\mathrm{Dehn}(\mathcal{G})$ as a *profinite Dehn multi-twist* of $\mathcal{G}$.

Proposition 4.5 (Equalities concerning the group of profinite Dehn multi-twists). *It holds that*

$$\begin{aligned}\mathrm{Dehn}(\mathcal{G}) &= \bigcap_{v\in\mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\Pi_v|}(\mathcal{G}) = \bigcap_{z\in\mathrm{VCN}(\mathcal{G})} \mathrm{Aut}^{|\Pi_z|}(\mathcal{G}) \\ &= \bigcap_{z\in\mathrm{VCN}(\mathcal{G})} \mathrm{Out}^{|\Pi_z|}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})\end{aligned}$$

[cf. Definitions 2.13; 2.6, (i); [CmbGC]*, Proposition 1.2, (ii)] — where we use the notation "$\Pi_{(-)}$" to denote a VCN-subgroup [cf. Definition 2.1, (i)] of* $\Pi_{\mathcal{G}}$ *associated to "$(-)$"* $\in \mathrm{VCN}(\mathcal{G})$.

Proof. The first equality follows immediately from the various definitions involved [cf. also [CmbGC], Proposition 1.2, (i)]. The second equality follows immediately from the fact that any edge-like subgroup is contained in a verticial subgroup. The third equality follows immediately from Proposition 2.7, (ii). This completes the proof of Proposition 4.5. Q.E.D.

Lemma 4.6 (Construction of certain homomorphisms). *Let* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, $e \stackrel{\mathrm{def}}{=} \widetilde{e}(\mathcal{G}) \in \mathrm{Node}(\mathcal{G})$. *Then the following hold:*

(i) *Let* $\alpha \in \mathrm{Dehn}(\mathcal{G})$ *be a profinite Dehn multi-twist of* $\mathcal{G}$ *and* $\widetilde{v} \in \mathcal{V}(\widetilde{e}) \subseteq \mathrm{Vert}(\widetilde{\mathcal{G}})$. *Write* $\widetilde{w}$ *for the* **unique** *element of the complement* $\mathcal{V}(\widetilde{e}) \setminus \{\widetilde{v}\}$ *[cf.* [NodNon], *Remark 1.2.1, (iii)]. Then there exists a* **unique** *lifting* $\alpha[\widetilde{v}] \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ *of* α *which preserves the verticial subgroup* $\Pi_{\widetilde{v}} \subseteq \Pi_{\mathcal{G}}$ *of* $\Pi_{\mathcal{G}}$ *associated to* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *and induces the* **identity automorphism** *of* $\Pi_{\widetilde{v}}$. *Moreover, this lifting* $\alpha[\widetilde{v}]$ **preserves** *the verticial subgroup* $\Pi_{\widetilde{w}} \subseteq \Pi_{\mathcal{G}}$ *of* $\Pi_{\mathcal{G}}$ *associated to* $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, *and there exists a* **unique** *element* $\delta_{\widetilde{e},\widetilde{v}} \in \Pi_{\widetilde{e}}$ *of the edge-like subgroup* $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$ *of* $\Pi_{\mathcal{G}}$ *associated to* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ *such that the restriction of* $\alpha[\widetilde{v}]$ *to* $\Pi_{\widetilde{w}}$ *is the inner automorphism determined by* $\delta_{\widetilde{e},\widetilde{v}} \in \Pi_{\widetilde{e}}$ $(\subseteq \Pi_{\widetilde{w}})$.

(ii) *For* $\widetilde{v} \in \mathcal{V}(\widetilde{e})$, *denote by* $\mathfrak{D}_{\widetilde{e},\widetilde{v}} \colon \mathrm{Dehn}(\mathcal{G}) \to \Lambda_{\mathcal{G}}$ *the composite of the map*

$$\mathrm{Dehn}(\mathcal{G}) \longrightarrow \Pi_{\widetilde{e}}$$

given by assigning $\alpha \mapsto \delta_{\widetilde{e},\widetilde{v}} \in \Pi_{\widetilde{e}}$ *[cf. (i)] and the isomorphism*

$$\Pi_e \xrightarrow[\sim]{\mathfrak{syn}_b} \Lambda_v \xrightarrow[\sim]{\mathfrak{syn}_v} \Lambda_{\mathcal{G}}$$

[cf. Corollary 3.9, (ii), (v)] — where we write $v \stackrel{\mathrm{def}}{=} \widetilde{v}(\mathcal{G})$ *and* b *for the branch of* e *determined by the unique branch of* $\widetilde{e}$ *which abuts to* $\widetilde{v}$. *Then the map* $\mathfrak{D}_{\widetilde{e},\widetilde{v}} \colon \mathrm{Dehn}(\mathcal{G}) \to \Lambda_{\mathcal{G}}$ *is a* **homomorphism** *of profinite groups which does* **not depend** *on the choice of the element* $\widetilde{v} \in \mathcal{V}(\widetilde{e})$, *i.e., if* $\widetilde{w} \in \mathcal{V}(\widetilde{e}) \setminus \{\widetilde{v}\}$, *then* $\mathfrak{D}_{\widetilde{e},\widetilde{v}} = \mathfrak{D}_{\widetilde{e},\widetilde{w}}$. *Moreover, the homomorphism* $\mathfrak{D}_{\widetilde{e},\widetilde{v}}$ $(= \mathfrak{D}_{\widetilde{e},\widetilde{w}})$ **depends only** *on* $e \in \mathrm{Node}(\mathcal{G})$, *i.e., it does* **not depend** *on the choice of the element* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ *such that* $\widetilde{e}(\mathcal{G}) = e$.

Proof. First, we verify assertion (i). The fact that there exists a *unique* lifting $\alpha[\widetilde{v}] \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ of α which preserves $\Pi_{\widetilde{v}}$ and induces the *identity automorphism* of $\Pi_{\widetilde{v}}$ follows immediately, in light of the *slimness* of $\Pi_{\widetilde{v}}$ [cf. [CmbGC], Remark 1.1.3] and the *commensurable terminality* of $\Pi_{\widetilde{v}}$ in $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Proposition 1.2, (ii)], from the fact that $\alpha \in \mathrm{Out}^{|\Pi_{\widetilde{v}}|}(\Pi_{\mathcal{G}})$ [cf. Proposition 4.5]. The fact that $\alpha[\widetilde{v}]$ *preserves* $\Pi_{\widetilde{w}}$ follows immediately, in light of the *graphicity* of $\alpha[\widetilde{v}]$, from the fact that $\Pi_{\widetilde{w}}$ is the *unique verticial subgroup* H of $\Pi_{\mathcal{G}}$ such that $H \neq \Pi_{\widetilde{v}}$ and $\Pi_{\widetilde{e}} \subseteq H$ [cf. [NodNon], Remark 1.2.1, (iii); [NodNon], Lemma 1.7], together with the fact that $\alpha[\widetilde{v}]$ preserves $\Pi_{\widetilde{v}}$, $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$. The fact that there exists a *unique* element $\delta_{\widetilde{e},\widetilde{v}} \in \Pi_{\widetilde{e}}$ of $\Pi_{\widetilde{e}}$ such that the restriction of $\alpha[\widetilde{v}]$ to $\Pi_{\widetilde{w}}$ is the inner automorphism determined by $\delta_{\widetilde{e},\widetilde{v}}$ follows immediately, in light of the *slimness* of $\Pi_{\widetilde{w}}$ [cf. [CmbGC], Remark 1.1.3] and the *commensurable terminality* of $\Pi_{\widetilde{e}}$ [cf. [CmbGC], Proposition 1.2, (ii)], from the fact that $\alpha \in \mathrm{Out}^{|\Pi_{\widetilde{w}}|}(\Pi_{\mathcal{G}})$ [cf. Proposition 4.5]. This completes the proof of assertion (i). Next, we verify assertion (ii). The fact that the map $\mathfrak{D}_{\widetilde{e},\widetilde{v}}$ is a *homomorphism* follows immediately from the various *uniqueness* properties discussed in assertion (i). The fact that the map $\mathfrak{D}_{\widetilde{e},\widetilde{v}}$ does *not depend* on the choice of the element $\widetilde{v} \in \mathcal{V}(\widetilde{e})$ follows immediately from Corollary 3.9, (vi). The fact that the homomorphism $\mathfrak{D}_{\widetilde{e},\widetilde{v}}$ does *not depend* on the choice of the element $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ such that $\widetilde{e}(\mathcal{G}) = e$ follows immediately from the definition of the map $\mathfrak{D}_{\widetilde{e},\widetilde{v}}$. This completes the proof of assertion (ii). Q.E.D.

Definition 4.7. For each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\mathfrak{D}_e \stackrel{\mathrm{def}}{=} \mathfrak{D}_{\widetilde{e},\widetilde{v}} \colon \mathrm{Dehn}(\mathcal{G}) \longrightarrow \Lambda_{\mathcal{G}}$$

for the homomorphism obtained in Lemma 4.6, (ii). [Note that it follows from Lemma 4.6, (ii), that this homomorphism depends only on $e \in \mathrm{Node}(\mathcal{G})$.] We shall write

$$\mathfrak{D}_{\mathcal{G}} \stackrel{\mathrm{def}}{=} \bigoplus_{e \in \mathrm{Node}(\mathcal{G})} \mathfrak{D}_e \colon \mathrm{Dehn}(\mathcal{G}) \longrightarrow \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} .$$

Theorem 4.8 (Properties of profinite Dehn multi-twists). *Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Then the following hold:*

(i) **(Normality)** $\mathrm{Dehn}(\mathcal{G})$ *is* **normal** *in* $\mathrm{Aut}(\mathcal{G})$.

(ii) **(Compatibility with generization)** *Let* $S \subseteq \mathrm{Node}(\mathcal{G})$. *Then — relative to the inclusion* $\mathrm{Aut}^S(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G}_{\rightsquigarrow S})$ *[cf. Definition 2.8] induced by the specialization outer isomorphism* $\Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\rightsquigarrow S}}$ *with respect to* S *[cf. Proposition 2.9, (ii)] — we have a diagram of inclusions*

$$\begin{array}{ccc} \mathrm{Dehn}(\mathcal{G}) & \hookleftarrow & \mathrm{Dehn}(\mathcal{G}_{\rightsquigarrow S}) \\ \cap & & \cap \\ \mathrm{Aut}^S(\mathcal{G}) & \hookrightarrow & \mathrm{Aut}(\mathcal{G}_{\rightsquigarrow S})\,. \end{array}$$

Moreover, if we regard $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow S})$ *as a subset of* $\mathrm{Node}(\mathcal{G})$, *then the above inclusion* $\mathrm{Dehn}(\mathcal{G}_{\rightsquigarrow S}) \hookrightarrow \mathrm{Dehn}(\mathcal{G})$ *fits into a commutative diagram of profinite groups*

$$\begin{array}{ccc} \mathrm{Dehn}(\mathcal{G}_{\rightsquigarrow S}) & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) \\ {\scriptstyle \mathfrak{D}_{\mathcal{G}_{\rightsquigarrow S}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{D}_{\mathcal{G}}} \\ \bigoplus_{\mathrm{Node}(\mathcal{G}_{\rightsquigarrow S})} \Lambda_{\mathcal{G}} & \longrightarrow & \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \end{array}$$

— where the lower horizontal arrow is the natural inclusion determined by the inclusion $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow S}) \hookrightarrow \mathrm{Node}(\mathcal{G})$ *and the natural isomorphism* $\Lambda_{\mathcal{G}_{\rightsquigarrow S}} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$ *[cf. Corollary 3.9, (i)].*

(iii) **(Compatibility with "surgery")** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of* $\mathbb{G}$, $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)], and* $T \subseteq \mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ *[cf. Definition 2.5, (ii)] an* **omittable** *[cf. Definition 2.4, (i)] subset of* $\mathrm{Cusp}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$. *Then the natural homomorphism*

$$\begin{array}{ccc} \mathrm{Aut}^{\mathbb{H}\succ S\bullet T}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}) \\ \alpha & \mapsto & \alpha_{((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}} \end{array}$$

[cf. Definitions 2.4, (ii); 2.14, (ii)] induces a homomorphism

$$\mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Dehn}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T})\,.$$

Moreover, if we regard $\mathrm{Node}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T})$ *as a subset of* $\mathrm{Node}(\mathcal{G})$, *then the above homomorphism* $\mathrm{Dehn}(\mathcal{G}) \to$

$\mathrm{Dehn}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T})$ *fits into a commutative diagram of profinite groups*

$$\begin{array}{ccc} \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Dehn}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}) \\ {\scriptstyle \mathfrak{D}_{\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{D}_{((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}}} \\ \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} & \longrightarrow & \bigoplus_{\mathrm{Node}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T})} \Lambda_{\mathcal{G}} \end{array}$$

— where the lower horizontal arrow is the natural projection, and we apply the natural isomorphism $\Lambda_{\mathcal{G}} \xrightarrow{\sim} \Lambda_{((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\bullet T}}$ *[cf. Corollary 3.9, (ii)].*

(iv) **(Structure of the group of profinite Dehn multi-twists)** *The homomorphism defined in Definition 4.7*

$$\mathfrak{D}_{\mathcal{G}} \colon \mathrm{Dehn}(\mathcal{G}) \longrightarrow \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$$

is an **isomorphism** *of profinite groups that is* **functorial**, *in* $\mathcal{G}$, *with respect to isomorphisms of semi-graphs of anabelioids of pro-*Σ *PSC-type. In particular,* $\mathrm{Dehn}(\mathcal{G})$ *is a* **finitely generated free** $\widehat{\mathbb{Z}}^{\boldsymbol{\Sigma}}$**-module of rank** $\mathbf{Node}(\mathcal{G})^{\sharp}$. *We shall refer to a nontrivial profinite Dehn multi-twist whose image* $\in \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$ *lies in a direct summand [i.e., in a single "*$\Lambda_{\mathcal{G}}$*"] as a* **profinite Dehn twist**.

(v) **(Conjugation action on the group of profinite Dehn multi-twists)** *The action of* $\mathrm{Aut}(\mathcal{G})$ *on* $\bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$

$$\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(\mathrm{Dehn}(\mathcal{G})) \xrightarrow{\sim} \mathrm{Aut}(\bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}})$$

determined by conjugation by elements of $\mathrm{Aut}(\mathcal{G})$ *[cf. (i)] and the isomorphism of (iv) coincides with the action of* $\mathrm{Aut}(\mathcal{G})$ *on* $\bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$ *determined by the action* $\chi_{\mathcal{G}}$ *of* $\mathrm{Aut}(\mathcal{G})$ *on* $\Lambda_{\mathcal{G}}$ *and the natural action of* $\mathrm{Aut}(\mathcal{G})$ *on the finite set* $\mathrm{Node}(\mathcal{G})$.

Proof. Assertions (i), (ii), and (iii) follow immediately from the various definitions involved. Next, we verify assertion (iv). The *functoriality* of the homomorphism $\mathfrak{D}_{\mathcal{G}}$ follows immediately from the various definitions involved. The rest of the proof of assertion (iv) is devoted to verifying that the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is an *isomorphism*. First, we *claim* that

> $(*_1)$: if $\mathcal{G}$ is *noncyclically primitive* [cf. Definition 4.1], then the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *injective*.

Indeed, this follows immediately from Lemma 4.2, together with the definition of the homomorphism $\mathfrak{D}_{\mathcal{G}}$. This completes the proof of the above *claim* $(*_1)$.

Next, we *claim* that

> $(*_2)$: if $\mathcal{G}$ is *cyclically primitive* [cf. Definition 4.1], then the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *injective*.

Indeed, let $\alpha \in \mathrm{Ker}(\mathfrak{D}_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ be an element of $\mathrm{Ker}(\mathfrak{D}_{\mathcal{G}})$. Since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Denote by $\alpha_\infty \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ the automorphism of $\mathcal{G}_\infty$ determined by α [cf. Lemma 4.3, (ii)]; for integers $a \leq b \in \mathbb{Z}$, denote by $\alpha_{[a,b]} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[a,b]})$ the automorphism of $\mathcal{G}_{[a,b]}$ obtained by restricting $\alpha_\infty \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$. Then since α is a *profinite Dehn multi-twist*, one may verify easily that $\alpha_{[a,b]}$ is a *profinite Dehn multi-twist* of $\mathcal{G}_{[a,b]}$. Thus, since $\mathcal{G}_{[a,a+1]}$ is *noncyclically primitive* [cf. Lemma 4.3, (iv)], it follows immediately from the fact that $\alpha \in \mathrm{Ker}(\mathfrak{D}_{\mathcal{G}})$, together with the *claim* $(*_1)$, that $\alpha_{[a,a+1]}$ is *trivial*. Moreover, for any $a < b \in \mathbb{Z}$, it follows — by applying *induction on* $b-a$ and considering, in light of the *claim* $(*_1)$, the various *generizations* [cf. assertion (ii)] of $\mathcal{G}_{[a,b]}$ with respect to sets of the form "$\mathrm{Node}(\mathcal{G}_{[a.b]}) \setminus \{e\}$" — that the profinite Dehn multi-twist $\alpha_{[a,b]}$, hence also the automorphism α_∞, is *trivial*. In particular, it holds that α is *trivial* [cf. Lemma 4.3, (ii)], as desired. This completes the proof of the above *claim* $(*_2)$.

Next, we *claim* that

> $(*_3)$: for arbitrary $\mathcal{G}$, the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *injective*.

We verify this *claim* $(*_3)$ by *induction on* $\mathrm{Node}(\mathcal{G})^\sharp$. If $\mathrm{Node}(\mathcal{G})^\sharp \leq 1$, then the *claim* $(*_3)$ follows formally from the *claims* $(*_1)$ and $(*_2)$. Now suppose that $\mathrm{Node}(\mathcal{G})^\sharp > 1$, and that the *induction hypothesis* is in force. Let $e \in \mathrm{Node}(\mathcal{G})$ be a node of $\mathcal{G}$. Write $\mathbb{H}$ for the *unique* sub-semi-graph of *PSC-type* of $\mathbb{G}$ whose set of vertices is $\mathcal{V}(e)$. Then one may verify easily that $S \stackrel{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}|_{\mathbb{H}}) \setminus \{e\}$ is *not of separating type* as a subset of $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$. Thus, since $(\mathcal{G}|_{\mathbb{H}})_{\succ S}$ has *precisely one* node, it follows immediately from assertion (iii), together with the *claims* $(*_1)$ and $(*_2)$, that the profinite Dehn multi-twist $\alpha_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}$ of $(\mathcal{G}|_{\mathbb{H}})_{\succ S}$ determined by $\alpha \in \mathrm{Dehn}(\mathcal{G})$ is *trivial*. In particular, it follows immediately from the definition of a generization [cf., especially, the definition of the anabelioids corresponding to the vertices of a generization given in Definition 2.8, (vi)], together with the definition of a profinite Dehn multi-twist, that the automorphism $\alpha_{\mathcal{G}_{\rightsquigarrow\{e\}}}$ of the generization $\mathcal{G}_{\rightsquigarrow\{e\}}$ determined by α [cf. Proposition 2.9, (ii)] is a *profinite Dehn multi-twist*. Therefore, since $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow\{e\}})^\sharp < \mathrm{Node}(\mathcal{G})^\sharp$, it follows immediately from assertion (ii), together with the induction hypothesis, that $\alpha_{\mathcal{G}_{\rightsquigarrow\{e\}}} \in \mathrm{Ker}(\mathfrak{D}_{\mathcal{G}_{\rightsquigarrow\{e\}}})$, hence also $\alpha \in \mathrm{Ker}(\mathfrak{D}_{\mathcal{G}})$, is *trivial*. This completes the proof of the *claim* $(*_3)$.

Next, we *claim* that

> $(*_4)$: if $\mathcal{G}$ is *noncyclically primitive* [cf. Definition 4.1], then the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *surjective*.

Indeed, this follows immediately from Lemma 4.2, together with the various definitions involved. This completes the proof of the *claim* $(*_4)$.

Next, we *claim* that

> $(*_5)$: if $\mathcal{G}$ is *cyclically primitive* [cf. Definition 4.1], then the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *surjective*.

Indeed, let $\lambda \in \Lambda_{\mathcal{G}}$ be an element of $\Lambda_{\mathcal{G}}$. Since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Then it follows immediately from Corollary 3.9, (ii), together with Lemma 4.3, (v), that for any integers $a \leq 0 < b \in \mathbb{Z}$, the natural morphisms $\mathcal{G}_{[0,0]} \to \mathcal{G}_{[a,b]}$ and $\mathcal{G}_{[0,0]} \to \mathcal{G}_\infty \to \mathcal{G}$ induce isomorphisms $\Lambda_{\mathcal{G}_{[a,b]}} \overset{\sim}{\leftarrow} \Lambda_{\mathcal{G}_{[0,0]}} \overset{\sim}{\to} \Lambda_{\mathcal{G}}$. By abuse of notation, write $\lambda \in \Lambda_{\mathcal{G}_{[a,b]}}$ for the element of $\Lambda_{\mathcal{G}_{[a,b]}}$ corresponding to $\lambda \in \Lambda_{\mathcal{G}}$. Now since $\mathcal{G}_{[0,1]}$ is *noncyclically primitive* [cf. Lemma 4.3, (iv)], it follows from the *claims* $(*_1)$, $(*_4)$ that there exists a *unique* profinite Dehn multi-twist $\lambda_{[0,1]} \in \mathrm{Dehn}(\mathcal{G}_{[0,1]})$ such that $\mathfrak{D}_{\mathcal{G}_{[0,1]}}(\lambda_{[0,1]}) = \lambda$.

Next, we *claim* that

> (†) : for any $a \leq 0 < b \in \mathbb{Z}$, there exists a [necessarily unique — cf. *claim* $(*_3)$] profinite Dehn multi-twist $\lambda_{[a,b]} \in \mathrm{Dehn}(\mathcal{G}_{[a,b]})$ such that $\mathfrak{D}_e(\lambda_{[a,b]}) = \lambda$ for every node $e \in \mathrm{Node}(\mathcal{G}_{[a,b]})$.

We verify this *claim* (†) by *induction on* $b-a$. If $b-a = 1$, or equivalently, $[a,b] = [0,1]$, then we have already shown the existence of a profinite Dehn multi-twist $\lambda_{[0,1]} \in \mathrm{Dehn}(\mathcal{G}_{[0,1]})$ of the desired type. Now suppose that $1 < b-a$, and that for $I \in \{[a,b-1],[a+1,b]\}$, there exists a profinite Dehn multi-twist $\lambda_I \in \mathrm{Dehn}(\mathcal{G}_I)$ such that $\mathfrak{D}_e(\lambda_I) = \lambda$ for every node $e \in \mathrm{Node}(\mathcal{G}_I)$. Then one may verify easily that $\mathrm{Node}(\mathcal{G}_I)$ may be regarded as a subset of $\mathrm{Node}(\mathcal{G}_{[a,b]})$, that $\mathcal{H}_{[a,b]} \overset{\mathrm{def}}{=} (\mathcal{G}_{[a,b]})_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_I)}$ is *noncyclically primitive*, and that, if one allows v to range over the [two] vertices of $\mathcal{H}_{[a,b]}$, then the resulting semi-graphs of anabelioids $(\mathcal{H}_{[a,b]})|_v$ are naturally isomorphic to $\mathcal{H}_I \overset{\mathrm{def}}{=} (\mathcal{G}_I)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_I)}$ and $\mathcal{G}_{[c_I,c_I]}$, where we write c_I for b (respectively, a) if $I = [a,b-1]$ (respectively, $I = [a+1,b]$).

Let $\Pi_{e_I} \subseteq \Pi_{\mathcal{H}_I}$ be a cuspidal subgroup of $\Pi_{\mathcal{H}_I}$ corresponding to the cusp e_I determined by the *unique* node of $\mathcal{H}_{[a,b]}$; $\Pi_{e_{[c_I,c_I]}} \subseteq \Pi_{\mathcal{G}_{[c_I,c_I]}}$ a cuspidal subgroup of $\Pi_{\mathcal{G}_{[c_I,c_I]}}$ corresponding to the cusp $e_{[c_I,c_I]}$ determined by the *unique* node of $\mathcal{H}_{[a,b]}$; $\widetilde{\lambda}_I \in \mathrm{Aut}(\Pi_{\mathcal{H}_I})$ a lifting of the outomorphism of $\Pi_{\mathcal{H}_I}$ determined by $\lambda_I \in \mathrm{Dehn}(\mathcal{G}_I) \hookrightarrow \mathrm{Aut}(\mathcal{H}_I)$ [cf. Proposition 2.9, (ii)] which preserves Π_{e_I} and induces the identity automorphism of Π_{e_I}. [Note that since $\lambda_I \in \mathrm{Dehn}(\mathcal{G}_I)$, one may verify easily that such a lifting $\widetilde{\lambda}_I \in \mathrm{Aut}(\Pi_{\mathcal{H}_I})$ exists.] Then for *any* element $\delta \in \Pi_{e_{[c_I,c_I]}}$ of $\Pi_{e_{[c_I,c_I]}}$, it follows immediately from Lemma 4.2 that by gluing — by means of the natural isomorphism $\Pi_{e_I} \overset{\sim}{\to} \Pi_{e_{[c_I,c_I]}}$ — the automorphism $\widetilde{\lambda}_I \in \mathrm{Aut}(\Pi_{\mathcal{H}_I})$ to the inner automorphism of $\Pi_{\mathcal{G}_{[c_I,c_I]}}$ by $\delta \in \Pi_{e_{[c_I,c_I]}}$, we obtain an outomorphism $\lambda_{[a,b]}[\delta]$ of $\Pi_{\mathcal{H}_{[a,b]}}$, which — in light of [CmbGC], Proposition 1.5, (ii), together with the fact that $\lambda_I \in \mathrm{Dehn}(\mathcal{G}_I)$ — is contained in

$$\mathrm{Dehn}(\mathcal{G}_{[a,b]}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[a,b]}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{H}_{[a,b]}) \subseteq \mathrm{Out}(\Pi_{\mathcal{H}_{[a,b]}})$$

[cf. Proposition 2.9, (ii)]. Now it follows immediately from the definition of the homomorphism "$\mathfrak{D}_e$" that the assignment $\delta \mapsto \mathfrak{D}_{e_{\mathcal{G}_{[a,b]}}}(\lambda_{a,b}[\delta])$ — where we write $e_{\mathcal{G}_{[a,b]}}$ for the node of $\mathcal{G}_{[a,b]}$ corresponding to the unique node of $\mathcal{H}_{[a,b]}$ — determines a *bijection* $\Pi_{e_{[c_I,c_I]}} \overset{\sim}{\to} \Lambda_{\mathcal{G}}$. Thus, since $\mathfrak{D}_e(\lambda_I) = \lambda$ for every node $e \in \mathrm{Node}(\mathcal{G}_I)$, we conclude that there exists a *unique* element $\delta \in \Pi_{e_{[c_I,c_I]}}$ of $\Pi_{e_{[c_I,c_I]}}$ such that $\mathfrak{D}_e(\lambda_{[a,b]}[\delta]) = \lambda$ for every node $e \in \mathrm{Node}(\mathcal{G}_{[a,b]})$. This completes the proof of the *claim* (†).

Write $\lambda_\infty \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ for the automorphism of $\mathcal{G}_\infty$ determined by the $\lambda_{[a,b]}$'s of the *claim* (†). Now since $\mathfrak{D}_e(\lambda_{[a,b]}) = \lambda$ for arbitrary $a < b \in \mathbb{Z}$ and $e \in \mathrm{Node}(\mathcal{G}_{[a,b]})$, one may verify easily, by applying the *claim* $(*_3)$, that the automorphism λ_∞ commutes with the natural action of $\pi_1^{\mathrm{top}}(\mathbb{G}) \simeq \mathbb{Z}$ on $\mathcal{G}_\infty$. Thus, the automorphism λ_∞ determines an automorphism $\lambda_{\mathcal{G}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ of $\mathcal{G}$ [cf. Lemma 4.3, (ii)]. Moreover, it follows immediately from the definition of $\lambda_{\mathcal{G}}$, together with the fact that $\mathfrak{D}_e(\lambda_{[a,b]}) = \lambda$ for arbitrary $a < b \in \mathbb{Z}$ and $e \in \mathrm{Node}(\mathcal{G}_{[a,b]})$, that $\lambda_{\mathcal{G}}$ is a *profinite Dehn multi-twist* such that $\mathfrak{D}_{\mathcal{G}}(\lambda_{\mathcal{G}}) = \lambda \in \Lambda_{\mathcal{G}}$. This completes the proof of the *claim* $(*_5)$.

Finally, we *claim* that

> $(*_6)$: for arbitrary $\mathcal{G}$, the homomorphism $\mathfrak{D}_{\mathcal{G}}$ is *surjective*.

For each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, it follows from assertion (ii) that we have a commutative diagram of profinite groups

$$\begin{array}{ccc} \mathrm{Dehn}(\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G}) \setminus \{e\}}) & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) \\ {\scriptstyle \mathfrak{D}_{\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G}) \setminus \{e\}}}} \big\downarrow & & \big\downarrow {\scriptstyle \mathfrak{D}_{\mathcal{G}}} \\ \Lambda_{\mathcal{G}} & \longrightarrow & \bigoplus_{e' \in \mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \end{array}$$

— where the lower horizontal arrow is the natural inclusion into the component indexed by e. Now since $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G}) \setminus \{e\}})^{\sharp} = 1$, it follows from the *claims* $(*_4)$, $(*_5)$ that the left-hand vertical arrow $\mathfrak{D}_{\mathcal{G}_{\rightsquigarrow \mathrm{Node}(\mathcal{G}) \setminus \{e\}}}$ in the above commutative diagram is *surjective*. Therefore, by allowing "e" to *vary* among the elements of $\mathrm{Node}(\mathcal{G})$, we conclude that $\mathfrak{D}_{\mathcal{G}}$ is *surjective*. This completes the proof of the *claim* $(*_6)$ — hence also, in light of the *claim* $(*_3)$ — of assertion (iv).

Finally, assertion (v) follows immediately from the various definitions involved, together with assertion (iv). This completes the proof of Theorem 4.8. Q.E.D.

Remark 4.8.1. In the notation of Theorem 4.8, denote by $\pi_1^{\mathrm{temp}}(\mathcal{G})$ the *tempered fundamental group* of $\mathcal{G}$ [cf. the discussion preceding [SemiAn], Proposition 3.6], by $\pi_1^{\mathrm{top}}(\mathbb{G})$ the [discrete] *topological fundamental group* of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$, by $\mathcal{G}_\infty \to \mathcal{G}$ the *connected tempered covering* of $\mathcal{G}$ corresponding to the natural surjection $\pi_1^{\mathrm{temp}}(\mathcal{G}) \twoheadrightarrow \pi_1^{\mathrm{top}}(\mathbb{G})$ [where we refer to [SemiAn], §3, concerning tempered coverings of a semi-graph of anabelioids], by $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ the group of automorphisms of $\mathcal{G}_\infty$ that induce the *identity automorphism* of the underlying semi-graph of $\mathcal{G}_\infty$, and by $\mathrm{Dehn}(\mathcal{G}_\infty) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)$ the group of "*profinite Dehn multi-twists*" of $\mathcal{G}_\infty$ — i.e., automorphisms of $\mathcal{G}_\infty$ which induce the *identity automorphism* on the underlying semi-graph of $\mathcal{G}_\infty$, as well as on the anabelioids of $\mathcal{G}_\infty$ corresponding to the vertices of $\mathcal{G}_\infty$. Then the following hold:

(i) The natural morphism $\mathcal{G}_\infty \to \mathcal{G}$ induces an exact sequence

$$1 \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \longrightarrow \pi_1^{\mathrm{temp}}(\mathcal{G}) \longrightarrow \pi_1^{\mathrm{top}}(\mathbb{G}) \longrightarrow 1 .$$

Moreover, the subgroup $\pi_1^{\mathrm{temp}}(\mathcal{G}_\infty) \subseteq \pi_1^{\mathrm{temp}}(\mathcal{G})$ of $\pi_1^{\mathrm{temp}}(\mathcal{G})$ is *characteristic*.

(ii) There exist natural *injections*

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty) , \quad \mathrm{Dehn}(\mathcal{G}) \hookrightarrow \mathrm{Dehn}(\mathcal{G}_\infty) ,$$

$$\pi_1^{\mathrm{top}}(\mathbb{G}) \hookrightarrow \mathrm{Aut}(\mathcal{G}_\infty)$$

— where the third injection is determined up to composition with a $\pi_1^{\mathrm{top}}(\mathbb{G})$-inner automorphism — which satisfy the equalities

$$Z_{\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_\infty)}(\pi_1^{\mathrm{top}}(\mathbb{G})) = \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})\,;$$

$$\mathrm{Dehn}(\mathcal{G}) = \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \cap \mathrm{Dehn}(\mathcal{G}_\infty)\,.$$

(iii) There exists a natural isomorphism

$$\mathrm{Dehn}(\mathcal{G}_\infty) \xrightarrow{\sim} \prod_{\mathrm{Node}(\mathcal{G}_\infty)} \Lambda_{\mathcal{G}}\,.$$

Indeed, assertion (i) (respectively, (ii)) follows immediately from a similar argument to the argument used in the proof of Lemma 4.3, (i) (respectively, Lemma 4.3, (ii)), together with the various definitions involved. On the other hand, the existence of the natural isomorphism asserted in assertion (iii) follows immediately from the fact that the various homomorphisms $\mathfrak{D}_{(\mathcal{G}_\infty)|_{\mathbb{H}}}$ — where $\mathbb{H}$ ranges over the sub-semi-graphs of *PSC-type* [cf. Definition 2.2, (i)] of the underlying semi-graph of $\mathcal{G}_\infty$, and we write $(\mathcal{G}_\infty)|_{\mathbb{H}}$ for the semi-graph of anabelioids obtained by restricting $\mathcal{G}_\infty$ to $\mathbb{H}$ [cf. the discussion preceding [SemiAn], Definition 2.2], which [as is easily verified] is of *pro-*Σ *PSC-type* — are *isomorphisms*. [Note that since $(\mathcal{G}_\infty)|_{\mathbb{H}}$ is of *pro-*Σ *PSC-type*, the fact that $\mathfrak{D}_{(\mathcal{G}_\infty)|_{\mathbb{H}}}$ is an isomorphism is a consequence of Theorem 4.8, (iv). However, since $\mathbb{H}$ is a *tree*, it follows from the *simple structure of* $\mathbb{H}$ that one may verify that $\mathfrak{D}_{(\mathcal{G}_\infty)|_{\mathbb{H}}}$ is an isomorphism in a fairly *direct* fashion, by arguing as in the proofs of the *claims* $(*_1)$, $(*_4)$ that appear in the proof of Theorem 4.8, (iv).]

In particular, it follows immediately from assertions (ii), (iii) that one may recover the natural isomorphism

$$\mathrm{Dehn}(\mathcal{G}) \xrightarrow{\sim} Z_{\prod_{\mathrm{Node}(\mathcal{G}_\infty)} \Lambda_{\mathcal{G}}}(\pi_1^{\mathrm{top}}(\mathbb{G})) \xrightarrow{\sim} \prod_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$$

of Theorem 4.8, (iv).

Definition 4.9. We shall write

$$\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$$

for the [closed] subgroup of "*glueable*" collections of outomorphisms of the direct product $\prod_{v\in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$ consisting of elements $(\alpha_v)_{v\in \mathrm{Vert}(\mathcal{G})}$ such that $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ [cf. Definition 3.8, (ii)] for any v, $w \in \mathrm{Vert}(\mathcal{G})$.

Proposition 4.10 (Properties of automorphisms that fix the underlying semi-graph).

(i) **(Factorization)** *The natural homomorphism*

$$\begin{array}{ccc} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) & \longrightarrow & \prod_{v\in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \\ \alpha & \mapsto & (\alpha_{\mathcal{G}|_v})_{v\in \mathrm{Vert}(\mathcal{G})} \end{array}$$

[cf. Definition 2.14, (ii); Remark 2.5.1, (ii)] factors through the closed subgroup $\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v\in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v)$.

(ii) **(Exact sequence relating profinite Dehn multi-twists and glueable outomorphisms)** *The resulting homomorphism* $\rho^{\mathrm{Vert}}_{\mathcal{G}} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Glu}(\mathcal{G})$ *[cf. (i)] fits into an* **exact sequence** *of profinite groups*

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \xrightarrow{\rho^{\mathrm{Vert}}_{\mathcal{G}}} \mathrm{Glu}(\mathcal{G}) \longrightarrow 1\,.$$

(iii) **(Surjectivity of cyclotomic characters)** *The restriction of the pro-*Σ *cyclotomic character* $\chi_{\mathcal{G}}$ *of* $\mathcal{G}$ *[cf. Definition 3.8, (ii)] to* $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$

$$\chi_{\mathcal{G}}|_{\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})} \colon \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow (\widehat{\mathbb{Z}}^{\Sigma})^*$$

— hence also $\chi_{\mathcal{G}}$ *— is* **surjective**.

(iv) **(Liftability of automorphisms)** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of* $\mathbb{G}$ *and* $S \subseteq \mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)]. Then the homomorphism*

$$\begin{array}{ccc} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}|_{\mathbb{H}})_{\succ S}) \\ \alpha & \mapsto & \alpha_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}} \end{array}$$

[cf. Definitions 2.5, (ii); 2.14, (ii)] is **surjective**.

Proof. Assertion (i) follows immediately from Corollary 3.9, (iv). Next, we verify assertion (ii). It follows immediately from the various definitions involved that $\mathrm{Ker}(\rho_{\mathcal{G}}^{\mathrm{Vert}}) = \mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$. Thus, to complete the proof of assertion (ii), it suffices to verify that the homomorphism $\rho_{\mathcal{G}}^{\mathrm{Vert}}$ is *surjective.*

Now we *claim* that

> $(*_1)$: if $\mathcal{G}$ is *noncyclically primitive* [cf. Definition 4.1], then the homomorphism $\rho_{\mathcal{G}}^{\mathrm{Vert}}$ is *surjective.*

Indeed, this follows immediately from Corollary 3.9, (v); Lemma 4.2, together with the various definitions involved. This completes the proof of the *claim* $(*_1)$.

Next, we *claim* that

> $(*_2)$: if $\mathcal{G}$ is *cyclically primitive* [cf. Definition 4.1], then the homomorphism $\rho_{\mathcal{G}}^{\mathrm{Vert}}$ is *surjective.*

Indeed, since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Then it follows immediately from the fact that $\mathrm{Vert}(\mathcal{G})^{\sharp} = 1$ [cf. Remark 4.1.1], together with Lemma 4.3, (v), that the composite of natural morphisms $\mathcal{G}_{[0,0]} \to \mathcal{G}_{\infty} \to \mathcal{G}$ determines a natural identification $\mathrm{Glu}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[0,0]})$. Let $\alpha = \alpha_{[0,0]} \in \mathrm{Glu}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[0,0]})$ be an element of $\mathrm{Glu}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[0,0]})$. For each $a \in \mathbb{Z}$, denote by $\alpha_{[a,a]} \in \mathrm{Aut}(\mathcal{G}_{[a,a]})$ the automorphism of $\mathcal{G}_{[a,a]}$ determined by conjugating the automorphism α of $\mathcal{G}_{[0,0]}$ by the isomorphism $\gamma_{\infty}^{a} \colon \mathcal{G}_{[0,0]} \overset{\sim}{\to} \mathcal{G}_{[a,a]}$ [cf. Lemma 4.3, (iii), (vi)]. Then for any $c < b \in \mathbb{Z}$, it follows from the various definitions involved that the various $\alpha_{[a,a]}$'s satisfy the *gluing condition* necessary to apply the *claim* $(*_1)$, hence that we may *glue them together* [cf. the proof of the *claim* $(*_3)$ below for more details concerning this sort of gluing argument] to obtain a(n) [not necessarily unique] element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[c,b]})$. Thus, by allowing $c < b \in \mathbb{Z}$ to vary, we obtain a(n) [not necessarily unique] element $\alpha_{\infty} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\infty})$ of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\infty})$. Now it follows immediately from the definition of α_{∞} that for any $\gamma \in \pi_1^{\mathrm{top}}(\mathbb{G})$, the automorphism $[\alpha_{\infty}, \gamma] \overset{\mathrm{def}}{=} \alpha_{\infty} \cdot \gamma \cdot \alpha_{\infty}^{-1} \cdot \gamma^{-1}$ of $\mathcal{G}_{\infty}$ is a "profinite Dehn multi-twist" of $\mathcal{G}_{\infty}$, i.e., $[\alpha_{\infty}, \gamma] \in \mathrm{Dehn}(\mathcal{G}_{\infty})$ [cf. Remark 4.8.1]. Moreover, one may verify easily that the assignment $\gamma \mapsto [\alpha_{\infty}, \gamma]$ determines a 1-cocycle $\pi_1^{\mathrm{top}}(\mathbb{G}) \to \mathrm{Dehn}(\mathcal{G}_{\infty})$. Thus, by Remark 4.8.1, (iii), together with the [easily verified] fact that

$$H^1(\mathbb{Z}, \prod_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Sigma}) = \{0\}$$

— where we take the action of $\mathbb{Z}$ on $\prod_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Sigma}$ to be the action determined by the *trivial action* of $\mathbb{Z}$ on $\widehat{\mathbb{Z}}^{\Sigma}$ and the action of $\mathbb{Z}$ on the index set $\mathbb{Z}$ given by addition — we conclude that there exists an element $\beta \in \mathrm{Dehn}(\mathcal{G}_\infty)$ such that the automorphism $\beta \circ \alpha_\infty$ commutes with the natural action of $\pi_1^{\mathrm{top}}(\mathbb{G})$ on $\mathcal{G}_\infty$. In particular, it follows from Lemma 4.3, (ii), that $\beta \circ \alpha_\infty$ determines an element $\alpha_{\mathcal{G}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$. Now since $\beta \in \mathrm{Dehn}(\mathcal{G}_\infty)$, it follows immediately from the various definitions involved that $\rho_{\mathcal{G}}^{\mathrm{Vert}}(\alpha_{\mathcal{G}}) = \alpha \in \mathrm{Glu}(\mathcal{G}) \overset{\sim}{\to} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{[0,0]})$. This completes the proof of the *claim* $(*_2)$.

Finally, we *claim* that

> $(*_3)$: for arbitrary $\mathcal{G}$, the homomorphism $\rho_{\mathcal{G}}^{\mathrm{Vert}}$ is *surjective*.

We verify this *claim* $(*_3)$ by *induction on* $\mathrm{Node}(\mathcal{G})^\sharp$. If $\mathrm{Node}(\mathcal{G})^\sharp \leq 1$, then this follows immediately from the *claims* $(*_1)$, $(*_2)$. Now suppose that $\mathrm{Node}(\mathcal{G})^\sharp > 1$, and that the *induction hypothesis* is in force. Let $e \in \mathrm{Node}(\mathcal{G})$ be a node of $\mathcal{G}$. Write $\mathbb{H}$ for the *unique* sub-semi-graph of *PSC-type* of $\mathbb{G}$ whose set of vertices is $\mathcal{V}(e)$. Then one may verify easily that $S \overset{\mathrm{def}}{=} \mathrm{Node}(\mathcal{G}|_{\mathbb{H}}) \setminus \{e\}$ is *not of separating type* as a subset of $\mathrm{Node}(\mathcal{G}|_{\mathbb{H}})$. Thus, since $(\mathcal{G}|_{\mathbb{H}})_{\succ S}$ has *precisely one* node, and $(\alpha_v)_{v \in \mathcal{V}(e)}$ may be regarded as an element of $\mathrm{Glu}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$, it follows from the *claims* $(*_1)$, $(*_2)$ that there exists an automorphism $\beta \in \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ of $(\mathcal{G}|_{\mathbb{H}})_{\succ S}$ such that $\rho^{\mathrm{Vert}}_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}(\beta) = (\alpha_v)_{v \in \mathcal{V}(e)} \in \mathrm{Glu}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$. Write $\beta_{\rightsquigarrow\{e\}} \in \mathrm{Aut}^{|\mathrm{grph}|}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\rightsquigarrow\{e\}})$ for the automorphism of $((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\rightsquigarrow\{e\}}$ determined by $\beta \in \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$ [cf. Proposition 2.9, (ii)]. Then it follows immediately from Corollary 3.9, (i), together with the definition of a generization [cf., especially, the definition of the anabelioids corresponding to the vertices of a generization given in Definition 2.8, (vi)], that the element

$$\begin{aligned} \gamma &\overset{\mathrm{def}}{=} (\beta_{\rightsquigarrow\{e\}}, (\alpha_v)_{v \notin \mathcal{V}(e)}) \\ &\in \mathrm{Aut}^{|\mathrm{grph}|}(((\mathcal{G}|_{\mathbb{H}})_{\succ S})_{\rightsquigarrow\{e\}}) \times \textstyle\prod_{v \notin \mathcal{V}(e)} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}|_v) \end{aligned}$$

may be regarded as an element of $\mathrm{Glu}(\mathcal{G}_{\rightsquigarrow\{e\}})$. Now since $\mathrm{Node}(\mathcal{G}_{\rightsquigarrow\{e\}})^\sharp < \mathrm{Node}(\mathcal{G})^\sharp$, it follows from the induction hypothesis that there exists an automorphism $\alpha_{\rightsquigarrow\{e\}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\rightsquigarrow\{e\}})$ of $\mathcal{G}_{\rightsquigarrow\{e\}}$ such that $\rho^{\mathrm{Vert}}_{\mathcal{G}_{\rightsquigarrow\{e\}}}(\alpha_{\rightsquigarrow\{e\}}) = \gamma \in \mathrm{Glu}(\mathcal{G}_{\rightsquigarrow\{e\}})$. On the other hand, since $\beta_{\rightsquigarrow\{e\}}$ arises from an element β of $\mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$, it follows immediately from [CmbGC], Proposition 1.5, (ii), that $\alpha_{\rightsquigarrow\{e\}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\rightsquigarrow\{e\}})$ is contained in the image of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{\rightsquigarrow\{e\}})$ [cf. Proposition 2.9, (ii)]. Moreover, since $\rho^{\mathrm{Vert}}_{(\mathcal{G}|_{\mathbb{H}})_{\succ S}}(\beta) = (\alpha_v)_{v \in \mathcal{V}(e)} \in \mathrm{Glu}((\mathcal{G}|_{\mathbb{H}})_{\succ S})$, it follows immediately from our original characterization of $\alpha_{\rightsquigarrow\{e\}}$ that $\rho^{\mathrm{Vert}}_{\mathcal{G}}(\alpha_{\rightsquigarrow\{e\}}) = (\alpha_v)_{v\in \mathrm{Vert}(\mathcal{G})} \in \mathrm{Glu}(\mathcal{G})$. Thus, we conclude that $\rho^{\mathrm{Vert}}_{\mathcal{G}}$ is *surjective*, as desired. This completes the proof of the *claim* $(*_3)$, hence also of assertion (ii).

Next, we verify assertion (iii). First, let us observe that one may verify easily that there exist a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}$ that is *totally degenerate* [cf. Definition 2.3, (iv)], a subset $S \subseteq \mathrm{Node}(\mathcal{H})$, and an isomorphism of semi-graphs of anabelioids $\mathcal{H}_{\rightsquigarrow S} \overset{\sim}{\to} \mathcal{G}$. Now since we have a natural injection $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{H}) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{H}_{\rightsquigarrow S}) \overset{\sim}{\to} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ [cf. Proposition 2.9, (ii)], it follows immediately from Corollary 3.9, (i), that to verify assertion (iii), by replacing $\mathcal{G}$ by $\mathcal{H}$, we may assume without loss of generality that $\mathcal{G}$ is *totally degenerate*. On the other hand, it follows immediately from assertion (ii), together with Corollary 3.9, (ii), that to verify assertion (iii), it suffices to verify the *surjectivity* of $\chi_{\mathcal{G}|_v}$ for each $v \in \mathrm{Vert}(\mathcal{G})$. Thus, to verify assertion (iii), by replacing $\mathcal{G}$ by $\mathcal{G}|_v$, we may assume without loss of generality that $\mathcal{G}$ is of *type* $(0, 3)$ [cf. Definition 2.3, (i)]. But assertion (iii) in the case where $\mathcal{G}$ is of *type* $(0, 3)$ follows immediately by considering the natural outer action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the field of rational numbers $\mathbb{Q}$ — where we use the notation $\overline{\mathbb{Q}}$ to denote an algebraic closure of $\mathbb{Q}$ — on the semi-graph of anabelioids of pro-Σ PSC-type associated to the tripod $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ over $\overline{\mathbb{Q}}$. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Write $\mathcal{H} \overset{\mathrm{def}}{=} (\mathcal{G}|_{\mathbb{H}})_{\succ S}$. Then it follows immediately from assertion (ii), together with Theorem 4.8, (iii), that the homomorphism $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{H})$ in question fits into a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) & \xrightarrow{\rho^{\mathrm{Vert}}_{\mathcal{G}}} & \mathrm{Glu}(\mathcal{G}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{H}) & \longrightarrow & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{H}) & \xrightarrow{\rho^{\mathrm{Vert}}_{\mathcal{H}}} & \mathrm{Glu}(\mathcal{H}) & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*. Now since the left-hand vertical arrow is *surjective* [cf. Theorem 4.8, (iii), (iv)], to verify assertion (iv), it suffices to verify the *surjectivity* of the right-hand vertical arrow. But this follows immediately from assertion (iii), together with the definition of "Glu(−)". This completes the proof of assertion (iv). Q.E.D.

§5. Comparison with scheme theory

In the present §, we discuss [cf. Proposition 5.6; Theorem 5.7; Corollaries 5.9, 5.10 below] the relationship between *intrinsic, group-theoretic* properties of profinite Dehn multi-twists [such as *length*, *nondegeneracy*, and *positive definiteness* — cf. Definitions 5.1; 5.8, (ii), (iii) below] and *scheme-theoretic characterizations* of properties of outer representations of pro-Σ PSC-type [such as *length*, *strict nodal nondegeneracy*, and *IPSC-ness* — cf. Definition 5.3, (ii) below; [NodNon], Definition 2.4, (i), (iii)]. The resulting theory leads naturally to a proof of the *graphicity* of C-admissible outomorphisms contained in the commensurator of the group of profinite Dehn multi-twists [cf. Theorem 5.14 below].

Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_{\mathcal{G}}$ for the [pro-Σ] fundamental group of $\mathcal{G}$, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}$.

Definition 5.1. Let $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$) be an outer representation of pro-Σ PSC-type [cf. [NodNon], Definition 2.1, (i)] which is of *NN-type* [cf. [NodNon], Definition 2.4, (iii)] and $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ an element of $\mathrm{Node}(\widetilde{\mathcal{G}})$. Write $\Pi_I \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$ [cf. the discussion entitled "*Topological groups*" in §0]; $\widetilde{v}, \widetilde{w} \in \mathrm{Vert}(\mathcal{G})$ for the two distinct elements of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}, \widetilde{w}\}$ [cf. [NodNon], Remark 1.2.1, (iii)]; $I_{\widetilde{e}}$, $I_{\widetilde{v}}$, $I_{\widetilde{w}} \subseteq \Pi_I$ for the inertia subgroups of Π_I associated to $\widetilde{e}$, $\widetilde{v}$, $\widetilde{w}$, respectively, i.e., the centralizers of $\Pi_{\widetilde{e}}, \Pi_{\widetilde{v}}, \Pi_{\widetilde{w}} \subseteq \Pi_I$ in Π_I, respectively [cf. [NodNon], Definition 2.2]. Then it follows from condition (3) of [NodNon], Definition 2.4, that the natural homomorphism $I_{\widetilde{v}} \times I_{\widetilde{w}} \to I_{\widetilde{e}}$ is an *open injection.* Write

$$\mathrm{lng}^{\Sigma}_{\mathcal{G}}(\widetilde{e}, \rho) \stackrel{\mathrm{def}}{=} [I_{\widetilde{e}} : I_{\widetilde{v}} \times I_{\widetilde{w}}]$$

for the index of $I_{\widetilde{v}} \times I_{\widetilde{w}}$ in $I_{\widetilde{e}}$; we shall refer to $\mathrm{lng}^{\Sigma}_{\mathcal{G}}(\widetilde{e}, \rho)$ as the Σ*-length of* $\widetilde{e}$ *with respect to* ρ. Note that it follows immediately from the various definitions involved that the Σ-length of $\widetilde{e}$ with respect to ρ *depends only* on $e \stackrel{\mathrm{def}}{=} \widetilde{e}(\mathcal{G}) \in \mathrm{Node}(\mathcal{G})$ and ρ. Write

$$\mathrm{lng}^{\Sigma}_{\mathcal{G}}(e, \rho) \stackrel{\mathrm{def}}{=} \mathrm{lng}^{\Sigma}_{\mathcal{G}}(\widetilde{e}, \rho)\,;$$

we shall refer to $\mathrm{lng}^{\Sigma}_{\mathcal{G}}(e, \rho)$ as the Σ*-length of* $e \in \mathrm{Node}(\mathcal{G})$ *with respect to* ρ.

Lemma 5.2 (Outer representations of SVA-type and profinite Dehn multi-twists). *Let* $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}}))$ *be an outer representation of pro-*Σ *PSC-type which is of* **SVA-type** *[cf.* [NodNon], *Definition 2.4, (ii)] and* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ *an element of* $\mathrm{Node}(\widetilde{\mathcal{G}})$. *Write* $\Pi_I \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$ *[cf. the discussion entitled "Topological groups" in §0];* $\widetilde{v}$, $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ *for the two distinct elements of* $\mathrm{Vert}(\widetilde{\mathcal{G}})$ *such that* $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}, \widetilde{w}\}$ *[cf.* [NodNon], *Remark 1.2.1, (iii)];* $I_{\widetilde{e}}$, $I_{\widetilde{v}}$, $I_{\widetilde{w}} \subseteq \Pi_I$ *for the inertia subgroups of* Π_I *associated to* $\widetilde{e}$, $\widetilde{v}$, $\widetilde{w}$, *respectively;* $e \stackrel{\mathrm{def}}{=} \widetilde{e}(\mathcal{G})$; $v \stackrel{\mathrm{def}}{=} \widetilde{v}(\mathcal{G})$. *Then the following hold:*

(i) **(Outer representations of SVA-type and profinite Dehn multi-twists)** *The outer representation* ρ *factors through the closed subgroup* $\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$. *By abuse of notation, write* ρ *for the resulting homomorphism* $I \to \mathrm{Dehn}(\mathcal{G})$.

(ii) **(Outer representations of SVA-type and homomorphisms of Dehn coordinates)** *The natural inclusions* $I_{\widetilde{v}}$, $I_{\widetilde{w}} \hookrightarrow I_{\widetilde{e}}$ *and the composite* $I_{\widetilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$ *determine a diagram of profinite groups*

$$\begin{array}{ccccccccc} & & & & I_{\widetilde{v}} \times I_{\widetilde{w}} & & & & \\ & & & & \downarrow & & & & \\ 1 & \longrightarrow & \Pi_{\widetilde{e}} & \longrightarrow & I_{\widetilde{e}} & \longrightarrow & I & \longrightarrow & 1 \end{array}$$

— where the lower horizontal sequence is **exact**, *and the closed subgroups* $I_{\widetilde{v}}$, $I_{\widetilde{w}} \subseteq I_{\widetilde{e}}$ *determine* **sections** *of the surjection* $I_{\widetilde{e}} \twoheadrightarrow I$, *respectively — hence also homomorphisms*

$$I \stackrel{\sim}{\leftarrow} I_{\widetilde{v}} \to I_{\widetilde{e}}/I_{\widetilde{w}} \stackrel{\sim}{\leftarrow} \Pi_{\widetilde{e}} = \Pi_e \overset{\mathfrak{syn}_{b_{\widetilde{v}}}}{\stackrel{\sim}{\to}} \Lambda_v \overset{\mathfrak{syn}_v}{\stackrel{\sim}{\to}} \Lambda_{\mathcal{G}}$$

— where the first "$\stackrel{\sim}{\leftarrow}$*" denotes the isomorphism given by the composite* $I_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$, *and* $b_{\widetilde{v}}$ *denotes the branch of* e *determined by the [unique] branch of* $\widetilde{e}$ *that abuts to* $\widetilde{v}$. *Moreover, the composite of these homomorphisms*

$$I \to \Lambda_{\mathcal{G}}$$

coincides *with the composite*

$$I \stackrel{\rho}{\longrightarrow} \mathrm{Dehn}(\mathcal{G}) \stackrel{\mathfrak{D}_e}{\longrightarrow} \Lambda_{\mathcal{G}}$$

[cf. (i); Definition 4.7]. In particular, if ρ *is of* **SNN-type** *[cf.* [NodNon], *Definition 2.4, (iii)], then the image of the*

composite $I \xrightarrow{\rho} \mathrm{Dehn}(\mathcal{G}) \xrightarrow{\mathfrak{D}_e} \Lambda_{\mathcal{G}}$ *coincides with* $\mathrm{lng}^{\Sigma}_{\mathcal{G}}(e,\rho) \cdot \Lambda_{\mathcal{G}} \subseteq \Lambda_{\mathcal{G}}$.

(iii) **(Centralizers and cyclotomic characters)** *Suppose that* ρ *is of* **SNN-type** *[cf.* [NodNon], *Definition 2.4, (iii)]. Let* $e \in \mathrm{Node}(\mathcal{G})$ *be a node of* $\mathcal{G}$. *Then* $\chi_{\mathcal{G}}(\alpha) = 1$ *[cf. Definition 3.8, (ii)] for any* $\alpha \in Z_{\mathrm{Aut}^{\{e\}}(\mathcal{G})}(\mathrm{Im}(\rho)) \subseteq \mathrm{Aut}^{\{e\}}(\mathcal{G})$ *[cf. Definition 2.6, (i)].*

Proof. Assertion (i) follows immediately from condition (2′) of [NodNon], Definition 2.4. Next, we verify assertion (ii). The fact that the natural inclusions $I_{\widetilde{v}}, I_{\widetilde{w}} \hookrightarrow I_{\widetilde{e}}$ and the composite $I_{\widetilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$ give rise to the diagram and homomorphisms of the first and second displays in the statement of assertion (ii) follows immediately from [NodNon], Lemma 2.5, (iv); condition (2′) of [NodNon], Definition 2.4. On the other hand, it follows immediately from the various definitions involved that the image of each $\beta \in I$ via the composite of $I \xleftarrow{\sim} I_{\widetilde{v}}$ with the action $I_{\widetilde{v}} \to \mathrm{Aut}(\Pi_{\mathcal{G}})$ given by conjugation coincides with the "$\alpha[\widetilde{v}]$" of Lemma 4.6, (i), in the case where one takes "α" to be $\rho(\beta)$. Thus, it follows immediately from the definition of $I_{\widetilde{w}}$ that the image of $\beta \in I$ via the composite $I \xleftarrow{\sim} I_{\widetilde{v}} \to I_{\widetilde{e}}/I_{\widetilde{w}} \xrightarrow{\sim} \Pi_{\widetilde{e}}$ coincides with the "$\delta_{\widetilde{e},\widetilde{v}}$" of Lemma 4.6, (i), in the case where one takes "α" to be $\rho(\beta)$. Therefore, it follows immediately from the definition of $\mathfrak{D}_e$ that the homomorphisms of the final two displays of assertion (ii) coincide. Thus, the final portion of assertion (ii) concerning ρ of *SNN-type* follows immediately from the definition of Σ-length. This completes the proof of assertion (ii). To verify assertion (iii), let us first observe that, by Theorem 4.8, (v), the conjugation action of $\alpha \in \mathrm{Aut}^{\{e\}}(\mathcal{G})$ on the $\Lambda_{\mathcal{G}} \subseteq \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \xleftarrow{\sim} \mathrm{Dehn}(\mathcal{G})$ indexed by $e \in \mathrm{Node}(\mathcal{G})$ is given by multiplication by $\chi_{\mathcal{G}}(\alpha)$. On the other hand, since $\mathbb{N} \ni \mathrm{lng}^{\Sigma}_{\mathcal{G}}(e,\rho) \neq 0$, it follows from the final portion of assertion (ii) that the projection of $\mathrm{Im}(\rho)$ to the coordinate indexed by e is *open*. Thus, the fact that α lies in the *centralizer* $Z_{\mathrm{Aut}^{\{e\}}(\mathcal{G})}(\mathrm{Im}(\rho))$ implies that $\chi_{\mathcal{G}}(\alpha) = 1$, as desired. This completes the proof of assertion (iii). Q.E.D.

Definition 5.3. Let R be a *complete discrete valuation ring* whose residue field is separably closed of characteristic $\notin \Sigma$; $\pi \in R$ a prime element of R; v_R the discrete valuation of R such that $v_R(\pi) = 1$; $S^{\log}$ the log scheme obtained by equipping $S \overset{\mathrm{def}}{=} \mathrm{Spec}\, R$ with the log structure defined by the maximal ideal $(\pi) \subseteq R$ of R; $s^{\log}$ the log scheme obtained by equipping the spectrum s of the residue field of R with the log structure induced by the log structure of $S^{\log}$ via the natural closed immersion $s \hookrightarrow S$; $X^{\log}$ a *stable log curve* over $S^{\log}$; $\mathcal{G}_{X^{\log}}$ the semi-graph of anabelioids of pro-Σ PSC-type determined by the special fiber $X_s^{\log} \stackrel{\text{def}}{=} X^{\log} \times_{S^{\log}} s^{\log}$of the stable log curve $X^{\log}$ [cf. [CmbGC], Example 2.5]; $I_{S^{\log}}$ ($\simeq \widehat{\mathbb{Z}}^{\Sigma}$) the maximal pro-$\Sigma$ completion of the log fundamental group $\pi_1(S^{\log})$ of $S^{\log}$.

(i) One may verify easily that the natural outer representation $I_{S^{\log}} \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ associated to the stable log curve $X^{\log}$ over $S^{\log}$ factors through $\mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \subseteq \mathrm{Aut}(\mathcal{G}_{X^{\log}})$. We shall write

$$\rho_{X_s^{\log}} \colon I_{S^{\log}} \longrightarrow \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$$

for the resulting homomorphism.

(ii) It follows from the well-known local structure of a stable log curve in a neighborhood of a node that for each node e of the special fiber of $X^{\log}$, there exists a nonzero element $a_e \neq 0$ of the maximal ideal $(\pi) \subseteq R$ such that the completion $\widehat{\mathcal{O}}_{X,e}$ of the local ring $\mathcal{O}_{X,e}$ at e is isomorphic to $R[[s_1, s_2]]/(s_1 s_2 - a_e)$ — where s_1, s_2 denote indeterminates. Write

$$\mathrm{lng}_{X^{\log}}(e) \stackrel{\text{def}}{=} v_R(a_e); \quad \mathrm{lng}^{\Sigma}_{X^{\log}}(e) \stackrel{\text{def}}{=} [\widehat{\mathbb{Z}}^{\Sigma} : \mathrm{lng}_{X^{\log}}(e) \cdot \widehat{\mathbb{Z}}^{\Sigma}].$$

We shall refer to $\mathrm{lng}_{X^{\log}}(e)$ as the *length* of e and to $\mathrm{lng}^{\Sigma}_{X^{\log}}(e)$ as the Σ-*length* of e. One verifies easily that $\mathrm{lng}_{X^{\log}}(e)$, hence also $\mathrm{lng}^{\Sigma}_{X^{\log}}(e)$, *depends only* on e, i.e., is *independent* of the choice of the isomorphism $\widehat{\mathcal{O}}_{X,e} \simeq R[[s_1, s_2]]/(s_1 s_2 - a_e)$.

Lemma 5.4 (Local geometric universal outer representations). *In the notation of Definition 5.3, suppose that $\mathcal{G}_{X^{\log}}$ is of type (g, r) [cf. Definition 2.3, (i); Remark 2.3.1]. Write $N \stackrel{\text{def}}{=} \mathrm{Node}(\mathcal{G}_{X^{\log}})^{\sharp}$ and $\sigma^{\log} \colon S^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_S$ [cf. the discussion entitled "Curves" in §0] for the classifying morphism of the stable log curve $X^{\log}$ over $S^{\log}$. Then the following hold:*

(i) **(Local structure of the moduli stack of pointed stable curves)** *Write $\widehat{\mathcal{O}}$ for the completion of the local ring of*

$(\overline{\mathcal{M}}_{g,r})_S$ at the image of the closed point of S via the underlying (1-)morphism of stacks σ of $\sigma^{\log}$ and $T^{\log}$ for the [fs] log scheme obtained by equipping $T \stackrel{\text{def}}{=} \operatorname{Spec} \widehat{\mathcal{O}}$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_S$. [Thus, we have a tautological strict [cf. [Illu]*, 1.2] (1-)morphism $T^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_S$.] Then there exists an isomorphism of R-algebras $R[[t_1, \cdots, t_{3g-3+r}]] \xrightarrow{\sim} \widehat{\mathcal{O}}$ such that the following hold:*

- *The log structure of the log scheme $T^{\log}$ is given by the following chart:*

$$\begin{array}{ccc} \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e & \longrightarrow & R[[t_1, \cdots, t_{3g-3+r}]] \xrightarrow{\sim} \widehat{\mathcal{O}} \\ (n_{e_1}, \cdots, n_{e_N}) & \mapsto & t_1^{n_{e_1}} \cdots t_N^{n_{e_N}} \end{array}$$

  — where we write $\mathbb{N}_e$ for the copy of $\mathbb{N}$ indexed by $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$.

- *For $1 \leq i \leq N$, the homomorphism of R-algebras $\widehat{\mathcal{O}} \to R$ induced by the morphism σ maps t_i to a_{e_i} [cf. Definition 5.3, (ii)].*

(ii) **(Log-scheme-theoretic description of log fundamental groups)** *Write $I_{T^{\log}}$ for the maximal pro-Σ quotient of the log fundamental group $\pi_1(T^{\log})$ of $T^{\log}$. Then we have natural isomorphisms*

$$I_{S^{\log}} \xrightarrow{\sim} \mathrm{Hom}\Big(\mathbb{N}^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1)\Big);$$

$$\begin{aligned} I_{T^{\log}} &\xrightarrow{\sim} \mathrm{Hom}\Big(\bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1)\Big) \\ &\xrightarrow{\sim} \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathrm{Hom}\Big(\mathbb{N}_e^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1)\Big), \end{aligned}$$

and the homomorphism $I_{S^{\log}} \to I_{T^{\log}}$ induced by the classifying morphism $\sigma^{\log}$ is the homomorphism obtained by applying the functor "$\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}((-)^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1))$" to the homomorphism of monoids

$$\begin{array}{ccc} \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e & \longrightarrow & \mathbb{N} \\ (n_{e_1}, \cdots, n_{e_N}) & \mapsto & \sum_{i=1}^{N} n_{e_i} \mathrm{lng}_{X^{\log}}(e_i) \end{array}.$$

(iii) **(Local geometric universal outer representations)** *The natural outer representation $I_{T^{\log}} \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ associated to the stable log curve over $T^{\log}$ determined* *by the tautological strict morphism* $T^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_S$ *factors through* $\mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \subseteq \mathrm{Aut}(\mathcal{G}_{X^{\log}})$*; thus, we have a homomorphism* $I_{T^{\log}} \to \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$*. Moreover, the homomorphism* $\rho_{X_s^{\log}} \colon I_{S^{\log}} \to \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ *factors as the composite of the homomorphism* $I_{S^{\log}} \to I_{T^{\log}}$ *induced by* $\sigma^{\log}$ *and this homomorphism* $I_{T^{\log}} \to \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$*.*

Proof. Assertion (i) follows immediately from the well-known local structure of the log stack $(\overline{\mathcal{M}}^{\log}_{g,r})_S$ [cf. [Knud], Theorem 2.7]. Assertion (ii) follows immediately from assertion (i), together with the well-known structure of the log fundamental groups of $S^{\log}$ and $T^{\log}$. Assertion (iii) follows immediately from the various definitions involved. Q.E.D.

Definition 5.5. In the notation of Definition 5.3, Lemma 5.4, we shall write $t^{\log}$ for the log scheme obtained by equipping the closed point t of T with the log structure naturally induced by the log structure of $T^{\log}$; $X_t^{\log}$ for the stable log curve over $t^{\log}$ corresponding to the natural strict morphism $t^{\log}$ ($\hookrightarrow T^{\log}$) $\to (\overline{\mathcal{M}}^{\log}_{g,r})_S$;

$$\rho^{\mathrm{univ}}_{X_t^{\log}} \colon I_{T^{\log}} \longrightarrow \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$$

for the homomorphism obtained in Lemma 5.4, (iii).

Proposition 5.6 (**Outer representations arising from stable log curves**)**.** *In the notation of Definition 5.3, Lemma 5.4, the following hold:*

(i) **(Compatibility of Σ-lengths)** *For each node* $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$ *of* $\mathcal{G}_{X^{\log}}$*, it holds that*

$$\mathrm{lng}^{\Sigma}_{\mathcal{G}_{X^{\log}}}(e, \rho_{X_s^{\log}}) = \mathrm{lng}^{\Sigma}_{X^{\log}}(e)$$

[cf. Definitions 5.1; 5.3, (ii)].

(ii) **(Isomorphicity of local geometric universal outer representations)** *The homomorphism*

$$\rho^{\mathrm{univ}}_{X_t^{\log}} \colon I_{T^{\log}} \longrightarrow \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$$

[cf. Definition 5.5] is an **isomorphism** *of profinite groups.*

(iii) **(Compatibility with generization)** *Let* $Q \subseteq \mathrm{Node}(\mathcal{G}_{X^{\log}})$ *be a subset of* $\mathrm{Node}(\mathcal{G}_{X^{\log}})$. *Then there exist a stable log curve* $Y^{\log}$ *over* $S^{\log}$ *and an isomorphism of semi-graphs of anabelioids* $(\mathcal{G}_{X^{\log}})_{\rightsquigarrow Q} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$ *that fit into a commutative diagram of profinite groups*

$$\begin{array}{ccc} I_{T_Y^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{Y_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{Y^{\log}}) \\ \downarrow & & \downarrow \\ I_{T_X^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{X_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \end{array}$$

— where we write $I_{T_X^{\log}}$, $I_{T_Y^{\log}}$ *for the "*$I_{T^{\log}}$*" associated to* $X^{\log}$, $Y^{\log}$, *respectively; the right-hand vertical arrow is the natural inclusion induced, via the isomorphism* $(\mathcal{G}_{X^{\log}})_{\rightsquigarrow Q} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$, *by the natural inclusion of Theorem 4.8, (ii); the left-hand vertical arrow is the injection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural projection of monoids*

$$\bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \twoheadrightarrow \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})} \mathbb{N}_e \, .$$

[Note that it follows immediately from the various definitions involved that $\mathrm{Node}(\mathcal{G}_{Y^{\log}}) \xleftarrow{\sim} \mathrm{Node}((\mathcal{G}_{X^{\log}})_{\rightsquigarrow Q})$ *may be regarded as a subset of* $\mathrm{Node}(\mathcal{G}_{X^{\log}})$.*]*

(iv) **(Compatibility with specialization)** *Let* $\mathcal{H}$ *be a semi-graph of anabelioids of pro-*Σ *PSC-type,* $Q \subseteq \mathrm{Node}(\mathcal{H})$, *and* $\mathcal{H}_{\rightsquigarrow Q} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$ *an isomorphism of semi-graphs of anabelioids. Then there exist a stable log curve* $Y^{\log}$ *over* $S^{\log}$ *and an isomorphism of semi-graphs of anabelioids* $\mathcal{H} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$ *that fit into a commutative diagram of profinite groups*

$$\begin{array}{ccc} I_{T_X^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{X_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \\ \downarrow & & \downarrow \\ I_{T_Y^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{Y_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{Y^{\log}}) \end{array}$$

— where we write $I_{T_X^{\log}}$, $I_{T_Y^{\log}}$ *for the "*$I_{T^{\log}}$*" associated to* $X^{\log}$, $Y^{\log}$, *respectively; the right-hand vertical arrow is the natural*

inclusion induced, via the isomorphisms $\mathcal{H}_{\rightsquigarrow Q} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$ *and* $\mathcal{H} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$*, by the natural inclusion of Theorem 4.8, (ii); the left-hand vertical arrow is the injection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural projection of monoids*

$$\bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})} \mathbb{N}_e \twoheadrightarrow \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \, .$$

[Note that it follows immediately from the various definitions involved that $\mathrm{Node}(\mathcal{G}_{X^{\log}}) \xleftarrow{\sim} \mathrm{Node}(\mathcal{H}_{\rightsquigarrow Q})$ *may be regarded as a subset of* $\mathrm{Node}(\mathcal{G}_{Y^{\log}}) \xleftarrow{\sim} \mathrm{Node}(\mathcal{H})$*.]*

(v) **(Input compatibility with "surgery")** *Let* $\mathbb{H}$ *be a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of the underlying semi-graph of* $\mathcal{G}_{X^{\log}}$*,* $Q \subseteq \mathrm{Node}((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)], and* $U \subseteq \mathrm{Cusp}(((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q})$ *[cf. Definition 2.5, (ii)] an* **omittable** *[cf. Definition 2.4, (i)] subset of* $\mathrm{Cusp}(((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q})$*. Then there exist a stable log curve* $Y^{\log}$ *over* $S^{\log}$ *and an isomorphism* $(((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q})_{\bullet U} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$ *[cf. Definition 2.4, (ii)] that fit into a commutative diagram of profinite groups*

$$\begin{array}{ccccc} I_{S^{\log}} & \longrightarrow & I_{T_X^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{X_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \\ \| & & \downarrow & & \downarrow \\ I_{S^{\log}} & \longrightarrow & I_{T_Y^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{Y_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{Y^{\log}}) \end{array}$$

— where we write $I_{T_X^{\log}}$*,* $I_{T_Y^{\log}}$ *for the "*$I_{T^{\log}}$*" associated to* $X^{\log}$*,* $Y^{\log}$*, respectively; the left-hand horizontal arrows are the homomorphisms induced by the classifying morphisms associated to* $X^{\log}$*,* $Y^{\log}$*, respectively; the right-hand vertical arrow is the natural surjection induced, via the isomorphism* $(((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q})_{\bullet U} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$*, by the natural surjection of Theorem 4.8, (iii); the middle vertical arrow is the surjection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural inclusion of monoids*

$$\bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})} \mathbb{N}_e \hookrightarrow \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \, .$$

[Note that it follows immediately from the various definitions involved that $\mathrm{Node}(\mathcal{G}_{Y^{\log}}) \overset{\sim}{\leftarrow} \mathrm{Node}((((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q})_{\bullet U})$ *may be regarded as a subset of* $\mathrm{Node}(\mathcal{G}_{X^{\log}})$*.]*

(vi) **(Output compatibility with "surgery")** *Let* $\mathcal{H}$ *be a semi-graph of anabelioids of pro-*Σ *PSC-type,* $\mathbb{K}$ *a sub-semi-graph of* **PSC-type** *[cf. Definition 2.2, (i)] of the underlying semi-graph of* $\mathcal{H}$*,* $Q \subseteq \mathrm{Node}(\mathcal{H}|_{\mathbb{K}})$ *[cf. Definition 2.2, (ii)] a subset of* $\mathrm{Node}(\mathcal{H}|_{\mathbb{K}})$ *that is* **not of separating type** *[cf. Definition 2.5, (i)],* $U \subseteq \mathrm{Cusp}((\mathcal{H}|_{\mathbb{K}})_{\succ Q})$ *[cf. Definition 2.5, (ii)] an* **omittable** *[cf. Definition 2.4, (i)] subset of* $\mathrm{Cusp}((\mathcal{H}|_{\mathbb{K}})_{\succ Q})$*, and* $((\mathcal{H}|_{\mathbb{K}})_{\succ Q})_{\bullet U} \overset{\sim}{\to} \mathcal{G}_{X^{\log}}$ *[cf. Definition 2.4, (ii)] an isomorphism of semi-graphs of anabelioids. Then there exist a stable log curve* $Y^{\log}$ *over* $S^{\log}$ *and an isomorphism of semi-graphs of anabelioids* $\mathcal{H} \overset{\sim}{\to} \mathcal{G}_{Y^{\log}}$ *that fit into a commutative diagram of profinite groups*

$$\begin{array}{ccccc} I_{S^{\log}} & \longrightarrow & I_{T_Y^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{Y_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{Y^{\log}}) \\ \| & & \downarrow & & \downarrow \\ I_{S^{\log}} & \longrightarrow & I_{T_X^{\log}} & \xrightarrow{\rho^{\mathrm{univ}}_{X_t^{\log}}} & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \end{array}$$

— where we write $I_{T_X^{\log}}$*,* $I_{T_Y^{\log}}$ *for the "*$I_{T^{\log}}$*" associated to* $X^{\log}$*,* $Y^{\log}$*, respectively; the left-hand horizontal arrows are the homomorphisms induced by the classifying morphisms associated to* $Y^{\log}$*,* $X^{\log}$*, respectively; the right-hand vertical arrow is the natural surjection induced, via the isomorphisms* $((\mathcal{H}|_{\mathbb{K}})_{\succ Q})_{\bullet U} \overset{\sim}{\to} \mathcal{G}_{X^{\log}}$ *and* $\mathcal{H} \overset{\sim}{\to} \mathcal{G}_{Y^{\log}}$*, by the natural surjection of Theorem 4.8, (iii); the middle vertical arrow is the surjection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural inclusion of monoids*

$$\bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \hookrightarrow \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})} \mathbb{N}_e .$$

[Note that it follows immediately from the various definitions involved that $\mathrm{Node}(\mathcal{G}_{X^{\log}}) \overset{\sim}{\leftarrow} \mathrm{Node}(((\mathcal{H}|_{\mathbb{K}})_{\succ Q})_{\bullet U})$ *may be regarded as a subset of* $\mathrm{Node}(\mathcal{G}_{Y^{\log}}) \overset{\sim}{\leftarrow} \mathrm{Node}(\mathcal{H})$*.]*

Proof. Assertion (i) follows immediately from the well-known local structure of a stable log curve in a neighborhood of a node. Next, we verify assertion (ii). By allowing "$\rho_{X_s^{\log}}$" to *vary* among the natural outer representations $I_{S^{\log}} \to \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ associated to stable log curves "$X^{\log}$" over $S^{\log}$ whose classifying morphisms "σ" *coincide* with the given σ on the closed point s of S, one concludes that the *surjectivity* of $\rho^{\mathrm{univ}}_{X_t^{\log}}$ follows immediately from the final portion of Lemma 5.2, (ii), concerning ρ of *SNN-type* [cf. also assertion (i); Theorem 4.8, (iv)]. [Here, we recall that $\rho_{X_s^{\log}}$ is of *IPSC-type* [cf. [NodNon], Definition 2.4, (i)], hence also of *SNN-type* [cf. [NodNon], Remark 2.4.2].] On the other hand, since both $\mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ and $I_{T^{\log}}$ are *free* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules of rank* $\mathrm{Node}(\mathcal{G}_{X^{\log}})^{\sharp}$ [cf. Theorem 4.8, (iv); Lemma 5.4, (ii)], assertion (ii) follows immediately from this *surjectivity* of $\rho^{\mathrm{univ}}_{X_t^{\log}}$. This completes the proof of assertion (ii).

Assertion (iii) (respectively, (iv)) follows immediately, in light of the well-known structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_S$ [cf. also the discussion entitled "*The Étale Fundamental Group of a Log Scheme*" in [CmbCsp], §0, concerning the *specialization isomorphism* on fundamental groups, as well as Remark 5.6.1 below], by considering a lifting to $S^{\log}$ of a stable log curve over $s^{\log}$ obtained by *deforming* the nodes of the special fiber $X_s^{\log} \stackrel{\mathrm{def}}{=} X^{\log} \times_{S^{\log}} s^{\log}$ corresponding to the nodes contained in Q (respectively, *degenerating* the moduli of $X_s^{\log}$ so as to obtain nodes corresponding to the nodes contained in Q) [cf. also Proposition 4.10, (iii)].

Next, we verify assertion (v). First, we observe that one may verify easily that if $\mathbb{H}$ is the underlying semi-graph of $\mathcal{G}_{X^{\log}}$, and $Q = \emptyset$, then the stable log curve $Y^{\log}$ over $S^{\log}$ obtained by omitting the cusps of $X^{\log}$ contained in U and the resulting natural isomorphism $(\mathcal{G}_{X^{\log}})_{\bullet U} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$ satisfy the conditions given in the statement of assertion (v). Thus, one verifies immediately that to verify assertion (v), we may assume without loss of generality that $U = \emptyset$.

Write $\mathcal{H} \stackrel{\mathrm{def}}{=} ((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q}$ and $V \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\mathcal{G}_{X^{\log}}) \setminus \mathrm{Vert}((\mathcal{G}_{X^{\log}})|_{\mathbb{H}}) \subseteq \mathrm{Vert}(\mathcal{G}_{X^{\log}})$. Denote by $(g_{\mathcal{H}}, r_{\mathcal{H}})$ the type of $\mathcal{H}$, and, for each $v \in V$, by (g_v, r_v) the type of v [cf. Definition 2.3, (i), (iii); Remark 2.3.1]. Then it follows immediately from the general theory of stable log curves that there exists a *"clutching (1-)morphism"* corresponding to the *operations* "$(-)|_{\mathbb{H}}$" and "$(-)_{\succ Q}$" [i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6]

$$\mathcal{N} \stackrel{\mathrm{def}}{=} (\overline{\mathcal{M}}_{g_{\mathcal{H}}, r_{\mathcal{H}}})_s \times_s \Big(\prod_{v \in V} (\mathcal{M}_{g_v, r_v})_s \Big) \longrightarrow (\overline{\mathcal{M}}_{g,r})_s$$

— where the fiber product "$\prod_{v\in V}$" is taken over s — that satisfies the following condition: write $\underline{\mathcal{N}}^{\log}$ for the log stack obtained by equipping the stack $\mathcal{N}$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_s$ via the above clutching morphism; then there exists an $s^{\log}$-*valued point* $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ of $\underline{\mathcal{N}}^{\log}$ such that the image of $\sigma^{\log}_{\mathcal{N}}$ via the natural strict (1-)morphism $\underline{\mathcal{N}}^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_s$ coincides with the $s^{\log}$-valued point of $(\overline{\mathcal{M}}^{\log}_{g,r})_s$ obtained by restricting the classifying morphism $\sigma^{\log} \in (\overline{\mathcal{M}}^{\log}_{g,r})_S(S^{\log})$ of $X^{\log}$ to $s^{\log}$. If, moreover, we write $Y^{\log}_s$ for the stable log curve over $s^{\log}$ corresponding to the image of $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ via the composite of (1-)morphisms

$$\underline{\mathcal{N}}^{\log} \longrightarrow \mathcal{N}^{\log} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s \times_s \Big(\prod_{v\in V} (\mathcal{M}_{g_v,r_v})_s \Big) \stackrel{\mathrm{pr}_1^{\log}}{\longrightarrow} (\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$$

— where the first arrow is the (1-)morphism of log stacks obtained by "forgetting" the portion of the log structure of $\underline{\mathcal{N}}^{\log}$ that arises from [the portion of the log structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_s$ determined by] the irreducible components of the divisor $(\overline{\mathcal{M}}_{g,r})_s \setminus (\mathcal{M}_{g,r})_s$ which contain the image of $\mathcal{N} \to (\overline{\mathcal{M}}_{g,r})_s$ — then one verifies immediately that, for any stable log curve $Y^{\log}$ over $S^{\log}$ that lifts $Y^{\log}_s$, there exists a *natural identification isomorphism* $\mathcal{H} = ((\mathcal{G}_{X^{\log}})|_{\mathbb{H}})_{\succ Q} \stackrel{\sim}{\to} \mathcal{G}_{Y^{\log}}$.

Next, let us observe that by applying the various definitions involved, together with the fact that the (1-)morphism $\underline{\mathcal{N}}^{\log} \to (\overline{\mathcal{M}}^{\log}_{g,r})_S$ is *strict*, one may verify easily that the restrictions of the natural (1-)morphisms of log stacks

$$(\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s \stackrel{\mathrm{pr}_1^{\log}}{\longleftarrow} \mathcal{N}^{\log} \longleftarrow \underline{\mathcal{N}}^{\log} \longrightarrow (\overline{\mathcal{M}}^{\log}_{g,r})_s$$

to a suitable étale neighborhood of the underlying morphism of stacks of $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ induce the following morphisms between the charts of $(\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$, $\mathcal{N}^{\log}$, $\underline{\mathcal{N}}^{\log}$, and $(\overline{\mathcal{M}}^{\log}_{g,r})_s$ determined by the chart of "$(\overline{\mathcal{M}}^{\log}_{g_\bullet,r_\bullet})_s$" given in Lemma 5.4, (i):

$$\bigoplus_{e\in\mathrm{Node}(\mathcal{H})} \mathbb{N}_e \stackrel{\sim}{\to} \Big(\bigoplus_{e\in\mathrm{Node}(\mathcal{H})} \mathbb{N}_e\Big) \oplus \{0\} \hookrightarrow \bigoplus_{e\in\mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \stackrel{\sim}{\leftarrow} \bigoplus_{e\in\mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e$$

— where we use the notation $\mathbb{N}_e$ to denote a copy of the monoid $\mathbb{N}$ indexed by e, and the "$\hookrightarrow$" is the natural inclusion determined by the natural inclusion $\mathrm{Node}(\mathcal{H}) \hookrightarrow \mathrm{Node}(\mathcal{G})$. Thus, by applying the functor "$\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}((-)^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1))$" to the homomorphism $\bigoplus_{e \in \mathrm{Node}(\mathcal{H})} \mathbb{N}_e \hookrightarrow \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e$ obtained by composing the morphisms of the above display and considering the [relevant] isomorphism of Lemma 5.4, (ii), we obtain a homomorphism $I_{T_X^{\log}} \to I_{T_Y^{\log}}$, which makes the *left-hand square* of the diagram in the statement of assertion (v) *commute*.

On the other hand, to verify the *commutativity of the right-hand square* of the diagram in the statement of assertion (v), let us observe that by Theorem 4.8, (iv), it suffices to verify that for any node $e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ of $\mathcal{G}_{Y^{\log}}$, the two composites

$$I_{T_X^{\log}} \stackrel{\rho^{\mathrm{univ}}_{X_t^{\log}}}{\longrightarrow} \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \stackrel{\mathfrak{D}_{e_X}}{\longrightarrow} \Lambda_{\mathcal{G}_{X^{\log}}} \stackrel{\sim}{\longrightarrow} \Lambda_{\mathcal{G}_{Y^{\log}}} ;$$

$$I_{T_X^{\log}} \longrightarrow I_{T_Y^{\log}} \stackrel{\rho^{\mathrm{univ}}_{Y_t^{\log}}}{\longrightarrow} \mathrm{Dehn}(\mathcal{G}_{Y^{\log}}) \stackrel{\mathfrak{D}_{e}}{\longrightarrow} \Lambda_{\mathcal{G}_{Y^{\log}}}$$

— where we write e_X for the node of $\mathcal{G}_{X^{\log}}$ corresponding to the node $e \in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ via the natural inclusion $\mathrm{Node}(\mathcal{G}_{Y^{\log}}) \hookrightarrow \mathrm{Node}(\mathcal{G}_{X^{\log}})$ — coincide. But this follows immediately by comparing the natural action of $I_{T_X^{\log}}$ on the portion of $\mathcal{G}_{X^{\log}}$ corresponding to $\{e_X\} \cup \mathcal{V}(e_X)$ with the natural action of $I_{T_Y^{\log}}$ on the portion of $\mathcal{G}_{Y^{\log}}$ corresponding to $\{e\} \cup \mathcal{V}(e)$. This completes the proof of assertion (v).

Finally, we verify assertion (vi). First, we observe that one may verify easily that if $\mathbb{K}$ is the underlying semi-graph of $\mathcal{H}$, and $Q = \emptyset$, then the stable log curve $Y^{\log}$ over $S^{\log}$ obtained by equipping $X^{\log}$ with *suitable cusps* satisfies, for a suitable choice of isomorphism $\mathcal{H} \stackrel{\sim}{\to} \mathcal{G}_{Y^{\log}}$, the conditions given in the statement of assertion (vi). Thus, one verifies immediately that to verify assertion (vi), we may assume without loss of generality that $U = \emptyset$.

Write $V \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\mathcal{H}) \setminus \mathrm{Vert}(\mathcal{H}|_{\mathbb{K}}) \subseteq \mathrm{Vert}(\mathcal{H})$. Denote by $(g_{\mathcal{H}}, r_{\mathcal{H}})$ the type of $\mathcal{H}$, and, for each $v \in V$, by (g_v, r_v) the type of v. Then it follows immediately from the general theory of stable log curves that there exists a *clutching "(1-)morphism"* corresponding to the *operations* "$(-)|_{\mathbb{K}}$" and "$(-)_{\succ Q}$" [i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6]

$$\mathcal{N} \stackrel{\mathrm{def}}{=} (\overline{\mathcal{M}}_{g,r})_s \times_s \Big(\prod_{v \in V} (\mathcal{M}_{g_v, r_v})_s \Big) \longrightarrow (\overline{\mathcal{M}}_{g_{\mathcal{H}}, r_{\mathcal{H}}})_s$$

— where the fiber product "$\prod_{v \in V}$" is taken over s — that satisfies the following condition: write $\underline{\mathcal{N}}^{\log}$ for the log stack obtained by equipping the stack $\mathcal{N}$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$ via the above clutching morphism; then there exists an $s^{\log}$*-valued point* $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ of $\underline{\mathcal{N}}^{\log}$ such that the image of $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ via the composite of (1-)morphisms

$$\underline{\mathcal{N}}^{\log} \longrightarrow \mathcal{N}^{\log} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}^{\log}_{g,r})_s \times_s \Big(\prod_{v \in V} (\mathcal{M}_{g_v,r_v})_s\Big) \stackrel{\text{pr}_1^{\log}}{\longrightarrow} (\overline{\mathcal{M}}^{\log}_{g,r})_s$$

— where the first arrow is the (1-)morphism of log stacks obtained by "forgetting" the portion of the log structure of $\underline{\mathcal{N}}^{\log}$ that arises from [the portion of the log structure of $(\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$ determined by] the irreducible components of the divisor $(\overline{\mathcal{M}}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s \setminus (\mathcal{M}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$ which contain the image of $\mathcal{N} \to (\overline{\mathcal{M}}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$ — coincides with the $s^{\log}$-valued point of $(\overline{\mathcal{M}}^{\log}_{g,r})_s$ obtained by restricting the classifying morphism $\sigma^{\log} \in (\overline{\mathcal{M}}^{\log}_{g,r})_S(S^{\log})$ of $X^{\log}$ to $s^{\log}$. If, moreover, we write $Y^{\log}_s$ for the stable log curve over $s^{\log}$ corresponding to the image of $\sigma^{\log}_{\mathcal{N}} \in \underline{\mathcal{N}}^{\log}(s^{\log})$ via the natural strict (1-)morphism $\underline{\mathcal{N}}^{\log} \to (\overline{\mathcal{M}}^{\log}_{g_{\mathcal{H}},r_{\mathcal{H}}})_s$, then one verifies immediately that, for any stable log curve $Y^{\log}$ over $S^{\log}$ that lifts $Y^{\log}_s$, there exist a *sub-semi-graph of PSC-type* $\mathbb{K}'$ of the underlying semi-graph of $\mathcal{G}_{Y^{\log}}$, a *subset* $Q' \subseteq \text{Node}((\mathcal{G}_{Y^{\log}})|_{\mathbb{K}'})$, and an *isomorphism of semi-graphs of anabelioids* $\mathcal{H} \stackrel{\sim}{\to} \mathcal{G}_{Y^{\log}}$ that satisfy the following conditions:

(a) $((\mathcal{G}_{Y^{\log}})|_{\mathbb{K}'})_{\succ Q'}$ may be naturally identified with $\mathcal{G}_{X^{\log}}$.

(b) The isomorphism $\mathcal{H} \stackrel{\sim}{\to} \mathcal{G}_{Y^{\log}}$ induces an isomorphism $\mathbb{K} \stackrel{\sim}{\to} \mathbb{K}'$ and a bijection $Q \stackrel{\sim}{\to} Q'$, hence also an isomorphism $(\mathcal{H}|_{\mathbb{K}})_{\succ Q} \stackrel{\sim}{\to} ((\mathcal{G}_{Y^{\log}})|_{\mathbb{K}'})_{\succ Q'}$.

(c) The automorphism of $\mathcal{G}_{X^{\log}}$ determined by the composite

$$\mathcal{G}_{X^{\log}} \stackrel{\sim}{\longleftarrow} (\mathcal{H}|_{\mathbb{K}})_{\succ Q} \stackrel{\sim}{\longrightarrow} ((\mathcal{G}_{Y^{\log}})|_{\mathbb{K}'})_{\succ Q'} \stackrel{\sim}{\longrightarrow} \mathcal{G}_{X^{\log}}$$

— where the first arrow is the isomorphism given in the statement of assertion (vi); the second arrow is the isomorphism of (b); the third arrow is the natural isomorphism arising from the natural identification of (a) — is contained in $\text{Aut}^{|\text{grph}|}(\mathcal{G}_{X^{\log}})$, and, moreover, the automorphism of $\Lambda_{\mathcal{G}_{X^{\log}}}$ induced by this automorphism of $\mathcal{G}_{X^{\log}}$ is the *identity automorphism* [cf. Proposition 4.10, (iii)].

Thus, by applying a similar argument to the argument used in the proof of assertion (v), one verifies easily that the stable log curve $Y^{\log}$ and the isomorphism $\mathcal{H} \overset{\sim}{\to} \mathcal{G}_{Y^{\log}}$ satisfy the conditions given in the statement of assertion (vi). This completes the proof of assertion (vi). Q.E.D.

Remark 5.6.1. Here, we take the opportunity to correct a minor misprint in the discussion entitled *"The Étale Fundamental Group of a Log Scheme"* in [CmbCsp], §0. In the third paragraph of this discussion, the field K should be defined as a *maximal* algebraic extension of $K_\circ$ among those extensions which are *unramified* over $U_{S_\circ}$ [i.e., but *not necessarily* over $R_\circ$].

Theorem 5.7 (Compatibility of scheme-theoretic and abstract combinatorial cyclotomic synchronizations). *Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$; Σ a nonempty set of prime numbers; R a complete discrete valuation ring whose residue field is separably closed of characteristic $\notin \Sigma$; $S^{\log}$ the log scheme obtained by equipping $S \overset{\mathrm{def}}{=} \mathrm{Spec}\, R$ with the log structure defined by its closed point; $X^{\log}$ a stable log curve of type (g, r) over $S^{\log}$; $\mathcal{G}_{X^{\log}}$ the semi-graph of anabelioids of pro-Σ PSC-type determined by the special fiber of the stable log curve $X^{\log}$ [cf.* [CmbGC], *Example 2.5]; $\widehat{\mathcal{O}}$ the completion of the local ring of $(\overline{\mathcal{M}}_{g,r})_S$ [cf. the discussion entitled "Curves" in §0] at the image of the closed point of S via the underlying (1-)morphism of stacks $\sigma \colon S \to (\overline{\mathcal{M}}_{g,r})_S$ of the classifying morphism of $X^{\log}$; $T^{\log}$ for the log scheme obtained by equipping $T \overset{\mathrm{def}}{=} \mathrm{Spec}\, \widehat{\mathcal{O}}$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}^{\log}_{g,r})_S$ [cf. the discussion entitled "Curves" in §0]; $I_{T^{\log}}$ the maximal pro-Σ quotient of the log fundamental group $\pi_1(T^{\log})$ of $T^{\log}$. Then there exists an* **isomorphism**

$$\mathfrak{syn}_{X^{\log}} \colon \Lambda^{\Sigma} \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{N}^{\mathrm{gp}}, \widehat{\mathbb{Z}}^{\Sigma}(1)) \overset{\sim}{\longrightarrow} \Lambda_{\mathcal{G}_{X^{\log}}}$$

[cf. Definition 3.8, (i)] such that the composite

$$\begin{array}{ccccc} I_{T^{\log}} & \overset{\sim}{\to} & \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \Lambda^{\Sigma}[e] & & \\ & {\scriptstyle \oplus\, \mathfrak{syn}_{X^{\log}}} \overset{\sim}{\to} & \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \Lambda_{\mathcal{G}_{X^{\log}}} & \overset{\mathfrak{D}_{\mathcal{G}_{X^{\log}}}}{\overset{\sim}{\leftarrow}} & \mathrm{Dehn}(\mathcal{G}_{X^{\log}}) \end{array}$$

[cf. Definitions 4.4; 4.7] — where we use the notation $\Lambda^{\Sigma}[e]$ to denote a copy of Λ^{Σ} indexed by $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$, and the first arrow is the [relevant] isomorphism of Lemma 5.4, (ii) — **coincides** *with the outer representation $\rho^{\mathrm{univ}}_{X^{\log}_t} \colon I_{T^{\log}} \to \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$ [cf. Definition 5.5] associated*

to the stable log curve over $T^{\log}$ *corresponding to the tautological strict (1-)morphism* $T^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_S$.

Proof. In light of Theorem 4.8, (ii), (iv); Proposition 5.6, (ii), by applying Proposition 5.6, (iii), to the various generizations of the form "$(\mathcal{G}_{X^{\log}})_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_{X^{\log}}) \setminus \{e\}}$", it follows immediately that for each node $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$, there exists a(n) [necessarily unique] *isomorphism*

$$\mathfrak{syn}_{X^{\log}}[e] \colon \Lambda^{\Sigma}[e] \xrightarrow{\sim} \Lambda_{\mathcal{G}_{X^{\log}}}$$

— where $\Lambda^{\Sigma}[e]$ is a copy of Λ^{Σ} indexed by $e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$ — such that the composite

$$I_{T^{\log}} \xrightarrow{\sim} \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \Lambda^{\Sigma}[e] \xrightarrow[\sim]{\bigoplus_e \mathfrak{syn}_{X^{\log}}[e]} \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_{X^{\log}})} \Lambda_{\mathcal{G}_{X^{\log}}} \xleftarrow[\sim]{\mathfrak{D}_{\mathcal{G}_{X^{\log}}}} \mathrm{Dehn}(\mathcal{G}_{X^{\log}})$$

— where the first "$\xrightarrow{\sim}$" is the [relevant] isomorphism of Lemma 5.4, (ii) — *coincides* with $\rho^{\mathrm{univ}}_{X_t^{\log}}$.

Thus, to complete the proof of Theorem 5.7, it suffices to verify that this isomorphism $\mathfrak{syn}_{X^{\log}}[e]$ is *independent* of the choice of e. Now if $\mathrm{Node}(\mathcal{G}_{X^{\log}})^{\sharp} \leq 1$, then this independence is immediate. Thus, suppose that $\mathrm{Node}(\mathcal{G}_{X^{\log}})^{\sharp} > 1$ and fix two *distinct* nodes $e_1, e_2 \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$ of $\mathcal{G}_{X^{\log}}$. The rest of the proof of Theorem 5.7 is devoted to verifying that

(‡): the two isomorphisms

$$\Lambda^{\Sigma}[e_1] \xrightarrow[\sim]{\mathfrak{syn}_{X^{\log}}[e_1]} \Lambda_{\mathcal{G}_{X^{\log}}}, \quad \Lambda^{\Sigma}[e_2] \xrightarrow[\sim]{\mathfrak{syn}_{X^{\log}}[e_2]} \Lambda_{\mathcal{G}_{X^{\log}}}$$

coincide.

Next, let us observe that one may verify easily that there exist

- a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}^*$,
- a sub-semi-graph of PSC-type $\mathbb{K}^*$ of the underlying semi-graph of $\mathcal{H}^*$,
- an omittable subset $Q^* \subseteq \mathrm{Cusp}((\mathcal{H}^*)|_{\mathbb{K}^*})$, and
- an isomorphism

$$((\mathcal{H}^*)|_{\mathbb{K}^*})_{\bullet Q^*} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$$

such that the subset $U^* \subseteq \mathrm{Node}(\mathcal{H}^*)$ corresponding, relative to the isomorphism $((\mathcal{H}^*)|_{\mathbb{K}^*})_{\bullet Q^*} \overset{\sim}{\to} \mathcal{G}_{X^{\log}}$, to the subset $\{e_1, e_2\} \subseteq \mathrm{Node}(\mathcal{G}_{X^{\log}})$ is *not of separating type*. Thus, it follows immediately from Proposition 5.6, (vi) — i.e., by replacing $X^{\log}$ (respectively, e_1, e_2) by the stable log curve "$Y^{\log}$" obtained by applying Proposition 5.6, (vi), to the isomorphism $((\mathcal{H}^*)|_{\mathbb{K}^*})_{\bullet Q^*} \overset{\sim}{\to} \mathcal{G}_{X^{\log}}$ (respectively, by the two nodes $\in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ corresponding to the two nodes $\in U^*$) — that to verify the above (‡), we may assume without loss of generality that the subset $\{e_1, e_2\} \subseteq \mathrm{Node}(\mathcal{G}_{X^{\log}})$ is *not of separating type*.

Thus, it follows immediately from Proposition 5.6, (iii) — i.e., by replacing $X^{\log}$ (respectively, e_1, e_2) by the stable log curve "$Y^{\log}$" obtained by applying Proposition 5.6, (iii), to $(\mathcal{G}_{X^{\log}})_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_{X^{\log}}) \setminus \{e_1, e_2\}}$ (respectively, by the two nodes $\in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ corresponding to e_1, e_2) — that to verify the above (‡), we may assume without loss of generality that $\mathrm{Node}(\mathcal{G}_{X^{\log}}) = \{e_1, e_2\}$, and that $\mathrm{Node}(\mathcal{G}_{X^{\log}}) = \{e_1, e_2\}$ is *not of separating type*. One verifies easily that these hypotheses imply that $\mathrm{Vert}(\mathcal{G}_{X^{\log}})^\sharp = 1$.

Next, let us observe that one may verify easily that there exist [cf. Fig. 6 below]

- a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}^\dagger$,
- two distinct cusps $c_1^\dagger$, $c_2^\dagger \in \mathrm{Cusp}(\mathcal{H}^\dagger)$ of $\mathcal{H}^\dagger$,
- three distinct nodes $f_1^\dagger$, $f_2^\dagger$, $f_3^\dagger \in \mathrm{Node}(\mathcal{H}^\dagger)$ of $\mathcal{H}^\dagger$, and
- an isomorphism

$$(\mathcal{H}^\dagger_{\rightsquigarrow \{f_1^\dagger, f_2^\dagger, f_3^\dagger\}})_{\bullet \{c_1^\dagger, c_2^\dagger\}} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$$

such that

- $\mathrm{Vert}(\mathcal{H}^\dagger) = \{v_1^\dagger, v_2^\dagger, v_3^\dagger, v_4^\dagger\}$;
- for $i \in \{1, 2\}$, if we write $e_i^\dagger \in \mathrm{Node}(\mathcal{H}^\dagger)$ for the node corresponding, relative to the isomorphism $(\mathcal{H}^\dagger_{\rightsquigarrow \{f_1^\dagger, f_2^\dagger, f_3^\dagger\}})_{\bullet \{c_1^\dagger, c_2^\dagger\}} \overset{\sim}{\to} \mathcal{G}_{X^{\log}}$, to $e_i \in \mathrm{Node}(\mathcal{G}_{X^{\log}})$, then it holds that $\mathcal{V}(e_i^\dagger) = \{v_i^\dagger\}$;
- $\mathcal{V}(f_1^\dagger) = \{v_1^\dagger, v_3^\dagger\}$, $\mathcal{V}(f_2^\dagger) = \{v_2^\dagger, v_3^\dagger\}$, $\mathcal{V}(f_3^\dagger) = \{v_3^\dagger, v_4^\dagger\}$;
- $\mathcal{V}(c_1^\dagger) = \mathcal{V}(c_2^\dagger) = \{v_4^\dagger\}$;
- for $i \in \{1, 2, 3\}$, $v_i^\dagger$ is of type $(0, 3)$ [cf. Definition 2.3, (iii)].

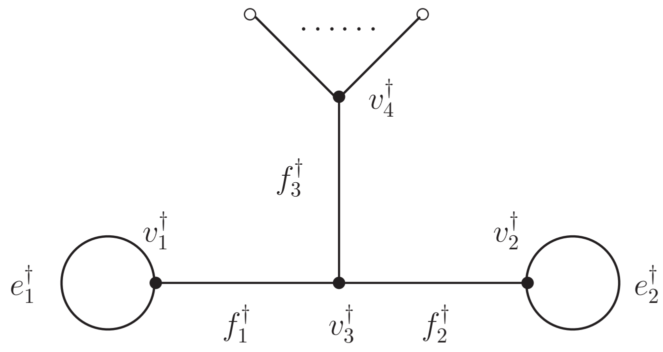

Figure 6: The underlying semi-graph of $\mathcal{H}^\dagger$

One verifies easily that these hypotheses imply that $(\mathcal{N}(v_1^\dagger) \cap \mathcal{N}(v_3^\dagger))^\sharp = (\mathcal{N}(v_2^\dagger) \cap \mathcal{N}(v_3^\dagger))^\sharp = 1$. Thus, it follows immediately from Proposition 5.6, (iv), (vi) — i.e., by replacing $X^{\log}$ (respectively, e_1, e_2) by the stable log curve "$Y^{\log}$" obtained by applying Proposition 5.6, (iv), (vi), to the isomorphism $(\mathcal{H}^\dagger_{\rightsquigarrow\{f_1^\dagger, f_2^\dagger, f_3^\dagger\}})_{\bullet\{c_1^\dagger, c_2^\dagger\}} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$ (respectively, by the two nodes $\in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ corresponding to the two nodes $e_1^\dagger$, $e_2^\dagger$) — that to verify the above (‡), we may assume without loss of generality that there exist vertices v_1, v_2, v_3 of $\mathcal{G}_{X^{\log}}$ such that

- for $i \in \{1, 2\}$, $\mathcal{V}(e_i) = \{v_i\}$;
- for $i \in \{1, 2, 3\}$, v_i is of type $(0, 3)$;
- $(\mathcal{N}(v_1) \cap \mathcal{N}(v_3))^\sharp = (\mathcal{N}(v_2) \cap \mathcal{N}(v_3))^\sharp = 1$.

Write $\mathbb{H}$ for the sub-semi-graph of PSC-type of the underlying semi-graph of $\mathcal{G}_{X^{\log}}$ whose set of vertices $= \{v_1, v_2, v_3\}$. Then one verifies easily that these hypotheses imply that $\mathrm{Node}((\mathcal{G}_{X^{\log}})|_{\mathbb{H}}) = \{e_1, e_2, f_1, f_2\}$, where we write $\{f_1\} = \mathcal{N}(v_1) \cap \mathcal{N}(v_3)$, $\{f_2\} = \mathcal{N}(v_2) \cap \mathcal{N}(v_3)$.

Thus, it follows immediately from Proposition 5.6, (v) — i.e., by replacing $X^{\log}$ (respectively, e_1, e_2) by the stable log curve "$Y^{\log}$" obtained by applying Proposition 5.6, (v), to $(\mathcal{G}_{X^{\log}})|_{\mathbb{H}}$ (respectively, by the two nodes $\in \mathrm{Node}(\mathcal{G}_{Y^{\log}})$ corresponding to e_1, e_2) — that to verify the above (‡), we may assume without loss of generality that there exist three distinct vertices v_1, v_2, v_3 of $\mathcal{G}_{X^{\log}}$ such that

- for $i \in \{1, 2\}$, $\mathcal{V}(e_i) = \{v_i\}$;

- for $i \in \{1, 2, 3\}$, v_i is of type $(0, 3)$;
- $\mathrm{Node}(\mathcal{G}_{X^{\log}}) = \{e_1, e_2, f_1, f_2\}$, where we write $\{f_1\} = (\mathcal{N}(v_1) \cap \mathcal{N}(v_3))$, $\{f_2\} = (\mathcal{N}(v_2) \cap \mathcal{N}(v_3))$.

One verifies easily that these hypotheses imply that there exists a cusp c of $\mathcal{G}_{X^{\log}}$ such that $\mathrm{Cusp}(\mathcal{G}_{X^{\log}}) = \{c\} = \mathcal{C}(v_3)$.

Then it follows immediately from the *explicit structure of* $\mathcal{G}_{X^{\log}}$ that there exists an automorphism τ of $X_t^{\log}$ [cf. Definition 5.5] such that the automorphism of $\mathrm{Node}(\mathcal{G}_{X^{\log}}) = \{e_1, e_2, f_1, f_2\}$ (respectively, $I_{T^{\log}} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}((\mathbb{N}_{e_1} \oplus \mathbb{N}_{e_2} \oplus \mathbb{N}_{f_1} \oplus \mathbb{N}_{f_2})^{\mathrm{gp}}, \widehat{\mathbb{Z}}^\Sigma(1))$ [cf. Lemma 5.4, (ii)]) induced by τ is given by mapping $e_1 \mapsto e_2$, $e_2 \mapsto e_1$, $f_1 \mapsto f_2$, $f_2 \mapsto f_1$, (respectively, by the corresponding permutation of factors of $\mathbb{N}_{e_1} \oplus \mathbb{N}_{e_2} \oplus \mathbb{N}_{f_1} \oplus \mathbb{N}_{f_2}$), and, moreover, τ preserves the cusp corresponding to c. Now it follows immediately from Corollary 3.9, (v), together with the fact that the automorphism of the anabelioid $(\mathcal{G}_{X^{\log}})_c$ corresponding to the cusp c induced by τ is the *identity automorphism* [cf. the argument used in the final portion of the proof of Corollary 3.9, (vi)], that the automorphism of $\Lambda_{\mathcal{G}_{X^{\log}}}$ induced by τ is the *identity automorphism*. Thus, by applying the evident *functoriality* of the homomorphism $\rho^{\mathrm{univ}}_{X_t^{\log}}$ with respect to the automorphism of $\mathcal{G}_{X^{\log}}$ induced by τ, one concludes immediately from the above description of τ, together with Theorem 4.8, (v), that the assertion (‡) holds. This completes the proof of Theorem 5.7. Q.E.D.

Definition 5.8. Let $\alpha \in \mathrm{Dehn}(\mathcal{G})$ be a profinite Dehn multi-twist of $\mathcal{G}$ and $u \in \Lambda_{\mathcal{G}}$ a topological generator of $\Lambda_{\mathcal{G}}$.

(i) Let $e \in \mathrm{Node}(\mathcal{G})$ be a node of $\mathcal{G}$. Then since $\Lambda_{\mathcal{G}}$ is a *free* $\widehat{\mathbb{Z}}^\Sigma$-*module of rank* 1 [cf. Definition 3.8, (i)], there exists a unique element $a_e \in \widehat{\mathbb{Z}}^\Sigma$ of $\widehat{\mathbb{Z}}^\Sigma$ such that $\mathfrak{D}_e(\alpha) = a_e u$. We shall refer to $a_e \in \widehat{\mathbb{Z}}^\Sigma$ as the *Dehn coordinate of* α *indexed by* e *with respect to* u.

(ii) We shall say that a profinite Dehn multi-twist $\alpha \in \mathrm{Dehn}(\mathcal{G})$ is *nondegenerate* if, for each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, the Dehn coordinate of α indexed by e with respect to u [cf. (i)] topologically generates an open subgroup of $\widehat{\mathbb{Z}}^\Sigma$. Note that it is immediate that if α is nondegenerate, then the Dehn coordinate ($\in \widehat{\mathbb{Z}}^\Sigma \xrightarrow{\sim} \prod_{l \in \Sigma} \mathbb{Z}_l \subseteq \prod_{l \in \Sigma} \mathbb{Q}_l$) of α indexed by e with respect to u is contained in $\prod_{l \in \Sigma} \mathbb{Q}_l^*$.

(iii) We shall say that a profinite Dehn multi-twist $\alpha \in \mathrm{Dehn}(\mathcal{G})$ is *positive definite* if α is nondegenerate [cf. (ii)], and, moreover, the following condition is satisfied: For each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, denote by $a_e \in \widehat{\mathbb{Z}}^{\Sigma}$ the Dehn coordinate of α indexed by e with respect to u [cf. (i)]. [Thus, $a_e \in \prod_{l \in \Sigma} \mathbb{Q}_l^*$ — cf. (ii).] Then for any e, $e' \in \mathrm{Node}(\mathcal{G})$, $a_e/a_{e'}$ is contained in the image of the diagonal map $\mathbb{Q}_{>0} \stackrel{\mathrm{def}}{=} \{ a \in \mathbb{Q} \mid a > 0 \} \hookrightarrow \prod_{l \in \Sigma} \mathbb{Q}_l^*$.

Remark 5.8.1. One may verify easily that the notions defined in Definition 5.8, (ii), (iii), are *independent* of the choice of the topological generator u of $\Lambda_{\mathcal{G}}$.

Corollary 5.9 (**Properties of outer representations of PSC–type and profinite Dehn multi-twists).** *Let Σ be a nonempty set of prime numbers and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ an outer representation of pro-Σ PSC-type [cf.* [NodNon]*, Definition 2.1, (i)]. Suppose that I is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}$. Then the following hold:*

(i) **(Outer representations of SVA-type and profinite Dehn multi-twists)** *The following three conditions are equivalent:*

(i-1) *ρ is of* **SVA-type** *[cf.* [NodNon]*, Definition 2.4, (ii)].*

(i-2) *The image of any topological generator of I is a* **profinite Dehn multi-twist** *[cf. Definition 4.4].*

(i-3) *There exists a topological generator of I whose image via ρ is a* **profinite Dehn multi-twist***.*

(ii) **(Outer representations of SNN-type and nondegenerate profinite Dehn multi-twists)** *The following three conditions are equivalent [cf. the related discussion of* [NodNon]*, Remark 2.14.1]:*

(ii-1) *ρ is of* **SNN-type** *[cf.* [NodNon]*, Definition 2.4, (iii)].*

(ii-2) *The image of any topological generator of I is a* **nondegenerate** *[cf. Definition 5.8, (ii)]* **profinite Dehn multi-twist***.*

(ii-3) *There exists a topological generator of I whose image via ρ is a* **nondegenerate profinite Dehn multi-twist***.*

(iii) **(Outer representations of IPSC-type and positive definite profinite Dehn multi-twists)** *The following three conditions are equivalent [cf. Remark 5.10.1 below; the related discussion of* [NodNon]*, Remark 2.14.1]:*

(iii-1) ρ *is of* **IPSC-type** *[cf.* [NodNon]*, Definition 2.4, (i)].*

(iii-2) *The image of any topological generator of* I *is a* **positive definite** *[cf. Definition 5.8, (iii)]* **profinite Dehn multi-twist***.*

(iii-3) *There exists a topological generator of* I *whose image via* ρ *is a* **positive definite profinite Dehn multi-twist***.*

(iv) **(Synchronization associated to outer representations of IPSC-type)** *Suppose that* ρ *is of* **IPSC-type***. Write*

$$(\widehat{\mathbb{Z}}^{\Sigma})^{+} \subseteq (\widehat{\mathbb{Z}}^{\Sigma})^{*}$$

for the intersection of the images of the diagonal map $\mathbb{Q}_{>0} \stackrel{\text{def}}{=} \{ a \in \mathbb{Q} \,|\, a > 0 \} \hookrightarrow \prod_{l \in \Sigma} \mathbb{Q}_l$ *and the composite of natural morphisms* $(\widehat{\mathbb{Z}}^{\Sigma})^{*} \hookrightarrow \widehat{\mathbb{Z}}^{\Sigma} \stackrel{\sim}{\to} \prod_{l \in \Sigma} \mathbb{Z}_l \subseteq \prod_{l \in \Sigma} \mathbb{Q}_l$*. [Thus, when* $\Sigma = \mathfrak{Primes}$*, it holds that* $(\widehat{\mathbb{Z}}^{\Sigma})^{+} = \{1\}$*.] Then there exists a* **natural** $(\widehat{\mathbb{Z}}^{\boldsymbol{\Sigma}})^{+}$**-orbit of isomorphisms** *of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*

$$\mathfrak{syn}_{\rho} \colon I \stackrel{\sim}{\longrightarrow} \Lambda_{\mathcal{G}}$$

that is **functorial***, in* ρ*, with respect to isomorphisms of outer representations of PSC-type [cf.* [NodNon]*, Definition 2.1, (ii)].*

(v) **(Compatibility of synchronizations with finite étale coverings)** *In the situation of (iv), let* $\Pi \subseteq \Pi_I \stackrel{\text{def}}{=} \Pi_{\mathcal{G}} \stackrel{\text{out}}{\rtimes} I$ *[cf. the discussion entitled "Topological groups" in §0] be an open subgroup of* Π_I *such that if we write* $\mathcal{H} \to \mathcal{G}$ *for the connected finite étale covering of* $\mathcal{G}$ *corresponding to* $\Pi \cap \Pi_{\mathcal{G}}$ *[so* $\Pi_{\mathcal{H}} = \Pi \cap \Pi_{\mathcal{G}}$*], then the outer representation* $\rho_{\Pi} \colon J \stackrel{\text{def}}{=} \Pi/\Pi_{\mathcal{H}} \to \mathrm{Out}(\Pi_{\mathcal{H}})$ *is of* **IPSC-type***. Then the diagram of* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules*

$$\begin{array}{ccc} J & \xrightarrow{\mathfrak{syn}_{\rho_{\Pi}}} & \Lambda_{\mathcal{H}} \\ \downarrow & & \downarrow \wr \\ I & \xrightarrow{\mathfrak{syn}_{\rho}} & \Lambda_{\mathcal{G}} \end{array}$$

— *where the left-hand vertical arrow is the natural inclusion; the right-hand vertical arrow is the isomorphism of Corollary 3.9, (iii)* — **commutes up to multiplication** *by an element* $\in \mathbb{Q}_{>0}$.

Proof. Assertion (i) follows immediately from condition (2′) of [NodNon], Definition 2.4. Next, we verify assertions (ii) and (iii). The implication

$$\text{(ii-1)} \Longrightarrow \text{(ii-2)}\ , \quad \text{(respectively,}\ \ \text{(iii-1)} \Longrightarrow \text{(iii-2))}$$

follows immediately from the final portion of Lemma 5.2, (ii), concerning ρ of *SNN-type* (respectively, Lemma 5.4, (ii); Theorem 5.7). The implications

$$\text{(ii-2)} \Longrightarrow \text{(ii-3)}\ , \quad \text{(iii-2)} \Longrightarrow \text{(iii-3)}$$

are immediate.

Next, we verify the implication

$$\text{(ii-3)} \Longrightarrow \text{(ii-1)}\,.$$

It follows from the implication (i-3) $\Rightarrow$ (i-1) that ρ is of *SVA-type*. Thus, to show the implication in question, it suffices to verify that ρ satisfies condition (3) of [NodNon], Definition 2.4. Let $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ be an element of $\mathrm{Node}(\widetilde{\mathcal{G}})$; $\Pi_I \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$ [cf. the discussion entitled "*Topological groups*" in §0]; $\widetilde{v}$, $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ the two distinct elements of $\mathrm{Vert}(\widetilde{\mathcal{G}})$ such that $\mathcal{V}(\widetilde{e}) = \{\widetilde{v}, \widetilde{w}\}$ [cf. [NodNon], Remark 1.2.1, (iii)]; $I_{\widetilde{e}}$, $I_{\widetilde{v}}$, $I_{\widetilde{w}} \subseteq \Pi_I$ the inertia subgroups of Π_I associated to $\widetilde{e}$, $\widetilde{v}$, $\widetilde{w}$, respectively. Then since the homomorphisms of the final two displays of Lemma 5.2, (ii), *coincide*, and $\Lambda_{\mathcal{G}_{X^{\log}}}$ and $I_{\widetilde{v}}$ are *free* $\widehat{\mathbb{Z}}^{\Sigma}$*-modules of rank* 1 [cf. Definition 3.8, (i); [NodNon], Lemma 2.5, (i)], it follows immediately from the definition of *nondegeneracy* that the composite of the second display of Lemma 5.2, (ii), is an *open injection*. Thus, it follows immediately that the natural homomorphism $I_{\widetilde{v}} \times I_{\widetilde{w}} \to I_{\widetilde{e}}$ has *open image*, and that $I_{\widetilde{v}} \cap I_{\widetilde{w}} = \{1\}$, i.e., that $I_{\widetilde{v}} \times I_{\widetilde{w}} \to I_{\widetilde{e}}$ is *injective*. That is to say, ρ satisfies condition (3) of [NodNon], Definition 2.4. This completes the proof of the implication in question.

Next, we verify the implication

$$\text{(iii-3)} \Longrightarrow \text{(iii-1)}\,.$$

Let $u \in \Lambda_{\mathcal{G}}$ be a topological generator of $\Lambda_{\mathcal{G}}$. Then it follows immediately from Lemma 5.4, (i), (ii), and Theorem 5.7 — by considering the stable log curve over $S^{\log}$ corresponding to a suitable homomorphism of R-algebras $\widehat{\mathcal{O}} \simeq R[[t_1, \cdots, t_{3g-g+r}]] \to R$ [cf. Lemma 5.4, (i)] — that to complete the proof of the implication in question, it suffices to verify that there exists a topological generator $\alpha \in I$ of I which satisfies the following condition $(*)$:

> $(*)$: The Dehn coordinates of $\rho(\alpha)$ with respect to u [cf. Definition 5.8, (i)] $\in \mathbb{N}_{\neq 0} \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$.

To this end, let $\alpha \in I$ be a topological generator of I such that $\rho(\alpha)$ is a *positive definite* profinite Dehn multi-twist of $\mathcal{G}$ [cf. condition (iii-3)]. For each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, denote by $a_e \in \widehat{\mathbb{Z}}^{\Sigma}$ the Dehn coordinate of $\rho(\alpha)$ indexed by e with respect to u. Now since $\rho(\alpha)$ is *nondegenerate*, it follows immediately from the definition of nondegeneracy that for each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, it holds that $a_e \in \mathbb{N}_{\neq 0} \cdot (\widehat{\mathbb{Z}}^{\Sigma})^*$. Thus, it follows immediately that for a *given* node $f \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, by replacing α by a suitable topological generator of I, we may assume without loss of generality that $a_f \in \mathbb{N}_{\neq 0}$. In particular, it follows immediately from the definition of *positive definiteness* that there exists an element $a \in \mathbb{N}_{\neq 0}$ such that for each node $e \in \mathrm{Node}(\mathcal{G})$ of $\mathcal{G}$, it holds that $a \cdot a_e \in \mathbb{N}_{\neq 0}$. Moreover, again by replacing α by a suitable topological generator of I, we may assume that every prime number dividing a belongs to Σ. But then it follows from the fact that $a_e \in \widehat{\mathbb{Z}}^{\Sigma} \cap (\frac{1}{a} \cdot \mathbb{N}_{\neq 0})$ that a_e is a positive rational number that is *integral* at every element of $\mathfrak{Primes}$, i.e., that $a_e \in \mathbb{N}_{\neq 0}$, as desired. In particular, the topological generator $\alpha \in I$ of I satisfies the above condition $(*)$. This completes the proof of the implication in question, hence also of assertions (ii) and (iii).

Next, we verify assertion (iv). It follows immediately from the final portion of Lemma 5.2, (ii), concerning ρ of *SNN-type* that for each $e \in \mathrm{Node}(\mathcal{G})$, the homomorphism $\mathfrak{syn}_\rho \colon I \to \Lambda_{\mathcal{G}}$ obtained by dividing the composite $I \xrightarrow{\rho} \mathrm{Dehn}(\mathcal{G}) \xrightarrow{\mathfrak{D}_e} \Lambda_{\mathcal{G}}$ by $\mathrm{lng}^{\Sigma}_{\mathcal{G}}(e, \rho)$ is an *isomorphism*. Moreover, by "translating into group theory" the *scheme-theoretic* content of Lemma 5.4, (ii), by means of the correspondence between *group-theoretic* and *scheme-theoretic* notions given in Proposition 5.6, (i); Theorem 5.7, one concludes that $\mathfrak{syn}_\rho$ is *independent* — up to multiplication by an element of $(\widehat{\mathbb{Z}}^{\Sigma})^+$ — of the choice of the node $e \in \mathrm{Node}(\mathcal{G})$. Now the *functoriality* of $\mathfrak{syn}_\rho$ follows immediately from the *functoriality* of the homomorphism $\mathfrak{D}_e$ [cf. Theorem 4.8, (iv)], together with the *group-theoreticity* of $\mathrm{lng}^{\Sigma}_{\mathcal{G}}(e, \rho)$. This completes the proof of assertion (iv).

Finally, assertion (v) follows immediately, in light of the *group-theoretic construction* of "$\mathfrak{syn}_\rho$" given in the proof of assertion (iv), from the various definitions involved. Q.E.D.

Remark 5.9.1.

(i) Corollary 5.9, (iv), may be regarded as a sort of *abstract combinatorial* analogue of the *cyclotomic synchronization* given in [GalSct], Theorem 4.3 [cf. also [AbsHyp], Lemma 2.5, (ii)].

(ii) It follows from Theorem 5.7 that one may think of the isomorphisms of Corollary 5.9, (iv), as a sort of *abstract combinatorial construction* of the various *identification isomorphisms* between the various copies of "$\widehat{\mathbb{Z}}^\Sigma(1)$" that appear in Lemma 5.4, (ii). Such identification isomorphisms are typically "taken for granted" in conventional discussions of scheme theory.

Remark 5.9.2.

(i) Consider the exact sequence of free $\widehat{\mathbb{Z}}^\Sigma$-modules

$$0 \longrightarrow M_{\mathcal{G}}^{\mathrm{vert}} \longrightarrow M_{\mathcal{G}} \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\mathrm{ab}} \longrightarrow M_{\mathcal{G}}^{\mathrm{comb}} \overset{\mathrm{def}}{=} M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}} \longrightarrow 0$$

— where we write $M_{\mathcal{G}}^{\mathrm{vert}} \subseteq M_{\mathcal{G}}$ for the $\widehat{\mathbb{Z}}^\Sigma$-submodule of $M_{\mathcal{G}}$ topologically generated by the images of the verticial subgroups of $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Remark 1.1.4]. Then one verifies easily that any profinite Dehn multi-twist $\alpha \in \mathrm{Dehn}(\mathcal{G})$ preserves and induces the identity automorphism on $M_{\mathcal{G}}^{\mathrm{vert}}$, $M_{\mathcal{G}}^{\mathrm{comb}}$. In particular, the homomorphism $M_{\mathcal{G}} \to M_{\mathcal{G}}$ obtained by considering the difference of the automorphism of $M_{\mathcal{G}}$ induced by α and the identity automorphism on $M_{\mathcal{G}}$ naturally determines [and is determined by!] a homomorphism

$$\alpha^{\mathrm{comb,vert}} \colon M_{\mathcal{G}}^{\mathrm{comb}} \longrightarrow M_{\mathcal{G}}^{\mathrm{vert}} .$$

Write $M_{\mathcal{G}}^{\mathrm{edge}} \subseteq M_{\mathcal{G}}^{\mathrm{vert}}$ for the $\widehat{\mathbb{Z}}^\Sigma$-submodule topologically generated by the image of the edge-like subgroups of $\Pi_{\mathcal{G}}$. Then the following two facts are well-known:

- If $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, then Poincaré duality $M_{\mathcal{G}} \overset{\sim}{\to} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(M_{\mathcal{G}}, \Lambda_{\mathcal{G}})$ determines an isomorphism $M_{\mathcal{G}}^{\mathrm{edge}} \overset{\sim}{\to} \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(M_{\mathcal{G}}^{\mathrm{comb}}, \Lambda_{\mathcal{G}})$ [cf. [CmbGC], Proposition 1.3].
- The natural homomorphism

$$\mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}^{\mathrm{vert}})$$

given by mapping $\alpha \mapsto \alpha^{\mathrm{comb,vert}}$ factors through the submodule $\mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}^{\mathrm{edge}}) \subseteq \mathrm{Hom}_{\widehat{\mathbb{Z}}^\Sigma}(M_{\mathcal{G}}^{\mathrm{comb}}, M_{\mathcal{G}}^{\mathrm{vert}})$.

[Indeed, this may be verified, for instance, by applying a similar argument to the argument used in the proof of [CmbGC], Proposition 1.3, involving *weights*.]

Thus, if $\mathrm{Cusp}(\mathcal{G}) = \emptyset$, then we obtain a homomorphism

$$\Omega_{\mathcal{G}} \colon \mathrm{Dehn}(\mathcal{G}) \longrightarrow M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{G}}, \widehat{\mathbb{Z}}^{\Sigma})$$

that is manifestly *functorial*, in $\mathcal{G}$, with respect to isomorphisms of semi-graphs of anabelioids of pro-Σ PSC-type. The matrices that appear in the image of this homomorphism $\Omega_{\mathcal{G}}$ are often referred to as *period matrices*.

(ii) Now let us recall that [CmbGC], Proposition 2.6, plays a *key role* in the proof of the combinatorial version of the Grothendieck conjecture given in [CmbGC], Corollary 2.7, (iii). Moreover, the proof of [CmbGC], Proposition 2.6, is essentially a formal consequence of the *nondegeneracy of the period matrix associated to a positive definite profinite Dehn multi-twist* — i.e., of the *injectivity* of the homomorphism

$$\alpha^{\mathrm{comb,vert}} \colon M_{\mathcal{G}}^{\mathrm{comb}} \longrightarrow M_{\mathcal{G}}^{\mathrm{vert}}$$

of (i) in the case where $\alpha \in \mathrm{Dehn}(\mathcal{G})$ is *positive definite* [cf. Corollary 5.9, (iii)].

(iii) In general, the period matrix associated to a profinite Dehn multi-twist may *fail to be nondegenerate* even if the profinite Dehn multi-twist is *nondegenerate*. Indeed, suppose that $\Sigma^{\sharp} = 1$, that $\mathcal{G}$ is the *double* [cf. [CmbGC], Proposition 2.2, (i)] of a semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{H}$ such that

$$(\mathrm{Vert}(\mathcal{H})^{\sharp}, \mathrm{Node}(\mathcal{H})^{\sharp}, \mathrm{Cusp}(\mathcal{H})^{\sharp}) = (1, 0, 2)\,.$$

Suppose, moreover, that $\mathcal{H}$ admits an automorphism which *permutes the two cusps* of $\mathcal{H}$ and *extends to an automorphism* ϕ of $\mathcal{G}$. [One verifies easily that such data exist.] Then one may verify easily that $\mathrm{Node}(\mathcal{G})^{\sharp} = 2$, that $\mathrm{Cusp}(\mathcal{G})^{\sharp} = 0$, and that the free $\widehat{\mathbb{Z}}^{\Sigma}$-module $M_{\mathcal{G}}^{\mathrm{comb}}$, hence also $M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{G}}, \widehat{\mathbb{Z}}^{\Sigma})$ [cf. (i)], is of *rank* 1. Now let us recall that the period matrix associated to a *positive definite* profinite Dehn multi-twist is necessarily *nondegenerate* [cf. Corollary 5.9, (iii); the proof of [CmbGC], Proposition 2.6]. Thus, since $\Sigma^{\sharp} = 1$, it follows immediately from the *functoriality* of $\Omega_{\mathcal{G}}$ [cf. (i)] and

$\mathfrak{D}_{\mathcal{G}}$ [cf. Theorem 4.8, (iv)] with respect to ϕ that the kernel of the composite of natural homomorphisms

$$\bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \overset{\mathfrak{D}_{\mathcal{G}}}{\underset{\sim}{\longleftarrow}} \mathrm{Dehn}(\mathcal{G}) \xrightarrow{\Omega_{\mathcal{G}}} M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{G}}, \widehat{\mathbb{Z}}^{\Sigma})$$

is a free $\widehat{\mathbb{Z}}^{\Sigma}$-submodule of $\bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$ of *rank* 1 that is *stabilized by* ϕ. On the other hand, since profinite Dehn multi-twists of the form $(u, u) \in \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$, where $u \in \Lambda_{\mathcal{G}}$, are [manifestly!] *positive definite*, we thus conclude that the kernel in question is equal to

$$\{ (u, -u) \in \bigoplus_{\mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \mid u \in \Lambda_{\mathcal{G}} \} .$$

In particular, any nonzero element of this kernel yields an example of a *nondegenerate* profinite Dehn multi-twist whose associated period matrix *fails to be nondegenerate*.

Corollary 5.10 (**Combinatorial/group-theoretic nature of scheme-theoreticity** Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; Σ a nonempty set of prime numbers; R a complete discrete valuation ring whose residue field k is separably closed of characteristic $\notin \Sigma$; $S^{\log}$ the log scheme obtained by equipping $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, R$ with the log structure determined by the maximal ideal of R; $x \in (\overline{\mathcal{M}}_{g,r})_S(k)$ a k-valued point of the **moduli stack of curves** $(\overline{\mathcal{M}}_{g,r})_S$ of type (g, r) over S [cf. the discussion entitled "*Curves*" in §0)**.** *; $\widehat{\mathcal{O}}$ the completion of the local ring of $(\overline{\mathcal{M}}_{g,r})_S$ at the image of x; $T^{\log}$ the log scheme obtained by equipping $T \stackrel{\mathrm{def}}{=} \mathrm{Spec}\, \widehat{\mathcal{O}}$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}_{g,r}^{\log})_S$ [cf. the discussion entitled "Curves" in §0]; $t^{\log}$ the log scheme obtained by equipping the closed point of T with the log structure induced by the log structure of $T^{\log}$; $X_t^{\log}$ the stable log curve over $t^{\log}$ corresponding to the natural strict (1-)morphism $t^{\log} \to (\overline{\mathcal{M}}_{g,r}^{\log})_S$; $I_{T^{\log}}$ the maximal pro-Σ quotient of the log fundamental group $\pi_1(T^{\log})$ of $T^{\log}$; $I_{S^{\log}}$ the maximal pro-Σ quotient of the log fundamental group $\pi_1(S^{\log})$ of $S^{\log}$; $\mathcal{G}_{X^{\log}}$ the semi-graph of anabelioids of pro-Σ PSC-type determined by the stable log curve $X_t^{\log}$ [cf.* [CmbGC]*, Example 2.5]; $\rho_{X_t^{\log}}^{\mathrm{univ}} \colon I_{T^{\log}} \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ the natural outer representation associated to $X_t^{\log}$ [cf. Definition 5.5]; I a profinite group; $\rho \colon I \to \mathrm{Aut}(\mathcal{G}_{X^{\log}})$ an* *outer representation of pro-*Σ *PSC-type [cf.* [NodNon], *Definition 2.1, (i)]. Then the following conditions are equivalent:*

(i) ρ *is of* **IPSC-type**.

(ii) *There exist a morphism of log schemes* $\phi^{\log}\colon S^{\log} \to T^{\log}$ *over* S *and an* **isomorphism of outer representations** *of pro-*Σ *PSC-type* $\rho \xrightarrow{\sim} \rho^{\mathrm{univ}}_{X_t^{\log}} \circ I_{\phi^{\log}}$ *[cf.* [NodNon], *Definition 2.1, (i)] — where we write* $I_{\phi^{\log}}\colon I_{S^{\log}} \to I_{T^{\log}}$ *for the homomorphism induced by* $\phi^{\log}$ *— i.e., there exist an* **automorphism** β *of* $\mathcal{G}_{X^{\log}}$ *and an isomorphism* $\alpha\colon I \xrightarrow{\sim} I_S^{\log}$ *such that the diagram*

$$\begin{array}{ccc} I & \xrightarrow{\rho} & \mathrm{Aut}(\mathcal{G}_{X^{\log}}) \\ {\scriptstyle\alpha}\downarrow\wr & & \downarrow\wr \\ I_{S^{\log}} & \xrightarrow{\rho_{X_t^{\log}} \circ I_{\phi^{\log}}} & \mathrm{Aut}(\mathcal{G}_{X^{\log}}) \end{array}$$

— where the right-hand vertical arrow is the automorphism of $\mathrm{Aut}(\mathcal{G}_{X^{\log}})$ *induced by* β *— commutes.*

(iii) *There exist a morphism of log schemes* $\phi^{\log}\colon S^{\log} \to T^{\log}$ *over* S *and an* **isomorphism** $\alpha\colon I \xrightarrow{\sim} I_S^{\log}$ *such that* $\rho = \rho^{\mathrm{univ}}_{X_t^{\log}} \circ I_{\phi^{\log}} \circ \alpha$ *— where we write* $I_{\phi^{\log}}\colon I_{S^{\log}} \to I_{T^{\log}}$ *for the homomorphism induced by* $\phi^{\log}$ *— i.e., the automorphism "*β*" of (ii) may be taken to be the* **identity**.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from the definition of the term "IPSC-type" [cf. [NodNon], Definition 2.4, (i)]. The implication (iii) $\Rightarrow$ (ii) is immediate. The implication (ii) $\Rightarrow$ (iii) follows immediately, in light of the *functoriality* asserted in Theorem 4.8, (iv), from Lemma 5.4, (i), (ii), and Theorem 5.7. Q.E.D.

Remark 5.10.1.

(i) The equivalence of Corollary 5.10 essentially amounts to the equivalence

"IPSC-type $\Longleftrightarrow$ positive definite"

which was discussed in [HM], Remark 2.14.1, *without proof.*

(ii) One way to understand the equivalence of Corollary 5.10 is as the statement that the property that an outer representation of PSC-type be of *scheme-theoretic origin* may be formulated purely in terms of *combinatorics/group theory*.

In the final portion of the present §5, we apply the theory developed so far [i.e., in particular, the equivalences of Corollary 5.9, (ii), (iii)] to derive results [cf. Theorem 5.14] concerning *normalizers* and *commensurators* of groups of profinite Dehn multi-twists.

Definition 5.11. Let $M \subseteq H \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ be closed subgroups of $\mathrm{Out}(\Pi_{\mathcal{G}})$. Suppose further that M is an *abelian pro-*Σ group [such as $\mathrm{Dehn}(\mathcal{G})$ — cf. Theorem 4.8, (iv)].

(i) We shall write

$$N_H^{\mathrm{scal}}(M) \subseteq N_H(M) \subseteq H$$

for the [closed] subgroup of H consisting of $\alpha \in H$ satisfying the following condition: $\alpha \in N_H(M)$, and, moreover, the action of α on M by conjugation coincides with the automorphism of M given by multiplication by an element of $(\widehat{\mathbb{Z}}^{\Sigma})^*$. We shall refer to $N_H^{\mathrm{scal}}(M)$ as the *scalar-normalizer* of M in H.

(ii) We shall write

$$C_H^{\mathrm{scal}}(M) \subseteq C_H(M) \subseteq H$$

for the subgroup of H consisting of $\alpha \in H$ satisfying the following condition: there exists an open $\widehat{\mathbb{Z}}^{\Sigma}$-submodule $M'_\alpha \subseteq M$ of M [possibly depending on α] such that the action of α on H by conjugation determines an automorphism of M'_α given by multiplication by an element of $(\widehat{\mathbb{Z}}^{\Sigma})^*$. We shall refer to $C_H^{\mathrm{scal}}(M)$ as the *scalar-commensurator* of M in H.

Lemma 5.12 (Scalar-normalizers and scalar-commensurators). *Let* $M \subseteq H \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ *be closed subgroups of* $\mathrm{Out}(\Pi_{\mathcal{G}})$*. Suppose further that* M *is an* **abelian pro-Σ** *group. Then:*

(i) *It holds that*

$$M \subseteq Z_H(M) \subseteq N_H^{\mathrm{scal}}(M) \subseteq C_H^{\mathrm{scal}}(M).$$

(ii) *If* $M' \subseteq M$ *is a* $\widehat{\mathbb{Z}}^{\Sigma}$*-submodule of* M*, then*

$$N_H^{\mathrm{scal}}(M) \subseteq N_H^{\mathrm{scal}}(M') \ ; \ C_H^{\mathrm{scal}}(M) \subseteq C_H^{\mathrm{scal}}(M').$$

If, moreover, $M' \subseteq M$ *is* **open** *in* M*, then*

$$C_H^{\mathrm{scal}}(M) = C_H^{\mathrm{scal}}(M').$$

Proof. These assertions follow immediately from the various definitions involved. Q.E.D.

Definition 5.13. Let $H \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ be a closed subgroup of $\mathrm{Out}(\Pi_{\mathcal{G}})$. Then we shall say that H is *IPSC-ample* (respectively, *NN-ample*) if H contains a positive definite (respectively, nondegenerate) [cf. Definition 5.8] profinite Dehn multi-twist $\in \mathrm{Dehn}(\mathcal{G})$.

Remark 5.13.1. It follows immediately from Theorem 4.8, (iv), that any open subgroup of $\mathrm{Dehn}(\mathcal{G})$ is *IPSC-ample*, hence also *NN-ample* [cf. Definition 5.13].

Theorem 5.14 (**Normalizers and commensurators of groups of profinite Dehn multi-twists).** *Let* Σ *be a nonempty set of prime numbers,* $\mathcal{G}$ *a semi-graph of anabelioids of pro-*Σ *PSC-type,* $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})$ *the group of* **group-theoretically cuspidal** *[cf.* [CmbGC]*, Definition 1.4, (iv)] outomorphisms of* $\Pi_{\mathcal{G}}$*, and* $M \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})$ *a closed subgroup of* $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})$ *which is* **abelian pro-**Σ*. Then the following hold:*

(i) *Suppose that one of the following two conditions is satisfied:*

(1) M *is* **IPSC-ample** *[cf. Definition 5.13].*

(2) M *is* **NN-ample** *[cf. Definition 5.13], and* $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$*.*

Then it holds that

$$N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$$

[cf. Definition 5.11]. If, moreover, $M \subseteq \mathrm{Dehn}(\mathcal{G})$ *[cf. Definition 4.4], then*

$$\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \subseteq N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$$

[cf. Definition 2.6, (i)]. In particular,

$$N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)\ ,\ \ C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)\ \subseteq\ \mathrm{Aut}(\mathcal{G})$$

are **open** *subgroups of* $\mathrm{Aut}(\mathcal{G})$.

(ii) *If* M *is an* **open** *subgroup of* $\mathrm{Dehn}(\mathcal{G})$*, then it holds that*

$$\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)\,.$$

If, moreover, $\mathrm{Node}(\mathcal{G}) \neq \emptyset$*, then*

$$\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \cap \mathrm{Ker}(\chi_{\mathcal{G}}) = Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$$

[cf. Definition 3.8, (ii)].

(iii) *It holds that*

$$\mathrm{Aut}(\mathcal{G}) = N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(\mathrm{Dehn}(\mathcal{G})) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(\mathrm{Dehn}(\mathcal{G}))\,.$$

Proof. First, we verify the inclusion $C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$ asserted in assertion (i). Suppose that condition (1) (respectively, (2)) is satisfied. Let $\alpha \in C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$. Then since $\alpha \in C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$, and M is *IPSC-ample* (respectively, *NN-ample*), it follows immediately that there exists an element $\beta \in M$ of M such that both β and $\alpha\beta\alpha^{-1} = \beta^{\lambda}$, where $\lambda \in (\widehat{\mathbb{Z}}^{\Sigma})^{*}$, are *positive definite* (respectively, *nondegenerate*) profinite Dehn multi-twists. Thus, the *graphicity* of α follows immediately from [NodNon], Remark 4.2.1, together with Corollary 5.9, (iii) (respectively, from [NodNon], Theorem A, together with Corollary 5.9, (ii)). This completes the proof of the inclusion $C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$, hence also, by Lemma 5.12, (i), of the two inclusions in the first display of assertion (i).

If, moreover, $M \subseteq \mathrm{Dehn}(\mathcal{G})$, then the inclusion $\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \subseteq N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ follows immediately from Theorem 4.8, (v). Thus, since

$\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G})$ is an *open* subgroup of $\mathrm{Aut}(\mathcal{G})$ [cf. Proposition 2.7, (iii)], it follows immediately that $N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$, hence also $C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$, is an *open* subgroup of $\mathrm{Aut}(\mathcal{G})$. This completes the proof of assertion (i).

Next, we verify the equality $\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ in the first display of assertion (ii). It follows immediately from Theorem 4.8, (i), that $\mathrm{Aut}(\mathcal{G}) \subseteq N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(\mathrm{Dehn}(\mathcal{G})) \subseteq C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$. Thus, to verify the equality $\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$, it suffices to verify the inclusion $C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$. To this end, let $\alpha \in C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$. Then it follows from Lemma 5.12, (ii), that

$$C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) = C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(\alpha \cdot M \cdot \alpha^{-1}) = \alpha \cdot C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \cdot \alpha^{-1} ,$$

i.e., $\alpha \in N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M))$. Thus, since $C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ is an *open* subgroup of $\mathrm{Aut}(\mathcal{G})$ [cf. assertion (i); Remark 5.13.1], we conclude that $\alpha \in C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(\mathrm{Aut}(\mathcal{G}))$. Thus, the fact that $\alpha \in \mathrm{Aut}(\mathcal{G})$ follows from the *commensurable terminality* of $\mathrm{Aut}(\mathcal{G})$ in $\mathrm{Out}(\Pi_{\mathcal{G}})$, i.e., the equality $\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}(\Pi_{\mathcal{G}})}(\mathrm{Aut}(\mathcal{G}))$ [cf. [CmbGC], Corollary 2.7, (iv)]. This completes the proof of the equality $\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$.

Next, we verify the equality

$$\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \ \cap \ \mathrm{Ker}(\chi_{\mathcal{G}}) = Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$$

in the second display of assertion (ii). Now it follows immediately from Theorem 4.8, (v), that $\mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \cap \mathrm{Ker}(\chi_{\mathcal{G}}) \subseteq Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$. Thus, to show the equality in question, it suffices to verify the inclusion $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \cap \mathrm{Ker}(\chi_{\mathcal{G}})$. To this end, let us observe that since $\mathrm{Aut}(\mathcal{G}) = C_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ [cf. the preceding paragraph], it holds that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$. Thus, since the action of $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ on M by conjugation preserves and induces the identity automorphism on the intersection of M with each direct summand of $\bigoplus_{e \in \mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \overset{\mathfrak{D}_{\mathcal{G}}}{\xleftarrow{\sim}} \mathrm{Dehn}(\mathcal{G})$ [i.e., each "$\Lambda_{\mathcal{G}}$"], it follows immediately from Theorem 4.8, (v), in light of our assumption that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$, that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G})|}(\mathcal{G}) \cap \mathrm{Ker}(\chi_{\mathcal{G}})$. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii), together with Theorem 4.8, (i). This completes the proof of Theorem 5.14. Q.E.D.

Remark 5.14.1. In the notation of Theorem 5.14, (i) (respectively, Theorem 5.14, (ii)), in general, the inclusion

$$C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$$

[hence, *a fortiori*, by the inclusions of the first display of Theorem 5.14, (i), the inclusion $N^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})$] (respectively, in general, the inclusion

$$N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M) \subseteq \mathrm{Aut}(\mathcal{G})\,)$$

is *strict*. Indeed, suppose that there exist a node $e \in \mathrm{Node}(\mathcal{G})$ and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{G})$ of $\mathcal{G}$ such that α does *not stabilize* e, and $\chi_{\mathcal{G}}(\alpha) = 1$. [For example, in the notation of the final paragraph of the proof of Theorem 5.7, the node e_1 and the automorphism induced by τ of $\mathcal{G}_{X^{\log}}$ satisfy these conditions.] Now fix a prime number $l \in \Sigma$; write

$$M \stackrel{\mathrm{def}}{=} l \cdot (\Lambda_{\mathcal{G}})_e \oplus \Big(\bigoplus_{f \neq e} \Lambda_{\mathcal{G}} \Big) \subseteq \bigoplus_{f \in \mathrm{Node}(\mathcal{G})} \Lambda_{\mathcal{G}} \stackrel{\mathfrak{D}_{\mathcal{G}}}{\stackrel{\sim}{\leftarrow}} \mathrm{Dehn}(\mathcal{G})$$

— where we use the notation $(\Lambda_{\mathcal{G}})_e$ to denote a copy of $\Lambda_{\mathcal{G}}$ indexed by $e \in \mathrm{Node}(\mathcal{G})$. Then M is an *open* subgroup of $\mathrm{Dehn}(\mathcal{G})$, hence also *IPSC-ample* [cf. Remark 5.13.1], but it follows immediately from Theorem 4.8, (v), that $\alpha \notin C^{\mathrm{scal}}_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$ (respectively, $\alpha \notin N_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}})}(M)$).

§6. Centralizers of geometric monodromy

In the present §, we study the *centralizer* of the image of certain *geometric monodromy* groups. As an application, we prove a *"geometric version of the Grothendieck conjecture"* for the *universal curve over the moduli stack* of pointed smooth curves [cf. Theorem 6.13 below].

Definition 6.1. Let Σ be a nonempty set of prime numbers and Π a pro-Σ surface group [cf. [MT], Definition 1.2]. Then we shall write

$$\mathrm{Out}^{\mathrm{C}}(\Pi) = \mathrm{Out}^{\mathrm{FC}}(\Pi) = \mathrm{Out}^{\mathrm{PFC}}(\Pi)$$

for the group of outomorphisms of Π which induce bijections on the set of cuspidal inertia subgroups of Π. We shall refer to an element of $\mathrm{Out}^{\mathrm{C}}(\Pi) = \mathrm{Out}^{\mathrm{FC}}(\Pi) = \mathrm{Out}^{\mathrm{PFC}}(\Pi)$ as a *C-*, *FC-*, or *PFC-admissible* outomorphism of Π.

Remark 6.1.1. In the notation of Definition 6.1, suppose that either $\Sigma^{\sharp} = 1$ or $\Sigma = \mathfrak{Primes}$. Then it follows from the various definitions involved that Π is equipped with a natural structure of *pro-Σ configuration space group* [cf. [MT], Definition 2.3, (i)]. Thus, the terms "*C-/FC-/PFC-admissible* outomorphism of Π" and the notation "$\mathrm{Out}^{\mathrm{C}}(\Pi) = \mathrm{Out}^{\mathrm{FC}}(\Pi)$" have already been defined in [CmbCsp], Definition 1.1, (ii), and Definition 1.4, (iii), of the present paper. In this case, however, one may verify easily that these definitions *coincide*.

Lemma 6.2 (**Extensions arising from log configuration spaces**). *Let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$; $0 < m < n$ positive integers; $\Sigma_{\mathrm{F}} \subseteq \Sigma_{\mathrm{B}}$ nonempty sets of prime numbers; k an algebraically closed field of characteristic zero;* $(\mathrm{Spec}\, k)^{\log}$ *the log scheme obtained by equipping* $\mathrm{Spec}\, k$ *with the log structure given by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a* **stable log curve** *of type (g, r) over* $(\mathrm{Spec}\, k)^{\log}$*. Suppose that $\Sigma_{\mathrm{F}} \subseteq \Sigma_{\mathrm{B}}$ satisfy one of the following two conditions:*

(1) *Σ_{F} and Σ_{B} determine* **PT-formations** *[i.e., are either of* **cardinality one** *or* **equal to $\mathfrak{Primes}$** *— cf.* [MT], *Definition 1.1, (ii)].*

(2) *$n - m = 1$ and $\Sigma_{\mathrm{B}} = \mathfrak{Primes}$.*

Write

$$X_n^{\log}\ ,\ \ X_m^{\log}$$

for the n-th, m-th **log configuration spaces** *of the stable log curve $X^{\log}$ [cf. the discussion entitled "Curves" in §0], respectively; Π_n, $\Pi_{\mathrm{B}} \stackrel{\mathrm{def}}{=} \Pi_m$ for the respective maximal pro-Σ_{B} quotients of the kernels of the natural surjections $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$, $\pi_1(X_m^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$; $\Pi_{n/m} \subseteq \Pi_n$ for the kernel of the surjection $\Pi_n \twoheadrightarrow \Pi_{\mathrm{B}} = \Pi_m$ induced by the projection $X_n^{\log} \to X_m^{\log}$ obtained by forgetting the last $(n-m)$ factors; Π_{F} for the maximal pro-Σ_{F} quotient of $\Pi_{n/m}$; Π_{T} for the quotient of Π_n by the kernel of the natural surjection $\Pi_{n/m} \twoheadrightarrow \Pi_{\mathrm{F}}$. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1\,,$$

which determines an outer representation

$$\rho_{n/m} \colon \Pi_{\mathrm{B}} \longrightarrow \mathrm{Out}(\Pi_{\mathrm{F}})\,.$$

Then the following hold:

(i) *The isomorphism class of the exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1$$

depends only *on* (g,r) *and the pair* $(\Sigma_{\mathrm{F}},\Sigma_{\mathrm{B}})$*, i.e., if* $1 \to \Pi_{\mathrm{F}}^{\bullet} \to \Pi_{\mathrm{T}}^{\bullet} \to \Pi_{\mathrm{B}}^{\bullet} \to 1$ *is the exact sequence "*$1 \to \Pi_{\mathrm{F}} \to \Pi_{\mathrm{T}} \to \Pi_{\mathrm{B}} \to 1$*" associated, with respect to the same* $(\Sigma_{\mathrm{F}},\Sigma_{\mathrm{B}})$*, to another stable log curve of type* (g,r) *over* $(\mathrm{Spec}\, k)^{\log}$*, then there exists a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}}^{\bullet} & \longrightarrow & \Pi_{\mathrm{T}}^{\bullet} & \longrightarrow & \Pi_{\mathrm{B}}^{\bullet} & \longrightarrow & 1 \end{array}$$

— where the vertical arrows are **isomorphisms** *which may be chosen to arise* **scheme-theoretically***.*

(ii) *The profinite group* Π_{B} *is equipped with a natural structure of pro-*Σ_{B} **configuration space group** *[cf.* [MT]*, Definition 2.3, (i)]. If, moreover,* $\Sigma_{\mathrm{F}} \subseteq \Sigma_{\mathrm{B}}$ *satisfies condition (1) (respectively, (2)), then the profinite group* Π_{F} *is equipped with a natural structure of pro-*Σ_{F} **configuration space group** *(respectively,* **surface group** *[cf.* [MT]*, Definition 1.2]).*

(iii) *The outer representation* $\rho_{n/m}\colon \Pi_{\mathrm{B}} \to \mathrm{Out}(\Pi_{\mathrm{F}})$ *factors through the closed subgroup* $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$ *[cf. Definition 6.1;* [CmbCsp]*, Definition 1.1, (ii)].*

Proof. Assertion (i) follows immediately by considering a suitable *specialization isomorphism* [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as Remark 5.6.1 of the present paper]. Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Lemma 6.2. Q.E.D.

Definition 6.3. In the notation of Lemma 6.2 in the case where

$$(m,n,\Sigma_{\mathrm{B}}) = (1,2,\mathfrak{Primes}),$$

let $x \in X(k)$ be a k-valued point of the underlying scheme X of $X^{\log}$.

(i) We shall denote by

$$\mathcal{G}$$

the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by the stable log curve $X^{\log}$; by

$$\mathcal{G}_x$$

the semi-graph of anabelioids of pro-Σ_{F} PSC-type determined by the geometric fiber of $X_2^{\log} \to X^{\log}$ over $x^{\log} \stackrel{\mathrm{def}}{=} x \times_X X^{\log}$; by $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}_x}$ the [pro-$\mathfrak{Primes}$, pro-Σ_{F}] fundamental groups of $\mathcal{G}$, $\mathcal{G}_x$, respectively. Thus, we have a natural outer isomorphism

$$\Pi_{\mathrm{B}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$$

and a natural $\mathrm{Im}(\rho_{2/1})$ ($\subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$)-torsor of outer isomorphisms

$$\Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x} .$$

Let us *fix* isomorphisms $\Pi_{\mathrm{B}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$, $\Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ that belong to these collections of isomorphisms.

(ii) Denote by

$$c^{\mathrm{F}}_{\mathrm{diag},x} \in \mathrm{Cusp}(\mathcal{G}_x)$$

the cusp of $\mathcal{G}_x$ [i.e., the cusp of the geometric fiber of $X_2^{\log} \to X^{\log}$ over $x^{\log}$] determined by the diagonal divisor of $X_2^{\log}$. For $v \in \mathrm{Vert}(\mathcal{G})$ (respectively, $c \in \mathrm{Cusp}(\mathcal{G})$) [i.e., which corresponds to an irreducible component (respectively, a cusp) of $X^{\log}$], denote by

$$v^{\mathrm{F}}_x \in \mathrm{Vert}(\mathcal{G}_x) \quad (\text{respectively, } c^{\mathrm{F}}_x \in \mathrm{Cusp}(\mathcal{G}_x))$$

the vertex (respectively, cusp) of $\mathcal{G}_x$ that corresponds naturally to $v \in \mathrm{Vert}(\mathcal{G})$ (respectively, $c \in \mathrm{Cusp}(\mathcal{G})$).

(iii) Let $e \in \mathrm{Edge}(\mathcal{G})$, $v \in \mathrm{Vert}(\mathcal{G})$, $S \subseteq \mathrm{VCN}(\mathcal{G})$, and $z \in \mathrm{VCN}(\mathcal{G})$. Then we shall say that x *lies on* e if the image of x is the cusp or node corresponding to $e \in \mathrm{Edge}(\mathcal{G})$. We shall say that x *lies on* v if x does not lie on any edge of $\mathcal{G}$, and, moreover, the image of x is contained in the irreducible component corresponding to $v \in \mathrm{Vert}(\mathcal{G})$. We shall write $x \curvearrowright S$ if x lies on some $s \in S$. We shall write $x \curvearrowright z$ if $x \curvearrowright \{z\}$.

Lemma 6.4 (**Cusps and vertices of fibers**). *In the notation of Definition 6.3, let* x, $x' \in X(k)$ *be* k*-valued points of* X*. Then the following hold:*

(i) *The isomorphism* $\Pi_{\mathcal{G}_x} \xrightarrow{\sim} \Pi_{\mathcal{G}_{x'}}$*, obtained by forming the composite of the isomorphisms* $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{x'}}$ *[cf. Definition 6.3,* *(i)] is* **group-theoretically cuspidal** *[cf.* [CmbGC], *Definition 1.4, (iv)].*

(ii) *The injection* $\mathrm{Cusp}(\mathcal{G}) \hookrightarrow \mathrm{Cusp}(\mathcal{G}_x)$ *given by mapping* $c \mapsto c_x^{\mathrm{F}}$ *determines a* **bijection**

$$\mathrm{Cusp}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_x) \setminus \{c_{\mathrm{diag},x}^{\mathrm{F}}\}$$

[cf. Definition 6.3, (ii)]. Moreover, if we regard $\mathrm{Cusp}(\mathcal{G})$ *as a subset of each of the sets* $\mathrm{Cusp}(\mathcal{G}_x)$, $\mathrm{Cusp}(\mathcal{G}_{x'})$ *by means of the above injections, then the bijection* $\mathrm{Cusp}(\mathcal{G}_x) \overset{\sim}{\to} \mathrm{Cusp}(\mathcal{G}_{x'})$ *determined by the* **group-theoretically cuspidal** *isomorphism* $\Pi_{\mathcal{G}_x} \overset{\sim}{\to} \Pi_{\mathcal{G}_{x'}}$ *of (i) maps* $c_{\mathrm{diag},x}^{\mathrm{F}} \mapsto c_{\mathrm{diag},x'}^{\mathrm{F}}$ *and induces the* **identity automorphism** *on* $\mathrm{Cusp}(\mathcal{G})$. *Thus, in the remainder of the present §, we shall* **omit** *the subscript "x" from the notation "c_x^{F}" and "$c_{\mathrm{diag},x}^{\mathrm{F}}$".*

(iii) *The injection* $\mathrm{Vert}(\mathcal{G}) \hookrightarrow \mathrm{Vert}(\mathcal{G}_x)$ *given by mapping* $v \mapsto v_x^{\mathrm{F}}$ *[cf. Definition 6.3, (ii)] is* **bijective** *if and only if* $x \curvearrowright \mathrm{Vert}(\mathcal{G})$ *[cf. Definition 6.3, (iii)]. If* $x \curvearrowright \mathrm{Edge}(\mathcal{G})$, *then the complement of the image of* $\mathrm{Vert}(\mathcal{G})$ *in* $\mathrm{Vert}(\mathcal{G}_x)$ *is of cardinality one; in this case, we shall write*

$$v_{\mathrm{new},x}^{\mathrm{F}} \in \mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G})$$

for the **unique element** *of* $\mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G})$.

(iv) *Suppose that* $x \curvearrowright \mathrm{Cusp}(\mathcal{G})$ *(respectively,* $\mathrm{Node}(\mathcal{G})$*). Then it holds that* $c_{\mathrm{diag}}^{\mathrm{F}} \in \mathcal{C}(v_{\mathrm{new},x}^{\mathrm{F}})$ *[cf. (iii)], and* $(\mathcal{C}(v_{\mathrm{new},x}^{\mathrm{F}})^\sharp, \mathcal{N}(v_{\mathrm{new},x}^{\mathrm{F}})^\sharp) = (2,1)$ *(respectively,* $= (1,2)$*). Moreover, for any element* $e^F \in \mathcal{N}(v_{\mathrm{new},x}^{\mathrm{F}})$, *it holds that* $\mathcal{V}(e^{\mathrm{F}})^\sharp = 2$.

Proof. These assertions follow immediately from the various definitions involved. Q.E.D.

Definition 6.5. In the notation of Definition 6.3:

(i) Write

$$\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G}) \sqcup \{c_{\mathrm{diag}}^{\mathrm{F}}\}$$

[cf. Definition 6.3, (ii); Lemma 6.4, (ii)].

(ii) Let $\alpha \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})$ be an *C-admissible* outomorphism of Π_{F} [cf. Definition 6.1; Lemma 6.2, (ii)]. Then it follows immediately from Lemma 6.4, (ii), that for any k-valued point $x \in X(k)$ of X, the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. (i)] obtained by conjugating the natural action of α on $\mathrm{Cusp}(\mathcal{G}_x)$ by the natural bijection $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_x)$ implicit in Lemma 6.4, (ii), does *not depend* on the choice of x. We shall refer to this automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ as the *automorphism of* $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *determined by* α. Thus, we have a natural homomorphism $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \to \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$.

(iii) For $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. (i)], we shall refer to a closed subgroup of Π_{F} obtained as the image — via the isomorphism $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}}$ [cf. Definition 6.3, (i)] for some k-valued point $x \in X(k)$ — of a cuspidal subgroup of $\Pi_{\mathcal{G}_x}$ associated to the cusp of $\mathcal{G}_x$ corresponding to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ as a *cuspidal subgroup of* Π_{F} *associated to* $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$. Note that it follows immediately from Lemma 6.4, (ii), that the Π_{F}-conjugacy class of a cuspidal subgroup of Π_{F} associated to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *depends only* on $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$, i.e., does *not depend* on the choice of x or on the choices of isomorphisms made in Definition 6.3, (i).

Lemma 6.6 (Images of VCN-subgroups of fibers). *In the notation of Definition 6.3, let* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$ *be a cuspidal subgroup of* Π_{F} *associated to* $c^{\mathrm{F}}_{\mathrm{diag}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *[cf. Definition 6.5, (i), (iii)],* $x \in X(k)$ *a* k*-valued point of* X*,* $z^{\mathrm{F}} \in \mathrm{VCN}(\mathcal{G}_x) \setminus \{c^{\mathrm{F}}_{\mathrm{diag}}\}$*, and* $\Pi_{z^{\mathrm{F}}} \subseteq \Pi_{\mathcal{G}_x}$ *a VCN-subgroup of* $\Pi_{\mathcal{G}_x}$ *associated to* z^{F}*. Write* $N_{\mathrm{diag}} \subseteq \Pi_{\mathrm{F}}$ *for the normal closed subgroup of* Π_{F} *topologically normally generated by* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}}$*. [Note that it follows immediately from Lemma 6.4, (i), that* N_{diag} *is normal in* Π_{T}*.] Then the following hold:*

(i) *Write* $\mathcal{G}^{\Sigma_{\mathrm{F}}}$ *for the semi-graph of anabelioids of pro-*Σ_{F} *PSC-type obtained by forming the* **pro-Σ_{F} completion** *of* $\mathcal{G}$ *[cf.* [SemiAn]*, Definition 2.9, (ii)]. Then there exists a* **natural outer isomorphism** $\Pi_{\mathrm{F}}/N_{\mathrm{diag}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\Sigma_{\mathrm{F}}}}$ *that satisfies the following conditions:*

- *Suppose that* $x \curvearrowright \mathrm{Vert}(\mathcal{G})$ *[cf. Definition 6.3, (iii)]. Then the* $\Pi_{\mathcal{G}^{\Sigma_{\mathrm{F}}}}$*-conjugacy class of the image of the composite*

$$\Pi_{z^{\mathrm{F}}} \hookrightarrow \Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}} \xrightarrow{\sim} \Pi_{\mathcal{G}^{\Sigma_{\mathrm{F}}}}$$

coincides with the $\Pi_{\mathcal{G}^{\Sigma_F}}$*-conjugacy class of any VCN-subgroup of* $\Pi_{\mathcal{G}^{\Sigma_F}}$ *associated to the element of* $\mathrm{VCN}(\mathcal{G}^{\Sigma_F}) = \mathrm{VCN}(\mathcal{G})$ *naturally determined by* z^F*.*

- *Suppose that* $x \curvearrowright e \in \mathrm{Edge}(\mathcal{G})$*, and that* $z^F \notin \{v^F_{\mathrm{new},x}\} \cup \mathcal{E}(v^F_{\mathrm{new},x})$ *(respectively,* $z^F \in \{v^F_{\mathrm{new},x}\} \cup \mathcal{E}(v^F_{\mathrm{new},x})$*) [cf. Lemma 6.4, (iii)]. Then the* $\Pi_{\mathcal{G}^{\Sigma_F}}$*-conjugacy class of the image of the composite*

$$\Pi_{z^F} \hookrightarrow \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_F \twoheadrightarrow \Pi_F/N_{\mathrm{diag}} \overset{\sim}{\rightarrow} \Pi_{\mathcal{G}^{\Sigma_F}}$$

coincides with the $\Pi_{\mathcal{G}^{\Sigma_F}}$*-conjugacy class of any VCN-subgroup of* $\Pi_{\mathcal{G}^{\Sigma_F}}$ *associated to the element of* $\mathrm{VCN}(\mathcal{G}^{\Sigma_F}) = \mathrm{VCN}(\mathcal{G})$ *natural determined by* z^F *(respectively, associated to* $e \in \mathrm{Edge}(\mathcal{G}^{\Sigma_F}) = \mathrm{Edge}(\mathcal{G})$*).*

(ii) *The image of the composite*

$$\Pi_{z^F} \hookrightarrow \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_F \twoheadrightarrow \Pi_F/N_{\mathrm{diag}}$$

is **commensurably terminal**.

(iii) *Suppose that either*

- $z^F \in \mathrm{Edge}(\mathcal{G}_x)$*,*

or

- $z^F = v^F_x$ *for* $v \in \mathrm{Vert}(\mathcal{G})$ *such that* x *does* **not lie** *on* v*.*

Then the composite

$$\Pi_{z^F} \hookrightarrow \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_F \twoheadrightarrow \Pi_F/N_{\mathrm{diag}}$$

is **injective**.

(iv) *Let* $\Pi_{(z')^F} \subseteq \Pi_{\mathcal{G}_x}$ *be a VCN-subgroup of* $\Pi_{\mathcal{G}_x}$ *associated to an element* $(z')^F \in \mathrm{VCN}(\mathcal{G}_x) \setminus \{c^F_{\mathrm{diag}}\}$*. Suppose that either*

- $x \curvearrowright \mathrm{Vert}(\mathcal{G})$*,*

or

- $x \curvearrowright \mathrm{Edge}(\mathcal{G})$*, and* z^F*,* $(z')^F \notin \{v^F_{\mathrm{new},x}\} \cup \mathcal{E}(v^F_{\mathrm{new},x})$*.*

Then if the Π_F/N_{diag}*-conjugacy classes of the images of* Π_{z^F}*,* $\Pi_{(z')^F} \subseteq \Pi_{\mathcal{G}_x}$ *via the composite*

$$\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_F \twoheadrightarrow \Pi_F/N_{\mathrm{diag}}$$

coincide, then $z^{\mathrm{F}} = (z')^{\mathrm{F}}$.

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from [CmbGC], Proposition 1.2, (ii), and assertion (i), together with the various definitions involved. Assertion (iii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iv) follows immediately from [CmbGC], Proposition 1.2, (i), and assertion (i), together with the various definitions involved. Q.E.D.

Lemma 6.7 (**Outomorphisms preserving the diagonal**). *In the notation of Definition 6.3, let* $H \subseteq \Pi_{\mathrm{B}}$ *be an open subgroup of* Π_{B}*,* $\widetilde{\alpha}$ *an automorphism of* $\Pi_{\mathrm{T}}|_H \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{T}} \times_{\Pi_{\mathrm{B}}} H$ *over* H*,* $\alpha_{\mathrm{F}} \in \mathrm{Out}(\Pi_{\mathrm{F}})$ *the outomorphism of* Π_{F} *determined by the restriction* $\widetilde{\alpha}|_{\Pi_{\mathrm{F}}}$ *of* $\widetilde{\alpha}$ *to* $\Pi_{\mathrm{F}} \subseteq \Pi_{\mathrm{T}}|_H$*, and* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$ *a cuspidal subgroup of* Π_{F} *associated to* $c^{\mathrm{F}}_{\mathrm{diag}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *[cf. Definition 6.5, (i), (iii)]. Then the following hold:*

(i) *Suppose that* $\widetilde{\alpha}$ *preserves* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$*. Then the automorphism of* $\Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ *[where we refer to the statement of Lemma 6.6 concerning* N_{diag}*] induced by* $\widetilde{\alpha}$ *is the* **identity automorphism***. If, moreover,* α_{F} *is* **C-admissible** *[cf. Definition 6.1; Lemma 6.2, (ii)], then the automorphism of* $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *induced by* α_{F} *[cf. Definition 6.5, (ii)] is the* **identity automorphism***.*

(ii) *Let* $e \in \mathrm{Edge}(\mathcal{G})$*,* $x \in X(k)$ *be such that* $x \curvearrowright e$*. Suppose that* α_{F} *is* **C-admissible***, and that* $\mathrm{Edge}(\mathcal{G}) = \{e\} \cup \mathrm{Cusp}(\mathcal{G})$*. Then it holds that* $\alpha_{\mathrm{F}} \in \mathrm{Aut}(\mathcal{G}_x)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{\mathrm{F}}))$*. If, moreover,* $\widetilde{\alpha}$ *preserves* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$*, then* $\alpha_{\mathrm{F}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ $(\subseteq \mathrm{Aut}(\mathcal{G}_x))$*.*

Proof. First, we verify assertion (i). Now let us observe that it follows immediately from a similar argument to the argument used in the proof of [CmbCsp], Proposition 1.2, (iii) — i.e., by considering the action of $\widetilde{\alpha}$ on the *decomposition subgroup* $D \subseteq \Pi_{\mathrm{T}}|_H$ of $\Pi_{\mathrm{T}}|_H$ associated to the *diagonal divisor* of $X_2^{\log}$ such that $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq D$, and applying the fact that $D = N_{\Pi_{\mathrm{T}}|_H}(\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}}) \subseteq \Pi_{\mathrm{T}}|_H$ — that $\widetilde{\alpha}$ induces the *identity automorphism* on some normal open subgroup $J \subseteq \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ of $\Pi_{\mathrm{F}}/N_{\mathrm{diag}}$. Thus, it follows immediately from the *slimness* [cf. [CmbGC], Remark

1.1.3] of $\Pi_{\mathcal{G}^{\Sigma_F}} \overset{\sim}{\leftarrow} \Pi_F/N_{diag} \hookrightarrow \mathrm{Aut}(J)$ that $\widetilde{\alpha}$ induces the *identity automorphism* on Π_F/N_{diag}. This completes the proof of the fact that $\widetilde{\alpha}$ induces the *identity automorphism* of Π_F/N_{diag}. On the other hand, if, moreover, α_F is *C-admissible*, then since $\widetilde{\alpha}$ induces the *identity automorphism* of Π_F/N_{diag}, it follows immediately from [CmbGC], Proposition 1.2, (i), applied to the cuspidal inertia subgroups of $\Pi_F/N_{diag} \overset{\sim}{\rightarrow} \Pi_{\mathcal{G}^{\Sigma_F}}$ [cf. Lemma 6.6, (i)] that the automorphism of $\mathrm{Cusp}^F(\mathcal{G})$ induced by α_F is the *identity automorphism*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\Pi_e \subseteq \Pi_{\mathcal{G}} \overset{\sim}{\leftarrow} \Pi_B$ be an edge-like subgroup associated to the edge $e \in \mathrm{Edge}(\mathcal{G})$. By abuse of notation, we shall write $H \cap \Pi_e \subseteq \Pi_B$ for the intersection of H with the image of Π_e in Π_B. Now since α_F is *C-admissible*, and $\widetilde{\alpha}$ is an automorphism of $\Pi_T|_H$ *over* H, it holds that $\alpha_F \in Z_{\mathrm{Out}^C(\Pi_F)}(\rho_{2/1}(H))$ [cf. the discussion entitled "*Topological groups*" in §0], hence also that $\alpha_F \in Z_{\mathrm{Out}^C(\Pi_F)}(\rho_{2/1}(H \cap \Pi_e))$. On the other hand, in light of the well-known structure of $X^{\log}$ in a neighborhood of the cusp or node corresponding to e, one verifies easily — by applying [NodNon], Proposition 2.14, together with our assumption that $\mathrm{Edge}(\mathcal{G}) = \{e\} \cup \mathrm{Cusp}(\mathcal{G})$ — that the image of the composite

$$\Pi_e \hookrightarrow \Pi_{\mathcal{G}} \overset{\sim}{\leftarrow} \Pi_B \overset{\rho_{2/1}}{\rightarrow} \mathrm{Out}(\Pi_F) \overset{\sim}{\rightarrow} \mathrm{Out}(\Pi_{\mathcal{G}_x}),$$

hence also the image $\rho_{2/1}(H \cap \Pi_e) \subseteq \mathrm{Out}(\Pi_F) \overset{\sim}{\rightarrow} \mathrm{Out}(\Pi_{\mathcal{G}_x})$, is *NN-ample* [cf. Definition 5.13; Theorem 5.9, (ii)]. Thus, since $c^F_{diag} \in \mathrm{Cusp}(\mathcal{G}_x) \neq \emptyset$, it follows immediately from Theorem 5.14, (i), that $\alpha_F \in \mathrm{Aut}(\mathcal{G}_x)$. This completes the proof of the fact that $\alpha_F \in \mathrm{Aut}(\mathcal{G}_x)$. Now suppose, moreover, that $\widetilde{\alpha}$ preserves $\Pi_{c^F_{diag}} \subseteq \Pi_F$. Then it follows from assertion (i) that α_F fixes the *cusps* of $\mathcal{G}_x$, hence that it fixes $v^F_{new,x}$. On the other hand, since $\widetilde{\alpha}$ induces the *identity automorphism* of Π_F/N_{diag} [cf. assertion (i)], it follows from Lemma 6.6, (iii), (iv), that α_F fixes the *vertices* of $\mathcal{G}_x$ that are $\neq v^F_{new,x}$, as well as [cf. [CmbGC], Proposition 1.2, (i)] the *branches of nodes* of $\mathcal{G}_x$ that *abut to such vertices*. Thus, $\alpha_F \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$, as desired. This completes the proof of assertion (ii). Q.E.D.

Lemma 6.8 (Triviality of certain outomorphisms). *In the notation of Definition 6.3, let* $\Pi_{c^F_{diag}} \subseteq \Pi_F$ *be a cuspidal subgroup of* Π_F *associated to* $c^F_{diag} \in \mathrm{Cusp}^F(\mathcal{G})$ *[cf. Definition 6.5, (i), (iii)],* $H \subseteq \Pi_B$ *an open subgroup of* Π_B*, and* $\alpha \in Z_{\mathrm{Out}^C(\Pi_F)}(\rho_{2/1}(H))$ *[cf. Definition 6.1;*

Lemma 6.2, (ii)]. Suppose that α preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$. Then α is the **identity outomorphism**.

Proof. The following argument is essentially the same as the argument applied in [CmbCsp], [NodNon] to prove [CmbCsp], Corollary 2.3, (ii); [NodNon], Corollary 5.3.

Let $\Pi_{\mathrm{T}}|_H \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{T}} \times_{\Pi_{\mathrm{B}}} H$ and $\widetilde{\alpha} \in \mathrm{Aut}_H(\Pi_{\mathrm{T}}|_H)$ a lifting of $\alpha \in Z_{\mathrm{Out^C}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H)) \subseteq Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H)) \stackrel{\sim}{\leftarrow} \mathrm{Aut}_H(\Pi_{\mathrm{T}}|_H)/\mathrm{Inn}(\Pi_{\mathrm{F}})$ [cf. the discussion entitled "*Topological groups*" in §0]. Since we have assumed that α preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$, it follows from Lemma 6.7, (i), (ii), that by replacing $\widetilde{\alpha}$ by a suitable Π_{F}-conjugate of $\widetilde{\alpha}$, we may assume without loss of generality that $\widetilde{\alpha}$ *preserves* $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_{\mathrm{F}}$, and, moreover, that

(a) the automorphism of $\Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ induced by $\widetilde{\alpha}$ is the *identity automorphism*;

(b) for $e \in \mathrm{Edge}(\mathcal{G})$, $x \in X(k)$ such that $x \curvearrowright e$, if $\mathrm{Edge}(\mathcal{G}) = \{e\} \cup \mathrm{Cusp}(\mathcal{G})$, then $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{\mathrm{F}})$).

Next, we *claim* that

$(*_1)$: if $(g, r) = (0, 3)$, then α is the *identity outomorphism*.

Indeed, write c_1, c_2, $c_3 \in \mathrm{Cusp}(\mathcal{G})$ for the three distinct cusps of $\mathcal{G}$; $v \in \mathrm{Vert}(\mathcal{G})$ for the *unique* vertex of $\mathcal{G}$. For $i \in \{1, 2, 3\}$, let $x_i \in X(k)$ be such that $x_i \curvearrowright c_i$. Next, let us observe that since our assumption that $(g, r) = (0, 3)$ implies that $\mathrm{Node}(\mathcal{G}) = \emptyset$, it follows immediately from (b) that for $i \in \{1, 2, 3\}$, the outomorphism α of $\Pi_{\mathcal{G}_{x_i}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ is $\in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{x_i})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{x_i}}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{\mathrm{F}})$). Next, let us fix a verticial subgroup $\Pi_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{x_2} \in \mathrm{Vert}(\mathcal{G}_{x_2})$ [cf. Definition 6.3, (ii)]. Then since $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{x_2})$, it follows immediately from the *commensurable terminality* of the image of the composite $\Pi_{v^{\mathrm{F}}_{x_2}} \hookrightarrow \Pi_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ [cf. Lemma 6.6, (ii)], together with (a), that there exists an N_{diag}-conjugate $\widetilde{\beta}$ of $\widetilde{\alpha}$ such that $\widetilde{\beta}(\Pi_{v^{\mathrm{F}}_{x_2}}) = \Pi_{v^{\mathrm{F}}_{x_2}}$. Thus, since the composite $\Pi_{v^{\mathrm{F}}_{x_2}} \hookrightarrow \Pi_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ is *injective* [cf. Lemma 6.6, (iii)], it follows immediately from (a) that $\widetilde{\beta}$ induces the *identity automorphism* on $\Pi_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}$. Next, let $\Pi_{c^{\mathrm{F}}_1} \subseteq \Pi_{\mathrm{F}}$ be a cuspidal subgroup of Π_{F} associated to $c_1 \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. Definition 6.5, (iii)] which is contained in $\Pi_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}$; $\Pi_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi_{\mathcal{G}_{x_3}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ a verticial subgroup associated to $v^{\mathrm{F}}_{x_3} \in \mathrm{Vert}(\mathcal{G}_{x_3})$ that contains $\Pi_{c^{\mathrm{F}}_1} \subseteq \Pi_{\mathrm{F}}$. Then since $\widetilde{\beta}$ induces the *identity automorphism* on $\Pi_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi_{\mathcal{G}_{x_2}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$, it follows from the inclusion $\Pi_{c^{\mathrm{F}}_1} \subseteq \Pi_{v^{\mathrm{F}}_{x_2}}$ that $\widetilde{\beta}(\Pi_{c^{\mathrm{F}}_1}) = \Pi_{c^{\mathrm{F}}_1}$. Thus, since the verticial subgroup $\Pi_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ is the *unique* verticial subgroup of $\Pi_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{x_3} \in \mathrm{Vert}(\mathcal{G}_{x_3})$ which contains $\Pi_{c^{\mathrm{F}}_1}$ [cf. [CmbGC], Proposition 1.5, (i)], it follows immediately from the fact that $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{x_3})$ that $\widetilde{\beta}(\Pi_{v^{\mathrm{F}}_{x_3}}) = \Pi_{v^{\mathrm{F}}_{x_3}}$. In particular, since the composite $\Pi_{v^{\mathrm{F}}_{x_3}} \hookrightarrow \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ is *injective* [cf. Lemma 6.6, (iii)], it follows immediately from (a) that $\widetilde{\beta}$ induces the *identity automorphism* on $\Pi_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$. On the other hand, since Π_{F} is topologically generated by $\Pi_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi_{\mathcal{G}_{x_2}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ and $\Pi_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ [cf. [CmbCsp], Lemma 1.13], this implies that $\widetilde{\beta}$ induces the *identity automorphism* on Π_{F}. This completes the proof of the *claim* $(*_1)$.

Next, we *claim* that

> $(*_2)$: for arbitrary (g, r), α is the *identity outomorphism.*

Indeed, we verify the *claim* $(*_2)$ by *induction on* $3g-3+r$. If $3g-3+r = 0$, i.e., $(g, r) = (0, 3)$, then the *claim* $(*_2)$ amounts to the *claim* $(*_1)$. Now suppose that $3g-3+r > 1$, and that the *induction hypothesis* is in force. Since $3g-3+r > 1$, one verifies easily that there exists a stable log curve $Y^{\log}$ of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$ such that $Y^{\log}$ has *precisely one node*. Thus, it follows immediately from Lemma 6.2, (i), that to verify the *claim* $(*_2)$, by replacing $X^{\log}$ by $Y^{\log}$, we may assume without loss of generality that $\mathrm{Node}(\mathcal{G})^{\sharp} = 1$. Let e be the *unique* node of $\mathcal{G}$ and $x \in X(k)$ such that $x \curvearrowright e$. Now let us observe that since $\mathrm{Node}(\mathcal{G})^{\sharp} = 1$, and $e \in \mathrm{Node}(\mathcal{G})$, it follows from (b) that $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\leftarrow} \mathrm{Out}(\Pi_{\mathrm{F}}))$. Write $\{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2\} = \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},x})$ [cf. Lemma 6.4, (iv)]. Also, for $i \in \{1, 2\}$, denote by $v_i \in \mathrm{Vert}(\mathcal{G})$ the vertex of $\mathcal{G}$ such that $(v_i)^{\mathrm{F}}_x \in \mathrm{Vert}(\mathcal{G}_x)$ is the *unique* element of $\mathcal{V}(e^{\mathrm{F}}_i) \setminus \{v^{\mathrm{F}}_{\mathrm{new},x}\}$ [cf. Lemma 6.4, (iv)]; by $\mathbb{H}_i$ the sub-semi-graph of PSC-type of the underlying semi-graph $\mathbb{G}_x$ of $\mathcal{G}_x$ whose set of vertices $= \{v^{\mathrm{F}}_{\mathrm{new},x}, (v_i)^{\mathrm{F}}_x\}$; and by $S_i \overset{\mathrm{def}}{=} \mathrm{Node}((\mathcal{G}_x)|_{\mathbb{H}_i}) \setminus \{e^{\mathrm{F}}_i\} \subseteq \mathrm{Node}((\mathcal{G}_x)|_{\mathbb{H}_i})$ the complement of $\{e^{\mathrm{F}}_i\}$. [Thus, if $\mathcal{G}$ is *noncyclically primitive* (respectively, *cyclically primitive*) [cf. Definition 4.1], then $\mathbb{H}_i \neq \mathbb{G}_x$ and $S_i = \emptyset$ (respectively, $\mathbb{H}_i = \mathbb{G}_x$ and $S_i = \{e^{\mathrm{F}}_{3-i}\}$). In particular, $S_i \subseteq \mathrm{Node}((\mathcal{G}_x)|_{\mathbb{H}_i})$ is *not of separating type*.]

Next, let us observe that to complete the proof of the above *claim* $(*_2)$, it suffices to verify that

(†): $\alpha \in \mathrm{Dehn}(\mathcal{G}_x)$, and, moreover, for $i \in \{1,2\}$, α is contained in the kernel of the natural surjection $\mathrm{Dehn}(\mathcal{G}_x) \twoheadrightarrow \mathrm{Dehn}(((\mathcal{G}_x)|_{\mathbb{H}_i})_{\succ S_i})$ [cf. Theorem 4.8, (iii), (iv)].

Indeed, since [as is easily verified] $\mathrm{Node}(\mathcal{G}_x) = \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},x}) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2\}$, it follows immediately from Theorem 4.8, (iii), (iv), that

$$\bigcap_{i=1}^{2} \mathrm{Ker}\Big(\mathrm{Dehn}(\mathcal{G}_x) \twoheadrightarrow \mathrm{Dehn}(((\mathcal{G}_x)|_{\mathbb{H}_i})_{\succ S_i})\Big) = \{1\}\,.$$

In particular, the implication (†) $\Rightarrow$ $(*_2)$ holds. The remainder of the proof of the *claim* $(*_2)$ is devoted to verifying the above (†).

For $i \in \{1,2\}$, let $\Pi_{(v_i)^{\mathrm{F}}_x} \subseteq \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ be a verticial subgroup of $\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ associated to the vertex $(v_i)^{\mathrm{F}}_x \in \mathcal{V}(e^{\mathrm{F}}_i) \setminus \{v^{\mathrm{F}}_{\mathrm{new},x}\}$. Then since $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$, it follows that $\widetilde{\alpha}$ preserves the Π_{F}-conjugacy class of $\Pi_{(v_i)^{\mathrm{F}}_x} \subseteq \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$. Thus, since the image of the composite $\Pi_{(v_i)^{\mathrm{F}}_x} \hookrightarrow \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ is *commensurably terminal* [cf. Lemma 6.6, (ii)], it follows immediately from (a) that there exists an N_{diag}-conjugate $\widetilde{\beta}_i$ [which *may depend* on $i \in \{1,2\}$!] of $\widetilde{\alpha}$ such that $\widetilde{\beta}_i(\Pi_{(v_i)^{\mathrm{F}}_x}) = \Pi_{(v_i)^{\mathrm{F}}_x}$. Therefore, since the composite $\Pi_{(v_i)^{\mathrm{F}}_x} \hookrightarrow \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{\mathrm{diag}}$ is *injective* [cf. Lemma 6.6, (iii)], it follows from (a) that $\widetilde{\beta}_i$ induces the *identity automorphism* of $\Pi_{(v_i)^{\mathrm{F}}_x}$.

Next, let $\Pi_{e^{\mathrm{F}}_i} \subseteq \Pi_{(v_i)^{\mathrm{F}}_x}$ be a nodal subgroup of $\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ associated to $e^{\mathrm{F}}_i \in \mathrm{Node}(\mathcal{G}_x)$ that is contained in $\Pi_{(v_i)^{\mathrm{F}}_x}$; $\Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i} \subseteq \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ a verticial subgroup [which *may depend* on $i \in \{1,2\}$!] associated to $v^{\mathrm{F}}_{\mathrm{new},x} \in \mathrm{Vert}(\mathcal{G}_x)$ which contains $\Pi_{e^{\mathrm{F}}_i}$:

$$\Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i} \supseteq \Pi_{e^{\mathrm{F}}_i} \subseteq \Pi_{(v_i)^{\mathrm{F}}_x} \quad \subseteq \quad \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}\,.$$

Then since $\widetilde{\beta}_i$ preserves and induces the *identity automorphism* on $\Pi_{(v_i)^{\mathrm{F}}_x}$, it follows from the inclusion $\Pi_{e^{\mathrm{F}}_i} \subseteq \Pi_{(v_i)^{\mathrm{F}}_x}$ that $\widetilde{\beta}_i(\Pi_{e^{\mathrm{F}}_i}) = \Pi_{e^{\mathrm{F}}_i}$. Moreover, since $\Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i}$ is the *unique* verticial subgroup of $\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{\mathrm{new},x}$ which contains $\Pi_{e^{\mathrm{F}}_i}$ [cf. [CmbGC], Proposition 1.5, (i)], it follows immediately from the fact that $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ that $\widetilde{\beta}_i(\Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i}) = \Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i}$. Thus, $\widetilde{\beta}_i$ preserves the closed subgroup $\Pi_{\mathrm{F}_i} \subseteq \Pi_{\mathrm{F}}$ of Π_{F} obtained by forming the image of the natural homomorphism

$$\varinjlim \Big(\Pi_{v^{\mathrm{F}}_{\mathrm{new},x};i} \hookleftarrow \Pi_{e^{\mathrm{F}}_i} \hookrightarrow \Pi_{(v_i)^{\mathrm{F}}_x}\Big) \longrightarrow \Pi_{\mathrm{F}}$$

— where the inductive limit is taken in the category of pro-Σ_{F} groups. Now one may verify easily that the Π_{F}-conjugacy class of $\Pi_{\mathrm{F}_i} \subseteq \Pi_{\mathrm{F}}$ *coincides* with the Π_{F}-conjugacy class of the image of the natural outer injection $\Pi_{((\mathcal{G}_x)|_{\mathbb{H}_i})_{\succ S_i}} \hookrightarrow \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ discussed in Proposition 2.11; in particular, Π_{F_i} is *commensurably terminal* in Π_{F} [cf. Proposition 2.11]. Moreover, by applying a similar argument to the argument used in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., by considering the portion of the underlying scheme X_2 of $X_2^{\log}$ corresponding to the underlying scheme $(X_{v_i})_2$ of the 2-nd log configuration space $(X_{v_i})_2^{\log}$ of the stable log curve $X_{v_i}^{\log}$ determined by $\mathcal{G}|_{v_i}$], one concludes that there exists a verticial subgroup $\Pi_{v_i} \subseteq \Pi_{\mathcal{G}} \overset{\sim}{\leftarrow} \Pi_{\mathrm{B}}$ associated to $v_i \in \mathrm{Vert}(\mathcal{G})$ such that the outer representation of Π_{v_i} on Π_{F} determined by the composite $\Pi_{v_i} \hookrightarrow \Pi_{\mathrm{B}} \overset{\rho_{2/1}}{\to} \mathrm{Out}(\Pi_{\mathrm{F}})$ preserves the Π_{F}-conjugacy class of $\Pi_{\mathrm{F}_i} \subseteq \Pi_{\mathrm{F}}$ [so we obtain a natural outer representation $\Pi_{v_i} \to \mathrm{Out}(\Pi_{\mathrm{F}_i})$ — cf. Lemma 2.12, (iii)], and, moreover, that if we write $\Pi_{\mathrm{T}_i} \overset{\mathrm{def}}{=} \Pi_{\mathrm{F}_i} \overset{\mathrm{out}}{\rtimes} \Pi_{v_i}$ $(\subseteq \Pi_{\mathrm{T}})$ [cf. the discussion entitled "*Topological groups*" in §0], then Π_{T_i} is naturally isomorphic to the "Π_{T}" obtained by taking "$\mathcal{G}$" to be $\mathcal{G}|_{v_i}$.

Now since $\widetilde{\beta}_i(\Pi_{\mathrm{F}_i}) = \Pi_{\mathrm{F}_i}$, and $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H))$, one may verify easily that the outomorphism of Π_{F_i} determined by $\widetilde{\beta}_i|_{\Pi_{\mathrm{F}_i}}$ [cf. Lemma 2.12, (iii)] is $\in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}_i})}(\rho_{2/1}(H \cap \Pi_{v_i}))$ — where, by abuse of notation, we write $H \cap \Pi_{v_i} \subseteq \Pi_{\mathrm{B}}$ for the intersection of H with the image of Π_{v_i} in Π_{B}. Therefore, since the quantity "$3g-3+r$" associated to $\mathcal{G}|_{v_i}$ is $< 3g-3+r$, by considering a similar diagram to the diagram in [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1, (x), and applying the induction hypothesis, we conclude that $\widetilde{\beta}_i|_{\Pi_{\mathrm{F}_i}}$ is a Π_{F_i}*-inner automorphism.* In particular, it follows immediately [by allowing $i \in \{1,2\}$ to vary] that the outomorphism α is $\in \mathrm{Dehn}(\mathcal{G}_x)$, and, moreover — by considering the natural identification outer isomorphism $\Pi_{\mathrm{F}_i} \overset{\sim}{\to} \Pi_{((\mathcal{G}_x)|_{\mathbb{H}_i}))_{\succ S_i}}$ — that α is contained in the kernel of the natural surjection $\mathrm{Dehn}(\mathcal{G}_x) \twoheadrightarrow \mathrm{Dehn}(((\mathcal{G}_x)|_{\mathbb{H}_i}))_{\succ S_i})$, as desired. This completes the proof of (†), hence also of Lemma 6.8. Q.E.D.

Definition 6.9. In the notation of Definition 6.3:

(i) Suppose that $2g-2+r > 1$, i.e., $(g,r) \notin \{(0,3),(1,1)\}$. Then we shall write

$$A_{g,r} \overset{\mathrm{def}}{=} \{1\} \subseteq \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$$

[cf. Definition 6.5, (i)].

(ii) Suppose that $(g, r) = (1, 1)$. Then we shall write

$$(\mathbb{Z}/2\mathbb{Z} \simeq) \quad A_{g,r} \stackrel{\text{def}}{=} \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))\,.$$

(iii) Suppose that $(g, r) = (0, 3)$. Then we shall write

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq) \quad A_{g,r} \subseteq \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$$

for the subgroup of $\mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$ obtained as the image of the subgroup of the symmetric group on 4 letters

$$\{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \subseteq \mathfrak{S}_4$$

via the isomorphism $\mathfrak{S}_4 \overset{\sim}{\to} \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$ arising from a bijection $\{1, 2, 3, 4\} \overset{\sim}{\to} \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$. [Note that since the above subgroup of $\mathfrak{S}_4$ is *normal*, the subgroup $A_{g,r} \subseteq \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$ does *not depend* on the choice of the bijection $\{1, 2, 3, 4\} \overset{\sim}{\to} \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$.]

Lemma 6.10 (Permutations of cusps arising from certain C-admissible outomorphisms). *In the notation of Definition 6.3, let $H \subseteq \Pi_{\mathrm{B}}$ be an open subgroup of Π_{B}. Then the following hold:*

(i) *The composite*

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H)) \hookrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \to \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$$

[cf. Definition 6.5, (ii)] **factors** *through the subgroup $A_{g,r} \subseteq \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$ [cf. Definition 6.9], hence determines a homomorphism*

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1})) \longrightarrow A_{g,r}\,.$$

(ii) *The composite*

$$\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \longrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1})) \longrightarrow A_{g,r}$$

of the natural homomorphism

$$\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \longrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1}))$$

with the homomorphism of (i) is an **isomorphism**. *In particular, the homomorphism* $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1})) \to A_{g,r}$ *of (i) is a* **split surjection** *[cf. the discussion entitled "Topological groups" in §0].*

Proof. First, we verify assertion (i). If $(g, r) = (1, 1)$, then since $A_{g,r} = \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$, assertion (i) is immediate. On the other hand, if $r = 0$, then since $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})^{\sharp} = 1$, assertion (i) is immediate. Thus, in the remainder of the proof of assertion (i), we suppose that $(g, r) \neq (1, 1)$, $r \geq 1$.

Now we verify assertion (i) in the case where $r = 1$. Let us observe that it follows immediately from Lemma 6.2, (i), that by replacing $X^{\log}$ by a suitable stable log curve of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$, we may assume without loss of generality [cf. our assumption that $r = 1$, which implies that $(g, r) \neq (0, 3)$] that $\mathcal{G}$ is *cyclically primitive* [cf. Definition 4.1]. Let $c \in \mathrm{Cusp}(\mathcal{G})$ be the *unique* cusp of $\mathcal{G}$, $e \in \mathrm{Node}(\mathcal{G})$ the *unique* node of $\mathcal{G}$, $x \in X(k)$ such that $x \curvearrowright e$, and $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H))$. Then let us observe that it follows immediately from our assumption that $\mathcal{G}$ is *cyclically primitive* of type $(g, r) \neq (1, 1)$ (respectively, the various definitions involved) that the vertex of $\mathcal{G}_x$ to which c^{F} (respectively, $c^{\mathrm{F}}_{\mathrm{diag}}$) abuts is *not of type* $(0, 3)$ (respectively, is of *type* $(0, 3)$). Moreover, it follows immediately from Lemma 6.7, (ii), that the outomorphism α of $\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ is $\in \mathrm{Aut}(\mathcal{G}_x)$. Thus, we conclude that the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ induced by α is the *identity automorphism*. This completes the proof of assertion (i) in the case where $r = 1$.

Next, we verify assertion (i) in the case where $r > 1$. Let us observe that it follows immediately from Lemma 6.2, (i), that by replacing $X^{\log}$ by a suitable stable log curve of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$, we may assume without loss of generality that $\mathrm{Node}(\mathcal{G}) = \emptyset$. Let $v \in \mathrm{Vert}(\mathcal{G})$ be the *unique* vertex of $\mathcal{G}$ [cf. our assumption that $\mathrm{Node}(\mathcal{G}) = \emptyset$] and $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_{2/1}(H))$. Now let us observe that for any $c \in \mathrm{Cusp}(\mathcal{G})$, $x \in X(k)$ such that $x \curvearrowright c$, it follows immediately from the various definitions involved that $\mathrm{Vert}(\mathcal{G}_x) = \{v^{\mathrm{F}}_x, v^{\mathrm{F}}_{\mathrm{new},x}\}$; $\mathcal{C}(v^{\mathrm{F}}_{\mathrm{new},x}) = \{c^{\mathrm{F}}, c^{\mathrm{F}}_{\mathrm{diag}}\}$; $\mathcal{C}(v^{\mathrm{F}}_x) = \mathrm{Cusp}(\mathcal{G}_x) \setminus \{c^{\mathrm{F}}, c^{\mathrm{F}}_{\mathrm{diag}}\}$; v^{F}_x is of type (g, r); $v^{\mathrm{F}}_{\mathrm{new},x}$ is of type $(0, 3)$. Moreover, it follows immediately from Lemma 6.7, (ii), that the outomorphism α of $\Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}}$ is $\in \mathrm{Aut}(\mathcal{G}_x)$. Thus, if $(g, r) \neq (0, 3)$, then since v^{F}_x is of type (g, r), and $v^{\mathrm{F}}_{\mathrm{new},x}$ is of type $(0, 3)$, it follows immediately that α induces the *identity automorphism* on $\mathrm{Vert}(\mathcal{G}_x)$, hence that α preserves the subset $\{c, c^{\mathrm{F}}_{\mathrm{diag}}\} \subseteq \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ corresponding to $\mathcal{C}(v^{\mathrm{F}}_{\mathrm{new},x}) = \{c^{\mathrm{F}}, c^{\mathrm{F}}_{\mathrm{diag}}\}$. In particular, if $(g, r) \neq (0, 3)$, (respectively, $(g, r) = (0, 3)$), then — by allowing "c" to *vary* among the elements of

$\mathrm{Cusp}(\mathcal{G})$ — one may verify easily that the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ induced by α is the *identity automorphism* (respectively, satisfies the condition that

> for any subset $S \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ of *cardinality* 2, the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ induced by α determines an automorphism of the set $\{S, \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}) \setminus S\} \subseteq 2^{\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})}$,

hence, by Lemma 6.11 below, is contained in $A_{g,r} \subseteq \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})))$. This completes the proof of assertion (i) in the case where $r > 1$, hence also of assertion (i).

Next, we verify assertion (ii). One verifies easily that the composite of natural homomorphisms

$$\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \to \mathrm{Aut}_{\Pi_{\mathrm{B}}}(\Pi_{\mathrm{T}})/\mathrm{Inn}(\Pi_{\mathrm{F}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1}))$$

[cf. the discussion entitled "*Topological groups*" in §0] factors through $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1})) \subseteq Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1}))$. In particular, we obtain a natural homomorphism $\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \to Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1}))$. Now the fact that the composite

$$\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \to Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{2/1})) \hookrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \to \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}))$$

determines a *surjection* $\mathrm{Aut}_{X^{\log}}(X_2^{\log}) \twoheadrightarrow A_{g,r}$ is well-known and easily verified. To verify that this surjection is *injective*, observe that an element of the kernel of this surjection determines an automorphism of the *trivial family* $X^{\log} \times_{(\mathrm{Spec}\, k)^{\log}} X^{\log} \to X^{\log}$ over $X^{\log}$ that *preserves* the image of the *diagonal*. On the other hand, since the *relative tangent bundle* of this trivial family has *no nonzero global sections*, one concludes immediately that such an automorphism is *constant*, i.e., arises from a single automorphism of the fiber $X^{\log}$ over $(\mathrm{Spec}\, k)^{\log}$ that is *compatible with the diagonal*, hence [as is easily verified] equal to the *identity automorphism*, as desired. This completes the proof of assertion (ii). Q.E.D.

Lemma 6.11 (**A subgroup of the symmetric group on 4 letters).** *Write* $G \subseteq \mathfrak{S}_4$ *for the subgroup of the symmetric group on* 4 *letters* $\mathfrak{S}_4$ *consisting of* $g \in \mathfrak{S}_4$ *such that*

> $(*)$*: for any subset* $S \subseteq \{1, 2, 3, 4\}$ *of* **cardinality 2***, the automorphism* g *of* $\{1, 2, 3, 4\}$ *determines an automorphism of the set* $\{S, \{1, 2, 3, 4\} \setminus S\} \subseteq 2^{\{1,2,3,4\}}$.

Then

$$G = \{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}\,.$$

Proof. First, let us observe that one may verify easily that

$$\{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \subseteq G\,.$$

Thus, to verify Lemma 6.11, it suffices to verify that $G^\sharp = 4$. Next, let us observe that it follows immediately from the condition $(*)$ that for any element $g \in G$, it holds that $g^4 = \mathrm{id}$; in particular, by the *Sylow Theorem*, together with the fact that $\mathfrak{S}_4^\sharp = 2^3 \cdot 3$, we conclude that G is a 2-group. Thus, to verify Lemma 6.11, it suffices to verify that $G^\sharp \neq 8$. Next, let us observe that it follows immediately from the condition $(*)$ that $G \subseteq \mathfrak{S}_4$ is *normal*. Thus, if $G^\sharp = 8$, then since $\mathfrak{S}_4^\sharp = 2^3 \cdot 3$, and $(1\,2) \in \mathfrak{S}_4$ is of *order* 2, again by the *Sylow Theorem*, we conclude that $(1\,2) \in G$, in contradiction to the fact that $(1\,2)$ does *not satisfy* the condition $(*)$. This completes the proof of Lemma 6.11. Q.E.D.

Theorem 6.12 (**Centralizers of geometric monodromy groups arising from configuration spaces**). *Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; $0 < m < n$ positive integers; $\Sigma_\mathrm{F} \subseteq \Sigma_\mathrm{B}$ nonempty sets of prime numbers; k an algebraically closed field of characteristic zero; $(\mathrm{Spec}\, k)^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k$ with the log structure given by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a* **stable log curve** *of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$. Suppose that $\Sigma_\mathrm{F} \subseteq \Sigma_\mathrm{B}$ satisfy one of the following two conditions:*

(1) *Σ_F and Σ_B determine* **PT-formations** *[i.e., are either of* **cardinality one** *or* **equal to $\mathfrak{Primes}$** *— cf.* [MT]*, Definition 1.1, (ii)].*

(2) *$n - m = 1$ and $\Sigma_\mathrm{B} = \mathfrak{Primes}$.*

Write

$$X_n^{\log}\ ,\ \ X_m^{\log}$$

for the n-th, m-th **log configuration spaces** *of the stable log curve $X^{\log}$ [cf. the discussion entitled "Curves" in §0], respectively; Π_n, $\Pi_\mathrm{B} \stackrel{\mathrm{def}}{=} \Pi_m$ for the respective maximal pro-Σ_B quotients of the kernels of the natural surjections $\pi_1(X_n^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$, $\pi_1(X_m^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$; $\Pi_{n/m} \subseteq \Pi_n$ for the kernel of the surjection $\Pi_n \twoheadrightarrow \Pi_\mathrm{B} = \Pi_m$ induced by the projection $X_n^{\log} \to X_m^{\log}$ obtained by forgetting the last $(n - m)$*

factors; Π_{F} *for the maximal pro-*Σ_{F} *quotient of* $\Pi_{n/m}$*;* Π_{T} *for the quotient of* Π_n *by the kernel of the natural surjection* $\Pi_{n/m} \twoheadrightarrow \Pi_{\mathrm{F}}$*. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1\,,$$

which determines an outer representation

$$\rho_{n/m} \colon \Pi_{\mathrm{B}} \longrightarrow \mathrm{Out}(\Pi_{\mathrm{F}})\,.$$

Then the following hold:

(i) *Let* $H \subseteq \Pi_{\mathrm{B}}$ *be an open subgroup of* Π_{B}*. Recall that* $X_n^{\log} \to X_m^{\log}$ *may be regarded as the* $(n-m)$*-th log configuration space of the family of stable log curves* $X_{m+1}^{\log} \to X_m^{\log}$ *over* $X_m^{\log}$*. Then the composite of natural homomorphisms*

$$\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \longrightarrow \mathrm{Aut}_{X_m^{\log}}(X_n^{\log}) \longrightarrow \mathrm{Aut}_{\Pi_{\mathrm{B}}}(\Pi_{\mathrm{T}})/\mathrm{Inn}(\Pi_{\mathrm{F}})$$
$$\stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\mathrm{Im}(\rho_{n/m})) \subseteq Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H))$$

— where the first arrow is the homomorphism arising from the functoriality of the construction of the log configuration space; the third arrow is the isomorphism appearing in the discussion entitled "Topological groups" in §0 — determines an **isomorphism**

$$\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H))$$

— where we write $\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})$ *for the group of* **FC-admissible** *[cf. Definition 6.1;* [CmbCsp]*, Definition 1.1, (ii)] outomorphisms of* Π_{F} *[cf. Lemma 6.2, (ii)]. Here, we recall that the automorphism group* $\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log})$ *is isomorphic to*

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \textit{if } (g,r,m) = (0,3,1); \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } (g,r,m) = (1,1,1); \\ \{1\} & \textit{if } (g,r,m) \notin \{(0,3,1),(1,1,1)\}. \end{cases}$$

(ii) *The isomorphism of (i) and the natural inclusion* $\mathfrak{S}_{n-m} \hookrightarrow Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H))$ *— where we write* $\mathrm{Out}^{\mathrm{PFC}}(\Pi_{\mathrm{F}})$ *for the group of* **PFC-admissible** *[cf. Definitions 1.4, (iii); 6.1] outomorphisms of* Π_{F} *[cf. Lemma 6.2, (ii)] — determine an* **isomorphism**

$$\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \times \mathfrak{S}_{n-m} \stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H))\,.$$

(iii) *Let H be a closed subgroup of* $\mathrm{Out}^{\mathrm{PFC}}(\Pi_{\mathrm{F}})$ *that contains an open subgroup of* $\mathrm{Im}(\rho_{n/m}) \subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$. *Then H is* **almost slim** *[cf. the discussion entitled "Topological groups" in §0]. If, moreover,* $H \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})$, *and* $(g, r, m) \notin \{(0, 3, 1), (1, 1, 1)\}$, *then H is* **slim** *[cf. the discussion entitled "Topological groups" in §0].*

Proof. First, we verify assertion (i). We begin by observing that the description of the automorphism group $\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log})$ given in the statement of assertion (i) follows immediately from Lemma 6.10, (ii). Next, let us observe that

$(*_1)$: to verify assertion (i), it suffices to verify assertion (i) in the case where $\Sigma_{\mathrm{B}} = \mathfrak{Primes}$.

Indeed, this follows immediately from the various definitions involved. Thus, in the remainder of the proof of assertion (i), we suppose that $\Sigma_{\mathrm{B}} = \mathfrak{Primes}$.

Next, we *claim* that

$(*_2)$: the composite homomorphism of assertion (i) determines an *injection*

$$\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H)) \, .$$

Indeed, one verifies easily that the composite as in assertion (i) factors through $Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H))$. On the other hand, by considering the action of $\mathrm{Aut}_{X_m^{\log}}(X_{m+1}^{\log})$ on the set of conjugacy classes of cuspidal inertia subgroups of suitable subquotients [arising from fiber subgroups] of Π_{F}, it follows immediately that the composite as in assertion (i) is *injective* [cf. Lemma 6.10, (ii)]. This completes the proof of the *claim* $(*_2)$.

Next, we *claim* that

$(*_3)$: the injection of $(*_2)$ is an *isomorphism*.

Indeed, it follows immediately from the various definitions involved that if $N_{\mathrm{B}} \subseteq \Pi_{\mathrm{B}}$ is a fiber subgroup of Π_{B} of length 1 [cf. Lemma 6.2, (ii); [MT], Definition 2.3, (iii)], then the natural surjection $\Pi_{\mathrm{T}} \times_{\Pi_{\mathrm{B}}} N_{\mathrm{B}} \twoheadrightarrow N_{\mathrm{B}}$ may be regarded as the "$\Pi_{\mathrm{T}} \twoheadrightarrow \Pi_{\mathrm{B}}$" obtained by taking "$(g, r, m, n)$" to be $(g, r + m - 1, 1, n - m + 1)$. Thus, by applying the inclusion $Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H)) \subseteq Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_{\mathrm{F}})}(\rho_{n/m}(H \cap N_{\mathrm{B}}))$ and replacing $\Pi_{\mathrm{T}} \twoheadrightarrow \Pi_{\mathrm{B}}$ by $\Pi_{\mathrm{T}} \times_{\Pi_{\mathrm{B}}} N_{\mathrm{B}} \twoheadrightarrow N_{\mathrm{B}}$, we may assume without loss of generality that $m = 1$. On the other hand, it follows immediately from the various definitions involved that if $N_{\mathrm{F}} \subseteq \Pi_{\mathrm{F}}$ is a fiber subgroup of Π_{F} of length $n - 2$, then the natural surjection $\Pi_{\mathrm{T}}/N_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{B}}$ may be regarded as the "$\Pi_T \twoheadrightarrow \Pi_B$" obtained by taking "$(g, r, m, n)$" to be $(g, r, 1, 2)$. Thus, since the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_F) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_F/N_F)$ is *injective* [cf. [NodNon], Theorem B], by replacing $\Pi_T \twoheadrightarrow \Pi_B$ by $\Pi_T/N_F \twoheadrightarrow \Pi_B$, we may assume without loss of generality that $(m, n) = (1, 2)$. In particular — in light of our assumption that $\Sigma_B = \mathfrak{Primes}$ [cf. $(*_1)$] — we are in the situation of Definition 6.3.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_F)}(\rho_{n/m}(H))$. Then it follows immediately from Lemma 6.10, (ii), that there exists an element β of the image of the injection of $(*_2)$ such that $\alpha \circ \beta \in Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_F)}(\rho_{n/m}(H))$ induces the *identity automorphism* of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. Definition 6.5, (i), (ii)]. In particular, $\alpha \circ \beta$ preserves the Π_F-conjugacy class of a cuspidal subgroup $\Pi_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi_F$ of Π_F associated to $c^{\mathrm{F}}_{\mathrm{diag}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. Definition 6.5, (iii)]. Thus, it follows from Lemma 6.8 that $\alpha \circ \beta$ is the *identity outomorphism* of Π_F. In particular, we conclude that the injection of $(*_2)$ is *surjective*. This completes the proof of the *claim* $(*_3)$, hence also of assertion (i).

Next, we verify assertion (ii). First, let us observe that by considering the action of $Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_F)}(\rho_{n/m}(H))$ on the set of fiber subgroups of Π_F of length 1, we obtain an exact sequence of profinite groups

$$1 \longrightarrow Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_F)}(\rho_{n/m}(H)) \longrightarrow Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_F)}(\rho_{n/m}(H)) \longrightarrow \mathfrak{S}_{n-m}\,.$$

Now by considering the action of $\mathfrak{S}_{n-m}$ on $X^{\log}_n$ over $X^{\log}_m$ obtained by permuting the first $n-m$ factors of $X^{\log}_n$, we obtain a section $\mathfrak{S}_{n-m} \hookrightarrow Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_F)}(\rho_{n/m}(H))$ of the third arrow in the above exact sequence; in particular, the third arrow is *surjective*. On the other hand, it follows from [NodNon], Theorem B, that the image of the section $\mathfrak{S}_{n-m} \hookrightarrow Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_F)}(\rho_{n/m}(H))$ *commutes with* $Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_F)}(\rho_{n/m}(H))$. Thus, the composite of natural homomorphisms

$$\mathrm{Aut}_{X^{\log}_m}(X^{\log}_{m+1}) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{FC}}(\Pi_F)}(\rho_{n/m}(H)) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{FPC}}(\Pi_F)}(\rho_{n/m}(H))$$

[cf. assertion (i)] and the section $\mathfrak{S}_{n-m} \hookrightarrow Z_{\mathrm{Out}^{\mathrm{PFC}}(\Pi_F)}(\rho_{n/m}(H))$ determine an isomorphism as in the statement of assertion (ii). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions, (i), (ii). This completes the proof of Theorem 6.12. Q.E.D.

Remark 6.12.1. By considering a suitable *specialization isomorphism*, one may replace the expression "k an algebraically closed field of characteristic zero" in the statement of Theorem 6.12 by the expression "k an algebraically closed field of characteristic $\notin \Sigma_B$".

Theorem 6.13 (**Centralizers of geometric monodromy groups arising from moduli stacks of pointed curves).** *Let* (g, r) *be a pair of nonnegative integers such that* $2g-2+r>0$*;* Σ *a nonempty set of prime numbers;* k *an algebraically closed field of characteristic zero. Write* $\Pi_{\mathcal{M}_{g,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{M}_{g,r})_k)$ *for the étale fundamental group of the moduli stack* $(\mathcal{M}_{g,r})_k$ *[cf. the discussion entitled "Curves" in §0];* $\Pi_{g,r}$ *for the maximal pro-*Σ *quotient of the kernel* $N_{g,r}$ *of the natural surjection* $\pi_1((\mathcal{C}_{g,r})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{g,r})_k) = \Pi_{\mathcal{M}_{g,r}}$ *[cf. the discussion entitled "Curves" in §0];* $\Pi_{\mathcal{C}_{g,r}}$ *for the quotient of the étale fundamental group* $\pi_1((\mathcal{C}_{g,r})_k)$ *of* $(\mathcal{C}_{g,r})_k$ *by the kernel of the natural surjection* $N_{g,r} \twoheadrightarrow \Pi_{g,r}$*. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{g,r} \longrightarrow \Pi_{\mathcal{C}_{g,r}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow 1\,,$$

which determines an outer representation

$$\rho_{g,r}\colon \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\Pi_{g,r})\,.$$

Then the following hold:

(i) *The profinite group* $\Pi_{g,r}$ *is equipped with a natural structure of pro-*Σ **surface group** *[cf.* [MT], *Definition 1.2].*

(ii) *Let* $H \subseteq \Pi_{\mathcal{M}_{g,r}}$ *be an open subgroup of* $\Pi_{\mathcal{M}_{g,r}}$*. Suppose that*

$$2g-2+r>1, \text{ i.e., } (g,r) \notin \{(0,3),(1,1)\}.$$

Then the composite of natural homomorphisms

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{g,r}}}(\Pi_{\mathcal{C}_{g,r}})/\mathrm{Inn}(\Pi_{g,r})$$

$$\stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}(\Pi_{g,r})}(\mathrm{Im}(\rho_{g,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,r})}(\rho_{g,r}(H))$$

[cf. the discussion entitled "Topological groups" in §0] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H))$$

[cf. (i); Definition 6.1]. Here, we recall that the automorphism group $\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k)$ *is isomorphic to*

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) = (0,4); \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) \in \{(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,r) \notin \{(0,4),(1,2),(2,0)\}\,. \end{cases}$$

(iii) *Let* $H \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ *be a closed subgroup of* $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ *that contains an open subgroup of* $\mathrm{Im}(\rho_{g,r}) \subseteq \mathrm{Out}(\Pi_{g,r})$*. Suppose that*

$$2g - 2 + r > 1, \text{ i.e., } (g,r) \notin \{(0,3),(1,1)\}.$$

Then H *is* **almost slim** *[cf. the discussion entitled "Topological groups" in §0]. If, moreover,*

$$2g - 2 + r > 2, \text{ i.e., } (g,r) \notin \{(0,3),(0,4),(1,1),(1,2),(2,0)\},$$

then H *is* **slim** *[cf. the discussion entitled "Topological groups" in §0].*

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). First, we recall that the description of the automorphism group $\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k)$ given in the statement of assertion (ii) is well-known [cf., e.g., [CorHyp], Theorem B, if $2g-2+r > 2$, i.e., $(g,r) \notin \{(0,4),(1,2),(2,0)\}$]. Next, we *claim* that

> $(*_1)$: the composite homomorphism of assertion (ii) determines an injection
>
> $$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H))\,.$$

Indeed, one verifies easily that the composite as in assertion (ii) factors through $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H))$. Thus, the *claim* $(*_1)$ follows immediately from the well-known fact that any *nontrivial* automorphism of a hyperbolic curve over an algebraically closed field of characteristic $\notin \Sigma$ induces a *nontrivial* outomorphism of the maximal pro-Σ quotient of the étale fundamental group of the hyperbolic curve [cf., e.g., [LocAn], the proof of Theorem 14.1]. This completes the proof of the *claim* $(*_1)$.

Next, we *claim* that

> $(*_2)$: if $r > 0$, then the injection of $(*_1)$ is an *isomorphism*.

Indeed, write $N \subseteq \Pi_{\mathcal{M}_{g,r}}$ for the kernel of the surjection $\Pi_{\mathcal{M}_{g,r}} \twoheadrightarrow \pi_1((\mathcal{M}_{g,r-1})_k)$ determined by the (1-)morphism $(\mathcal{M}_{g,r})_k \to (\mathcal{M}_{g,r-1})_k$ obtained by forgetting the last section. Then it follows immediately from the various definitions involved that there exists a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,r} & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \end{array}$$

— where the upper sequence is the exact sequence obtained by pulling back the exact sequence $1 \to \Pi_{g,r} \to \Pi_{\mathcal{C}_{g,r}} \to \Pi_{\mathcal{M}_{g,r}} \to 1$ by the natural inclusion $N \hookrightarrow \Pi_{\mathcal{M}_{g,r}}$; the lower sequence is the exact sequence "$1 \to \Pi_{\mathrm{F}} \to \Pi_{\mathrm{T}} \to \Pi_{\mathrm{B}} \to 1$" obtained by applying the procedure given in the statement of Theorem 6.12 in the case where $(m, n, \Sigma_{\mathrm{F}}, \Sigma_{\mathrm{B}}) = (1, 2, \Sigma, \mathfrak{Primes})$ to a stable log curve of type $(g, r-1)$ over $(\mathrm{Spec}\, k)^{\log}$; the vertical arrows are *isomorphisms*. Thus, it follows immediately from Theorem 6.12, (i), that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H \cap N))$ is isomorphic to the automorphism group $\mathrm{Aut}_{X^{\log}}(X_2^{\log})$ for the stable log curve $X^{\log}$ over $(\mathrm{Spec}\, k)^{\log}$ *of type* $(g, r-1)$. In particular, by the *claim* $(*_1)$, we obtain that

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H))$$
$$\subseteq Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H \cap N)) \stackrel{\sim}{\leftarrow} \mathrm{Aut}_{X^{\log}}(X_2^{\log})\,.$$

Thus, by comparing $(\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k))^{\sharp}$ with $\mathrm{Aut}_{X^{\log}}(X_2^{\log})^{\sharp}$ [cf. Theorem 6.12, (i)], we conclude that the injection of the *claim* $(*_1)$ is an *isomorphism*. This completes the proof of the *claim* $(*_2)$. Moreover, it follows immediately from the proof of the *claim* $(*_2)$ that

> $(*_3)$: if $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{0,4})}(\rho_{0,4}(H))$ induces the *identity automorphism* on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_{0,4}$, then α is the *identity outomorphism* of $\Pi_{0,4}$.

In light of the *claim* $(*_2)$, in the remainder of the proof of assertion (ii), we assume that

$$r = 0, \text{ hence also that } g \geq 2.$$

For $x \in (\overline{\mathcal{M}}_{g,0})_k(k)$, write

$$\mathcal{G}_x$$

for the semi-graph of anabelioids of pro-Σ PSC-type associated to the geometric fiber of $(\overline{\mathcal{C}}_{g,0}^{\log})_k \to (\overline{\mathcal{M}}_{g,0}^{\log})_k$ over $x^{\log} \stackrel{\mathrm{def}}{=} x \times_{(\overline{\mathcal{M}}_{g,0})_k} (\overline{\mathcal{M}}_{g,0}^{\log})_k$; thus, we have a natural $\mathrm{Im}(\rho_{g,0})$ $(\subseteq \mathrm{Out}(\Pi_{g,0}))$-torsor of outer isomorphisms $\Pi_{g,0} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$. Let us *fix* an isomorphism $\Pi_{g,0} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$ that belongs to this collection of isomorphisms. Moreover, for $x \in (\overline{\mathcal{M}}_{g,0})_k(k)$, we shall say that x satisfies the condition $(\dagger)$ if

$(\dagger_1)$ $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2\}$; $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \cdots, e_{g+1}\}$;

$(\dagger_2)$ $\mathcal{N}(v_1) = \mathcal{N}(v_2) = \mathrm{Node}(\mathcal{G}_x)$;

$(\dagger_3)$ v_1 and v_2 are of type $(0, g+1)$;

we shall say that x satisfies the condition (‡) if

($\ddagger_1$) $\mathrm{Vert}(\mathcal{G}_x) = \{v_1^*, v_2^*, w^*\}$; $\mathrm{Node}(\mathcal{G}_x) = \{e_1^*, e_2^*, \cdots, e_{g+1}^*, f^*\}$;

($\ddagger_2$) $\mathcal{N}(v_1^*) = \{e_1^*, e_2^*, \cdots, e_{g+1}^*\}$; $\mathcal{N}(v_2^*) = \{e_1^*, e_2^*, \cdots, e_{g-1}^*, f^*\}$; $\mathcal{N}(w^*) = \{e_g^*, e_{g+1}^*, f^*\}$;

($\ddagger_3$) v_1^* is of type $(0, g+1)$, v_2^* is of type $(0, g)$, and w^* is of type $(0, 3)$.

Let us observe that one may verify easily that there exists a k-valued point $x \in (\overline{\mathcal{M}}_{g,0})_k(k)$ that satisfies (†); if, moreover, $g > 2$, then there exists a k-valued point $x \in (\overline{\mathcal{M}}_{g,0})_k(k)$ that satisfies (‡).

Let $x \in (\overline{\mathcal{M}}_{g,0})_k(k)$ be a k-valued point. Then we *claim* that

> ($*_4$): if x satisfies (†), and, relative to the isomorphism $\Pi_{g,0} \overset{\sim}{\to} \Pi_{\mathcal{G}_x}$ *fixed* above,
>
> $$\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H))$$
>
> determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0})$), then for any $e \in \mathrm{Node}(\mathcal{G}_x)$, the image α_e of α via the natural inclusion $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_x)_{\rightsquigarrow\{e\}})$ [cf. Proposition 2.9, (ii)] satisfies
>
> $$\alpha_e \in \mathrm{Dehn}((\mathcal{G}_x)_{\rightsquigarrow\{e\}})\,.$$

Indeed, let $e \in \mathrm{Node}(\mathcal{G}_x)$ and $y \in (\overline{\mathcal{M}}_{g,0})_k(k)$ a k-valued point such that $\mathcal{G}_y$ corresponds to $(\mathcal{G}_x)_{\rightsquigarrow\{e\}}$ [cf. the special fibers of the stable log curves over "$S^{\log}$" that appear in Proposition 5.6, (iii)]. Write $v \in \mathrm{Vert}(\mathcal{G}_y)$ for the *unique* vertex of $\mathcal{G}_y$. [Note that it follows from the definition of the condition (†) that $\mathrm{Vert}(\mathcal{G}_y)^\sharp = 1$.] Then it follows immediately from the general theory of stable log curves that there exist a *"clutching (1-)morphism"* corresponding to the *operation* of resolving the nodes of $\mathcal{G}_y$ [i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6]

$$(\mathcal{M}_{0,2g})_k \longrightarrow (\overline{\mathcal{M}}_{g,0})_k$$

and a k-valued point $\widetilde{y} \in (\mathcal{M}_{0,2g})_k(k)$ such that the image of $\widetilde{y}$ via the above clutching morphism coincides with y, and, moreover, $\mathcal{G}_{\widetilde{y}}$ is naturally isomorphic to $(\mathcal{G}_y)|_v$. Write $(\underline{\mathcal{M}}_{0,2g}^{\log})_k$ for the log stack obtained by equipping $(\mathcal{M}_{0,2g})_k$ with the log structure induced by the log structure of $(\overline{\mathcal{M}}_{g,0}^{\log})_k$ via the above clutching morphism. Then one verifies easily that the composite

$$\Pi_{\underline{\mathcal{M}}_{0,2g}} \stackrel{\text{def}}{=} \pi_1((\underline{\mathcal{M}}^{\log}_{0,2g})_k) \longrightarrow \pi_1((\overline{\mathcal{M}}^{\log}_{g,0})_k) \stackrel{\sim}{\longleftarrow} \Pi_{\mathcal{M}_{g,0}} \stackrel{\rho_{g,0}}{\longrightarrow} \mathrm{Out}(\Pi_{g,0})$$

— where the first arrow is the outer homomorphism induced by the above clutching morphism, and the second arrow is the outer isomorphism obtained by applying the "*log purity theorem*" to the natural (1-)morphism $(\mathcal{M}_{g,0})_k \hookrightarrow (\overline{\mathcal{M}}^{\log}_{g,0})_k$ [cf. [ExtFam], Theorem B] — factors through $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_y) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_y}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0})$. Moreover, the resulting homomorphism $\Pi_{\underline{\mathcal{M}}_{0,2g}} \to \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_y)$ fits into a commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_{\underline{\mathcal{M}}_{0,2g}} & \longrightarrow & \Pi_{\mathcal{M}_{0,2g}} \\ \downarrow & & \downarrow \\ \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_y) & \xrightarrow{\rho^{\mathrm{Vert}}_{\mathcal{G}_y}} & \mathrm{Glu}(\mathcal{G}_y) = \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_y)|_v) \end{array}$$

[cf. Definition 4.9; Proposition 4.10, (ii)] — where the upper horizontal arrow is the outer homomorphism induced by the (1-)morphism $(\underline{\mathcal{M}}^{\log}_{0,2g})_k \to (\mathcal{M}_{0,2g})_k$ obtained by forgetting the log structure. Moreover, one verifies easily that there exists a natural outer isomorphism $\Pi_{(\mathcal{G}_y)|_v} \stackrel{\sim}{\to} \Pi_{0,2g}$ such that the homomorphism $\Pi_{\mathcal{M}_{0,2g}} \to \mathrm{Out}(\Pi_{0,2g})$ obtained by conjugating the outer action implicit in the right-hand vertical arrow of the above diagram $\Pi_{\mathcal{M}_{0,2g}} \to \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_y)|_v) \subseteq \mathrm{Out}(\Pi_{(\mathcal{G}_y)|_v})$ by the outer isomorphism $\Pi_{(\mathcal{G}_y)|_v} \stackrel{\sim}{\to} \Pi_{0,2g}$ *coincides* with $\rho_{0,2g}$. Thus, by considering the image in $\Pi_{\mathcal{M}_{0,2g}}$ of the inverse image of $H \subseteq \Pi_{\mathcal{M}_{g,0}}$ in $\Pi_{\underline{\mathcal{M}}_{0,2g}}$ [cf. the diagrams of the above displays], it follows immediately from the *claims* $(*_2)$ [in the case where "(g,r)"$= (0,2g)$] and $(*_3)$ [in the case where $g = 2$], together with the various definitions involved, that if $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H))$ determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0}))$, then the image of α via

$$\begin{array}{ccccc} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) & \hookrightarrow & \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_x)_{\rightsquigarrow\{e\}}) & \stackrel{\sim}{\to} & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_y) \\ & \stackrel{\rho^{\mathrm{Vert}}_{\mathcal{G}_y}}{\twoheadrightarrow} & \mathrm{Glu}(\mathcal{G}_y) & = & \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_y)|_v) \end{array}$$

[cf. Proposition 2.9, (ii)] is *trivial*. In particular, it follows from Proposition 4.10, (ii), that the image α_e of α via $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_x)_{\rightsquigarrow\{e\}})$ satisfies $\alpha_e \in \mathrm{Dehn}((\mathcal{G}_x)_{\rightsquigarrow\{e\}})$. This completes the proof of the *claim* $(*_4)$.

Next, we *claim* that

($*_5$): if x satisfies (†), and $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H))$ determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0})$), then α is the *identity outomorphism* of $\Pi_{g,0}$.

Indeed, it follows from the *claim* ($*_4$) that

$$\alpha \in \bigcap_{e \in \mathrm{Node}(\mathcal{G}_x)} \mathrm{Im}\Big(\mathrm{Dehn}((\mathcal{G}_x)_{\rightsquigarrow\{e\}}) \to \mathrm{Dehn}(\mathcal{G}_x)\Big)$$

[cf. Theorem 4.8, (ii)]. On the other hand, it follows immediately from Theorem 4.8, (ii), (iv), that the right-hand intersection is $= \{1\}$. This completes the proof of the *claim* ($*_5$).

Next, we *claim* that

($*_6$): we have

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G}_x)|}(\mathcal{G}_x) \ (\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0}))\,;$$

if, moreover, x satisfies (‡), then

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)\,.$$

Indeed, it follows immediately from Proposition 5.6, (ii), together with the definition of $x^{\log} = x \times_{(\overline{\mathcal{M}}_{g,0})_k} (\overline{\mathcal{M}}^{\log}_{g,0})_k$, that the composite

$$\pi_1(x^{\log}) \longrightarrow \pi_1((\overline{\mathcal{M}}^{\log}_{g,0})_k) \overset{\sim}{\longleftarrow} \Pi_{\mathcal{M}_{g,0}} \overset{\rho_{g,0}}{\longrightarrow} \mathrm{Out}(\Pi_{g,0})$$

— where the second arrow is the outer isomorphism obtained by applying the "*log purity theorem*" to the natural (1-)morphism $(\mathcal{M}_{g,0})_k \hookrightarrow (\overline{\mathcal{M}}^{\log}_{g,0})_k$ [cf. [ExtFam], Theorem B] — determines a *surjection* $\pi_1(x^{\log}) \twoheadrightarrow \mathrm{Dehn}(\mathcal{G}_x)$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\leftarrow} \mathrm{Out}(\Pi_{g,0})$) [i.e., which induces an *isomorphism* between the respective maximal pro-Σ quotients]. Thus, it follows immediately from the various definitions involved that there exists an open subgroup $M \subseteq \mathrm{Dehn}(\mathcal{G}_x)$ such that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_x})}(M)$ relative to the identification $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0}) \overset{\sim}{\to} \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_x})$ arising from our choice of an isomorphism $\Pi_{g,0} \overset{\sim}{\to} \Pi_{\mathcal{G}_x}$. Therefore, the inclusion $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G}_x)|}(\mathcal{G}_x)$ follows immediately from Theorem 5.14, (ii). This completes the proof of the inclusion $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G}_x)|}(\mathcal{G}_x)$. On the other hand, if, moreover, x satisfies (‡), then it follows immediately from the definition of the condition (‡) that $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) = \mathrm{Aut}^{|\mathrm{Node}(\mathcal{G}_x)|}(\mathcal{G}_x)$. In particular, we obtain that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$. This completes the proof of the *claim* $(*_6)$.

Next, we *claim* that

> $(*_7)$: if x satisfies (†), then for any $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H))$, there exists an element β of the image of the injection of $(*_1)$ such that the outomorphism $\alpha \circ \beta$ of $\Pi_{g,0} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ is $\in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \xleftarrow{\sim} \mathrm{Out}(\Pi_{g,0}))$.

Indeed, suppose that $g > 2$. Then by the definitions of (†), (‡), one may verify easily that there exist $y \in (\overline{\mathcal{M}}_{g,0})_k(k)$ and $f \in \mathrm{Node}(\mathcal{G}_y)$ such that y satisfies (‡), and, moreover, $\mathcal{G}_x$ corresponds to $(\mathcal{G}_y)_{\rightsquigarrow\{f\}}$ [cf. Proposition 5.6, (iv)]. Thus, it follows immediately from the *claim* $(*_6)$ that $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,0}(H)) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_y) \hookrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ [cf. Proposition 2.9, (ii)], i.e., so we may take β to be the *identity outomorphism*. This completes the proof of the *claim* $(*_7)$ in the case where $g > 2$.

Next, suppose that $g = 2$. Write $\mathbb{G}_x$ for the underlying semi-graph of $\mathcal{G}_x$ and $\mathrm{Aut}^{|\mathrm{Node}|}(\mathbb{G}_x)$ for the group of automorphisms of $\mathbb{G}_x$ which induce the *identity automorphism* of the set of nodes of $\mathbb{G}_x$. Then one may verify easily from the *explicit structure of* $\mathbb{G}_x$ [cf. the definition of the condition (†)] that $\mathrm{Aut}^{|\mathrm{Node}|}(\mathbb{G}_x)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus, since the automorphism group $\mathrm{Aut}_{(\mathcal{M}_{2,0})_k}((\mathcal{C}_{2,0})_k)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it follows immediately from the *claim* $(*_6)$, together with the various definitions involved, that — to complete the proof of the *claim* $(*_7)$ in the case where $g = 2$ — it suffices to verify that the composite of natural homomorphisms

$$\mathrm{Aut}_{(\mathcal{M}_{2,0})_k}((\mathcal{C}_{2,0})_k) \longrightarrow \mathrm{Aut}(\mathcal{G}_x) \longrightarrow \mathrm{Aut}(\mathbb{G}_x)$$

factors through $\mathrm{Aut}^{|\mathrm{Node}|}(\mathbb{G}_x) \subseteq \mathrm{Aut}(\mathbb{G}_x)$ and is *injective*. Now the fact that the composite in question factors through $\mathrm{Aut}^{|\mathrm{Node}|}(\mathbb{G}_x) \subseteq \mathrm{Aut}(\mathbb{G}_x)$ follows immediately from the *claim* $(*_6)$, applied to elements of the image of the injection of $(*_1)$. On the other hand, the *injectivity* of the composite in question follows immediately from the *injectivity* of the natural homomorphism $\mathrm{Aut}_{(\mathcal{M}_{2,0})_k}((\mathcal{C}_{2,0})_k) \to \mathrm{Aut}(\mathcal{G}_x)$ [cf. the proof of the *claim* $(*_1)$] and the *claim* $(*_5)$. This completes the proof of the *claim* $(*_7)$ in the case where $g = 2$, hence also — in light of the above proof of the *claim* $(*_7)$ in the case where $g > 2$ — of the *claim* $(*_7)$. Thus, the *surjectivity* of the injection of $(*_1)$ follows immediately from the *claims* $(*_5)$ and $(*_7)$. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii). This completes the proof of Theorem 6.13. Q.E.D.

Remark 6.13.1. In the notation of Theorem 6.13, since $\Pi_{\mathcal{M}_{0,3}} = \{1\}$, it is immediate that a similar result to the results stated in Theorem 6.13, (ii), (iii), does *not hold in the case where* $(g, r) = (0, 3)$. On the other hand, it is not clear to the authors at the time of writing whether or not a similar result to the results stated in Theorem 6.13, (ii), (iii), *holds in the case where* $(g, r) = (1, 1)$. Nevertheless, we are able to obtain a *conditional result* concerning the centralizer of the geometric monodromy group *in the case where* $(g, r) = (1, 1)$ [cf. Theorem 6.14, (iii), (iv) below].

Theorem 6.14 (Centralizers of geometric monodromy groups arising from moduli stacks of punctured semi-ellptic curves). *In the notation of Theorem 6.13, write* $(\mathcal{C}^{\pm}_{1,1})_k$ *for the stack-theoretic quotient of* $(\mathcal{C}_{1,1})_k$ *by the natural action of* $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ *over the moduli stack* $(\mathcal{M}_{1,1})_k$*;* $\Pi^{\pm}_{1,1}$ *for the maximal pro-*Σ *quotient of the kernel* $N^{\pm}_{1,1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\pi_1((\mathcal{C}^{\pm}_{1,1})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{1,1})_k) = \Pi_{\mathcal{M}_{1,1}})$ *of the natural surjection* $\pi_1((\mathcal{C}^{\pm}_{1,1})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{1,1})_k) = \Pi_{\mathcal{M}_{1,1}}$*;* $\Pi_{\mathcal{C}^{\pm}_{1,1}}$ *for the quotient of the étale fundamental group* $\pi_1((\mathcal{C}^{\pm}_{1,1})_k)$ *of the stack* $(\mathcal{C}^{\pm}_{1,1})_k$ *by the kernel of the natural surjection* $N^{\pm}_{1,1} \twoheadrightarrow \Pi^{\pm}_{1,1}$*. Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi^{\pm}_{1,1} \longrightarrow \Pi_{\mathcal{C}^{\pm}_{1,1}} \longrightarrow \Pi_{\mathcal{M}_{1,1}} \longrightarrow 1\,,$$

which determines an outer representation

$$\rho^{\pm}_{1,1} \colon \Pi_{\mathcal{M}_{1,1}} \longrightarrow \mathrm{Out}(\Pi^{\pm}_{1,1})\,.$$

Write $\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1})$ *for the group of outomorphisms of* $\Pi^{\pm}_{1,1}$ *which induce bijections on the set of cuspidal inertia subgroups of* $\Pi^{\pm}_{1,1}$*. Suppose that*

$$2 \in \Sigma.$$

Then the following hold:

(i) *The profinite group* $\Pi^{\pm}_{1,1}$ *is* **slim** *[cf. the discussion entitled "Topological groups" in §0].*

(ii) *Let* $H \subseteq \Pi_{\mathcal{M}_{1,1}}$ *be an open subgroup of* $\Pi_{\mathcal{M}_{1,1}}$*. Then the composite of natural homomorphisms*

$$\begin{aligned}\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}^{\pm}_{1,1})_k) &\longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{1,1}}}(\Pi_{\mathcal{C}^{\pm}_{1,1}})/\mathrm{Inn}(\Pi^{\pm}_{1,1}) \\ &\stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}(\Pi^{\pm}_{1,1})}(\mathrm{Im}(\rho^{\pm}_{1,1})) \subseteq Z_{\mathrm{Out}(\Pi^{\pm}_{1,1})}(\rho^{\pm}_{1,1}(H))\end{aligned}$$

[cf. (i); the discussion entitled "Topological groups" in §0] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}^{\pm}_{1,1})_k) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1})}(\rho^{\pm}_{1,1}(H))\,.$$

Here, we recall that $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}^{\pm}_{1,1})_k) = \{1\}$.

(iii) *Let* $H \subseteq \Pi_{\mathcal{M}_{1,1}}$ *be an open subgroup of* $\Pi_{\mathcal{M}_{1,1}}$*. Then the composite of natural homomorphisms*

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{1,1}}}(\Pi_{\mathcal{C}_{1,1}})/\mathrm{Inn}(\Pi_{1,1})$$
$$\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{1,1})}(\mathrm{Im}(\rho_{1,1})) \subseteq Z_{\mathrm{Out}(\Pi_{1,1})}(\rho_{1,1}(H))$$

[cf. Theorem 6.13, (i); the discussion entitled "Topological groups" in §0] determines an **injection**

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,1})}(\rho_{1,1}(H))\,.$$

Moreover, the image of this injection is **centrally terminal** *in* $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,1})}(\rho_{1,1}(H))$ *[cf. the discussion entitled "Topological groups" in §0]. Here, we recall that* $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \simeq \mathbb{Z}/2\mathbb{Z}$.

(iv) *The composite of natural homomorphisms*

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{1,1}}}(\Pi_{\mathcal{C}_{1,1}})/\mathrm{Inn}(\Pi_{1,1})$$
$$\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{1,1})}(\mathrm{Im}(\rho_{1,1}))$$

[cf. Theorem 6.13, (i); the discussion entitled "Topological groups" in §0] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,1})}(\mathrm{Im}(\rho_{1,1}))\,.$$

Proof. Assertion (i) follows immediately from a similar argument to the argument used in the proof of [MT], Proposition 1.4. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us recall that the description of the automorphism group $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}^{\pm}_{1,1})_k)$ given in the statement of assertion (ii) is well-known and easily verified. Write $\mathcal{E} \to (\mathcal{M}_{1,1})_k$ for the family of elliptic curves determined by the family of hyperbolic curves $(\mathcal{C}_{1,1})_k \to (\mathcal{M}_{1,1})_k$ of type $(1,1)$; $\mathcal{U} \to (\mathcal{C}_{1,1})_k$ for the restriction of the finite étale covering $\mathcal{E} \to \mathcal{E}$ over $(\mathcal{M}_{1,1})_k$ given by *multiplication by* 2 to $(\mathcal{C}_{1,1})_k \subseteq \mathcal{E}$. Then one verifies easily that the action of

$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ on $(\mathcal{C}_{1,1})_k$ *lifts naturally* to an action [i.e., given by *"multiplication by* ± 1*"*] on $\mathcal{U}$ over $(\mathcal{M}_{1,1})_k$. Write $\mathcal{P}$ for the stack-theoretic quotient of $\mathcal{U}$ by the action of $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ on $\mathcal{U}$; $\Pi_{\mathcal{P}/\mathcal{M}}$ for the maximal pro-Σ quotient of the kernel of the natural surjection $\pi_1(\mathcal{P}) \twoheadrightarrow \pi_1((\mathcal{M}_{1,1})_k)$;

$$\rho_{\mathcal{P}/\mathcal{M}} \colon \Pi_{\mathcal{M}_{1,1}} \longrightarrow \mathrm{Out}(\Pi_{\mathcal{P}/\mathcal{M}})$$

for the natural pro-Σ outer representation arising from the family of hyperbolic curves $\mathcal{P} \to (\mathcal{M}_{1,1})_k$. Thus, since $2 \in \Sigma$, one verifies easily that $\Pi_{\mathcal{P}/\mathcal{M}}$ may be regarded as a normal open subgroup of $\Pi^{\pm}_{1,1}$. Now let us observe that one verifies easily that

> $(*_1)$: $\mathcal{P} \to (\mathcal{M}_{1,1})_k$ is a *family of hyperbolic curves of type* $(0,4)$. If, moreover, we denote by $\mathcal{T} \to (\mathcal{M}_{1,1})_k$ the connected finite étale covering that trivializes the finite étale covering determined by the four cusps of $\mathcal{P} \to (\mathcal{M}_{1,1})_k$, then the classifying (1-)morphism $\mathcal{T} \to (\mathcal{M}_{0,4})_k$ of $\mathcal{P} \times_{(\mathcal{M}_{1,1})_k} \mathcal{T} \to \mathcal{T}$ [which is well-defined up to the natural action of $\mathfrak{S}_4$ on $(\mathcal{M}_{0,4})_k$] is *dominant*.

Now we *claim* that

> $(*_2)$: every element of $\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1})$ preserves the normal open subgroup $\Pi_{\mathcal{P}/\mathcal{M}} \subseteq \Pi^{\pm}_{1,1}$.

Indeed, let us observe that one verifies easily that the natural surjections $\Pi^{\pm}_{1,1} \twoheadrightarrow \Pi^{\pm}_{1,1}/\Pi_{1,1}$, $\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}}$ determine an isomorphism

$$(\Pi^{\pm}_{1,1})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} (\Pi^{\pm}_{1,1}/\Pi_{1,1}) \times (\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}}) \, .$$

Moreover, it follows immediately from the various definitions involved that the natural action of $(\Pi^{\pm}_{1,1})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathbb{Z}/2\mathbb{Z}$ on the set of conjugacy classes of cuspidal inertia subgroups of the kernel of the natural surjection $\Pi^{\pm}_{1,1} \twoheadrightarrow (\Pi^{\pm}_{1,1})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathbb{Z}/2\mathbb{Z}$ [which is equipped with a natural structure of pro-Σ surface group of type $(1,4)$] *factors through* $(\Pi^{\pm}_{1,1})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathbb{Z}/2\mathbb{Z} \overset{\sim}{\to} (\Pi^{\pm}_{1,1}/\Pi_{1,1}) \times (\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}}) \overset{\mathrm{pr}_2}{\twoheadrightarrow} (\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}})$, and that the resulting action of $(\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}})$ is *faithful*. Thus, we conclude that every element of $\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1})$ preserves the normal open subgroup $\Pi_{\mathcal{P}/\mathcal{M}} \subseteq \Pi^{\pm}_{1,1}$. This completes the proof of the *claim* $(*_2)$.

To verify assertion (ii), take an element $\alpha^{\pm} \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1})}(\rho^{\pm}_{1,1}(H))$. Then it follows from the *claim* $(*_2)$ that $\alpha^{\pm}$ naturally determines an element $\alpha_{\mathcal{P}} \in \mathrm{Aut}(\Pi_{\mathcal{P}/\mathcal{M}})/\mathrm{Inn}(\Pi^{\pm}_{1,1})$. Let us fix a lifting $\beta \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{P}/\mathcal{M}})$

of $\alpha_{\mathcal{P}}$. Next, let us observe that since $\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}}$ is *finite*, to verify assertion (ii), by replacing H by an open subgroup of $\Pi_{\mathcal{M}_{1,1}}$ contained in H, we may assume without loss of generality that β *commutes with* $\rho_{\mathcal{P}/\mathcal{M}}(H) \subseteq \mathrm{Out}(\Pi_{\mathcal{P}/\mathcal{M}})$, i.e., $\beta \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{P}/\mathcal{M}})}(\rho_{\mathcal{P}/\mathcal{M}}(H))$. Then it follows immediately from Theorem 6.13, (ii), in the case where $(g,r) = (0,4)$, together with $(*_1)$, that β is contained in the image of the natural injection $\Pi^{\pm}_{1,1}/\Pi_{\mathcal{P}/\mathcal{M}} \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{P}/\mathcal{M}})$ obtained by conjugation. Thus, $\alpha_{\mathcal{P}}$, hence also — by the *manifest injectivity* [cf. assertion (i)] of the homomorphism $\mathrm{Out}^{\mathrm{C}}(\Pi^{\pm}_{1,1}) \to \mathrm{Aut}(\Pi_{\mathcal{P}/\mathcal{M}})/\mathrm{Inn}(\Pi^{\pm}_{1,1})$ implicit in the content of the *claim* $(*_2)$ — $\alpha^{\pm}$, is *trivial*. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, recall that the description of $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ given in the statement of assertion (iii) is well-known and easily verified. Next, let us observe that the fact that the composite in the statement of assertion (iii) determines an *injection*

$$\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,1})}(\rho_{1,1}(H))$$

follows immediately from a similar argument to the argument used in the proof of the *claim* $(*_1)$ in the proof of Theorem 6.13, (ii), together with the various definitions involved. Next, let us observe that by applying the natural outer isomorphism $\Pi^{\pm}_{1,1} \xrightarrow{\sim} \Pi_{1,1} \overset{\mathrm{out}}{\rtimes} \mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$, we obtain an exact sequence of profinite groups

$$1 \longrightarrow \mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \longrightarrow Z_{\mathrm{Out}(\Pi_{1,1})}(\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)) \longrightarrow \mathrm{Out}(\Pi^{\pm}_{1,1})$$

— where we regard $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ as a closed subgroup of $\mathrm{Out}(\Pi_{1,1})$ by means of the injection "$\hookrightarrow$" of the above display. Thus, the *central terminality* asserted in the statement of assertion (iii) follows immediately, in light of the above exact sequence, from assertion (ii). This completes the proof of assertion (iii).

Finally, we verify assertion (iv). It follows immediately from assertion (iii) that the image of the homomorphism $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,1})}(\mathrm{Im}(\rho_{1,1}))$ determined by the composite in the statement of assertion (iv) is *centrally terminal*. On the other hand, as is well-known, this image of $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k)$ in $\mathrm{Out}(\Pi_{1,1})$ is contained in $\mathrm{Im}(\rho_{1,1}) \subseteq \mathrm{Out}(\Pi_{1,1})$. [Indeed, recall that there exists a natural outer isomorphism $\mathrm{SL}_2(\mathbb{Z})^\wedge \xrightarrow{\sim} \Pi_{\mathcal{M}_{1,1}}$, where we write $\mathrm{SL}_2(\mathbb{Z})^\wedge$ for the profinite completion of $\mathrm{SL}_2(\mathbb{Z})$, such that the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})^\wedge$

in $\mathrm{Out}(\Pi_{1,1})$ coincides with the image of the *unique nontrivial* element of $\mathrm{Aut}_{(\mathcal{M}_{1,1})_k}((\mathcal{C}_{1,1})_k) \simeq \mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Out}(\Pi_{1,1})$.] Now assertion (iv) follows immediately. This completes the proof of assertion (iv). Q.E.D.

Remark 6.14.1. The authors hope to be able to address the issue of whether or not a similar result to the results stated in Theorem 6.13, (ii), (iii), holds for other families of pointed curves [e.g., the universal curves over moduli stacks of *hyperelliptic curves* or more general *Hurwitz stacks*] in a sequel to the present paper.

References

[André] Y. André, On a geometric description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic avatar of $\widehat{\mathrm{GT}}$, *Duke Math. J.* **119** (2003), no. **1**, 1-39.

[Hsh] Y. Hoshi, Absolute anabelian cuspidalizations of configuration spaces of proper hyperbolic curves over finite fields, *Publ. Res. Inst. Math. Sci.* **45** (2009), no. **3**, 661-744.

[NodNon] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations, *Hiroshima Math. J.* **41** (2011), no. **3**, 275-342.

[Illu] L. Illusie, An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology, *Astérisque* **279** (2002), 271-322.

[JR] M. Jarden and J. Ritter, Normal automorphisms of absolute Galois groups of $\mathfrak{p}$-adic fields. *Duke Math. J.* **47** (1980), no. **1**, 47-56.

[Knud] F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $\mathcal{M}_{g,n}$. *Math. Scand.* **52** (1983), no. **2**, 161-199.

[Mln] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, **33**, Princeton University Press, Princeton, N.J, 1980.

[CorHyp] S. Mochizuki, Correspondences on Hyperbolic Curves, *J. Pure Appl. Algebra* **131** (1998), no. **3**, 227-244.

[LocAn] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), no. **2**, 319-423.

[ExtFam] S. Mochizuki, Extending families of curves over log regular schemes, *J. Reine Angew. Math.* **511** (1999), 43-71.

[AbsHyp] S. Mochizuki, The absolute anabelian geometry of hyperbolic curves, *Galois theory and modular forms*, 77-122, Dev. Math, **11**, Kluwer Acad. Publ, Boston, MA, 2004.

[GalSct] S. Mochizuki, Galois sections in absolute anabelian geometry, *Nagoya Math. J.* **179** (2005), 17-45.

[SemiAn] S. Mochizuki, Semi-graphs of anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), no. **1**, 221-322.

[CmbGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math J.* **59** (2007), 455-479.

[AbsCsp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), no. **3**, 451-539.

[CmbCsp] S. Mochizuki, On the Combinatorial Cuspidalization of Hyperbolic Curves, *Osaka J. Math.* **47** (2010), 651-715.

[MT] S. Mochizuki and A. Tamagawa, The Algebraic and Anabelian Geometry of Configuration Spaces, *Hokkaido Math. J.* **37** (2008), no. **1**, 75-131.

[NSW] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Second edition, Grundlehren der Mathematischen Wissenschaften, **323**, Springer-Verlag, Berlin, 2008.

[NS] N. Nikolov and D. Segal, Finite index subgroups in profinite groups, *C. R. Math. Acad. Sci. Paris* **337** (2003), no. **5**, 303-308.

(Yuichiro Hoshi) *Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN*
E-mail address: `yuichiro@kurims.kyoto-u.ac.jp`

(Shinichi Mochizuki) *Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN*
E-mail address: `motizuki@kurims.kyoto-u.ac.jp`