ON THE ESSENTIAL LOGICAL STRUCTURE OF INTER-UNIVERSAL TEICHMÜLLER THEORY IN TERMS OF LOGICAL AND "$\wedge$"/LOGICAL OR "$\vee$" RELATIONS:

REPORT ON THE OCCASION OF THE PUBLICATION OF THE FOUR MAIN PAPERS ON INTER-UNIVERSAL TEICHMÜLLER THEORY

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Contents:

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Introduction

In the present paper, we give a detailed exposition of the **essential logical structure** of inter-universal Teichmüller theory in terms of elementary **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*. One important topic in this exposition is the issue of **"redundant copies"**, i.e., the issue of how the **arbitrary identification** of copies of *isomorphic mathematical objects* that appear in the various constructions of inter-universal Teichmüller theory impacts — and indeed *invalidates* — the *essential logical structure* of inter-universal Teichmüller theory. This issue continues, at least at the time of writing of the present paper, to be a focal point of **fundamental misunderstandings** and **entirely unnecessary confusion** concerning inter-universal Teichmüller theory in certain sectors of the mathematical community [cf. the discussion of Example 2.4.5].

We begin, in §1, by reporting on various *non-mathematical* aspects of the situation surrounding inter-universal Teichmüller theory, such as the issue of *"redundant copies"*. Perhaps the most central portion of this discussion of non-mathematical aspects of the situation surrounding inter-universal Teichmüller theory concerns the **long-term, historical importance** of producing *detailed, explicit, mathematically substantive*, and *readily accessible* **written documentation** of the **essential logical structure** of issues under debate [cf. §1.5].

We then proceed, in §2, to discuss *elementary aspects* of the mathematics surrounding the essential logical structure of inter-universal Teichmüller theory. Our discussion of these elementary aspects, which concerns mathematics at the *advanced undergraduate* or *beginning graduate* level and does *not* require any advanced knowledge of *anabelian geometry* or *inter-universal Teichmüller theory*, focuses on the close relationship between

· **integration** and **differentiation** on — i.e., so to speak, the *"differential geometry"* of — the **real line** [cf. §2.1, §2.2],

· the *geometry* of **adjacent closed intervals** of the **real line** and the **loops** that arise by *identifying* various closed subspaces of such closed intervals [cf. §2.3], and

· **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"* [cf. §2.4].

One *important unifying theme* that relates these seemingly disparate topics is the theme of **"carry operations"**, which appear in the various **arithmetic**, **geometric**, and **Boolean-logical** situations discussed in §2 [cf. Example 2.4.6].

The preparatory topics of §2 lead naturally to the detailed exposition of the *essential logical structure* of inter-universal Teichmüller theory given in §3. From a strictly rigorous point of view, this exposition assumes a substantial level of knowledge and understanding of the technicalities of inter-universal Teichmüller theory [which are surveyed, for instance, in [Alien]], although the essential mathematical content of most of the issues discussed may in fact be understood at the level of the elementary considerations discussed in §2. The **essential logical structure** of inter-universal Teichmüller theory may be represented **symbolically** as follows:

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots) \\ & \Longrightarrow & A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \ \ldots \ \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ \ldots \ \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \ \ldots) \\ & & \vdots \end{array}$$

[cf. the discussion of ($\wedge$($\dot{\vee}$)-Chn) in §3.10]. [Here, "$\dot{\vee}$" denotes the Boolean operator *exclusive-OR*, i.e., *"XOR"*.] Indeed, §3 is devoted, for the most part, to giving a detailed exposition of various aspects of this *symbolic representation*, such as the following:

- the **logical AND "$\wedge$'s"** in the above display may be understood as corresponding to the **Θ-link** of inter-universal Teichmüller theory and are closely related to the **multiplicative** structures of the rings that appear in the domain and codomain of the Θ-link;
- the **logical XOR "$\dot{\vee}$'s"** in the above display may be understood as corresponding to various *indeterminacies* that arise mainly from the $\mathfrak{log}$-*Kummer-correspondence*, i.e., from sequences of iterates of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which may be thought of as a device for constructing **additive** log-shells.

This appearance of *logical AND "$\wedge$'s"* and *logical XOR "$\dot{\vee}$'s"* is of interest in that it is reminiscent of the well-known description of the **"carry-addition"** operation on Teichmüller representatives of the *truncated Witt ring* $\mathbb{Z}/4\mathbb{Z}$ in terms of **Boolean addition** "$\dot{\vee}$" and **Boolean multiplication** "$\wedge$" in the *field* $\mathbb{F}_2$ and may be understood as a sort of **"Boolean intertwining"** that *mirrors*, in a *remarkable* fashion, the **"arithmetic intertwining"** between **addition** and **multiplication** in number fields and local fields which is, in some sense, the main object of study in inter-universal Teichmüller theory [cf. the discussion of Example 2.4.6, (iii); the discussion surrounding (TrHrc) in §3.10].

Section 1: General summary for non-specialists

We begin with an overall summary of *non-mathematical aspects* of the situation surrounding [IUTchI-IV], which may be of interest to both *non-mathematicians* and *mathematicians*. We also refer to [FsADT], [FKvid], [FsDss], [FsPio] for a discussion of various aspects of this situation from slightly different points of view.

§1.1. Publication of [IUTchI-IV]

The four main papers [IUTchI-IV] on inter-universal Teichmüller theory (IUT) were accepted for publication in the Publications of the Research Institute for Mathematical Sciences (PRIMS) on February 5, 2020. This was announced at an online video news conference held at Kyoto University on April 3, 2020. The four papers were subsequently published in several special volumes of PRIMS, a leading international journal in the field of mathematics with a distinguished history dating back over half a century.

The refereeing for these Special Volumes was overseen by an Editorial Board for the Special Volumes chaired by Professors Masaki Kashiwara and Akio Tamagawa. [Needless to say, as the author of these four papers, I was completely excluded from the activities of this Editorial Board for the Special Volumes.] Professor Kashiwara, a professor emeritus at RIMS, Kyoto University, is a global leader in the fields of algebraic analysis and representation theory. Professor Tamagawa, currently a professor at RIMS, Kyoto University, is a leading pioneer in the field of anabelian geometry and related research in arithmetic geometry. Here, it should be noted that, to a substantial extent,

> **inter-universal Teichmüller theory** *arose as an extension/application — developed by the author in the highly mathematically stimulating environment at RIMS, Kyoto University, over the course of roughly two decades [i.e., 1992 - 2012] — of precisely the sort of* **anabelian geometry** *that was pioneered by Tamagawa.*

It is for this reason that PRIMS stood out among mathematics journals worldwide as the most appropriate — i.e., in the sense of being by far the most [and indeed perhaps the only truly] *technically qualified* — journal for the task of refereeing and publishing the four papers [IUTchI-IV] on inter-universal Teichmüller theory.

Both Professors Kashiwara and Tamagawa have an outstandingly high international reputation, built up over distinguished careers that span several decades. It is entirely inconceivable that any refereeing process overseen by these mathematicians might be conducted relative to anything less than the highest mathematical standards, free of any inappropriate non-mathematical considerations. In an article in the Asahi Shimbun [a major Japanese newspaper] published shortly after the announcement of April 3, 2020, Professor Tamagawa is quoted as saying that he has

"100 percent confidence in the refereeing"

that was done for the four papers [IUTchI-IV].

In another article in the Asahi Shimbun [also published shortly after the announcement of April 3, 2020], Professors Shigefumi Mori, a professor emeritus at RIMS, Kyoto University, and Nobushige Kurokawa, a professor emeritus at the Tokyo Institute of Technology, express their expectations about the possibility of applying inter-universal Teichmüller theory to other unsolved problems in number theory.

In particular, the results proven in these four papers [IUTchI-IV] may now be quoted in the mathematical literature as results proven in papers that have been published in a leading international journal in the field of mathematics after undergoing an exceptionally thorough [seven and a half year long] refereeing process.

§**1.2.** <u>**Redundancy assertions of the "redundant copies school" (RCS)**</u>

Unfortunately, it has been brought to my attention that, despite the developments discussed in §1.1, **fundamental misunderstandings** concerning the mathematical content of inter-universal Teichmüller theory persist in certain sectors of the mathematical community. These misunderstandings center around a certain *oversimplification* — which is *patently flawed*, i.e., leads to an immediate contradiction — of inter-universal Teichmüller theory. This oversimplified version of inter-universal Teichmüller theory is based on assertions of **redundancy** concerning various **multiple copies** of certain mathematical objects that appear in inter-universal Teichmüller theory. In the present paper, I shall refer to the *school of thought* [i.e., in the sense of a "collection of closely interrelated ideas"] constituted by these assertions as

the **"RCS"**, i.e., *"redundant copies school [of thought]"*.

One fundamental reason for the use of this term "RCS" [i.e., "redundant copies school [of thought]"] in the present paper, as opposed to proper names of mathematicians, is to emphasize the importance of concentrating on **mathematical content**, as opposed to *non-mathematical* — i.e., such as *social*, *political*, or *psychological* — aspects or interpretations of the situation.

Thus, in a word, the central assertions of the RCS may be summarized as follows:

> Various **multiple copies** of certain mathematical objects in inter-universal Teichmüller theory are **redundant** and hence may be identified with one another. On the other hand, once one makes such **identifications**, one obtains an immediate **contradiction**.

In the present paper, I shall refer to redundancy in the sense of the assertions of the RCS as **"RCS-redundancy"**, to the identifications of RCS-redundant copies that appear in the assertions of the RCS as **"RCS-identifications"**, and to the oversimplified version of inter-universal Teichmüller theory obtained by implementing the RCS-identifications as **"RCS-IUT"**.

As discussed in [Rpt2018] [cf., especially, [Rpt2018], §18], there is *absolutely no doubt* that

> *RCS-IUT is indeed a meaningless and absurd theory that leads immediately to a contradiction.*

A more technical discussion of this contradiction, in the language of inter-universal Teichmüller theory, is given in §3.1 below, while a digested version in more elementary language of the technical discussion of §3 may be found in Example 2.4.5 below.

Rather, the *fundamental misunderstandings* underlying the RCS lie in the assertions of *RCS-redundancy*. The usual sense of the word "redundant" suggests that there should be some sort of **equivalence**, or **close logical relationship**, between the original version of the theory [i.e., IUT] and the theory obtained [i.e., RCS-IUT] by implementing the RCS-identifications of RCS-redundant objects. In fact, however,

> implementing the RCS-identifications of RCS-redundant objects **radically alters/invalidates** the **essential logical structure** of IUT

in such a fundamental way that it seems entirely unrealistic to verify any sort of "close logical relationship" between IUT and RCS-IUT.

A more technical discussion of the *three main types* of *RCS-redundancy/RCS-identification* — which we refer to as "(RC-FrÉt)", "(RC-$\mathfrak{log}$)", and "(RC-Θ)" — is given, in the language of inter-universal Teichmüller theory, in §3.2, §3.3, §3.4, below. In fact, however, the **essential mathematical content** of these three main types of RCS-redundancy/RCS-identification is **entirely elementary** and lies well within the framework of *undergraduate-level mathematics*. A discussion of this essentially elementary mathematical content is given in §2.3, §2.4 below [cf., especially, Example 2.4.5].

One important consequence of the technical considerations discussed in §3 below is the following:

> from the point of view of the **logical relationships** between various assertions of the RCS, the **most fundamental type** of RCS-redundancy is (RC-Θ).

That is to say, (RC-Θ) may be understood as the **logical cornerstone** of the various assertions of the RCS.

§1.3. Qualitative assessment of assertions of the RCS

As discussed in detail in §3.4 below [cf. also §2.3, §2.4],

> implementing the **logical cornerstone RCS-identification** of (RC-Θ) **completely invalidates** the crucial **logical AND "$\wedge$" property** satisfied by the Θ-link — a property that underlies that the **entire logical structure** of inter-universal Teichmüller theory.

In particular, understanding the issue of how the RCS treats this *fundamental conflict* between the RCS-identification of (RC-Θ) and the crucial $\wedge$*-property* of the Θ-link is central to the issue of assessing the assertions of the RCS.

In March 2018, discussions were held at RIMS with two adherents of the RCS concerning, in particular, (RC-Θ) [cf. [Rpt2018], [Dsc2018]]. Subsequent to these discussions, after a few e-mail exchanges, these two adherents of the RCS informed me via e-mail in August 2018 — in response to an e-mail that I sent to them in which I stated that I was prepared to continue discussing inter-universal Teichmüller theory with them, but that I had gotten the impression that they were not interested in continuing these discussions — that indeed they were *not interested in continuing these discussions concerning inter-universal Teichmüller theory.* In the same e-mail, I also stated that perhaps it might be more productive to continue these discussions of inter-universal Teichmüller theory via *different participants* [i.e., via *"representatives"* of the two sides] and encouraged them to suggest *possible candidates* for doing this, but they never responded to this portion of my e-mail. [Incidentally, it should be understood that I have no objection to making these e-mail messages public, but will refrain from doing so in the absence of explicit permission from the two recipients of the e-mails.]

Since March 2018, I have spent a tremendous amount of time discussing the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above with quite a number of mathematicians. Moreover, over the past two years, many mathematicians [including myself!] with whom I have been in contact have devoted a quite substantial amount of time and effort to analyzing and discussing certain 10pp. manuscripts written by adherents of the RCS — indeed to such an extent that by now, many of us can cite numerous key passages in these manuscripts by memory. More recently, one mathematician with whom I have been in contact has made a quite intensive study of the mathematical content of recent blog posts by adherents of the RCS.

Despite all of these efforts, the **only justification** for the **logical cornerstone RCS-identification** of (RC-Θ) that we [i.e., I myself, together with the many mathematicians with whom I have discussed these issues] could find either in oral explanations during the discussions of March 2018 or in subsequent written records produced by adherents of the RCS [i.e., such as the 10pp. manuscripts referred to above or various blog posts] were statements of the form

"I don't see why not".

[I continue to find it *utterly bizarre* that such justifications of the assertions of the RCS appear to be taken seriously by some professional mathematicians.] In particular, we were *unable to find any detailed mathematical discussion* by adherents of the RCS of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above. That is to say, in summary,

> the **mathematical justification** for the **"redundancy"** asserted in the logical cornerstone assertion (RC-Θ) of the RCS remains a **complete mystery** to myself, as well as to all of the mathematicians that I have consulted concerning this issue.

Put another way, the response of all of the mathematicians with whom I have had technically meaningful discussions concerning the assertions of the RCS was *completely uniform* and *unanimous*, i.e., to the effect that these assertions of the RCS were **obviously completely mathematically inaccurate/absurd**, and that they had no idea why adherents of the RCS continued to make such manifestly absurd assertions. In particular, it should be emphasized that

> *I continue to search for a professional mathematician [say, in the field of arithmetic geometry] who feels that he/she understands the mathematical content of the assertions of the RCS and is willing to discuss this mathematical content with me or other mathematicians with whom I am in contact*

[cf. the text at the beginning of [Dsc2018]].

In this context, one important observation that should be kept in mind is the following [cf. the discussion of [Rpt2018], §18]:

(UndIg) There is a *fundamental difference* between

(UndIg1) criticism of a mathematical theory that is based on a **solid, technically accurate understanding** of the content and logical structure of the theory and

(UndIg2) criticism of a mathematical theory that is based on a **fundamental ignorance** of the content and logical structure of the theory.

An elementary classical example of this sort of difference is discussed in §2.1 below.

In the case of the RCS, the lack of any thorough mathematical discussion of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned above in the various oral/written explanations set forth by adherents of the RCS demonstrates, in a definitive way, that *none of the adherents of the RCS has a solid, technically accurate understanding of the logical structure of inter-universal Teichmüller theory in its original form*, i.e., in particular, of the central role played in this logical structure by the "$\wedge$-property" of the Θ-link. Put another way, the only logically consistent explanation of this state of affairs is that the theory "RCS-IUT" that adherents of the RCS have in mind, i.e., the theory that is the object of their criticism, is simply a **completely different** — and **logically unrelated** — **theory** from the theory constituted by inter-universal Teichmüller theory in its original form.

Finally, it should be mentioned that although some people have asserted parallels between the assertions of the RCS and the *fundamental error in the first version of Wiles's proof of the Modularity Conjecture in the mid-1990's*, this analogy is **entirely inappropriate** for numerous reasons. Indeed, as is well-known, nothing even remotely close to the phenomena discussed thus far in the present §1.3 occurred in the case of the error in the first version of Wiles's proof. The fact that there was indeed a fatal error in the first version of Wiles's proof was never disputed in any way by any of the parties involved; the only issue that arose was the issue of whether or not the proof could be fixed. By contrast, *no essential errors* have been found in inter-universal Teichmüller theory, since the four preprints [IUTchI-IV] on inter-universal Teichmüller theory were released in August 2012. That is to say, in a word, the assertions of the RCS are nothing more than **meaningless, superficial misunderstandings** of inter-universal Teichmüller theory on the part of people who are clearly *not operating on the basis of a solid, technically accurate understanding of the mathematical content and essential logical structure* of inter-universal Teichmüller theory.

§**1.4.** <u>**The importance of extensive, long-term interaction**</u>

In general, the **transmission of mathematical ideas** between individuals who share a sufficient stock of **common mathematical culture** may be achieved in a relatively efficient way and in a relatively brief amount of time. Typical examples of this sort of situation in the context of interaction between professional mathematicians include

- *one-hour mathematical lectures*,
- *week-long mathematical lecture series*, and
- *informal mathematical discussions for several days to a week.*

In the context of mathematical education, typical examples include

- *written or oral mathematical examinations* and
- *mathematics competitions.*

The successful operation of each of these examples relies, in an essential way, on a **common framework of mathematical culture** that is shared by the various participants in the activity under consideration.

On the other hand, in the case of a **fundamentally new** area of research, such as inter-universal Teichmüller theory, which evolved out of research over the past quarter of a century concerning *absolute anabelian geometry*, certain types of *categories arising from arithmetic geometry*, and certain arithmetic aspects of *theta functions*, the collection of mathematicians who share such a sufficient stock of common mathematical culture tends to be relatively small in number. In particular, for most mathematicians — even many arithmetic geometers or anabelian geometers — short-term interaction of the sort that occurs in the various typical examples mentioned above is **far from sufficient** to achieve an effective transmission of mathematical ideas. That is to say, no matter how mathematically talented the participants in such platforms of interaction may be, it takes time for the participants to

· *analyze and sort out numerous mutual misunderstandings*,
· *develop effective techniques of communication* that can transcend such misunderstandings, and
· *digest and absorb new ideas and modes of thought.*

Depending on the mathematical content under consideration, as well as on the mathematical talent, mathematical background, and time constraints of the participants, this painstaking process of analysis/development/digestion/absorption may require

> *patiently sustained efforts to continue constructive, orderly mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] over a period of* **months** *or even* **years**

to reach fruition. Indeed, my experience in exposing the ideas of inter-universal Teichmüller theory to numerous mathematicians over the past decade suggests strongly that, in the case of inter-universal Teichmüller theory, it is difficult to expedite this process to the extent that it can be satisfactorily achieved in less than half a year or so.

In particular, in the case of inter-universal Teichmüller theory, a week-long session of discussions such as the discussions held at RIMS in March 2018 with two adherents of the RCS [cf. [Rpt2018], [Dsc2018]] is *far from sufficient.* This is something that I emphasized, both orally during these discussions and in e-mails to these two adherents of the RCS during the summer of 2018 subsequent to these discussions.

§**1.5.** **The historical significance of detailed, explicit, accessible records**

As was discussed in §1.3, I *continue to search* for a professional mathematician [say, in the field of arithmetic geometry] who purports to understand the **mathematical justification** for the **RCS-redundancy** asserted in the logical cornerstone assertion (RC-Θ) — i.e., in particular, who has confronted the mathematical content of the **fundamental "(RC-Θ) vs. $\wedge$-property" conflict** mentioned in §1.3 — and who is prepared to discuss this mathematical content with me or other mathematicians with whom I am in contact. Of course,

> a **detailed, explicit, mathematically substantive**, and **readily accessible written exposition**

— i.e., as an alternative to direct mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] — of the *mathematical justification* for the logical cornerstone assertion (RC-Θ) would also be quite welcome [cf. the discussion of [Rpt2014], (7)]. Moreover, in this context, it should be *emphasized* that such a detailed, explicit, mathematically substantive, and readily accessible written exposition would be of great value not only for *professional mathematicians and graduate students* who are involved with inter-universal Teichmüller theory at the **present time**, but also for scholars in the **[perhaps distant!] future**.

In general, it cannot be overemphasized that maintaining such detailed, explicit, mathematically substantive, and readily accessible written records is

of **fundamental importance** to the **development** of **mathematics**.

Indeed, as was discussed in the final portion of [Rpt2018], §3, from a **historical** point of view, it is only by maintaining such written records that the field of mathematics can avoid the sort of well-known and well-documented **confusion** that lasted for so many centuries concerning *"Fermat's Last Theorem"*. Moreover, it is fascinating to re-examine, from the point of view of a modern observer, the *intense debates* that occurred, during the time of *Galileo*, concerning the *theory of heliocentrism* or, during the time of *Einstein*, concerning the *theory of relativity*. Again, it cannot be overemphasized that

such historical re-examinations are technically possible precisely because of the existence of **detailed, explicit, mathematically substantive**, and **readily accessible written expositions** of the **logical structure** underlying the various central assertions that arose in the debate.

In this context, it should be noted that, from a historical point of view, one pattern that typically underlies the **formidable deadlocks** that tend to occur in such debates is the point of view, on the part of parties opposed to a newly developed theory, that

(CmSn) it is a **"matter of course"** or **"common sense"** — i.e., in the language of the above discussion, a matter that is so *profoundly self-evident* that any *"decent, reasonable observer"* would undoubtedly find *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of its **logical structure** to be **entirely unnecessary** — that the issues under consideration can be completely resolved within some existing, familiar framework of thought without the introduction of the newly developed theory, which is regarded as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense*.

In fact, however,

(OvDlk) ultimately, the *only meaningful technical tool* that humanity can apply to develop the **cultural infrastructure** necessary to **overcome such deadlocks** is precisely the production of *detailed, explicit, mathematically substantive, and readily accessible* **written expositions** of the **logical structure** underlying points of view that are regarded by certain parties as being a *"matter of common sense"* [cf. the discussion of (UndIg) in §1.3; the discussion of §2.1 below; the discussion of (FxRng), (FxEuc), (FxFld) in §3.1 below].

§1.6. <u>The importance of further dissemination</u>

One fundamental and frequently discussed theme in the further development of inter-universal Teichmüller theory is the issue of increasing the number of professional mathematicians who have a **solid, technically accurate understanding** of the details of inter-universal Teichmüller theory. Indeed, this issue is in some sense the central topic of [Rpt2013], [Rpt2014]. As discussed in §1.4, in order to achieve such a solid, technically accurate understanding of the theory, it is necessary to devote a substantial amount of *time* and *effort* over a period of roughly half a year to two or three years, depending on various factors. It also requires the participation of professional mathematicians or graduate students who are

· *sufficiently familiar with numerous more classical theories in arithmetic geometry* [cf. the discussion of [Alien], §4.1, (ii); [Alien], §4.4, (ii)],

· *sufficiently well motivated and enthusiastic* about studying inter-universal Teichmüller theory, and

· *sufficiently mathematically talented*, and who have a

· *sufficient amount of time* to devote to studying the theory.

As a result of quite substantial dissemination efforts not only on my part, but also on the part of many other mathematicians, the number of professional mathematicians who have achieved a sufficiently detailed understanding of inter-universal Teichmüller theory to make independent, well informed, definitive statements concerning the theory that may be confirmed by existing experts on the theory [cf. also the discussion of §1.7 below] is roughly on the order of 10. It is worth noting that although this collection of mathematicians is centered around RIMS, Kyoto University, it includes mathematicians of many nationalities and of age ranging from around 30 to around 60. One recent example demonstrated quite dramatically that it is quite possible to achieve a solid mathematical understanding of inter-universal Teichmüller theory as a graduate student by studying on one's own, outside of Japan, and with essentially zero contact with RIMS, except for a very brief period of a few months at the final stage of the student's study of inter-universal Teichmüller theory.

Finally, we observe, in the context of the discussion [cf. §1.3, §1.4, §1.5] of the assertions of the RCS, that another point that should be emphasized is that it is also of fundamental importance to

> *increase the number of professional mathematicians [say, in the field of arithmetic geometry] who have a* **solid technical understanding** *of the mathematical content of the* **assertions** *of the* **RCS**, *and who are prepared to discuss this mathematical content with members of the* **"IUT community"**

[i.e., with mathematicians who are substantially involved in mathematical research and/or dissemination activities concerning inter-universal Teichmüller theory]. Here, we note in passing that such a *solid technical understanding* of the mathematical content of the assertions of the RCS is *by no means "equivalent"* to expressions of support for the RCS on the basis of *non-mathematical* — i.e., such as *social*, *political*, or *psychological* — reasons. In this context, it should also be emphasized and understood [cf. the discussion of [Rpt2014], (7)] that *both*

- producing *detailed, explicit, mathematically substantive*, and *readily accessible written expositions* of the *mathematical justification* of assertions of the RCS [such as (RC-Θ)!], i.e., as discussed in §1.5, *and*

- increasing the number of professional mathematicians [say, in the field of arithmetic geometry] who have a *solid technical understanding* of the mathematical content of the assertions of the RCS, and who are prepared to discuss this mathematical content with members of the IUT community

are in the **interest** *not only* of the IUT community, but of the RCS *as well*.

§1.7. The notion of an "expert"

One topic that sometimes arises in the context of discussions of *dissemination* of inter-universal Teichmüller theory [i.e., as in §1.6], is the following issue:

> What is the *definition* of, or *criterion* for, being an **"expert"** on inter-universal Teichmüller theory?

In a word, it is very difficult to give a brief, definitive answer, e.g., in the form of a straightforward, easily applicable criterion, to this question. On the other hand, in this context, it should also be pointed out that the difficulties that arise in the case of inter-universal Teichmüller theory are, in fact, not so qualitatively different from the difficulties that arise in answering the *analogous question* for *mathematical theories other than inter-universal Teichmüller theory*. These difficulties arise throughout the daily life of professional mathematicians in numerous contexts, such as the following:

(Ev1) preparing suitable *exercises or examination problems to educate and evaluate students*,
(Ev2) *evaluating junior mathematicians*,
(Ev3) *refereeing/evaluating mathematical papers* for journals.

From my point of view, as the author of [IUTchI-IV], one *fundamental criterion* that I always keep in mind — not only the in case of [IUTchI-IV], but also in the case of other papers that I have written, as well as when I am involved in the various types of *evaluation procedures* (Ev1) $\sim$ (Ev3) discussed above — is the issue of

> *the extent to which the level of understanding of the mathematician in question enables the mathematician to "stand on his/her own two feet" with regard to various assertions concerning the theory, on the basis of independent, logical reasoning, without needing to be "propped up" or corrected by me or other known experts in the theory.*

I often refer to this criterion as the criterion of **autonomy** of **understanding**. Of course, from a strictly rigorous point of view, this criterion is, in some sense, not so "well-defined" and, in many contexts, difficult to apply in a straightforward fashion. On the other hand, in the past, various mathematicians involved with inter-universal Teichmüller theory have demonstrated such an *autonomous level of understanding* in the following ways:

(Atm1) the ability to detect various *minor errors/oversights* in [IUTchI-III];

(Atm2) the ability to propose *new, insightful ways of thinking* about various aspects of inter-universal Teichmüller theory;

(Atm3) the ability to propose *ways of modifying inter-universal Teichmüller theory* so as to yield *stronger or more efficient* versions of the theory;

(Atm4) the ability to produce *technically accurate oral or written expositions* of inter-universal Teichmüller theory;

(Atm5) the ability to supervise or direct *new mathematicians* — i.e., by training/educating professional mathematicians or graduate students with regard to inter-universal Teichmüller theory — who, in due time, demonstrate various of the four types of ability (Atm1) $\sim$ (Atm4) discussed above.

Of course, just as in the case of other mathematical theories, different experts demonstrate their expertise in different ways. That is to say, experts in inter-universal Teichmüller theory often demonstrate their expertise with respect to some of these five types of ability (Atm1) $\sim$ (Atm5), but not others.

In this context, it should be pointed out that one aspect of inter-universal Teichmüller theory that is currently still under development is the analogue for inter-universal Teichmüller theory of (Ev1), i.e., preparing suitable *exercises* for mathematicians studying inter-universal Teichmüller theory. This point of view may be seen in the discussion in the final portion of the Introduction to [Alien], as well as in the discussion of *"valuable pedagogical tools"* in [Rpt2018], §17. Indeed, many of the technical issues discussed in [Rpt2018], §15, may easily be reformulated as *"exercises"* or, alternatively, as *"examination problems"* for evaluating the level of understanding of mathematicians in the process of studying inter-universal Teichmüller theory.

§**1.8.** **Fabricated versions spawn fabricated dramas**

As discussed in §1.6, §1.7, by now there is a substantial number of mathematicians who have attained a *thorough, accurate, and automous understanding* of inter-universal Teichmüller theory. In each of the cases of such mathematicians that I have observed thus far, such an understanding of the theory was achieved essentially by means of a *thorough study of the original papers [IUTchI-IV]*, followed by a period of *constructive discussions with one or more existing experts* that typically lasted roughly from two to six months to sort out and resolve various "bugs" in the mathematician's understanding of the theory that arose when the mathematician studied the original papers on his/her own [cf. the discussion of §1.4].

On the other hand, there is also a growing collection of mathematicians who have a somewhat *inaccurate* and *incomplete* — and indeed often quite *superficial* — *understanding* of certain aspects of the theory. This in and of itself is not problematic — that is to say, so long as the mathematician in question maintains an *appropriate level of self-awareness* of the inaccurate and incomplete nature of his/her level of understanding of the theory — and indeed is a phenomenon that often occurs as abstract mathematical theories are disseminated.

Unfortunately, however, a certain portion of this collection of mathematicians [i.e., whose understanding of the theory is inaccurate and incomplete] have exhibited a tendency to

> assert/justify the validity of their inaccurate and incomplete understanding of the theory by means of **"reformulations"** or **"simplifications"** of the theory, which are in fact *substantively different from* and have *no directly logical relationship* to [e.g., are *by no means "equivalent"* to!] the original theory.

Indeed, the version, referred to in the present paper as "RCS-IUT" [cf. §1.2], that arises from implementing the assertions of the RCS appears to be the most famous of these fabricated versions of inter-universal Teichmüller theory. On the other hand, other, less famous fabricated versions of inter-universal Teichmüller theory have also come to my attention in recent years.

Here, before proceeding, we note that, in general, reformulations or simplifications of a mathematical theory are *not necessarily problematic*, i.e., so long as they are indeed based on a *thorough and accurate understanding* of the original theory and, moreover, can be shown to have a *direct logical relationship* to the original theory.

The authoring of fabricated versions of inter-universal Teichmüller theory appears to be motivated, to a substantial extent, by a deep desire to recast inter-universal Teichmüller theory in a *"simplified"* form that is *much closer* to the sort of mathematics with which the author of the fabricated version *is already familiar/feels comfortable*. On the other hand, this phenomenon of producing fabricated versions also appears to have been

> substantially fueled by numerous *grotesquely distorted mass media reports* and *comments* on the English-language internet that blithely paint inter-universal Teichmüller theory as a sort of *cult religion*, *fanatical political movement*, *mystical philosophy*, or *vague sketch/proposal for a mathematical theory*.

Moreover, another unfortunate tendency, of which RCS-IUT is perhaps the most egregious example, is for fabricated versions of inter-universal Teichmüller theory to

> *spawn* **lurid social/political dramas** *revolving around the content of the fabricated version, which in fact have essentially nothing to do with the content of inter-universal Teichmüller theory.*

Such lurid dramas then spawn further grotesquely distorted mass media reports and comments on the English-language internet, which then reinforce and enhance the social/political status of the fabricated version. Here, it should be emphasized that such **vicious spirals** have little [or nothing] to do with substantive mathematical content and indeed serve only to mass-produce **unnecessary confusion** that is entirely counterproductive, from the point of the view of charting a sound, sustainable course in the future development of the field of mathematics [cf. the discussion of [Alien], §4.4, (iv)].

In fact, of course, inter-universal Teichmüller theory is neither a religion, nor a political movement, nor a mystical philosophy, nor a vague sketch/proposal for a mathematical theory. Rather, it should be emphasized that

> *inter-universal Teichmüller theory is a rigorously formulated mathematical theory that has been verified countless times by quite a number of mathematicians, has undergone an exceptionally thorough seven and a half year long refereeing process, and was subsequently published in a leading international journal in the field of mathematics.*

[cf. the discussion of §1.1]. In particular, in order to avoid the sort of *vicious spirals* referred to above, it is of the utmost importance

> to concentrate, in discussions of inter-universal Teichmüller theory, on **substantive mathematical content**, as opposed to *non-mathematical* — such as *social*, *political*, or *psychological* — aspects or interpretations of the situation.

As discussed in §1.2, this is the main reason for the use of the term "RCS" in the present paper.

§**1.9.** <u>**Geographical vs. mathematical proximity**</u>

Historically, mathematical interaction between professional mathematicians relied on physical meetings or the exchange of hardcopy documents. Increasingly, however, advances in information technology have made it possible for mathematical interaction between professional mathematicians to be conducted electronically, by means of e-mail or online video communication. Of course, this does not imply that physical meetings or the exchange of hardcopy documents — especially in situations where physical meetings or the exchange of hardcopy documents do indeed **function** in a **meaningful** way, from the point of view of those involved — should necessarily be eschewed.

On the other hand, physical meetings between participants who live in distant regions requires **travel**. Moreover, travel, depending on the situations of the participants, can be a *highly taxing enterprise*. Indeed, travel, as well as lodging accommodations, typically requires the expenditure of a quite substantial amount of money, as well as **physical** and **mental effort** on the part of those involved. This *effort* can easily climb to *unmanageable* [i.e., from the point of view of certain of the participants] *proportions*, especially when *substantial cultural* — i.e., either in *mathematical* or in *non-mathematical* culture, or in *both* — *differences* are involved. The current situation involving the COVID-19 pandemic adds yet another dimension to the *reckoning*, from the point of view of the participants, of the physical and mental effort that must be expended in order to travel. As a result,

> *when, from the point of view of at least one of the key participants, the amount of* **effort**, **time**, *and/or* **money** *that must be expended to travel clearly exceeds, by a substantial margin — i.e., "$\ggg$" — the* **gain** *[i.e., relative to various mathematical or non-mathematical criteria of the key*

> *participant in question] that appears likely to be obtained from the travel under consideration, it is highly probable that the travel under consideration will end up simply not taking place.*

One "classical" example of this phenomenon "$\ggg$" is the relative scarcity of professional mathematicians in Europe or North America who travel to Japan frequently [e.g., at least once a year] or for substantial periods to time.

I have, at various times in my career, been somewhat surprised by assertions on the part of some mathematicians to the effect that travel should somehow be *forced* on mathematicians, i.e., to the effect that some sort of *coercion* may somehow *"override"* the *fundamental inequality* "$\ggg$" that exists as a result of the circumstances in which a mathematician finds him/herself in. In my experience, although this sort of coercion to travel may result in some sort of superficial influence in the very short term, *it can never succeed in the long term.* That is to say, the *fundamental circumstances* that give rise to the *fundamental inequality* "$\ggg$" can never be altered by means of such coercive measures to travel [cf. the discussion of [Rpt2014], (8)].

In this context, I was most impressed by the following two *concrete examples*, which came to my attention recently. In describing these examples, I have often used *somewhat indirect expressions*, in order to protect the privacy of the people involved.

Example 1.9.1: The insufficiency of geographical proximity. This example concerns the results obtained in a paper written in the fall of 2019 by a graduate student (St1) from country (Ct1). This student (St1) showed his paper to a prominent senior researcher (Pf1) at a university in country (Ct2) in a certain area of number theory. The education and career of this researcher (Pf1) was conducted entirely at universities in countries (Ct2), (Ct3), and (Ct4). This researcher (Pf1) informed (St1) of his very positive evaluation of the *originality* of the results obtained in the paper by (St1). Another prominent senior researcher (Pf2) in a certain area of number theory was informed by (Pf1) of the paper by (St1). This researcher (Pf2), who works at a university in country (Ct2) in close physical proximity to (Pf1), also took a generally positive position with regard to the paper by (St1). On the other hand, several months subsequent to this interaction between (Pf1) and (St1), a junior researcher (Pf3), who is originally from country (Ct5), but currently works at a university in country (Ct2) in close physical proximity to (Pf1) and (Pf2), informed student (St1) [via e-mail contacts between (Pf3) and (St1)'s advisor] that

> the results of the paper by (St1) are in fact **"well-known"** and **essentially contained in papers published in the 1990's by (Pf4)**, a prominent senior researcher in country (Ct6).

[To be more precise, in fact the results of the paper by (St1) are not entirely contained in the papers by (Pf4) in the sense that the paper by (St1) contains certain numerically explicit estimates that are not contained in the papers by (Pf4).] Country (Ct6) is in close physical proximity to country (Ct3), and in fact, one of the research advisors of researcher (Pf4), when (Pf4) was a graduate student, was a prominent researcher (Pf5) who is originally from country (Ct7), but has pursued his career as a mathematician mainly in countries (Ct3) and (Ct6). Here, it should be pointed out that (Pf1), (Pf2), and (Pf4) are very close in age, and that (Pf1) received his undergraduate education in country (Ct3) at one of the universities that played in prominent role in the career of (Pf5). The paper by student (St1) concerns mathematics that has been *studied extensively* by — and indeed forms one of the *central themes* of the research of — both (Pf1) and (Pf4), but from *very different points of view*, using *very different techniques*, since (Pf1) and (Pf4) work in substantially *different areas of number theory*. On the other hand, at *no time* during the initial several months of interaction between (Pf1), (Pf2), and (St1) was the work of (Pf4) mentioned. That is to say, (Pf1) and (Pf2) discussed the results obtained in the paper by (St1) in a way that can only be explained by the hypothesis that

> (Pf1) and (Pf2) were, at the time, **entirely unaware** of the very **close relationship** between the results obtained in the paper by (St1) and the **papers in the 1990's by (Pf4)**.

— i.e., despite the **numerous opportunities** afforded by **close physical proximity**, as well as proximity of age, for substantial interaction between (Pf1) and (Pf4). The paper by student (St1) is currently submitted for publication to a certain mathematical journal. Student (St1) recently received a referee's report for his paper, which apparently [i.e., judging from the comments made in the referee's report] was written by a mathematician working in an area of number theory close to (Pf1). This referee's report also makes *no mention* of the papers in the 1990's by (Pf4) and the fact that the results obtained in the paper by (St1) appear, with the exception of certain numerically explicit estimates, to be essentially contained in these papers of (Pf4). Finally, it should be mentioned that each official language of each of these countries (Ct1), (Ct2), (Ct3), (Ct4), (Ct5), (Ct6), (Ct7) belongs to the European branch of the Indo-European family of languages, and that at least six of the ten pairs of countries in the list (Ct2), (Ct3), (Ct4), (Ct6), (Ct7) share a common official language [i.e., with the other country in the pair].

<u>Example 1.9.2</u>**: The remarkable potency of mathematical proximity.** This example concerns the study of inter-universal Teichmüller theory by a graduate student (St2), who is originally from country (Ct8), but was enrolled in the doctoral program in mathematics at a university in country (Ct9) under the supervision of a senior faculty member (Pf6), who is originally from country (Ct10). This graduate student (St2) began his study of inter-universal Teichmüller theory as a graduate student and continued his study during his years as a graduate student with **essentially no mathematical contact** with any researchers who are significantly involved with inter-universal Teichmüller theory, except for his advisor (Pf6) and one mid-career researcher (Pf7) from country (Ct11). Here, we remark that the official language of each of these countries (Ct8), (Ct9), (Ct10), (Ct11) belongs to the European branch of the Indo-European family of languages. In particular, with the exception of a few very brief e-mail exchanges with me and a brief two-week long stay at RIMS in 2016 to participate in a workshop on IUT, this student (St2) had **essentially no mathematical contact**, prior to the fall of 2019, with **any researchers at Kyoto University** who are involved with inter-universal Teichmüller theory. Even in these circumstances,

> this student was able *not only* to achieve a **very technically sound understanding** of inter-universal Teichmüller theory on his own, by reading [IUTchI-IV] and making use of various resources, activities, and contacts within country (Ct9), *but also* to succeed, as a graduate student, in making **highly nontrivial original research contributions** to a certain mild generalization of inter-universal Teichmüller theory, as well as to certain related aspects of anabelian geometry.

My first [i.e., with the exception of a few very brief e-mail exchanges prior to this] mathematical contact with this student (St2) was in the fall of 2019. Although this student (St2) initially had some technical questions concerning aspects of inter-universal Teichmüller theory that he was unable to understand on his own, after a few relatively brief discussions in person with me, he was able to find answers to these technical questions in a relatively short period of time [roughly a month or two] without much trouble.

Section 2: Elementary mathematical aspects of "redundant copies"

The *essence* of the central mathematical assertions of the RCS revolves, perhaps somewhat remarkably, around *quite elementary considerations* that lie well within the framework of undergraduate-level mathematics. Before examining, in §3, the assertions of the RCS in the technical terminology of inter-universal Teichmüller theory, we pause to give a detailed exposition of these elementary considerations.

§**2.1.** <u>**The history of limits and integration**</u>

The classical notion of **integration** [e.g., for continuous real-valued functions on the real line], as well as the more fundamental, but closely related notion of a **limit**, have a long history, dating back [at least] to the 17-th century. Initially, these notions did not have rigorous definitions, i.e., were not *"well-defined"*, in the sense understood by mathematicians today. The lack of such rigorous definitions frequently led, up until around the end of the 19-th century, to **"contradictions"** or **"paradoxes"** in mathematical work — such as *Grandi's series*

$$\sum_{n=0}^{\infty} (-1)^n$$

— concerning integrals or limits.

Ultimately, of course, the theory of limits and integrals *evolved*, especially during the period starting from around the mid-19-th century and lasting until around the early 20-th century, to the extent that such *"contradictions/paradoxes"* could be resolved in a definitive way. This process of *evolution* involved, for instance, in the case of integration, first the introduction of the *Riemann integral* and later the introduction of the *Lebesgue integral*, which made it possible to integrate functions — such as, for instance, the *indicator function* on the real line of the *subset of rational numbers* — whose Riemann integral is not well-defined.

Here, it should be noted that at various key points during this *evolution* of the notions of limits and integration, the central *"contradictions/paradoxes"* that, at times, led to substantial criticism and confusion arose from a **solid, technically accurate understanding** of the content and logical structure of the assertions — such as, for instance, various possible approaches to computing the value of Grandi's series — at the center of these "contradictions/paradoxes". It is precisely for this reason that such criticism and confusion ultimately lead to substantive refinements in the theory that were sufficient to resolve the original "contradictions/paradoxes" in a definitive way.

> Such *constructive episodes in the history of mathematics* — which may be studied by scholars today precisely because of the existence of *detailed, explicit, mathematically substantive, and readily accessible written records*! [cf. the discussion of §1.5] — stand in *stark contrast* to [cf. the discussion of (UndIg) in §1.3] criticism of a mathematical theory that is based on a **fundamental ignorance** of the content and logical structure of the theory, such as the following *"false contradiction"* in the theory of integration, which may be observed in some students who are still in an initial stage with regard to their study of the notion of integration.

Example 2.1.1: False contradiction in the theory of integration. Consider the following computation of the definite integral of a real-valued function on the real line

$$\int_0^1 x^n \; dx \;=\; \tfrac{1}{n+1}$$

for n a positive integer. Suppose that one takes the [drastically oversimplified and manifestly absurd, from the point of view of any observer who has an accurate understanding of the theory of integration!] point of view that the *most general possible interpretation* of the equation of the above display is one in which the following three *symbols*

$$\text{“}\int_0^1\text{”}, \quad \text{“}x\text{”}, \quad \text{“}dx\text{”}$$

are allowed to be *arbitrary positive real numbers* a, b, c. Here, we note that in the case of "dx", such a substitution "$dx \mapsto c$" could be *"justified"* by quoting *conventional "ϵ-δ" treatments* of the theory of limits and integrals, in which *infinitesimals* — i.e., such as "dx" or "ϵ" — are allowed to be *arbitrary positive real numbers*, which are regarded as being *"arbitrarily small"*, and observing that any positive real number c is indeed much smaller than "most other positive real numbers" [such as $1000 \cdot c$, etc.]. On the other hand, by substituting the values $n = 1, 2, 3$, one obtains relations

$$abc = 1, \quad ab^2c = \tfrac{1}{2}, \quad ab^3c = \tfrac{1}{3}.$$

The first two of these relations imply that $b = \frac{1}{2}$ [so $b^2 = \frac{1}{4}$], while the first and third relations imply that $b^2 = \frac{1}{3} \neq \frac{1}{4}$ — a *"contradiction"*!

§2.2. Derivatives and integrals

In the context of the historical discussion of integration in §2.1, it is interesting to recall the **fundamental theorem of calculus**, i.e., the result to the effect that, roughly speaking, the operations of *integration* and *differentiation* of functions [i.e., real-valued functions on the real line satisfying suitable conditions] are *inverse* to one another. Thus, from a certain point of view,

> the *"essential information"* contained in a *function* may be understood as being **"essentially equivalent"** to the *"essential information"* contained in the *derivative* of the function

— that is to say, since one may always go back and forth at will between a function and its derivative by integrating and then differentiating. This point of view might then tempt some observers to conclude that

> any mathematical proof that relies, in an essential way, on consideration of the *derivative* of a function must be *fundamentally flawed* since any information that might possibly be extracted from the derivative of the function should already be available [cf. the *"essential equivalence"* discussed above] from the function prior to passing to the derivative, i.e., in *"contradiction"* to the essential dependence of the proof on passing to the derivative.

Alternatively, this point of view may be summarized in the following way:

> the *"essential equivalence"* discussed above implies that any usage, in a mathematical proof, of the derivative of a function is necessarily inherently **redundant** in nature.

In fact, of course, such "pseudo-mathematical reasoning" is itself **fundamentally flawed**. Two examples of well-known proofs in arithmetic geometry that depend, in a essential way, on passing to the derivative will be discussed in the final portion of §3.2 below. These examples are in fact *closely related to the mathematics that inspired* inter-universal Teichmüller theory [cf. the discussion in the final portion of §3.2 below]. One central aspect of the situations discussed in §3.2 below is the exploitation of properties of [various more abstract analogues of] the derivative of a function that **differ**, in a very **substantive, qualitative** way, from the properties of the original function. One important example of this sort of situation is the *validity/invalidity* of various **symmetry properties**. This phenomenon may be observed in the following elementary example.

Example 2.2.1: Symmetry properties of derivatives. The real-valued function

$$f(x) = x$$

on the real line is *not invariant* [i.e., *not symmetric*] with respect to translations by an arbitrary constant $c \in \mathbb{R}$. That is to say, in general, it is not the case that $f(x + c) = f(x)$. On the other hand, the derivative $f'(x) = 1$ of this function is **manifestly invariant/symmetric** with respect to such translations.

§2.3. Line segments vs. loops

By comparison to the examples given in §2.1, §2.2, the following *elementary geometric examples* are much more *closely technically related* to the assertions of the RCS concerning inter-universal Teichmüller theory.

<u>Example 2.3.1</u>: **Endpoints of an oriented line segment.**

(i) Write

$$\mathbb{I} \stackrel{\text{def}}{=} [0,1] \subseteq \mathbb{R}$$

for the *closed unit interval* [i.e., the set of nonnegative real numbers ≤ 1] in the real line $\mathbb{R}$. Thus, $\mathbb{I}$ is equipped with a natural *topology* [i.e., induced by the topology of $\mathbb{R}$], hence can be regarded as a *topological space*, indeed more specifically, as a *one-dimensional topological manifold with boundary* that is equipped with a natural *orientation* [i.e., induced by the usual orientation of $\mathbb{R}$]. Write

$$\alpha \stackrel{\text{def}}{=} \{0\}, \quad \beta \stackrel{\text{def}}{=} \{1\}$$

for the topological spaces [consisting of a single point!] determined by the two endpoints of $\mathbb{I}$. Thus, α and β are *isomorphic as topological spaces*. In certain situations that occur in category theory, it is often customary to replace a given category by a full subcategory called a *skeleton*, which is *equivalent* to the given category, but also satisfies the property that any two isomorphic objects in the skeleton are equal. This point of view of working with *skeletal categories* [i.e., categories which are their own skeletons] is motivated by the idea that *nonequal isomorphic objects* are *"redundant"*. Of course, there are indeed various situations in which nonequal isomorphic objects are redundant in the sense that working with skeletal categories, as opposed to arbitrary categories, does not result in any substantive difference in the mathematics under consideration.

(ii) On the other hand, if, in the present discussion of $\mathbb{I}$, α, β — which one may visualize as follows

$$\underset{\alpha}{\bullet} \overset{\mathbb{I}}{\text{———}>\text{———}} \underset{\beta}{\bullet}$$

— one **identifies** α and β, then one obtains a *new topological space*, that is to say, more specifically, an *oriented one-dimensional topological manifold* [whose orientation is induced by the orientation of $\mathbb{I}$]

$$\mathbb{L} \stackrel{\text{def}}{=} \mathbb{I}/\langle \alpha \sim \beta \rangle$$

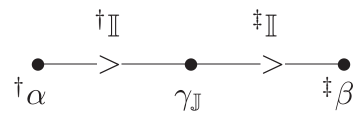
that is homeomorphic to the unit circle, i.e., may be visualized as a **loop**. Write $\gamma_{\mathbb{L}} \subseteq \mathbb{L}$ for the image of $\alpha \subseteq \mathbb{I}$, or, equivalently, $\beta \subseteq \mathbb{I}$, in $\mathbb{L}$. As is well-known from elementary topology, the topological space $\mathbb{L}$ is *structurally/qualitatively very different* from the topological space $\mathbb{I}$. For instance, whereas $\mathbb{I}$ has a *trivial fundamental group*, $\mathbb{L}$ has a *nontrivial fundamental group* [isomorphic to the additive group of integers $\mathbb{Z}$]. In particular,

> it is *by no means* the case that the fact that α and β are isomorphic as topological spaces implies a sort of **"redundancy"** to the effect that any mathematical argument involving $\mathbb{I}$ [cf. the above observation concerning fundamental groups!] is **entirely equivalent** to a corresponding mathematical argument in which α and β are **identified**, i.e., in which "$\mathbb{I}$" is *replaced* by "$\mathbb{L}$".

(iii) In this context, we observe that the [one-dimensional oriented topological manifold with boundary] $\mathbb{I}$ does **not** admit any **symmetries** that *switch* α and β. Moreover, even if one passes to the quotient $\mathbb{I} \twoheadrightarrow \mathbb{L}$, the [one-dimensional oriented topological manifold] $\mathbb{L}$ does **not** admit any **symmetries** that *reverse the orientation* of $\mathbb{L}$.

<u>Example 2.3.2</u>: **Gluing of adjacent oriented line segments.**

(i) A similar elementary geometric situation to the situation discussed in Example 2.3.1, but which is technically a bit more similar to the situation that arises in inter-universal Teichmüller theory may be given as follows. We begin with **two distinct copies** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ of $\mathbb{I}$. Thus, ${}^{\dagger}\mathbb{I}$ has *endpoints* ${}^{\dagger}\alpha$, ${}^{\dagger}\beta$ [i.e., corresponding respectively to the endpoints α, β of $\mathbb{I}$]; similarly, ${}^{\ddagger}\mathbb{I}$ has *endpoints* ${}^{\ddagger}\alpha$, ${}^{\ddagger}\beta$. We then proceed to form a *new topological space* $\mathbb{J}$ by **gluing** ${}^{\dagger}\mathbb{I}$ to ${}^{\ddagger}\mathbb{I}$ via the *unique isomorphism of topological spaces* ${}^{\dagger}\beta \xrightarrow{\sim} {}^{\ddagger}\alpha$. Thus, ${}^{\dagger}\beta$ and ${}^{\ddagger}\alpha$ are **identified** in $\mathbb{J}$. Let us write $\gamma_{\mathbb{J}}$ for the one-pointed topological space obtained by identifying ${}^{\dagger}\beta$ and ${}^{\ddagger}\alpha$. Thus, $\mathbb{J}$ may be visualized as follows:



(ii) Observe that the gluing operation that gave rise to $\mathbb{J}$ is such that we may regard ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ as *subspaces* ${}^{\dagger}\mathbb{I} \subseteq \mathbb{J}$, ${}^{\ddagger}\mathbb{I} \subseteq \mathbb{J}$ of $\mathbb{J}$. Since each of these subspaces ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ of $\mathbb{J}$ is naturally isomorphic to $\mathbb{I}$, one may take the point of view, as in the discussion of Example 2.3.1, that these two subspaces are **"redundant"** and hence should be **identified** with one another [say, via the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$] to form a *new topological space*

$$\mathbb{M} \stackrel{\text{def}}{=} \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \sim {}^{\ddagger}\mathbb{I} \rangle$$

— where we observe that the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$ determine a *natural isomorphism* of topological spaces $\mathbb{M} \xrightarrow{\sim} \mathbb{L} = \mathbb{I}/\langle \alpha \sim \beta \rangle$, with the **loop** $\mathbb{L}$ considered in Example 2.3.1. Write $\gamma_{\mathbb{M}} \subseteq \mathbb{M}$ for the image of $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ in $\mathbb{M}$. Thus, the natural isomorphism $\mathbb{M} \xrightarrow{\sim} \mathbb{L}$ maps $\gamma_{\mathbb{M}}$ isomorphically onto $\gamma_{\mathbb{L}}$. On the other hand, just as in the situation discussed in Example 2.3.1,

> it is *by no means* the case that the fact that ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ are [in fact, naturally] isomorphic as topological spaces implies a sort of **"redundancy"** to the effect that any mathematical argument involving $\mathbb{J}$ is **entirely equivalent** to a corresponding mathematical argument in which ${}^{\dagger}\mathbb{I}$ and ${}^{\ddagger}\mathbb{I}$ are **identified** [say, via the natural isomorphisms of ${}^{\dagger}\mathbb{I}$, ${}^{\dagger}\mathbb{I}$ with $\mathbb{I}$], i.e., in which "$\mathbb{J}$" is *replaced* by "$\mathbb{M}$".

Indeed, for instance, one verifies immediately, just as in the situation of Example 2.3.1, that the fundamental groups of $\mathbb{J}$ and $\mathbb{M}$ are *not isomorphic*. That is to say, just as in the situation of Example 2.3.1, the topological space $\mathbb{J}$ is *structurally/qualitatively very different* from the topological space $\mathbb{M}$.

§2.4. Logical AND "$\wedge$" vs. logical OR "$\vee$"

The essential mathematical content of the *elementary geometric examples* discussed in §2.3 may be reformulated in terms of the *symbolic logical relators* AND "$\wedge$" and OR "$\vee$". This reformulation renders the elementary geometric examples of §2.3 in a form that is even more *directly technically related* to various central aspects of the assertions of the RCS concerning inter-universal Teichmüller theory.

Example 2.4.1: "$\wedge$" vs. "$\vee$" for adjacent oriented line segments.

(i) Recall the situation discussed in Example 2.3.2. Thus, $\mathbb{J} \supseteq {}^{\dagger}\mathbb{I} \supseteq {}^{\dagger}\beta = \gamma_{\mathbb{J}} = {}^{\ddagger}\alpha \subseteq {}^{\ddagger}\mathbb{I} \subseteq \mathbb{J}$, i.e.,

(AOL1) the following condition **holds**:

$$\Big(\gamma_{\mathbb{J}} = {}^{\dagger}\beta \subseteq {}^{\dagger}\mathbb{I}\Big) \quad \wedge \quad \Big(\gamma_{\mathbb{J}} = {}^{\ddagger}\alpha \subseteq {}^{\ddagger}\mathbb{I}\Big).$$

On the other hand, if one **identifies** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$, then one obtains a topological space $\mathbb{M} \xrightarrow{\sim} \mathbb{L}$, i.e., a **loop**. Here, "$\xrightarrow{\sim}$" denotes the *natural isomorphism* discussed in Example 2.3.2, (ii). Now suppose that we are given a connected subspace

$$\gamma_{\mathbb{I}} \subseteq \mathbb{I}$$

whose image in the quotient $\mathbb{I} \twoheadrightarrow \mathbb{L} = \mathbb{I}/\langle \alpha \sim \beta \rangle$ *coincides* with $\gamma_{\mathbb{L}} \subseteq \mathbb{L}$, i.e., with the image of $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ via the composite of the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M} = \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \sim {}^{\ddagger}\mathbb{I} \rangle$ with the natural isomorphism $\mathbb{M} \xrightarrow{\sim} \mathbb{L}$. Then *observe* that

(AOL2) the following condition **holds**: $\gamma_{\mathbb{I}} \in \{\alpha, \beta\}$, i.e.,

$$\Big(\gamma_{\mathbb{I}} = \beta \subseteq \mathbb{I}\Big) \quad \vee \quad \Big(\gamma_{\mathbb{I}} = \alpha \subseteq \mathbb{I}\Big).$$

Of course,

(AOL3) the essential mathematical content discussed in this condition (AOL2) may be *formally* described as a condition involving the *AND relator* "$\wedge$":

$$\Big(\beta \in \{\alpha, \beta\}\Big) \quad \wedge \quad \Big(\alpha \in \{\alpha, \beta\}\Big).$$

But the essential mathematical content of the *OR relator* "$\vee$" statement in (AOL2) remains unchanged.

(ii) On the other hand, [unlike the case with $\gamma_{\mathbb{J}}$!]

(AOL4) the following condition does **not** hold:

$$\Big(\gamma_{\mathbb{I}} = \beta \subseteq \mathbb{I}\Big) \quad \wedge \quad \Big(\gamma_{\mathbb{I}} = \alpha \subseteq \mathbb{I}\Big).$$

That is to say, in summary, the operation of **identifying** ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ — e.g., on the grounds of **"redundancy"** [cf. the discussion of Example 2.3.2] — has the effect of passing from a situation in which

the **AND relator** *"$\wedge$" holds [cf. (AOL1)]*

to a situation in which

the **OR relator** *"$\vee$" holds [cf. (AOL2), (AOL3)],* but
the **AND relator** *"$\wedge$" does* **not** *hold [cf. (AOL4)]!*

(iii) It turns out that this phenomenon — i.e., of an identification of "redundant copies" leading to a passage from the *validity* of an "$\wedge$" relation to the *validity* of an "$\vee$" relation coupled with the *invalidity* of an "$\wedge$" relation — forms a **very precise model** of the situation that arises in the assertions of the RCS concerning inter-universal Teichmüller theory [cf. the discussion of §3.2, §3.4 below].

Example 2.4.2: Differentials on oriented line segments.

(i) In the situation of Example 2.4.1, one way to understand the gap between (AOL1) and (AOL4) — i.e., the *central issue* of whether the *AND relator* "$\wedge$" *holds* or does *not hold* — is to think in terms of the restriction to $\mathbb{I} \subseteq \mathbb{R}$ of the *coordinate function* "x" of Example 2.2.1. Indeed,

(AOD1) one may interpret (AOL4) as the statement that the coordinate functions "x" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$ do **not glue together** to form a single, well-defined $\mathbb{R}$-valued function on $\mathbb{J}$ [that is to say, since it is not clear whether the value of such a function on $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$ should be 0 or 1, i.e., such a function is *not well-defined* on $\gamma_{\mathbb{J}} \subseteq \mathbb{J}$];

(AOD2) on the other hand, (AOL1) may be interpreted as the statement that such a function [i.e., obtained by *gluing together* the coordinate functions "x" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$] **can indeed be defined** if one regards its values as being [not in a *single copy* of $\mathbb{R}$, but rather] in the set ${}^{\dagger,\ddagger}\mathbb{R}$ obtained by gluing together *two distinct copies* ${}^{\dagger}\mathbb{R}$, ${}^{\ddagger}\mathbb{R}$ of $\mathbb{R}$ by *identifying* ${}^{\dagger}1 \in {}^{\dagger}\mathbb{R}$ with ${}^{\ddagger}0 \in {}^{\ddagger}\mathbb{R}$.

(ii) On the other hand, if, instead of considering the coordinate function "x", one considers the *differential* "dx" associated to this coordinate function [cf. the discussion of Example 2.2.1], then one observes immediately that

(AOD3) the **differentials** "dx" on the two copies ${}^{\dagger}\mathbb{I}$, ${}^{\ddagger}\mathbb{I}$ that constitute $\mathbb{J}$ do indeed **glue together** to form a single, well-defined differential on $\mathbb{J}$ that, moreover, is *compatible* with the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M} = \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \ \sim \ {}^{\ddagger}\mathbb{I} \rangle$ in the sense that, as is easily verified, it arises as the pull-back, via this quotient map $\mathbb{J} \twoheadrightarrow \mathbb{M}$, of a [smooth] differential on the [smooth manifold constituted by the] loop $\mathbb{M}$.

Note, moreover, that the gluings and compatibility of (AOD3) may be achieved without considering functions or differentials valued in some sort of complicated [i.e., by comparison to $\mathbb{R}$!] set such as ${}^{\dagger,\ddagger}\mathbb{R}$.

(iii) It turns out [cf. the discussion of Example 2.4.1, (iii)] that the phenomenon discussed in (AOD3) is *closely related* to the situation that arises in inter-universal Teichmüller theory [cf. the discussion of §3.2 below].

Example 2.4.3: Representation via subgroup indices of "$\wedge$" vs. "$\vee$".

(i) Let A be an *abelian group* and $B_1, B_2 \subseteq A$ *subgroups* of A such that $B_1 \cap B_2$ has *finite index* in B_1 and B_2. Then one may define a positive rational number, which we call the *index* of B_2 relative to B_1,

$$[B_1 : B_2] \stackrel{\mathrm{def}}{=} [B_1 : B_1 \cap B_2]/[B_2 : B_1 \cap B_2] \in \mathbb{Q}_{>0}.$$

Thus, $[B_1 : B_2] \cdot [B_2 : B_1] = 1$; when $B_2 \subseteq B_1$, this notion of index coincides with the usual notion of the index of B_2 in B_1.

(ii) Let n be a positive integer ≥ 2. Consider the *diagram of group homomorphisms*

$$G_1 \xrightarrow{n\cdot} G_2 \xrightarrow{n\cdot} G_3$$

— where, for $i = 1, 2, 3$, G_i denotes a copy of [the additive group of rational integers] $\mathbb{Z}$, and the arrows are given by *multiplication* by n. For $i = 1, 2, 3$, write $G_i^{\mathbb{Q}} \stackrel{\mathrm{def}}{=} G_i \otimes_{\mathbb{Z}} \mathbb{Q}$ for the tensor product of G_i over $\mathbb{Z}$ with $\mathbb{Q}$. Then observe that this diagram induces a *diagram of group isomorphisms*

$$G_1^{\mathbb{Q}} \xrightarrow{\sim} G_2^{\mathbb{Q}} \xrightarrow{\sim} G_3^{\mathbb{Q}}$$

— i.e., in which the arrows are *isomorphisms*. Let us use these isomorphisms to identify the groups $G_i^{\mathbb{Q}}$, for $i = 1, 2, 3$, and denote the resulting group by $G_*^{\mathbb{Q}}$.

(iii) Observe that the *first diagram* of (ii) is *structurally reminiscent* of the object $\mathbb{J}$ discussed in Examples 2.3.2, 2.4.1, 2.4.2, i.e., if one regards

· the *first arrow* of the first diagram of (ii) as corresponding to ${}^{\dagger}\mathbb{I}$,
· the *second arrow* of the first diagram of (ii) as corresponding to ${}^{\ddagger}\mathbb{I}$, and
· G_1, G_2, and G_3 as corresponding to ${}^{\dagger}\alpha$, ${}^{\dagger}\beta = {}^{\ddagger}\alpha$, and ${}^{\ddagger}\beta$, respectively.

Here, we observe that G_2 appears **simultaneously** as the *codomain* of the arrow $G_1 \xrightarrow{n\cdot} G_2$ **AND** [cf. (AOL1)!] as the *domain* of the arrow $G_2 \xrightarrow{n\cdot} G_3$. Moreover, we may consider *indices* of G_1, G_2, and G_3 as *subgroups* of $G_*^{\mathbb{Q}}$

$$[G_2 : G_1] = [G_1 : G_2]^{-1} = n; \quad [G_3 : G_2] = [G_2 : G_3]^{-1} = n;$$
$$[G_3 : G_1] = [G_1 : G_3]^{-1} = n^2$$

in a **consistent** fashion, i.e., in a fashion that does **not** give rise to any **contradictions**.

(iv) On the other hand, suppose that we *delete* the *"distinct labels"* G_1, G_2, G_3 from the copies of $\mathbb{Z}$ considered in the *first diagram* of (ii). This yields a *diagram*

$$\mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z}$$

in which the *second arrow* may be regarded as a *copy* of the *first arrow*. This situation might motivate some observers to conclude that these two arrows are **"redundant"** and hence should be **identified** with one another — cf. the discussion of the quotient $\mathbb{J} \twoheadrightarrow \mathbb{M}$ in Example 2.3.2, (ii) — to form a *diagram*

$$\overset{n\cdot}{\curvearrowright} \mathbb{Z}$$

consisting of a *single copy* of $\mathbb{Z}$ and the *endomorphism* of this single copy of $\mathbb{Z}$ given by *multiplication* by n. At first glance, this operation of **identification** may appear to give rise to various **"contradictions"** in the computation of the *index*, i.e., such as

$$1 = [G_1 : G_1] = [\mathbb{Z} : \mathbb{Z}] = [G_2 : G_1] = n \geq 2$$

and so on. In fact, however, if one takes into account the *OR relator* "$\vee$" [but not the *AND relator* "$\wedge$"!] relations that one obtains upon executing the identification operation in question [cf. (AOL2), (AOL4)!], then one concludes that [after executing the identification operation in question!] each of the indices $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, *may only be computed up to multiplication by an integral power of* n, i.e., that

> each index $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, is only well-defined as **"some indeterminate element"** of $n^{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \{n^m \mid m \in \mathbb{Z}\} \subseteq \mathbb{Q}_{>0}$.

In particular, in fact there is **no contradiction**.

Example 2.4.4: Logical "$\wedge/\vee$" vs. "narrative $\wedge/\vee$". Consider the following argument concerning a *natural number* $x \in \{1, 3\}$, i.e., a natural number for which it holds that $(x = 1) \ \vee \ (x = 3)$:

(Nar1) Suppose that $x = 3$. Then it follows that $x = 3 > 2$. That is to say, we *conclude* that $x > 2$.

(Nar2) Since $(x = 1) \ \vee \ (x = 3)$, we may consider the case $x = 1$. Then, by applying the conclusion of (Nar1), we conclude that $1 = x > 2$, i.e., that $1 > 2$ — a *contradiction*!

Of course, this argument is *completely fallacious*! On the other hand, it yields a readily understood concrete example of the *absurdity* that arises when, as is in effect done in (Nar2), *logical OR "$\vee$"* is *confused* with *logical AND "$\wedge$"*! In various contexts, this sort of confusion can arise from the *ambiguity of various narrative expressions* that appear in the discussion of a mathematical argument. This sort of ambiguity can lead to a situation in which

> a **"narrative AND $\wedge$"** — i.e., the fact that in a particular narrative exposition of an argument, one performs **both** the task of considering the case "$x = 3$" [cf. (Nar1)] **and** the task of considering the case "$x = 1$" [cf. (Nar2)] — is mistakenly construed as a **logical AND "$\wedge$"**.

In a similar vein, one may consider situations in which the roles played by "$\wedge$" and "$\vee$" are *reversed*, i.e., in which a **"narrative OR $\vee$"** — i.e., the fact that in a particular narrative exposition of an argument, one's attention is concentrated

either on the task of considering one situation **or** on the task of considering another situation — is mistakenly construed as a **logical OR "$\vee$"**. Indeed, it appears that one *fundamental cause*, in the context of the *essential logical structure* of inter-universal Teichmüller theory [cf. the discussion of Example 2.4.5 below!], of the confusion on the part of some mathematicians between logical AND "$\wedge$" and logical OR "$\vee$" lies precisely in this sort of confusion between **"narrative $\wedge/\vee$"** and **logical "$\wedge/\vee$"**.

Example 2.4.5: Numerical representation of "$\wedge$" vs. "$\vee$".

(i) A slightly more sophisticated numerical representation of the difference between "$\wedge$" and "$\vee$" — which in fact *mirrors the* **essential logical structure** *of inter-universal Teichmüller theory in a very direct fashion* — may be given as follows [cf. [Alien], Example 3.11.4]. Indeed, the *essential logical flow* of inter-universal Teichmüller theory may be summarized as follows:

· one starts with from the definition of an object called the **Θ-link**;
· one then constructs a complicated apparatus that is referred to as the **multiradial representation of the Θ-pilot** [cf. [IUTchIII], Theorem 3.11];
· finally, one derives a **final numerical estimate** [cf. [IUTchIII], Corollary 3.12] in an essentially straightforward fashion from the multiradial representation of the Θ-pilot.

(ii) An elementary model of this essential logical flow may be given by means of *real numbers* $A, B \in \mathbb{R}_{>0}$ and $\epsilon, N \in \mathbb{R}$ such that $0 \leq \epsilon \leq 1$ in the following way:

· **Θ-link**:

$$\left(N \stackrel{\mathrm{def}}{=} -2B\right) \quad \wedge \quad \left(N \stackrel{\mathrm{def}}{=} -A\right);$$

· **multiradial representation of the Θ-pilot**:

$$\left(N = -2A + \epsilon\right) \quad \wedge \quad \left(N = -A\right);$$

· **final numerical estimate**:

$$-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A \leq 1.$$

Thus, the definition of the Θ-*link* and the construction of the *multiradial representation of the* Θ-*pilot* are *meaningful/nontrivial* precisely on account of the validity of the **AND relator** "$\wedge$", which is rendered possible, in the definition of the Θ-link, precisely by allowing the real numbers A, B to be [*a priori*] **distinct** real numbers — cf. (AOL1) vs. (AOL4), where we think in terms of the *correspondences*

$$B \quad \longleftrightarrow \quad {}^{\dagger}\mathbb{I}, \quad N \quad \longleftrightarrow \quad \gamma_{\mathbb{J}}, \quad A \quad \longleftrightarrow \quad {}^{\ddagger}\mathbb{I}$$

$$-2B \quad \longleftrightarrow \quad {}^{\dagger}\beta, \quad -A \quad \longleftrightarrow \quad {}^{\ddagger}\alpha, \quad -2A \quad \longleftrightarrow \quad {}^{\ddagger}\beta.$$

The passage from the *multiradial representation of the* Θ*-pilot* to the *final numerical estimate* is then *immediate/straightforward/logically transparent*.

(iii) By contrast, if, in the elementary numerical model of (ii), one *replaces* "$\wedge$" by "$\vee$", then our elementary numerical model of the logical structure of inter-universal Teichmüller theory takes the following form:

· **"$\vee$" version of Θ-link**:

$$\left(N \stackrel{\mathrm{def}}{=} -2B\right) \quad \vee \quad \left(N \stackrel{\mathrm{def}}{=} -A\right) \qquad \text{[cf. } \left(N \stackrel{\mathrm{def}}{=} -2A\right) \quad \vee \quad \left(N \stackrel{\mathrm{def}}{=} -A\right)\text{];}$$

· **"$\vee$" version of multiradial representation of the Θ-pilot**:

$$\left(N = -2A + \epsilon\right) \quad \vee \quad \left(N = -A\right);$$

· **final numerical estimate**:

$$-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A \leq 1.$$

That is to say, the use of *distinct* real numbers A, B in the definition of the "$\vee$" version of Θ-link seems **entirely superfluous** [cf. (AOL2), relative to the correspondences discussed in (ii)]. This motivates one to **identify** A and B — i.e., to suppose *"for the sake of simplicity"* that $A = B$ — which then has the effect of rendering the definition of the original "$\wedge$" version of the Θ-link *invalid/self-contradictory* [cf. (AOL4), relative to the correspondences discussed in (ii)]. Once one *identifies* A and B, i.e., once one supposes "for the sake of simplicity" that $A = B$, the passage from the "$\vee$" version of Θ-link to the resulting "$\vee$" version of the *multiradial representation of the* Θ*-pilot* then seems **entirely meaningless/devoid of any interesting content**. The passage from the resulting meaningless "$\vee$" version of the *multiradial representation of the* Θ*-pilot* to the *final numerical estimate* then seems **abrupt/mysterious/entirely unjustified**, i.e., put another way, looks as if

> one **erroneously replaced** the "$\vee$" in the meaningless "$\vee$" version of the multiradial representation of the Θ-pilot by an "$\wedge$" **without any mathematical justification whatsoever**.

It is precisely this **pernicious chain of misunderstandings** emanating from the **"redundancy"** assertions of the RCS that has given rise to a substantial amount of unnecessary confusion concerning inter-universal Teichmüller theory.

Example 2.4.6: Carry operations in arithmetic, geometry, and Boolean logic.

(i) Observe that if, in the situation of Example 2.4.3, (ii), one focuses one's attention on the subset $D_{4-i} \subseteq G_i$ in the copy of $\mathbb{Z}$ denoted by G_i, where $i = 1, 2, 3$, corresponding to $\{0, 1, 2, \ldots, n-1\} \subseteq \mathbb{Z}$, then the situation considered in Example

2.4.3, (ii), closely resembles the situation that arises in *elementary arithmetic computations* — such as *addition* and *multiplication* — involving **base n expansions of natural numbers**. That is to say, one may think of

· D_1 as the *first digit*, i.e., when $n = 10$, the **"ones digit"**,
· D_2 as the *second digit*, i.e., when $n = 10$, the **"tens digit"**, and
· D_3 as the *third digit*, i.e., when $n = 10$, the **"hundreds digit"**

of such an expansion. When performing such elementary arithmetic computations — such as addition and multiplication — involving base n expansions of natural numbers, recall that it is of *fundamental importance* to take into account the various **carry operations** that occur. In particular, we observe that

> the use of **distinct labels** for **distinct digits** plays a *fundamental role* in elementary arithmetic computations involving *base n expansions of natural numbers*

— cf. the *distinct labels* "G_1", "G_2", "G_3" in the discussion of Example 2.4.3, (ii), (iii). This situation is reminiscent of the important role played by the **distinct labels** "A", "B" in the **"Θ-link"** of Example 2.4.5, (ii). Note, morever, that

> **deletion/confusion** of these **distinct labels** for **distinct digits** has the effect of **completely invalidating**, at least in the usual "strict sense", elementary arithmetic computations involving *base n expansions of natural numbers*

— cf. the situation considered in Example 2.4.3, (iv). On the other hand, if one restricts one's attention to a **specific computational algorithm** [involving, say, addition and multiplication operations], then in fact it is often possible — i.e., depending on the content of the specific computational algorithm under consideration — to obtain

> **estimates** to the effect that applying the algorithm either *with* or *without* the use of *distinct labels* for *distinct digits* in the base n expansions of the natural numbers involved in fact yields the **same result**, up to some **explicitly bounded discrepancy**.

[For instance, when $n = 10$, any algorithm that only involves *addition* and *multiplication* operations yields the *same result modulo* 9 $(= 10 - 1)$, regardless of whether or not one uses *distinct labels* for *distinct digits* in decimal expansions of natural numbers.] Such estimates are reminiscent of the **"multiradial representation"** of Example 2.4.5, (ii).

(ii) Observe that the discussion of

· **adjacent oriented line segments** "${}^{\dagger}\mathbb{I}$", "${}^{\ddagger}\mathbb{I}$" and
· **oriented loops** "$\mathbb{L}$", "$\mathbb{M}$"

in Examples 2.3.1, 2.3.2, 2.4.1, 2.4.2 [cf. also the discussion of Examples 2.4.3, (iii); 2.4.5, (ii)] may be regarded as a sort of **limiting case** of the discussion of **base n expansions of natural numbers** in (i) above, i.e., if one

· considers the *real numbers* obtained by *dividing* the natural numbers $\leq 2n$ in the discussion of (i) above *by* n and then
· passes to the *limit* $n \to +\infty$.

That is to say, in summary,

> the **adjacency** of the *oriented line segments* "$^{\dagger}\mathbb{I}$", "$^{\ddagger}\mathbb{I}$" may be understood as a sort of **continuous, geometric representation** of the **carry operation** that appears in elementary arithmetic computations involving *base n expansions of natural numbers.*

(iii) From the point of view of discussions of the **logical structure** of mathematical arguments represented in terms of **Boolean operators** such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*, it is of interest to consider the discussion of (i) above in the **binary** case, i.e., the case $n = 2$. We begin our discussion of the binary case by recalling the following well-known facts:

· **multiplication** in the *field* $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the **Boolean operator AND "$\wedge$"**;

· **addition** in the *field* $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the **Boolean operator XOR** [i.e., *"exclusive OR"*], which we denote by "$\dot{\vee}$";

· **"carry-addition"** in the *truncated ring of Witt vectors* $\mathbb{F}_2 \times \mathbb{F}_2$ — i.e., addition of two elements of $\mathbb{F}_2 \overset{\sim}{\to} \{0\} \times \mathbb{F}_2 \subseteq \mathbb{F}_2 \times \mathbb{F}_2$, regarded as *Teichmüller representatives* in the truncated ring of Witt vectors $\mathbb{Z}/4\mathbb{Z}$, that is to say, an addition operation in which one allows for the **carry operation** [cf. the discussion of (i)!] to the *first factor* of $\mathbb{F}_2 \times \mathbb{F}_2$ — may be regarded as corresponding to an operator that we shall refer to as the **"COR", or "carry-OR", operator** and denote by "$\ddot{\vee}$"; thus, we have

$$\ddot{\vee} = (\wedge, \dot{\vee})$$

[so $0 \ \ddot{\vee} \ 0 \ = \ (0, 0)$; $1 \ \ddot{\vee} \ 0 \ = \ 0 \ \ddot{\vee} \ 1 \ = \ (0, 1)$; $1 \ \ddot{\vee} \ 1 \ = \ (1, 0)$].

These well-known facts case may be summarized as follows:

($\ddot{\vee}= \wedge \dot{\vee}$) **Conventional mixed-characteristic/"carry" addition** in $\mathbb{Z}$ considered modulo 4 — i.e., "$\ddot{\vee}$" — may be described in terms of the *"splitting"* of the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ determined by the *"Teichmüller representatives"* $0, 1 \in \mathbb{Z}$ via the equation

$$\ddot{\vee} \ = \ (\wedge, \dot{\vee})$$

— i.e., which exhibits "$\ddot{\vee}$" as an operation obtained by *"stacking"* **multiplication** *"$\wedge$" in $\mathbb{F}_2$ on top of* **addition** *"$\dot{\vee}$" in* $\mathbb{F}_2$. Here, we note that this splitting via Teichmüller representatives $0, 1 \in \mathbb{Z}$ is *compatible* with the **multiplicative** structures in $\mathbb{Z}$ and $\mathbb{F}_2$, but **not** with the **additive** structures in $\mathbb{Z}$ and $\mathbb{F}_2$. Put another way, one may think of the ring structures of $\mathbb{Z}$ and $\mathbb{F}_2$ as structures that share a *common multiplicative structure* [cf. "$\wedge$"!], but do *not* share a *common additive structure* [cf. "$\dot{\vee}$"!].

These observations will be of *fundamental importance* in the theory developed in §3 [cf., especially, the discussion at the beginning of §3.10].

(iv) Some readers may object to the comparisons and analogies between inter-universal Teichmüller theory and the **"mathematics of carry operations"** discussed in (i), (ii), and (iii) as being inappropriate on the grounds that this mathematics of carry operations is much too *"trivial"* to be of any substantive interest. On the other hand, we observe that the mathematics of carry operations discussed in (i), (ii), and (iii) is *intimately intertwined* with numerous important developments in the *history of mathematics*. With regard to the content of (i), we recall that the use of **place-value decimal numerals**, i.e., that make use of notation for *zero*, appears to date back to **Indian** texts and inscriptions from the 7-th to 9-th centuries AD. Such numerals also reached the **Arab** world during this period, but this **Hindu-Arabic numeral system** apparently only became widely used in **Europe** during the late middle ages, between the 13-th and 15-th centuries. In this context, one noteworthy development was the book *Liber Abaci* published by the Italian mathematician *Fibonacci* in 1202, which promoted the use of the Hindu-Arabic numeral system in Europe. Here, it is useful to recall that, by comparison to earlier numeral systems, such as the *Greco-Roman* and *Babylonian* systems, *place-value decimal numerals* not only facilitate elementary arithmetic computations — i.e., via the systematic use of **carry operations**, as discussed in (i)! — but also

> make it possible to express **all** — hence, in particular, *infinitely many* — **natural numbers** by means of **finitely many symbols**

— i.e., unlike earlier numerical systems, in which only *finitely many natural numbers* could be expressed using finitely many symbols. This *revolutionary importance* of the development of place-value decimal numerals in India was recognized, for instance, in writings of the 18-th century French mathematician *Laplace*. In this context, it is also of interest to observe that the discussion in (ii) of the interpretation of the discussion of (i) in terms of *line segments* is reminiscent of the discussion of the *"Euclidean algorithm"* in *Euclid's Elements*, in which numbers are often represented as *lengths of line segments*. Finally, we recall that the Boolean aspects discussed in (iii) played an important role in the [well-known!] development of *modern digital computers* in the 20-th century.

Section 3: The logical structure of inter-universal Teichmüller theory

In the present §3, we give a detailed exposition of the **essential logical structure** *of inter-universal Teichmüller theory*, with a special focus on issues related to *RCS-redudancy*. From a strictly rigorous point of view, this exposition assumes a substantial level of knowledge and understanding of the technical details of inter-universal Teichmüller theory [which are surveyed, for instance, in [Alien]]. On the other hand, in a certain *qualitative sense*, the discussion of the present §3 may in fact be understood, at a relatively elementary level, via the analogies that we discuss with the topics covered in §2. Indeed, in this context, it should be emphasized that, despite the relatively novel nature of the set-up of inter-universal Teichmüller theory,

> the *essential mathematical content* that lies at the heart of all of the issues covered in the present §3 concerns entirely well-known mathematics at the *advanced undergraduate or beginning graduate* level [i.e., the topics covered in §2].

§**3.1. One-dimensionality via identification of RCS-redundant copies**

Inter-universal Teichmüller theory concerns the explicit description of the relationship between various possible *intertwinings* — namely,

the **"Θ"-** and **"q-" intertwinings**

— between the **two underlying combinatorial/arithmetic dimensions** of a ring [cf., e.g., [Alien], §2.11; [Alien], §3.11, (v), as well as the discussion of §3.9 below]. There are many different ways of thinking about these two underlying combinatorial/arithmetic dimensions of a ring; one way to understand these two dimensions is to think of them as corresponding, respectively, to the **unit group** and **value group** of the various local fields that appear as completions of a number field at one of its valuations.

In more technical language, this sort of decomposition into unit groups and value groups may be seen in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** that appear in the **Θ-link** of inter-universal Teichmüller theory. Thus, if one thinks in terms of such $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips, then inter-universal Teichmüller theory may be summarized as follows:

(2-Dim) The main content of inter-universal Teichmüller theory is an **explicit description**, up to certain relatively mild indeterminacies, of the **Θ-intertwining** on the [*two-dimensional!*] $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** that appear in the **Θ-link** in terms of the **q-intertwining** on these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips by means of the **$\mathfrak{log}$-link** and various types of **Kummer theory** that are used to relate **Frobenius-like** and **étale-like** structures.

In particular, the essential mathematical content of inter-universal Teichmüller theory concerns an *a priori* **variable** relationship between the **two underlying combinatorial/arithmetic dimensions** of a ring.

Put another way, if one arbitrarily *"crushes"* these two dimensions into a *single dimension* — i.e., in more technical language, assumes that

(1-Dim) there exists a **consistent choice** of a **fixed relationship** between these two dimensions of (2-Dim), so that these two dimensions may, in effect, be regarded as a **single dimension**

— then one immediately obtains a *superficial contradiction.* This is not a "new" observation, but rather, in some sense, the *starting point* of inter-universal Teichmüller theory, i.e., the *initial motivation* for regarding the relationship between the two underlying combinatorial/arithmetic dimensions of a ring as being *variable*, rather than fixed.

One *central assertion* of the RCS [which appears, for instance, in certain 10pp. manuscripts written by adherents of the RCS] is to the effect that the *existence*, as in (1-Dim), of a *consistent choice of a fixed relationship* between the two dimensions of (2-Dim) may be derived as a consequence — i.e., in more succinct notation,

$$\text{(RC-FrÉt), (RC-}\mathfrak{log}\text{), (RC-}\Theta\text{)} \quad \text{“}\Longrightarrow\text{”} \quad \text{(1-Dim)}$$

— of certain **"redundant copies assertions"**, as follows:

(RC-FrÉt) the **Frobenius-like** and **étale-like** versions of objects in inter-universal Teichmüller theory are **"redundant"**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory;

(RC-$\mathfrak{log}$) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on either side of the **$\mathfrak{log}$-link** in inter-universal Teichmüller theory are **"redundant"**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory;

(RC-Θ) the **($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters** on either side of the **Θ-link** in inter-universal Teichmüller theory are **"redundant"**, i.e., may be **identified** with one another without affecting the essential logical structure of the theory.

In the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4], we discuss in more detail the **falsity** of each of these "RCS-redundancy" assertions [i.e., (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ)].

Here, it should be noted that this *falsity* of (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ) is *by no means a difficult or subtle issue*, but rather a sort of matter of *"belaboring the intuitively obvious"* from the point of view of mathematicians who are thoroughly familiar with inter-universal Teichmüller theory. Nevertheless, as discussed in [Rpt2018], §17, it is a *pedagogically meaningful exercise* to write out and discuss the details surrounding this sort of issue. Moreover, as discussed in §1.5 of the present paper, it is desirable from a *historical* point of view to produce detailed, explicit, and readily accessible written expositions concerning this sort of issue.

This state of affairs prompts the following *question*:

Why do adherents of the RCS continue to insist on asserting the validity of these assertions (RC-FrÉt), (RC-$\mathfrak{log}$), (RC-Θ)?

Any sort of complete, definitive answer to this question lies beyond the scope of the present paper. On the other hand, it seems natural to conjecture that one *fundamental motivation* for these assertions of RCS-redundancy may be found in the fact that

(FxRng) many arithmetic geometers have *only experienced* working in situations where **all schemes** — or, alternatively, **rings** — that appear in a theory are regarded as belonging to a **single category** that is **fixed** throughout the theory, hence are related to another via **ring homomorphisms**, i.e., in such a way that the **ring structure** of the various rings involved is always respected [cf. the discussion of §1.5, as well as the discussion of §3.8 below].

It is not difficult to imagine that the *heuristics and intuition* that result from years [or decades!] of immersive experience in such mathematical situations could create a *mindset* that is fertile ground for the RCS-redundancy assertions that will be discussed in detail in the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4].

Finally, we observe that this situation is, in certain respects, reminiscent of the situation that occurred in the late 19-th century with regard to such novel [i.e., at the time] notions as the notion of an **abstract manifold** or an **abstract Riemann surface**. That is to say,

(FxEuc) from the point of view of anyone for whom it is a *"matter of course"* or *"common sense"* that all geometry must take place within *some* **fixed, static ambient Euclidean space** — such as, for instance, the **complex plane** — such *abstract geometric notions* as the notion of an abstract manifold or abstract Riemann surface might come across as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense* [cf. the discussion of §1.5; the discussion of [IUTchI], §I2].

In this context, it is of interest — especially from a historical point of view — to recall that, in some sense, the *most fundamental classical example* of such an abstract geometry is the Riemann surface that arises by applying the technique of **analytic continuation** to the **complex logarithm**, i.e., which may be regarded as a sort of *distant ancestor* [cf. the discussion of [IUTchI], Remark 5.1.4; [Alien], §3.3, (vi)] of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory. Another [in fact closely related!] *fundamental classical example* of such an abstract geometry is the **hyperbolic geometry** of the **upper half-plane**, which may also be regarded as a sort of *distant ancestor* of numerous aspects of inter-universal Teichmüller theory [cf. (InfH) and Example 3.3.2 in §3.3 below, as well as the discussion of [IUTchI], Remark 6.12.3, (iii); [IUTchIII], Remark 2.3.3, (ix), (x)].

Another *historically important* instance of this sort of situation may be seen in the introduction, in the early 19-th century, of **Galois groups** — i.e., of [finite] automorphism groups of **abstract fields** — as a tool for investigating the roots of polynomial equations. That is to say,

(FxFld) until the advent of *Galois groups/abstract fields*, the issue of investigating the roots of polynomial equations was always regarded — again as a *"matter of course"* or *"common sense"* — as an issue of investigating various *"exotic numbers"* inside *some* **fixed, static ambient field** such as the field of complex numbers; moreover, from this more classical "common sense" point of view, the idea of working with *automorphisms of abstract fields* — i.e., fields that are *not constrained* [since such constraints would rule out the existence of nontrivial automorphisms!] to be treated as subsets of some fixed, static ambient field — might come across as *deeply disturbing* and *unlikely to be of use in any substantive mathematical sense* [cf. the discussion of §1.5].

On the other hand, from the point of view of inter-universal Teichmüller theory, this *radical transition*

$$\textbf{roots as concrete numbers} \quad \rightsquigarrow \quad \textbf{Galois groups/abstract fields}$$

that occurred in the early 19-th century may be regarded as a sort of *distant ancestor* of the *transition*

$$\textbf{Galois groups/abstract fields} \quad \rightsquigarrow \quad \textbf{abstract groups/anabelian algorithms}$$

that occurs in inter-universal Teichmüller theory [cf. the discussion at the beginning of §3.2 below; the discussion of §3.8 below; the discussion of the final portion of [Alien], §4.4, (i)].

§**3.2. RCS-redundancy of Frobenius-like/étale-like versions of objects**

We begin by recalling that ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters — i.e., lattice points in the log-theta-lattice — give rise to both *Frobenius-like* and *étale-like* objects. Whereas the datum of a *Frobenius-like* object *depends*, *a priori*, on the coordinates "(n, m)" of the ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater from which it arises, *étale-like* objects satisfy various **[horizontal/vertical] coricity** properties to the effect that they map *isomorphically* to corresponding objects in a *vertically* [in the case of *vertical coricity*] or *horizontally* [in the case of *horizontal coricity*] neighboring ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater of the log-theta-lattice [cf., e.g., the discussion of [Alien], §2.7, (ii), (iii), (iv); [Alien], §2.8, $2^{\mathrm{Fr/\acute{e}t}}$; [Alien], §3.3, (ii), (vi), (vii); [Rpt2018], §15]. Here, we recall that *étale-like* objects correspond, for the most part, to

> **arithmetic fundamental groups** — such as, for instance, the *étale fundamental group* "$\pi_1(X)$" of a hyperbolic curve X over a number field or mixed characteristic local field

— or, more generally, to objects that may be *reconstructed* from such arithmetic fundamental groups, so long as the object is regarded as being *equipped with auxiliary data* consisting of the *arithmetic fundamental group* from which it was reconstructed, together with the *reconstruction algorithm* that was applied to reconstruct the object. Here, we recall that, in this context, it is of *fundamental importance* that these arithmetic fundamental groups be treated simply as *abstract topological groups* [cf. the discussion of §3.8 below for more details]. **Étale-like** objects also satisfy a

> *crucial* **symmetry** *property* with respect to **permutation** of **adjacent vertical lines** of the log-theta-lattice

[cf. [Alien], §3.2; the discussion surrounding Fig. 3.12 in [Alien], §3.6, (i)]. That is to say, in summary,

> the crucial **coricity/symmetry** properties satisfied by **étale-like** objects — which are, in essence, a formal consequence of treating the arithmetic fundamental groups that appear as **abstract topological groups** [cf. the discussion of §3.8 below for more details] — play a central role in the **multiradial algorithms** of inter-universal Teichmüller theory [i.e., [IUTchIII], Theorem 3.11] and are **not** satisfied by **Frobenius-like** objects

— cf., e.g., the discussion of [Alien], §2.7, (iii); [Alien], §3.1, (iii); [Alien], Example 3.2.2; [Rpt2018], §15, (LbΘ), (Lb$\mathfrak{log}$), (LbMn), (EtFr), (EtΘ), (Et$\mathfrak{log}$), (EtMn).

On the other hand, once one implements the **RCS-identifications** discussed in (RC-$\mathfrak{log}$), (RC-Θ), there is, in effect, **"only one"** ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater in the log-theta-lattice, so all issues of determining relationships between corresponding objects in ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters at *distinct coordinates "(n, m)"* of the log-theta-lattice appear, at first glance, to have been *"trivially resolved"*. Put another way,

> once one implements the **RCS-identifications** of (RC-$\mathfrak{log}$), (RC-Θ), even *Frobenius-like* objects appear, at first glance, to satisfy **all possible coricity/symmetry properties**, i.e., at a more symbolic level,

$$\text{(RC-}\mathfrak{log}\text{), (RC-}\Theta\text{)} \quad \text{“}\Longrightarrow\text{”} \quad \text{(RC-FrÉt).}$$

In particular, the assertions of the RCS discussed in §3.1 and the present §3.2 may be summarized, at a symbolic level, as follows:

$$\text{(RC-}\mathfrak{log}\text{), (RC-}\Theta\text{)} \quad \text{“}\Longrightarrow\text{”} \quad \text{(RC-FrÉt), (RC-}\mathfrak{log}\text{), (RC-}\Theta\text{)} \quad \text{“}\Longrightarrow\text{”} \quad \text{(1-Dim).}$$

In fact, however, the RCS-identifications of (RC-$\mathfrak{log}$), (RC-Θ) do *not* resolve such issues [i.e., of relating corresponding objects in ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters at *distinct coordinates "(n,m)"* of the log-theta-lattice] *at all* [cf. the discussion of *symmetries* in Example 2.3.1, (iii)!], but rather merely have the effect of

> *translating/reformulating* such issues of relating corresponding objects in ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters at *distinct coordinates "(n,m)"* of the log-theta-lattice into issues of tracking the effect on objects in ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters as one **moves along the paths** constituted by various composites of Θ- and $\mathfrak{log}$-links.

On the other hand, at a *purely formal level*,

> the discussion given above of the **falsity** of (RC-FrÉt) — i.e., as a consequence of the crucial **coricity/symmetry** properties discussed above — is, in some sense, predicated on the **falsity** of (RC-$\mathfrak{log}$), (RC-Θ).

This *falsity* of (RC-$\mathfrak{log}$), (RC-Θ) will be discussed in detail in §3.3, §3.4, below.

In this context, it is useful to observe that the situation surrounding the **Θ-link** and (RC-Θ), (RC-FrÉt) (respectively, the **$\mathfrak{log}$-link** and (RC-$\mathfrak{log}$), (RC-FrÉt)) is **structurally reminiscent** of the object $\mathbb{J}$ discussed in Examples 2.3.2, 2.4.1, 2.4.2 [cf. also the correspondences discussed in Example 2.4.5, (ii); the discussion of [IUTchIII], Remark 1.2.2, (vi), (vii)], i.e., if one regards

(StR1) the *domain* of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\dagger}\mathbb{I}$,
(StR2) the *codomain* of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\ddagger}\mathbb{I}$,
(StR3) the *gluing data* — i.e., a certain *$\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip* (respectively, *$\mathcal{F}$-prime-strip*) — that arises from the *domain* ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to ${}^{\dagger}\beta = \gamma_{\mathbb{J}}$,
(StR4) the *gluing data* — i.e., a certain *$\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip* (respectively, *$\mathcal{F}$-prime-strip*) — that arises from the *codomain* ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater of the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to $\gamma_{\mathbb{J}} = {}^{\ddagger}\alpha$,
(StR5) the *étale-like* objects that are *coric* with respect to the Θ- (respectively, $\mathfrak{log}$-) *link* as corresponding to the *glued differential* discussed in (AOD3), and
(StR6) the *RCS-identification* of (RC-Θ) (respectively, (RC-$\mathfrak{log}$)) as corresponding to the operation of passing to the quotient

$$\mathbb{J} \quad \twoheadrightarrow \quad \mathbb{M} \; = \; \mathbb{J}/\langle {}^{\dagger}\mathbb{I} \; \sim \; {}^{\ddagger}\mathbb{I} \rangle \quad \stackrel{\sim}{\to} \quad \mathbb{L} \; = \; \mathbb{I}/\langle \alpha \sim \beta \rangle.$$

This **strong structural similarity** will play an important role in the discussion of §3.3, §3.4, below.

Finally, we observe that the portion, i.e., (StR5), of this *strong structural similarity* involving the *glued differential* discussed in (AOD3) is particularly of interest in the context of the discussion of [Alien], §2. That is to say, as discussed in the first paragraph of [Alien], §2.6, **étale-like** objects in inter-universal Teichmüller theory play an analogous role to the role played by **tangent bundles/sheaves of differentials** in the

· special case of the *invariance of the height under isogenies between abelian varieties* [due to Faltings] discussed in [Alien], §2.3, §2.4 [cf. also [Rpt2018], §16, (DiIsm)], as well as in the

· discussion of *differentiation* of *p-adic liftings of the Frobenius morphism* given in [Alien], §2.5.

The *efficacy* of the technique of considering induced maps on *differentials* in the various examples discussed in [Alien], §2.3, §2.4, §2.5, is also notable in the context of the discussion of the *fundamental theorem of calculus* in §2.2, as well as in the context of (RC-FrÉt) and (StR5).

§**3.3.** <u>**RCS-redundant copies in the domain/codomain of the $\mathfrak{log}$-link**</u>

The **Θ-link** of inter-universal Teichmüller theory is defined, in the style of *classical complex Teichmüller theory* [cf. Example 3.3.1 below; [IUTchI], Remark 3.9.3], as a **deformation** of the **ring structure** in a ($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theater that depends, in an essential way, on the **splitting** into **unit groups** and **value groups** of the various localizations of the number field involved. On the other hand, the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory [i.e., in essence, the p-adic logarithm at primes of the number field of residue characteristic p] has the effect of **juggling/rotating** these unit groups and value groups, e.g., by mapping *units* to *non-units* [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. In particular,

> there is **no natural way** to relate the *two* Θ*-links* [i.e., the *two horizontal arrows* in the following diagram] that emanate from the *domain* and *codomain* of the $\mathfrak{log}$*-link* [i.e., the *left-hand vertical arrow* in the following diagram]
>
> $$\begin{tikzcd} \bullet \arrow[r, "\Theta"] & \bullet \\ \bullet \arrow[u, "\mathfrak{log}"] \arrow[r, "\Theta"] & \bullet \arrow[u, dotted, dash, "??"] \end{tikzcd}$$

— that is to say, there is *no natural candidate* for "??" [i.e., such as, for instance, an *isomorphism* or the $\mathfrak{log}$*-link* between the two bullets "•" on the *right-hand side* of the diagram] that makes the diagram *commute*. Indeed, it is an easy exercise, in the style of [Rpt2018], §15, (LbΘ), (Lb$\mathfrak{log}$), (LbMn), to show that *neither* of these candidates for "??" [i.e., an *isomorphism* or the $\mathfrak{log}$*-link*] yields a commutative diagram.

Thus, in summary, any *identification* of the *domain* and *codomain* of the $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)!] yields a situation in which the *local splittings* into *unit groups* and *value groups* of the resulting identified "•'s" are *no longer well-defined*. In particular,

> any such **identification** of the *domain* and *codomain* of the $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)!] yields a situation in which the **Θ-link** is **not well-defined**

— i.e., a situation in which *the apparatus of inter-universal Teichmüller theory completely ceases to function* — cf. the discussion of the *definition of the* Θ*-link* in the latter half of [Alien], §3.3, (ii). This discussion may be summarized, at a symbolic level, as follows:

$$\textit{definition} \text{ of the } \Theta\text{-link} \quad \Longrightarrow \quad \textit{falsity} \text{ of (RC-}\mathfrak{log}\text{)}.$$

Next, we observe [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)] that the *non-existence* of a *solution* for "??" in the above diagram [i.e., that makes the diagram *commute*] amounts, at a *structural* level, to essentially the *same* phenomenon as the **incompatibility** of the **dilations** that appear in *classical complex Teichmüller theory* with **multiplication** by **non-real roots of unity** [cf. Example 3.3.1 below]. Write $\mathbb{R}$, $\mathbb{C}$, respectively, for the topological fields of real and complex numbers. Then as observed in the discussion of the latter half of [Alien], §3.3, (ii) [cf., especially, the discussion surrounding [Alien], Fig. 3.6]:

(InfH) this *structural similarity* is consistent with the *analogy* discussed in *loc. cit.* between

- the **"infinite H"** portion of the log-theta-lattice consisting of the *two vertical lines* [i.e., of $\mathfrak{log}$-links] on either side of a *horizontal arrow* [i.e., a Θ-link] of the log-theta-lattice and

- the elementary theory surrounding the **bijection**

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL_2^+(\mathbb{R}) / \mathbb{C}^\times & \overset{\sim}{\to} & [0,1) \\ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) & \mapsto & \frac{\lambda - 1}{\lambda + 1} \end{array}$$

— where $\lambda \in \mathbb{R}_{\geq 1}$; $GL_2^+(\mathbb{R})$ denotes the group of 2×2 real matrices of positive determinant; $\mathbb{C}^\times$ denotes the multiplicative group of $\mathbb{C}$, which we regard as a subgroup of $GL_2^+(\mathbb{R})$ via the assignment $a + ib \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, for $a, b \in \mathbb{R}$ such that $(a, b) \neq (0, 0)$; the domain of the bijection is the set of double cosets.

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & & \bullet & & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & \supseteq & \bullet & \xrightarrow{\Theta} & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
\dots & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \bullet & \xrightarrow{\Theta} & \dots & & \bullet & & \bullet \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}$$

That is to say,

- the **dilation** $\left(\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}\right)$ — cf. the *dilations* that appear in classical complex Teichmüller theory, i.e., as reviewed in Example 3.3.1 below — corresponds to the **Θ-link** portion of an *"infinite H"* [cf. Example 3.3.2, (iii), below], while

- the two copies of the group of **toral rotations** "$\mathbb{C}^{\times}$" [e.g., by roots of unity in $\mathbb{C}^{\times}$] on either side of "$GL_2^+(\mathbb{R})$" — which may be thought of as a representation of the *holomorphic structures* in the domain and codomain of the *dilation* $\left(\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}\right)$ [cf. the discussion of Example 3.3.1 below] — correspond, respectively, to the two *vertical lines* of $\mathfrak{log}$**-links** in the *"infinite H"* on either side of the Θ-link [cf. the discussion of Example 3.3.2, (iv), below].

<u>Example 3.3.1</u>: **Classical complex Teichmüller theory.** Let $\lambda \in \mathbb{R}_{>1}$. Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory

$$\begin{array}{ccccccc} & \Lambda : \mathbb{C} & \to & \mathbb{C} & & & \\ \mathbb{C} \ni z = x + iy & \mapsto & \zeta & = & \xi + i\eta \stackrel{\mathrm{def}}{=} x + \lambda \cdot iy & \in & \mathbb{C} \end{array}$$

— where $x, y \in \mathbb{R}$. Let $n \geq 2$ be an integer, ω a *primitive n-th root of unity*. Write $(\omega \in)\ \mu_n \subseteq \mathbb{C}$ for the group of n-th roots of unity. Then *observe* that

> if $n \geq 3$, then there does *not* exist $\omega' \in \mu_n$ such that $\Lambda(\omega \cdot z) = \omega' \cdot \Lambda(z)$ for all $z \in \mathbb{C}$.

[Indeed, this *observation* follows immediately from the fact that if $n \geq 3$, then $\omega \notin \mathbb{R}$.] That is to say, in words,

> Λ is **not compatible** with multiplication by μ_n unless $n = 2$ [in which case $\omega = -1$].

This *incompatibility* with **"indeterminacies"** arising from multiplication by μ_n, for $n \geq 3$, may be understood as one fundamental reason for the *special role* played by **square differentials** [i.e., as opposed to n-th power differentials, for $n \geq 3$] in classical complex Teichmüller theory [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)].

<u>Example 3.3.2</u>: **The Jacobi identity for the classical theta function.**

(i) Write $z = x + iy$ for the standard coordinate on the **upper half-plane** $\mathfrak{H} \stackrel{\mathrm{def}}{=} \{z = x + iy \in \mathbb{C} \mid y > 0\}$. Recall the **theta function** on $\mathfrak{H}$

$$\Theta(q) \stackrel{\mathrm{def}}{=} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}.$$

— where we write $q \stackrel{\text{def}}{=} e^{2\pi iz}$. Restricting to the *imaginary axis* [i.e., $x = 0$] yields a function

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}.$$

— where we write $t \stackrel{\text{def}}{=} y$.

(ii) Next, let us observe that

$$\iota \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})$$

maps $z \mapsto -z^{-1}$, hence $iy \mapsto iy^{-1}$, i.e., $t \mapsto t^{-1}$, while, for $\lambda \in \mathbb{R}_{\geq 1}$,

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})$$

maps $z \mapsto \lambda \cdot z$, hence $iy \mapsto i\lambda \cdot y$, i.e., $t \mapsto \lambda \cdot t$.

(iii) Next, we observe the following:

- As $t \to +\infty$, the terms in the series for $\theta(t)$ are **rapidly decreasing**, and $\theta(t) \to +0$. In particular, the series for $\theta(t)$ is relatively **easy to compute**.
- As $t \to +0$, the terms in the series for $\theta(t)$ **decrease very slowly**, and $\theta(t) \to +\infty$. In particular, the series for $\theta(t)$ is **very difficult to compute**.

Thus, in summary, the *"flow/dilation"* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$ along the *imaginary axis* may be regarded as a sort of *"link"*, in the context of the **theta function** $\theta(t)$, between *small values* [i.e., $\theta(t) \to +0$ as $t \to +\infty$] and *large values* [i.e., $\theta(t) \to +iy$ as $t \to +0$]. That is to say, this flow/dilation along the imaginary axis behaves in a way that is *strongly reminiscent* of the **Θ-link** of inter-universal Teichmüller theory [cf. the discussion of (InfH)].

(iv) The **Jacobi identity** for the **theta function** $\theta(t)$

$$\theta(t) \ = t^{-\frac{1}{2}} \cdot \theta(t^{-1})$$

allows one to analyze the behavior of $\theta(t)$ as $t \to +0$, which is *very difficult to compute* [cf. (iii)], in terms of the behavior of $\theta(t)$ as $t \to +\infty$, which is relatively *easy to compute* [cf. (iii)] — cf. the discussion of the *Jacobi identity* in [Pano], §3, §4; [Alien], §4.1, (i). Observe that this identity may be understood as a sort of **invariance** with respect to ι [cf. (ii)], up to a certain *easily computed factor* [i.e., $t^{-\frac{1}{2}}$]. Note that ι *"juggles"*, or *"rotates/permutes"*, the two dimensions of $\mathbb{R}^2$. This aspect of ι is strongly reminiscent of the **$\mathfrak{log}$-link** of inter-universal Teichmüller theory, which *"juggles"*, or *"rotates/permutes"*, the two underlying dimensions of the ring structures in a vertical column of the log-theta-lattice [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. By contrast, we note that the theta function $\theta(t)$ does **not** satisfy any interesting properties of **invariance** with respect to the **dilations** $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Finally, we recall that, in any *vertical line* of $\mathfrak{log}$-links in the log-theta-lattice,

· the *discrepancy* between the *[holomorphic] Frobenius-like* copies of objects on either side of a $\mathfrak{log}$-link [cf. (RC-$\mathfrak{log}$)], as well as

· the *discrepancy* between *[holomorphic] Frobenius-like* copies of objects and *[holomorphic] étale-like* copies of objects [cf. (RC-FrÉt)],

may be understood as the extent to which the diagram of arrows that constitutes the **$\mathfrak{log}$-Kummer-correspondence** associated to this vertical line of $\mathfrak{log}$-links **fails to commute**.

This failure to commute may be *estimated* by means of the **indeterminacy (Ind3)**, i.e., by interpreting this failure to commute as a sort of **"upper semi-commutativity"**. This indeterminacy (Ind3) is *highly nontrivial* and, in particular, gives rise to the **inequality** that appears in the final computation of log-volumes in inter-universal Teichmüller theory [cf. [IUTchIV], Corollary 3.12]. In this context, it is important to recall that the theory surrounding this indeterminacy (Ind3) depends, in an essential way, on the *absolute anabelian geometry* of [AbsTopIII], §1, i.e., which allows one to reconstruct a hyperbolic curve X over a number field or mixed characteristic local field from the *abstract profinite group* determined by the étale fundamental group $\pi_1(X)$ of the curve. That is to say, in summary, this *absolute anabelian geometry* allows one to show that

> the **discrepancies** between the various *[holomorphic] Frobenius-like* and *[holomorphic] étale-like* copies of objects in a vertical line of $\mathfrak{log}$-links [cf. (RC-$\mathfrak{log}$), (RC-FrÉt)] in the log-theta-lattice are **"bounded by"** the [relatively mild] indeterminacy (Ind3).

On the other hand, this *absolute anabelian geometry* most certainly does *not* imply that these discrepancies are trivial/non-existent, i.e., as asserted in (RC-$\mathfrak{log}$), (RC-FrÉt) — cf. the discussion of the **falsity** of (RC-$\mathfrak{log}$), (RC-FrÉt) in §3.2 and the present §3.3.

§3.4. RCS-redundant copies in the domain/codomain of the Θ-link

The **Θ-link** of inter-universal Teichmüller theory

$$\bullet \quad \xrightarrow{\Theta} \quad \bullet$$

is defined as a **gluing** between the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater "•" in the *domain* of the arrow and the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater "•" in the *codomain* of the arrow along $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* "$*$" that arise from the **Θ-pilot object** "Θ-$\mathfrak{plt}$" in the *domain* and the ***q*-pilot object** "$\mathfrak{q}$-$\mathfrak{plt}$" in the *codomain*. Here, it is important to note that this gluing is obtained by regarding these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips "$*$" as being known *only up to isomorphism*. This point of view, i.e., of regarding these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips "$*$" as being known only up to isomorphism, is implemented formally by taking the gluing to be the *full poly-isomorphism* — i.e., the set of all isomorphisms — between the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips arising from the *domain* and *codomain* of the Θ-link. Here, we recall that

· $\mathfrak{q}$-$\mathfrak{plt}$ essentially amounts to the arithmetic line bundle determined by [the ideal generated by] some $2l$-th root $\underline{\underline{q}}$ of the q-parameters at the valuations $\in \underline{\mathbb{V}}^{\mathrm{bad}}$, while

- $\Theta\text{-}\mathfrak{plt}$ essentially amounts to the collection of arithmetic line bundles determined by [the ideals generated by] the collection $\{\underline{\underline{q}}^{j^2}\}$, as j ranges over the integers $1, \ldots, l^* \stackrel{\mathrm{def}}{=} \frac{l-1}{2}$ [where l is the prime number that appears in the *initial* Θ-*data* under consideration].

Also, we recall that each ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater "$\bullet$" gives rise to an associated model "$\mathfrak{Ring}$" of the ring/scheme theory surrounding the elliptic curve under consideration. In the following discussion, we shall write

- ${}^\dagger\bullet$ for the "$\bullet$" in the *domain* of the Θ-link,
- ${}^\ddagger\bullet$ for the "$\bullet$" in the *codomain* of the Θ-link,
- $\square$ for an *arbitrary element* of the set consisting of "$\dagger$", "$\ddagger$", and the "*empty symbol*" [i.e., no symbol at all],
- ${}^\square\Theta\text{-}\mathfrak{plt} \in {}^\square\mathfrak{Ring}$ for the Θ-pilot arising from the collection "$\{{}^\square\underline{\underline{q}}^{j^2}\}$" that appears in the model of ring/scheme theory associated to ${}^\square\bullet$, and
- ${}^\square\mathfrak{q}\text{-}\mathfrak{plt} \in {}^\square\mathfrak{Ring}$ for the q-pilot arising from the "${}^\square\underline{\underline{q}}$" that appears in the model of ring/scheme theory associated to ${}^\square\bullet$.

Finally, we recall that since, for $j \neq 1$, the *valuation* [at each valuation $\in \underline{\mathbb{V}}^{\mathrm{bad}}$] of $\underline{\underline{q}}^{j^2}$ *differs* from that of $\underline{\underline{q}}$, the arithmetic degrees of the line bundles constituted by $\mathfrak{q}\text{-}\mathfrak{plt}$ and $\Theta\text{-}\mathfrak{plt}$ *differ*.

Thus, at a more formal level, the above description of the *gluing* that constitutes the Θ-link may be summarized as follows:

$$ {}^\dagger\mathfrak{Ring} \ni {}^\dagger\Theta\text{-}\mathfrak{plt} \ \leftarrow: \ * \ :\rightarrow \ {}^\ddagger\mathfrak{q}\text{-}\mathfrak{plt} \in {}^\ddagger\mathfrak{Ring} $$

$$ \mathfrak{Ring} \ni \mathfrak{q}\text{-}\mathfrak{plt} \neq \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring} $$

[where "$\leftarrow:$" and "$:\rightarrow$" denote the assignments that consitute the *gluing* discussed above].

In this context, we note the following **fundamental observation**, which underlies the **entire logical structure** of inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 3.12.2, ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$); [Alien], §3.11, (iv)]:

(AOΘ1) the following condition **holds**:

$$ \left(* \ :\rightarrow \ {}^\dagger\Theta\text{-}\mathfrak{plt} \in {}^\dagger\mathfrak{Ring} \right) \ \wedge \ \left(* \ :\rightarrow \ {}^\ddagger\mathfrak{q}\text{-}\mathfrak{plt} \in {}^\ddagger\mathfrak{Ring} \right). $$

By contrast, if one simply **deletes** the **distinct labels** "$\dagger$", "$\ddagger$" [cf. (RC-Θ)!], then

(AOΘ2) the following condition **holds**:

$$ \left(* \ :\rightarrow \ \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring} \right) \ \vee \ \left(* \ :\rightarrow \ \mathfrak{q}\text{-}\mathfrak{plt} \in \mathfrak{Ring} \right). $$

Of course,

(AOΘ3) the essential mathematical content discussed in this condition (AOΘ2) may be *formally* described as a condition involving the *AND relator* "$\wedge$":

$$\left(\mathfrak{q}\text{-}\mathfrak{plt} \in \{\mathfrak{q}\text{-}\mathfrak{plt}, \Theta\text{-}\mathfrak{plt}\}\right) \quad \wedge \quad \left(\Theta\text{-}\mathfrak{plt} \in \{\mathfrak{q}\text{-}\mathfrak{plt}, \Theta\text{-}\mathfrak{plt}\}\right).$$

On the other hand, precisely as a *consequence* of the fact [discussed above] that $\mathfrak{Ring} \ni \mathfrak{q}\text{-}\mathfrak{plt} \neq \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring}$,

(AOΘ4) the following condition does **not** hold:

$$\left(* \ :\to \ \Theta\text{-}\mathfrak{plt} \in \mathfrak{Ring}\right) \ \wedge \ \left(* \ :\to \ \mathfrak{q}\text{-}\mathfrak{plt} \in \mathfrak{Ring}\right).$$

That is to say, the operation of **identifying** $^{\dagger}\bullet$, $^{\ddagger}\bullet$ [hence also $^{\dagger}\mathfrak{Ring}$, $^{\ddagger}\mathfrak{Ring}$] — e.g., on the grounds of **"redundancy"** [i.e., as asserted in (RC-Θ)!] — by **deleting** the **distinct labels** "†", "‡" has the effect of passing from a situation in which

the **AND relator** *"$\wedge$" holds [cf. (AOΘ1)]*

to a situation in which

the **OR relator** *"$\vee$" holds [cf. (AOΘ2), (AOΘ3)],* but
the **AND relator** *"$\wedge$" does* **not** *hold [cf. (AOΘ4)]!*

In particular, relative to the *correspondences*

$$^{\dagger}\bullet, {}^{\dagger}\mathfrak{Ring} \quad \longleftrightarrow \quad {}^{\dagger}\mathbb{I}; \qquad * \quad \longleftrightarrow \quad \gamma_{\mathbb{J}}; \qquad {}^{\ddagger}\bullet, {}^{\ddagger}\mathfrak{Ring} \quad \longleftrightarrow \quad {}^{\ddagger}\mathbb{I}$$

$$^{\dagger}\Theta\text{-}\mathfrak{plt} \quad \longleftrightarrow \quad {}^{\dagger}\beta; \qquad {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \quad \longleftrightarrow \quad {}^{\ddagger}\alpha$$

[cf. the correspondences (StR1) $\sim$ (StR6) discussed in §3.2; the correspondences discussed in Example 2.4.5, (ii); the discussion of [Alien], §3.11, (iv)], one obtains very precise **structural resemblances**

$$\begin{array}{ccc} (AO\Theta1) & \longleftrightarrow & (AOL1), \\ (AO\Theta2) & \longleftrightarrow & (AOL2), \\ (AO\Theta3) & \longleftrightarrow & (AOL3), \\ (AO\Theta4) & \longleftrightarrow & (AOL4) \end{array}$$

with the situation discussed in Example 2.4.1, (i), (ii). Thus, in summary,

the **falsity** of (RC-Θ) may be understood as a consequence of the **falsity** [cf. (AOΘ4)] of the crucial **AND relator** "$\wedge$" in the **absence** of **distinct labels**, in stark contrast to the **truth** [cf. (AOΘ1)] of the crucial **AND relator** "$\wedge$" as an essentially *tautological consequence* of the use of the **distinct labels** "†", "‡".

In the context of the *central role* played in the logical structure of inter-universal Teichmüller theory by the **validity** of (AOΘ1), it is important to note [cf. the property discussed in (AOΘ4)!] that

(NoRng) there does **not** exist an isomorphism of **ring structures** $^{\dagger}\mathfrak{Ring} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{Ring}$ that induces, on value groups of corresponding local rings, the *desired assignment* $\{{}^{\dagger}\underline{\underline{q}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}$ [i.e., that appears in the Θ-link].

On the other hand, if, instead of considering the *full ring structures* of $^{\dagger}\mathfrak{Ring}$, $^{\ddagger}\mathfrak{Ring}$, one considers [cf. the discussion of [Rpt2018], §6]

- certain suitable subquotients — i.e., in the notation of [Alien], §3.3, (vii), (a^q), (a^{Θ}), "$\mathcal{O}^{\times}_{\underline{k}}$" — of the **underlying multiplicative monoids** of corresponding local fields, as well as
- the absolute Galois groups — i.e., in the notation of [Alien], §3.3, (vii), (a^q), (a^{Θ}), "G_k" — associated to corresponding local rings, regarded as **abstract topological groups** [that is to say, **not** as Galois groups, or equivalently/alternatively, as groups of field automorphisms! — cf. the discussion of §3.8 below],

then one obtains structures — i.e., the structures that constitute the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the Θ-link — that are **simultaneously associated** [as "underlying structures"] to *both* $^{\dagger}\mathfrak{Ring}$ *and* $^{\ddagger}\mathfrak{Ring}$ via *isomorphisms* [i.e., of certain suitable *multiplicative monoids* equipped with actions by certain suitable *abstract topological groups*] that restrict, on the subquotient monoids that correspond to the respective value groups, to the *desired assignment* $\{{}^{\dagger}\underline{\underline{q}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}$. It is this *crucial simultaneity* that yields, as a tautological consequence, the **validity** of the **AND relator** "$\wedge$" in (AOΘ1).

Working, as in the discussion above, with *multiplicative monoids* equipped with actions by *abstract topological groups*, necessarily gives rise to certain **indeterminacies**, called (Ind1), (Ind2), that play an important role in inter-universal Teichmüller theory. Certain aspects of these indeterminacies (Ind1), (Ind2) will be discussed in more detail in §3.5 below. In this context, we recall that one central assertion of the RCS [cf. the discussion of (SSInd), (SSId) in [Rpt2018], §7, §10] is to the effect that

(NeuRng) these indeterminacies (Ind1), (Ind2) may be *eliminated*, without affecting the essential logical structure of inter-universal Teichmüller theory, by taking the *multiplicative monoids* and *abstract topological groups* that appear in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips of the above discussion to be equipped with **rigidifications** by regarding them as arising from some **fixed "neutral" ring structure** $^{\boxdot}\mathfrak{Ring}$.

On the other hand, as discussed in (NoRng) above, there does **not** exist any ring structure that is *compatible* [i.e., in the sense discussed in (NoRng)], with the desired assignment $\{{}^{\dagger}\underline{\underline{q}}^{j^2}\} \mapsto {}^{\ddagger}\underline{\underline{q}}$. That is to say, in summary,

(NeuORInd) working with such a *fixed "neutral" ring structure* $^{\boxdot}\mathfrak{Ring}$ as in (NeuRng) means *either* that

(NeuORInd1) there is **no relationship** between "$*$" and $^{\boxdot}\mathfrak{Ring}$ [cf. the situation discussed in [Rpt2018], §10, (SSId)], *or* that

(NeuORInd2) the relationship between "$*$" and $^{\boxdot}\mathfrak{Ring}$ is *always necessarily* *subject* to an **indeterminacy** [cf. (AOΘ2), (AOΘ3)!]

$$\left(* \ :\to \ {}^{\boxdot}\Theta\text{-}\mathfrak{plt} \in {}^{\boxdot}\mathfrak{Ring} \right) \ \vee \ \left(* \ :\to \ {}^{\boxdot}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\boxdot}\mathfrak{Ring} \right).$$

Here, we observe that whichever of these "options"/"indeterminacies" that appear in (NeuORInd) [i.e., (NeuORInd1), (NeuORInd2)] one chooses to adopt, one is forced to contend with an indeterminacy that is, in some sense, *much more drastic* than the relatively mild indeterminacies (Ind1), (Ind2) whose *elimination* formed the *original motivation* for the introduction of ${}^{\boxdot}\mathfrak{Ring}$!

Finally, we *observe* that this *much more drastic indeterminacy* (NeuORInd) means [cf. the discussion of Example 2.4.4!] that throughout any argument, one must always take the position that the *only possible relationship* between "$*$" and ${}^{\boxdot}\Theta$-$\mathfrak{plt}$, ${}^{\boxdot}\mathfrak{q}$-$\mathfrak{plt}$ is one in which

(PltRel) *"$*$" maps either to* ${}^{\boxdot}\Theta$-$\mathfrak{plt}$ *or — i.e., "$\vee$"! — to* ${}^{\boxdot}\mathfrak{q}$-$\mathfrak{plt}$, *but* **not both***!*

Since ${}^{\ddagger}\mathfrak{Ring}$ may be thought of as a ring structure in which "$*$" *tautologically* maps to ${}^{\ddagger}\Theta$-$\mathfrak{plt}$, while ${}^{\dagger}\mathfrak{Ring}$ may be thought of as a ring structure in which "$*$" *tautologically* maps to ${}^{\ddagger}\mathfrak{q}$-$\mathfrak{plt}$, one may rephrase the above *observation* as the observation that one must always take the position that the *only possible relationship* between ${}^{\boxdot}\mathfrak{Ring}$, on the one hand, and ${}^{\ddagger}\mathfrak{Ring}$, ${}^{\dagger}\mathfrak{Ring}$, on the other, is one in which

(RngRel) *the ring structure* ${}^{\boxdot}\mathfrak{Ring}$ *is identified either with the ring structure* ${}^{\ddagger}\mathfrak{Ring}$ *or — i.e., "$\vee$"! — with the ring structure* ${}^{\dagger}\mathfrak{Ring}$, *but* **not both***!*

At this point, let us recall [cf., e.g., the discussion of §3.5, §3.11, below; [Rpt2018], §9, (GIUT), (ΘCR)] that

> inter-universal Teichmüller theory requires, in an essential way, the use of the **$\mathfrak{log}$-links**, hence, in particular, [in order to define the *power series* of the various p-adic logarithm functions that constitute these $\mathfrak{log}$-links!] the *ring structures* ${}^{\dagger}\mathfrak{Ring}$, ${}^{\ddagger}\mathfrak{Ring}$ on **both sides** — i.e., "$\wedge$"! — of the Θ-link

[cf. the discussion surrounding (InfH) of the two *vertical lines* of $\mathfrak{log}$-*links* in the *"infinite H"* on either side of the Θ-link]. In particular, we conclude formally that

> *it is impossible to implement the arguments of inter-universal Teichmüller theory once this sort of* **much more drastic indeterminacy** *(NeuORInd) has been imposed.*

§3.5. Indeterminacies and pilot discrepancy

As discussed in §3.4, the **Θ-link** involves a **gluing**

$$\{ {}^{\dagger}\underline{\underline{q}}^{j^2} \} \ \mapsto \ {}^{\ddagger}\underline{\underline{q}}$$

that identifies ${}^{\ddagger}\underline{\underline{q}}$ [i.e., $2l$-th roots of the q-*parameters* at primes of multiplicative reduction of the [copy belonging to ${}^{\ddagger}\mathfrak{Ring}$ of the] elliptic curve under consideration] with elements, i.e., the ${}^{\dagger}\underline{\underline{q}}^{j^2}$'s, which, when $j \neq 1$, have *different valuations* from the valuation of ${}^{\ddagger}\underline{\underline{q}}$.

On the other hand, in inter-universal Teichmüller theory, by applying the **multiradial representation** of [IUTchIII], Theorem 3.11, which involves various **indeterminacies** (Ind1), (Ind2), (Ind3), and then forming [cf. [IUTchIII], Corollary 3.12, and its proof] the **holomorphic hull** of the union of possible images of the Θ-pilot in this multiradial representation,

(ΘGl) one may treat **both** sides of the **Θ-link gluing** of the above display as belonging to a **single ring theory** without disturbing [cf. the *crucial* **AND relator "$\wedge$"** property discussed in §3.4!] the gluing.

Alternative ways to understand the *essential mathematical content* of (ΘGl) include the following:

(NonInf) One may think of (ΘGl) as a statement concerning the **mutually non-interference** or **simultaneous executatibility** of the **Kummer theories** surrounding the **q-pilot** and **Θ-pilot** relative to the **gluing** of *abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips* constituted by the Θ-link, i.e., when the Kummer theory surrounding the q-pilot is held **fixed**, and one allows the Kummer theory surrounding the Θ-pilot to be subject to various **indeterminacies**.

(Cohab) One may think of (ΘGl) as a statement concerning the **"cohabitation"**, or **"coexistence"**, of the **q-pilot** and **Θ-pilot** — relative to the **gluing** of *abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips* constituted by the Θ-link — within the **common container** obtained by applying the multiradial representation of the Θ-pilot, forming the **holomorphic hull** [relative to the holomorphic structure [i.e., ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater] that gave rise to the q-pilot under consideration], and finally taking **log-volumes**.

In this context, it is important to recall that this *sort of phenomenon* — i.e.,

of **computations** of **global degrees/heights** of elliptic curves in situations where a certain **"confusion"**, up to suitable **indeterminacies**, is allowed between *q-parameters* of the elliptic curves and certain *large positive powers* of these q-parameters [i.e., as in (ΘGl)]

— may be seen in various *classical examples* such as

· the proof by Faltings of the **invariance** of **heights** of *abelian varieties* under **isogeny** [cf. the discussion of [Alien], §2.3, §2.4],

· the classical proof in characteristic zero of the *geometric version* of the *Szpiro inequality* via the *Kodaira-Spencer morphism*, phrased in terms of the theory of **crystals** [cf. the discussion of [Alien], §3.1, (v)], and

· **Bogomolov's proof** over the complex numbers of the *geometric version* of the *Szpiro inequality* [cf. the discussion of [Alien], §3.10, (vi)]

— cf. also the discussion of [Rpt2018], §16.

Unfortunately, however, the situation summarized above in (ΘGl) has resulted in certain frequently voiced **misunderstandings** by some mathematicians. One such *frequently voiced misunderstanding* is to the effect that

(CnfInd1+2) the situation summarized in (ΘGl) may be explained as a consequence of a **"confusion"** between q-parameters and large positive powers of these q-parameters that results from the indeterminacies (Ind1), (Ind2).

In fact, however, as discussed in Example 3.5.1, (iii), below,

> at least in the case of q-parameters of sufficiently small valuation [i.e., sufficiently large positive *order*, in the sense of *loc. cit.*], such a **"confusion"** [i.e., between q-parameters and large positive powers of these q-parameters] can **never occur** as a consequence of (Ind1), (Ind2), i.e., both of which amount to *automorphisms* of the [underlying topological module of the] *log-shells* involved

[cf. also the discussion of (ΘInd) in [Rpt2018], §11]. In this context, we note that this misunderstanding (CnfInd1+2) appears to be caused in many cases, at least in part, by a more *general misunderstanding* concerning the operation of *passage to underlying structures* [cf. Example 3.5.2 below]. A more detailed discussion of the operation of *passage to underlying structures* may be found in §3.9 below.

As discussed in [Rpt2018], §11, the "confusion" summarized in (ΘGl) occurs in inter-universal Teichmüller theory as a consequence not only of the **local indeterminacies** (Ind1), (Ind2), (Ind3), but also of the constraints imposed by the **global realified Frobenioid** portions of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the Θ-link. In this context, it is of *particular importance* to observe that

(CnfInd3) the indeterminacy **(Ind3)**, which constrains one to restrict one's attention to **upper bounds** [i.e., but **not** lower bounds!] on the log-volume that is the subject of the computation of [IUTchIII], Corollary 3.12, already by itself — i.e., *without considering (Ind1), (Ind2), or global realified Frobenioids!* [cf. the discussion of (Ind3>1+2) in §3.11 below] — is **sufficient to account** for the possibility of a **"confusion"** of the sort summarized in (ΘGl) [i.e., between q-parameters and large positive powers of these q-parameters].

Indeed, the indeterminacy (Ind3) is *defined* in precisely such a way as to **identify** the ideals generated by *arbitrary positive powers* of the q-parameters.

Example 3.5.1: Bounded nature of log-shell automorphism indeterminacies.

(i) Write $\mathbb{Z}_p$ for the ring of p-adic integers, for some prime number p; $\mathbb{Q}_p$ for the field of fractions of $\mathbb{Z}_p$. Let M be a finitely generated free $\mathbb{Z}_p$-module, which, in the following discussion, we shall think of as being embedded in $M_{\mathbb{Q}_p} \stackrel{\text{def}}{=} M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$;

$$\alpha: \ M \ \stackrel{\sim}{\to} \ M$$

an *automorphism* of the $\mathbb{Z}_p$-module M. For $n \in \mathbb{Z}$, write

$$\mathcal{U}(M,n) \ \stackrel{\text{def}}{=} \ \{x \in M_{\mathbb{Q}_p} \mid x \in p^n \cdot M, \ x \notin p^{n+1} \cdot M\} \ \subseteq \ M_{\mathbb{Q}_p}.$$

Then observe that α induces a *bijection*

$$\mathcal{U}(M,n) \ \stackrel{\sim}{\to} \ \mathcal{U}(M,n)$$

for every $n \in \mathbb{Z}$.

(ii) In the notation of (i), suppose, for simplicity, that p is *odd.* Let K be a finite field extension of $\mathbb{Q}_p$. Write $\mathcal{O}_K \subseteq K$ for the ring of integers of K; $\mathcal{O}_K^\times \subseteq \mathcal{O}_K$ for the group of units of $\mathcal{O}_K$; $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for the maximal ideal of K; $\mathcal{I}_K \subseteq K$ for the *log-shell* associated to K [cf., e.g., the discussion of [IUTchIII], Remark 1.2.2, (i)], i.e., the result of multiplying by p^{-1} the image $\log_p(\mathcal{O}_K^\times)$ of $\mathcal{O}_K^\times$ by the p-adic logarithm $\log_p(-)$. Thus,

$$\mathcal{O}_K \subseteq \mathcal{I}_K \subseteq p^{-c} \cdot \mathcal{O}_K$$

for some nonnegative integer of c that depends only on the isomorphism class of the field K [cf. [IUTchIV], Proposition 1.2, (i)]. In particular, there exists a positive integer s that depends only on the isomorphism class of the field K such that for any *automorphism*

$$\phi: \mathcal{I}_K \xrightarrow{\sim} \mathcal{I}_K$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ and any $n \in \mathbb{Z}$, it holds that

$$\phi(\mathcal{U}(\mathcal{O}_K, n)) \subseteq \bigcup_{i=-s}^{s} \mathcal{U}(\mathcal{O}_K, n+i)$$

[where i ranges over the integers between $-s$ and s].

(iii) In the situation of (ii), we define the *order* of a nonzero element $x \in K$ to be the unique $n \in \mathbb{Z}$ such that $x \in \mathfrak{m}^n$, $x \notin \mathfrak{m}^{n+1}$. One thus concludes from the final portion of the discussion of (ii) that there exists a positive integer t that depends only on the isomorphism class of the field K such that for any *automorphism*

$$\phi: \mathcal{I}_K \xrightarrow{\sim} \mathcal{I}_K$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ and any nonzero element $q \in \mathcal{O}_K$ [i.e., such as the *q-parameter* of a Tate curve over K!], the absolute value of the *difference* between the *orders* of q and $\phi(q)$ is $\leq t$, i.e., in words,

> automorphisms of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ only give rise to **bounded discrepancies** in the orders of nonzero elements of $\mathcal{O}_K$.

Example 3.5.2: Automorphisms of underlying structures. In general, *distinct auxiliary structures* on some *common underlying structure* are *not* necessarily mapped to one another by some *automorphism of the common underlying structure.* Concrete examples of this phenomenon may be found in quite substantial abundance and include, in particular, the following [cf. the discussion of [Rpt2018], §11]:

(i) The group structures of the **finite abelian groups** $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are *not* mapped to one another by any *isomorphism of sets*, despite the fact that the underlying sets of these two groups are indeed isomorphic to one another.

(ii) The *scheme structures* of **non-isomorphic algebraic curves** over a common algebraically closed field are *not* mapped to one another by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of algebraic curves over a common algebraically closed field are indeed isomorphic to one another.

(iii) The *holomorphic structures* of **non-isomorphic compact Riemann surfaces** of the *same genus* are *not* mapped to one another by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of such Riemann surfaces are indeed isomorphic to one another.

(iv) The *field structures* of **non-isomorphic mixed-characteristic local fields** [which, by *local class field theory*, may be regarded as [the formal union with "$\{0\}$" of] some suitable subquotient of their respective absolute Galois groups] are *not*, in general, mapped to one another by any *isomorphism of profinite groups* between the respective absolute Galois groups [cf., e.g., [Ymgt], §2, Theorem, for an example of this phenomenon].

§**3.6.** **Chains of logical AND relations**

From the point of view of the *simple qualitative model* of inter-universal Teichmüller theory given in Example 2.4.5, the discussion of §3.4 concerns the **AND relator "$\wedge$"** in the **"Θ-link"** portion of Example 2.4.5, (ii). On the other hand, strictly speaking, this portion of inter-universal Teichmüller theory only concerns the *initial definition* of the Θ*-link*. That is to say, the *bulk* of the theory developed in [IUTchI-III] concerns, from the point of view of the simple qualitative model of inter-universal Teichmüller theory given in Example 2.4.5, (ii), the **preservation** of the **AND relator "$\wedge$"** as one passes from

· the **"Θ-link"** portion of Example 2.4.5, (ii), to
· the **"multiradial representation"** portion of Example 2.4.5, (ii).

By contrast, the passage from the *"multiradial representation"* portion of Example 2.4.5, (ii), to the *"final numerical estimate"* portion of Example 2.4.5, (ii) — i.e., which corresponds to the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 — is [cf. the discussion of the final portion of Example 2.4.5, (ii)!] *relatively straightforward* [cf. the discussion of §3.10, §3.11, below].

At this point, it is perhaps of interest to consider **"typical symptoms"** of mathematicians who are operating under **fundamental misunderstandings** concerning the essential logical structure of inter-universal Teichmüller theory. Such typical symptoms, which are in fact closely related to one another, include the following:

(Syp1) a sense of **unjustified** and **acutely harsh abruptness** in the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 [cf. the discussion of the final portions of Example 2.4.5, (ii), (iii)!];

(Syp2) a desire to see the "proof" of some sort of *commutative diagram* or **"compatibility property"** to the effect that taking *log-volumes of pilot objects* in the domain and codomain of the Θ-link yields the *same real number* [a property which, in fact, can *never* be proved since it is *false*! — cf. the discussion of §3.5];

(Syp3) a desire to see the *inequality* of the *final numerical estimate* obtained as the result of concatenating some **chain of intermediate inequalities**, i.e., as is often done in proofs in real/complex/functional analysis or analytic number theory.

Here, it should be noted that (Syp2) and (Syp3) often occur as approaches to mitigating the "harsh abruptness" of (Syp1).

With regard to (Syp3), it should be emphasized that it is *entirely unrealistic* to attempt to obtain the *inequality* of the *final numerical estimate* as the result of concatenating some *chain of intermediate inequalities* since this is simply *not* the way in which the logical structure of inter-universal Teichmüller theory is organized. That is to say, in a word, the logical structure of inter-universal Teichmüller theory does *not* proceed by concatenating some sort of chain of intermediate inequalities. Rather,

($\wedge$-Chn) the logical structure of inter-universal Teichmüller theory proceeds by *observing a* **chain** *of* **AND relations** *"$\wedge$"*

[cf. the discussion of [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (c^{itw}), (f^{itw}); [Alien], §3.11, (iv), (v)]. As observed in Example 2.4.5, (ii), (iii), once one follows this *chain of AND relations "$\wedge$"* up to and including the *multiradial representation of the* Θ*-pilot* [i.e., [IUTchIII], Theorem 3.11], the passage to the *final numerical estimate* [i.e., [IUTchIII], Corollary 3.12] is *relatively straightforward* [i.e., as one might expect, from the use of the word "corollary"!].

One essentially formal consequence of ($\wedge$-Chn) is the following: Since the *definition* of the Θ*-link*, the construction of the *multiradial representation of the* Θ*-pilot*, and the ultimate passage to the *final numerical estimate* consist of a **finite number of steps**, one natural and effective way to **analyze/diagnose** [cf. the discussion of §1.4!] the precise content of **misunderstandings** of inter-universal Teichmüller theory is to determine

($\wedge$-Dgns) **precisely where** in the finite sequence of steps that appear is the **first step** at which the person feels that the **preservation** of the **crucial AND relator** "$\wedge$" is *no longer clear.*

In some sense, the *starting point* of the various *AND relations* "$\wedge$" that appear in the *multiradial algorithm* of [IUTchIII], Theorem 3.11, is the observation that

($\wedge$-Input) the **input data** for this multiradial algorithm consists solely of an **abstract** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**; moreover, this multiradial algorithm is **functorial** with respect to arbitrary isomorphisms between $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips

[cf. [IUTchIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i)]. This property ($\wedge$-Input) means that the multiradial algorithm may be applied to *any* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* that appears, or alternatively/equivalently, that *any* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* may serve as the *gluing data* [cf. the "$\gamma_{\mathbb{J}}$" in the analogies discussed in §3.2, (StR3), (StR4), as well as Example 2.4.5, (ii)!] between a *given situation* [i.e., such as the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theater in the *codomain* of the Θ-link!] and the *content of the multiradial algorithm.*

On the other hand, in order to conclude that the multiradial algorithm yields **output data** satisfying suitable *AND relations "$\wedge$"*, it is necessary also to examine in detail the content of this output data, i.e., in particular, in the context of the central **IPL** and **SHE** properties discussed in [IUTchIII], Remark 3.11.1, (iii), as well as the **chain of (sub)quotients** aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)]. In a word, the essential "principle" that is applied throughout the various steps of the multiradial algorithm in order to derive *new AND relations "$\wedge$"* from *old AND relations "$\wedge$"* is the following **"principle of extension of indeterminacies"**:

(ExtInd) If A, B, and C are propositions, then it holds [that $B \implies B \vee C$ and hence] that

$$A \wedge B \implies A \wedge (B \vee C).$$

One important tool that is frequently used in inter-universal Teichmüller theory in a fashion that is closely related to (ExtInd) is the notion of a *poly-morphism* [cf. the discussion of §3.7 below for more details].

In the context of (ExtInd), it is interesting to note that, from the point of view of the discussion of §3.4,

> the "$\vee$" that appears in the conclusion — i.e., $A \wedge (B \vee C)$ — of (ExtInd) may be understood as amounting to essentially the *same phenomenon* as the "$\vee$" that appears in (NeuORInd2) [e.g., by taking "C" to be A].

That is to say, instead of *generating AND relations "$\wedge$" tautologically* by means of the introduction of *distinct labels* [i.e., as in (AOΘ1)] — i.e., say, by introducing a *new distinct label* for "C" so as to conclude a tautological relation

$$A \wedge B \wedge C$$

— (ExtInd) allows one to *generate new AND relations "$\wedge$"* while *avoiding* the introduction of *new distinct labels*. As discussed in §3.4, this point of view [i.e., of avoiding the introduction of new distinct labels] leads inevitably to *OR relations "$\vee$"*, i.e., as in (NeuORInd2) or as in the conclusion "$A \wedge (B \vee C)$" of (ExtInd). As discussed above, the reason that one wishes to **avoid** the introduction of **new distinct labels** when applying (ExtInd) is precisely that

(sQLTL) one wishes to apply (ExtInd) to form **"(sub)quotients/splittings"** of the log-theta-lattice [cf. the title of [IUTchIII]!], i.e., to **project** the *vertical line* on the *left-hand side* of the *infinite "H"* portion of the log-theta-lattice onto the *vertical line* on the *right-hand side* of this infinite "H" by somehow achieving some sort of **"crushing together"** of distinct coordinates [i.e., "(n, m)", where $n, m \in \mathbb{Z}$] of the log-theta-lattice

[cf. the discussion of §3.11 below; [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, ($\mathrm{c}^{\mathrm{itw}}$), ($\mathrm{f}^{\mathrm{itw}}$); [Alien], §3.11, (iv), (v)).

At this point, it is of interest to note that there are, in some sense, *two ways* in which (ExtInd) is applied during the execution of the various steps of the multiradial algorithm [cf. the discussion of §3.10, §3.11, below, for more details]:

(ExtInd1) operations that consist of simply adding **more possibilites/indeterminacies** [which corresponds to passing from B to $B \vee C$) within some **fixed container**;

(ExtInd2) operations in which one **identifies** [i.e., *"crushes together"*, by passing from B to $B \vee C$] objects with **distinct labels**, at the cost of passing to a situation in which the object is regarded as being only known **up to isomorphism**.

Typical examples of (ExtInd1) include the *upper semi-continuity of (Ind3)*, as well as the passage to *holomorphic hulls*. Typically, such applications of (ExtInd1) play an important role in establishing various **symmetry** or **invariance** properties such as multiradiality. This sort of establishment of various *symmetry* or *invariance* properties by means of (ExtInd1) then allows one to apply **label crushing** operations as in (ExtInd2). Put another way,

· (ExtInd1) may be understood as a sort of operation whose purpose is to **prepare** suitable **descent data**, while

· (ExtInd2) may be thought of as a sort of actual **descent operation**, i.e., from data that *depends* on the specification of a member of some collection of *distinct labels* to data that is *independent* of such a label specification.

[We refer to the discussion of §3.8 below for more details on *foundational aspects* of (ExtInd2) and to the discussion of §3.9 below for more details concerning the notion of *"descent"*.] Typical examples of (ExtInd2) in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

· identifying "$\Pi_{\underline{v}}$"'s [where $\underline{v} \in \underline{\mathbb{V}}$] at *different vertical coordinates* [i.e., "(n, m)" and "(n, m')", for $n, m, m' \in \mathbb{Z}$] of the log-theta-lattice, which results in a *"$\Pi_{\underline{v}}$ regarded up to isomorphism"* that is labeled by a *new label* "$(n, \circ)$";

· identifying "$G_{\underline{v}}$"'s [where $\underline{v} \in \underline{\mathbb{V}}$] at *different horizontal or vertical coordinates* [i.e., "(n, m)" and "(n', m')", for $n, n', m, m' \in \mathbb{Z}$] of the log-theta-lattice, which results in a *"$G_{\underline{v}}$ regarded up to isomorphism"* that is labeled by a *new label* "$(\circ, \circ)$";

· identifying the *$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips in the Θ-link* that arise from the Θ- and q-pilot objects in distinct ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters [i.e., the ($\Theta^{\pm\mathrm{ell}}$NF-)Hodge theaters in the *domain* and *codomain* of the Θ-link] by working with these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips up to isomorphism.

In some sense, the most *nontrivial instances* of the application of (ExtInd) in the context of the multiradial algorithm occur in relation to the **$\mathfrak{log}$-Kummer correspondence** [i.e., in the *vertical line* on the *left-hand side* of the infinite "H"] and closely related operations of **Galois evaluation** [cf. the discussion of §3.11 below]. The *Kummer theories* that appear in this $\mathfrak{log}$-Kummer-correspondence — i.e., Kummer theories for

· *multiplicative monoids* of nonzero elements of rings of integers in *mixed-characteristic local fields*,

· *mono-theta environments/theta monoids*, and

· *pseudo-monoids of κ-coric functions*

— involve the construction of various **[Kummer] isomorphisms** between

· *Frobenius-like data* and
· *corresponding data constructed via anabelian algorithms from étale-like objects.*

The *output* of such algorithms typically involves constructing the *"corresponding data"* as **one possibility among many**. Here, we note that

> either of these Frobenius-like/étale-like versions of *"corresponding data"* is — unlike, for instance, the data that constitutes an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip*! — *sufficiently robust* that it **completely determines** [even when only regarded up to isomorphism!] the **[usual] embedding** of the **Θ-pilot**.

That is to say, taken as a whole, the **multiradial algorithm** — and, especially, the portion of the multiradial algorithm that involves the $\mathfrak{log}$-*Kummer correspondence* and closely related operations of *Galois evaluation* — plays the role of

> *exhibiting the Frobenius-like* Θ*-pilot as* **one possibility** *within a* **collection of possibilities** *constructed via anabelian algorithms from étale-like data.*

Thus, in this situation, one obtains the *crucial preservation of the AND relation "∧"* by applying (ExtInd) twice, i.e., by applying

· (ExtInd1) to the enlargement of the **collection of possibilities** under consideration and
· (ExtInd2) to the **Kummer isomorphisms** involved, when one passes from *Frobenius-like object labels* "(n, m)" [where $n, m \in \mathbb{Z}$] to *étale-like object labels* "$(n, \circ)$" [where $n \in \mathbb{Z}$].

This is precisely what is meant by the **chain of (sub)quotients** aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)] discussed above [cf. also the discussion of §3.10, §3.11, below].

§**3.7.** Poly-morphisms and logical AND relations

Poly-morphisms — i.e., sets of morphisms between objects — appear throughout inter-universal Teichmüller theory as a tool for facilitating

> the **explicit enumeration** of a **collection** of **possibilities**.

Composable ordered pairs of poly-morphisms [i.e., pairs for which the domain of the first member in the pair coincides with the codomain of the second member in the pair] may be *composed* by considering the set of morphisms obtained by composing the morphisms that belong to the sets of morphisms that constitute the given pair of poly-morphisms. Such compositions of poly-morphisms allow one to *keep track* — in a *precise* and *explicit* fashion — of *collections* of *possibilities* under consideration.

From the point of view of **chains of AND relations "∧"**, as discussed in §3.6,

> the **collections** of **possibilities** enumerated by **poly-morphisms** are to be understood as being related to one another via **OR relations "∨"**.

That is to say, poly-morphisms may be thought of as a sort of **indeterminacy**, which is used in inter-universal Teichmüller theory to produce structures that satisfy various **symmetry** or **invariance** properties, hence yield suitable **descent data** [cf. the discussion of (ExtInd1) in §3.6; the discussion of §3.9 below].

Thus, for instance, in the case of the *full poly-isomorphism* that constitutes the Θ-*link*, one may understand the **fundamental AND relation "$\wedge$"** of (AOΘ1) — which, for simplicity, we denote by

$$A \wedge B$$

[where A and B correspond, respectively, in the notation of the discussion of §3.4, to "$* \ :\to \ {}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt} \in {}^{\ddagger}\mathfrak{Ring}$" and "$* \ :\to \ {}^{\dagger}\Theta\text{-}\mathfrak{plt} \in {}^{\dagger}\mathfrak{Ring}$"] — may be understood as a relation "$A \wedge (B_1 \vee B_2 \vee \dots)$", i.e., a relation to the effect that

> if one **fixes** the q-pilot ${}^{\ddagger}\mathfrak{q}\text{-}\mathfrak{plt}$, then this q-pilot is **glued**, via the **Θ-link**, to the Θ-pilot ${}^{\dagger}\Theta\text{-}\mathfrak{plt}$ by means of *one isomorphism* [of the full poly-isomorphism that constitutes the Θ-link] or *another isomorphism*, or *yet another isomorphism*, etc.

[Here, the various possible gluings that constitute B are denoted by B_1, B_2, ...] In particular, as discussed in ($\wedge$-Chn), if one starts with the Θ-*link* and then considers various *subsequent logical AND relations "$\wedge$"* that arise — for instance, by considering various *composites of poly-morphisms*! — by applying (ExtInd), then

($\wedge$($\vee$)-Chn) the *essential logical structure* of inter-universal Teichmüller theory, as discussed in ($\wedge$-Chn), may be understood as follows:

$$\begin{aligned} A \wedge B \quad &= \quad A \wedge (B_1 \vee B_2 \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots) \\ &\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots \vee B_1'' \vee B_2'' \vee \dots) \\ &\qquad \vdots \end{aligned}$$

Finally, we recall that various *"classical examples"* of the notion of a poly-morphism include

- the collection of maps between *topological spaces* that constitutes a *homotopy class*, or *stable homotopy class*, of maps;
- the collection of morphisms between complexes that constitutes a morphism of the associated *derived category*;
- the collection of morphisms obtained by considering some sort of *orbit* by some sort of *group action* on the domain or codomain of a given morphism

[cf. the discussion of [Rpt2018], §13, (PMEx1), (PMEx2), (PMQut)]. Also, in this context, it is useful to recall [cf. the discussion of [Alien], §4.1, (iv)] that

- **gluings via poly-morphisms** are closely related to the sorts of gluings that occur in the construction of **algebraic stacks** [i.e., algebraic stacks which are *not algebraic spaces*].

§**3.8. Inter-universality and logical AND relations**

One fundamental aspect of inter-universal Teichmüller theory lies in the consideration of **distinct universes** that arise naturally when one considers **Galois categories** — i.e., *étale fundamental groups* — associated to various schemes. Here, it is important to note that, when phrased in this way,

> this fundamental aspect of inter-universal Teichmüller theory is, at least from the point of view of *mathematical foundations*, **no different** from the situation that arises in [SGA1].

On the other hand, the fundamental difference between the situation considered in [SGA1] and the situations considered in inter-universal Teichmüller theory lies in the fact that, whereas in [SGA1], the various distinct schemes that appear are related to one another by means of **morphisms of schemes or rings**,

> the various distinct schemes that appear in inter-universal Teichmüller theory are related to one another, in general, by means of relations — such as the $\mathfrak{log}$- and Θ-*links* — that are **non-ring/scheme-theoretic** in nature, i.e., in the sense that they do **not** arise from morphisms of schemes or rings.

In general, when considering relations between distinct mathematical objects, it is of fundamental importance to specify those mathematical structures that are **common** — i.e., in the terminology of inter-universal Teichmüller theory, **coric** — to the various distinct mathematical objects under consideration. Here, we observe that

> this notion of being "common"/"coric" to the various distinct mathematical objects under consideration constitutes, when formulated at a formal, symbolic level, a **logical AND relation "$\wedge$"**.

— cf. the discussion of §3.4, §3.5, §3.6, §3.7.

Thus, in the situations considered in [SGA1], the *ring/scheme structures* of the various distinct schemes that appear are **coric** and hence allow one to relate the universes/Galois categories/étale fundamental groups associated to these distinct schemes in a way that makes use of the *common* ring/scheme structures between these schemes. At a concrete level, this means that

> in the situations considered in [SGA1], étale fundamental groups may be related to one in such a way that the only *indeterminacies* that occur are **inner automorphism indeterminacies**.

Moreover, these inner automorphism indeterminacies are *by no means superfluous* — cf. the discussion of Example 3.8.1 below.

Example 3.8.1: Inner automorphism indeterminacies. The *unavoidability* of *inner automorphism indeterminacies* may be understood in very elementary terms, as follows.

(i) Let k be a *perfect field*; $\overline{k}$ an *algebraic closure* of k; $N \subseteq G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ a *normal closed subgroup* of G_k; $\sigma \in G_k$ such that the automorphism $\iota_\sigma : N \xrightarrow{\sim} N$

of N given by *conjugating* by σ is *not* inner. [One verifies immediately that, for instance, if k is a *number field* or a *mixed-characteristic local field*, then such N, σ do indeed exist.] Write $k_N \subseteq \overline{k}$ for the subfield of N-invariants of $\overline{k}$, $G_{k_N} \stackrel{\text{def}}{=} N \subseteq G_k$, $Q_N \stackrel{\text{def}}{=} G_k/G_{k_N}$. Then observe that this situation yields an example of a situation in which one may verify directly that

> the **functoriality** of the **étale fundamental group** only holds if one allows for **inner automorphism indeterminacies** in the definition of the étale fundamental group.

Indeed, let us first observe that the *"basepoints"* of k and k_N determined by $\overline{k}$ allows us to regard G_k and G_{k_N}, respectively, as the *étale fundamental groups* of k and k_N. Thus, if one assumes that the *functoriality* of the *étale fundamental group* holds *even in the absence of inner automorphism indeterminacies*, then the *commutative diagram of schemes*

$$\begin{array}{ccc} \mathrm{Spec}(k_N) & \xrightarrow{\sigma} & \mathrm{Spec}(k_N) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k) & \end{array}$$

[where the diagonal morphisms are the natural morphisms] induces a *commutative diagram of profinite groups*

$$\begin{array}{ccc} G_{k_N} & \xrightarrow{\iota_\sigma} & G_{k_N} \\ & \searrow \quad \swarrow & \\ & G_k & \end{array}$$

— which [since the natural inclusion $N = G_{k_N} \hookrightarrow G_k$ is *injective*!] implies that ι_σ is the *identity automorphism*, in contradiction to our assumption concerning σ!

(ii) The phenomenon discussed in (i) may be understood as a consequence of the fact that, whereas $\mathrm{Spec}(k)$ is **coric** in the *commutative diagram of schemes* that appears in (i) [i.e., in the sense that this diagram *does indeed commute*!], $\mathrm{Spec}(\overline{k})$ is **not coric** in the *diagram of schemes*

$$\begin{array}{ccc} & \mathrm{Spec}(\overline{k}) & \\ & \swarrow \quad \searrow & \\ \mathrm{Spec}(k_N) & \xrightarrow{\sigma} & \mathrm{Spec}(k_N) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k) & \end{array}$$

[where the diagonal morphisms are the natural morphisms], i.e., in the sense that the upper portion of this diagram *does not commute*!

(iii) Finally, we consider the *natural exact sequence*

$$1 \longrightarrow G_{k_N} \longrightarrow G_k \longrightarrow Q_N \longrightarrow 1$$

of profinite groups. Then observe that the *inner automorphisms indeterminacies* of G_k [cf. the discussion of (i), (ii)!] induce *outer automorphism indeterminacies* of G_{k_N} that will *not*, in general, be *inner*. That is to say,

> if one considers G_{k_N} in the context of this *natural exact sequence*, then one must in fact consider G_{k_N} [*not only* up to *inner automorphism indeterminacies*, i.e., as discussed in (i), (ii), *but also*] up to certain **outer automorphism indeterminacies**.

Relative to the point of view of the discussion of (ii), these *outer automorphism indeterminacies* may be understood as a consequence of the fact that, in the context of the *field extensions* $k \subseteq k_N \subseteq \overline{k}$ and the *automorphisms* of these field extensions induced by elements of G_k,

> the field k is **coric**, whereas the field k_N is **not coric**

— i.e., in the context of these field extensions and automorphisms of field extensions, the relationship of k to the various field extensions that appear is **constant** and **fixed**, whereas the relationship of k_N to the various field extensions that appear is **variable**, i.e., subject to **indeterminacies** arising from the action of elements of G_k.

Unlike the situations considered in [SGA1] [cf. the discussion of Example 3.8.1], in which the *ring/scheme structures* of the various distinct schemes that appear are **coric**, the *ring structures* of the rings that appear on either side of the $\mathfrak{log}$- and Θ-links of inter-universal Teichmüller theory — i.e., such as number fields or completions of number fields at various valuations — are **not coric** [with respect to the respective links]. This leads one naturally to consider **weaker structures** such as

- *sets* equipped with a *topology* and a continuous action of a *topological group*, in the case of the $\mathfrak{log}$-*link*, or
- *realified Frobenioids* or *topological monoids* equipped with a continuous action of a *topological group*, in the case of the Θ-*link*,

which are indeed **coric** [with respect to the respective links]. Indeed, it is precisely this sort of consideration that gave rise to the term **"inter-universal"**.

Here, we note that it is of fundamental importance that these *topological groups* [which typically in fact arise as *Galois groups* or *arithmetic fundamental groups* of schemes] be treated as **abstract topological groups**, rather than as Galois groups or arithmetic fundamental groups [cf. the discussion at the beginning of §3.2]. That is to say, to treat these topological groups as Galois groups or arithmetic fundamental groups requires the use of the **ring/scheme structures** involved, i.e., the use of structures which are *not available* since they are **not common/coric** to the rings/schemes that appear on *opposite sides* of the $\mathfrak{log}$-/Θ-link [cf. the discussion of [Alien], §2.10; [IUTchIII], Remarks 1.1.2, 1.2.4, 1.2.5; [IUTchIV], Remarks 3.6.1, 3.6.2, 3.6.3]. In this context, it is also of fundamental importance to observe that it is precisely because these topological groups must be treated as *abstract topological groups* that **anabelian** results play a *central role* in inter-universal Teichmüller theory.

One consequence of the constraint [discussed above] that one must typically work, in inter-universal Teichmüller theory, with structures that are substantially *weaker* than ring structures is the necessity, in inter-universal Teichmüller theory, of allowing for various **indeterminacies**, such as (Ind1), (Ind2), (Ind3), that are somewhat more involved than the relatively simple inner automorphism indeterminacies that occur in [SGA1]. Here, we recall that from the discussion of ($\wedge(\vee)$-Chn) in §3.7 that

> it is precisely the *numerous indeterminacies* that arise in inter-universal Teichmüller theory that give rise to the numerous **logical OR relations "$\vee$"** in the display of ($\wedge(\vee)$-Chn).

On the other hand, once one takes such indeterminacies into account, i.e.,

> once one consents to work with various objects "up to certain suitable indeterminacies" — e.g., by means of **poly-morphisms**, as discussed in §3.7 — it is natural to **identify**, by applying **(ExtInd2)** [as discussed in §3.6], objects that are related to one another by means of collections of isomorphisms [i.e., poly-isomorphisms] that are uniquely determined up to suitable indeterminacies.

Here, we observe that this sort of **(ExtInd2) identification** that occurs repeatedly in inter-universal Teichmüller theory [cf. the discussion of §3.6] may at first glance appear somewhat *novel*. In fact, however, from the point of view of *mathematical foundations* — i.e., just as in the discussion of *inter-universality* given above! — this sort of *(ExtInd2) identification* is *qualitatively* very similar to *numerous classical constructions*, such as the following:

· the notion of an *algebraic closure* of a field [cf. the discussion of Example 3.8.1];

· various categorical constructions such as *direct* and *inverse limits* [i.e., such as *fiber products* of schemes] that are defined by means of some sort of *universal property*.

That is to say, in each of the *classical constructions*, the *"output object"* is, strictly speaking, from the point of view of mathematical foundations, *not well-defined as a* **particular set**, but rather as a *collection of sets* [where we note that, typically, this "collection" is *not a set*!] that are related to one another — and hence, in common practice, **identified** with one another, in the fashion of (ExtInd2)! — via *unique* [modulo, say, some sort of well-defined indeterminacy] *isomorphisms* by means of some sort of *"universal" property*.

In this context, it is also important to note that, from a foundational point of view, the sort of *"(sub)quotient"* obtained by applying (ExtInd2) [cf. the discussion of *"(sub)quotients"* in (sQLTL) and indeed throughout §3.6] must be regarded, *a priori*, as a **formal (sub)quotient**, i.e., as some sort of *diagram of arrows*. That is to say, at least from an *a priori* point of view,

(NSsQ) any explicit construction of a **"naive set-theoretic (sub)quotient"** necessarily requires the use of some sort of **set-theoretic enumeration** of each of the individual [set-theoretic] objects that are identified, up to isomorphism, via an application of (ExtInd2). On the other hand, as is well-known, typically such set-theoretic enumerations — which often reduce, roughly speaking, to consideration of the *"set of all sets"*! — lead immediately to a *contradiction*.

Indeed, it is precisely this aspect of the constructions of inter-universal Teichmüller theory that motivated the author to include the discussion of **species** in [IUTchIV], §3.

Finally, we recall [cf. also the discussion of §3.10 below] that

(LVsQ) it is only in the *final portion* of inter-universal Teichmüller theory, i.e., once one obtains a *formal (sub)quotient* that forms a **"closed loop"**, that one may pass from this *formal (sub)quotient* to a *"coarse/set-theoretic (sub)quotient"* by taking the **log-volume**

[cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (ix); Step (x) of the proof of [IUTchIII], Corollary 3.12].

§**3.9.** Passage and descent to underlying structures

One fundamental aspect of inter-universal Teichmüller theory lies in the use of numerous **functorial algorithms** that consist of the construction

$$\textit{input data} \quad \rightsquigarrow \quad \textit{output data}$$

of certain *output data* associated to given *input data*. When one applies such functorial algorithms, there are *two ways* in which the output data may be treated [cf. [Alien], §2.7, (iii); the discussion of *"post-anabelian structures"* in [IUTchII], Remark 1.11.3, (iii), (v); [IUTchIII], Remark 1.2.2, (vii)] :

(UdOut) One may consider the *output data* **independently** of the given *input data* and *functorial algorithms* used to construct the output data. In this case, the output data may be regarded as a sort of **"underlying structure"** associated to the input data.

(InOut) One may consider the *output data* as data **equipped** with the additional structure constituted by the *input data*, together with the *functorial algorithm* that gave rise to the output data by applying the algorithm to the input data.

Typical examples of this phenomenon in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

(sQGOut) *Functorial algorithms that associate to* $\Pi_{\underline{v}}$ *[where* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] some* **subquotient group** *of* $\Pi_{\underline{v}}$*, such as, for instance, the quotient* $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$: In this sort of situation, treatment of the output data [i.e., subquotient group of $\Pi_{\underline{v}}$] according to (InOut) is indicated by a "$(\Pi_{\underline{v}})$" following the notation for the particular subquotient under consideration; by contrast, treatment of the output data [i.e., subquotient group of $\Pi_{\underline{v}}$] according to (UdOut) is indicated by the *omission* of this "$(\Pi_{\underline{v}})$".

(MnOut) *Functorial algorithms that associate to* $\Pi_{\underline{v}}$ *[where* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] some sort of [abelian]* **monoid** *equipped with a continuous action by* $\Pi_{\underline{v}}$*, such as, for* *instance, [data isomorphic to] various subquotient monoids [i.e., "$\mathcal{O}^{\triangleright}$", "$\mathcal{O}^{\times}$", "$\mathcal{O}^{\times\boldsymbol{\mu}}$", etc.] of the multiplicative monoid* $\overline{F}_{\underline{v}}^{\times}$: In this sort of situation, treatment of the output data [i.e., monoid equipped with an action by $\Pi_{\underline{v}}$] according to (InOut) is indicated by a "$(\Pi_{\underline{v}})$" following the notation for the particular monoid equipped with an action by $\Pi_{\underline{v}}$ under consideration; by contrast, treatment of the output data [i.e., monoid equipped with an action by $\Pi_{\underline{v}}$] according to (UdOut) is indicated by the *omission* of this "$(\Pi_{\underline{v}})$".

(PSOut) *Functorial algorithms that associate some sort of* **prime-strip** *to some sort of input data*: In this sort of situation, treatment of the output data [i.e., some sort of prime-strip] according to (InOut) is indicated by a "$(-)$" [where "$-$" is the given *input data*] following the notation for the particular prime-strip under consideration; by contrast, treatment of the output data [i.e., some sort of prime-strip] according to (UdOut) is indicated by the *omission* of this "$(-)$".

Perhaps the most central example of (PSOut) in inter-universal Teichmüller theory is the notion of the **"q-/Θ-intertwinings"** on an *$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip* [cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (ii)]:

(ItwOut) This terminology refers to the treatment of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip according to (InOut), relative to the functorial algorithm for constructing the **q-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [in the case of the "q-intertwining"] or the **Θ-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [in the case of the "Θ-intertwining"] from some *$\Theta^{\pm\mathrm{ell}}$NF- or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater.*

In any situation in which one considers a construction from the point of view of (UdOut) — that is to say, as a construction that produces *"underlying data"* [i.e., *"output data"*] from *"original data"* [i.e., *"input data"*]

$$\begin{array}{ccc} \textit{input data} & \rightsquigarrow & \textit{output data} \\ \| & & \| \\ \textit{original data} & & \textit{underlying data} \end{array}$$

— it is natural to consider the issue of **descent** to [a functorial algorithm in] the *underlying data* of a **functorial algorithm** in the *original data*. Here, we say that

a *functorial algorithm* Φ in the *original data* **descends** to a *functorial algorithm* Ψ in the *underlying data* if there exists a functorial isomorphism

$$\Phi \quad \overset{\sim}{\rightarrow} \quad \Psi|_{original\ data}$$

between Φ and the *restriction* of Ψ, i.e., relative to the given construction *original data* $\rightsquigarrow$ *underlying data.*

That is to say, roughly speaking, to say that the functorial algorithm Φ in the original data *descends* to the *underlying data* means, in essence, that although the construction constituted by Φ depends, *a priori*, on the **"finer"** *original data*, in fact, up to natural isomorphism, it only depends on the **"coarser"** *underlying data.*

One elementary example of the phenomenon of *descent* is the following:

Example 3.9.1: Categories of open subschemes. Let X be a *scheme*, T a *topological space*. Write

· $|X|$ for the *underlying topological space* of X,
· $\mathrm{Open}(X)$ for the category of *open subschemes* of X and *open immersions* over X,
· $\mathrm{Open}(T)$ for the category of *open subsets* of T and *open immersions* over T.

Then the *functorial algorithm*

$$X \quad \mapsto \quad \mathrm{Open}(X)$$

— defined, say, on the category of schemes and morphisms of schemes — is easily verified to *descend*, relative to the construction $X \rightsquigarrow |X|$, to the *functorial algorithm*

$$T \quad \mapsto \quad \mathrm{Open}(T)$$

— defined, say, on the category of topological spaces and continuous maps of topological spaces. That is to say, one verifies immediately that there is a *natural functorial isomorphism*

$$\mathrm{Open}(X) \quad \stackrel{\sim}{\to} \quad \mathrm{Open}(|X|)$$

[i.e., in this case, following the conventions employed in inter-universal Teichmüller theory, a *natural functorial isomorphism class of equivalences of categories* — cf. the discussion of "Monoids and Categories" in [IUTchI], §0].

Perhaps the most fundamental example, in the context of inter-universal Teichmüller theory, of this phenomenon of *descent* is the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

· The *topological multiplicative monoid* determined by the *topological ring* given by [the union with $\{0\}$ of] $\mathcal{O}^{\rhd}(\Pi_X)$ [cf. [Alien], Example 2.12.3, (iii)] — that is to say, a construction that, *a priori*, from the point of view of [AbsTopIII], Theorem 1.9; [AbsTopIII], Corollary 1.10, is a functorial algorithm in the *topological group*

$$\Pi_X$$

[i.e., "$\Pi_{\underline{v}}$", from the point of view (sQGOut)] — in fact **descends** [cf. the discussion at the beginning of [Alien], §2.12; the discussion of [Alien], Example 2.12.3, (i)], relative to passage to the *underlying quotient group* discussed in (SQGOut), to a functorial algorithm in the *topological group*

$$G_k$$

[i.e., "$G_{\underline{v}}$", from the point of view (sQGOut)].

Finally, we remark that often, in inter-universal Teichmüller theory, the output data of the functorial algorithm Φ of the above discussion is regarded *"stack-theoretically"*. That is to say, the output data is *not a single "set-theoretic object"*, but rather a collection [which is not necessarily a set!] of set-theoretic objects linked by uniquely determined poly-isomorphisms of some sort. Typically, this sort of situation arises when one applies **(ExtInd2)** — cf. the discussion of **(NSsQ)** in §3.8. The most *central example* of this phenomenon in inter-universal Teichmüller theory is the **multiradial algorithm** — and, especially, the portion of the multiradial algorithm that involves the $\mathfrak{log}$-*Kummer correspondence* and closely related operations of *Galois evaluation* — which plays the role of

> *exhibiting the Frobenius-like* Θ*-pilot as* **one possibility** *within a* **collection of possibilities** *constructed via anabelian algorithms from étale-like data*

[cf. the discussion at the end of §3.6, as well as the discussion of §3.10, §3.11, below]. That is to say, the $\mathfrak{log}$-*Kummer correspondence* and closely related operations of *Galois evaluation* exhibit the Frobenius-like Θ-pilot as *one possibility* within a *collection of possibilities* constructed via anabelian algorithms from étale-like data **not** in a **set-theoretic sense** [i.e., *one possibility/element* contained in a *set of possibilities*], but rather in a *"stack-theoretic sense"*, in accordance with various applications of (ExtInd2) [cf. the discussion at the end of §3.6], i.e., as

> *one* **possibility***, up to* **isomorphism***, within some [not necessarily set-theoretic!] collection of* **possibilities**.

As discussed in (LVsQ) in §3.8, one arrives at a *set-theoretic situation* — i.e., *one possibility/element* contained in a *set of possibilities* — only after one obtains a *"closed loop"*, which allows one to pass to a *"coarse/set-theoretic (sub)quotient"* by taking the **log-volume**.

§**3.10. Detailed description of the chain of logical AND relations**

We begin the present §3.10 with the following well-known and, in some sense, *essentially tautological observation*: Just as every form of data — i.e., ranging from *text files* and *webpages* to *audiovisual data* — that can be processed by a computer can, ultimately, be expressed as a [perhaps very long!] *chain* of "0's" and "1's", the well-known *functional completeness*, in the sense of *propositional calculus*, of the collection of *Boolean operators* consisting of *logical AND "$\wedge$"*, *logical OR "$\vee$"*, and *negation "$\neg$"* motivates the point of view that one can, in principle, express

> the **essential logical structure** of any **mathematical argument** or **theory** in terms of **elementary logical relations**, i.e., such as **logical AND "$\wedge$"**, **logical OR "$\vee$"**, and **negation** "$\neg$".

Indeed, it is precisely this point of view that formed the *central motivation* and *conceptual starting point* of the exposition given in the present paper.

From the point of view of the correspondence with the terminology and modes of expression that actually appear in [IUTchI-III] and [Alien], the representation given in the present paper of the *essential logical structure* of inter-universal Teichmüller theory in terms of *elementary logical relations*, i.e., such as *logical AND "$\wedge$"* and *logical OR "$\vee$"*, may be understood as follows:

· **Logical AND "$\wedge$"** corresponds to such terms as

  · *simultaneous execution* and
  · *gluing*

[cf. [IUTchIII], Remark 3.11.1, (ii); [IUTchIII], Remark 3.12.2, (ii), (c^{itw}), (f^{itw}); the final portion of [Alien], §3.7, (i); [Alien], §3.11, (iv)].

· **Logical OR "$\vee$"** corresponds to such terms as

  · *indeterminacies*,
  · *poly-morphisms*, and
  · *projection/(sub)quotient/splitting*

[cf. §3.7; the title of [IUTchIII]; [IUTchIII], Remark 3.9.5, (xiii), (ix); [Alien], §3.11, (v); [Alien], §4.1, (iv)].

Recall the *essential logical structure* of inter-universal Teichmüller theory summarized in ($\wedge(\vee)$-Chn)

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \vee B_2 \vee \dots) \\ & \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots) \\ & \Longrightarrow & A \wedge (B_1 \vee B_2 \vee \dots \vee B_1' \vee B_2' \vee \dots \vee B_1'' \vee B_2'' \vee \dots) \\ & & \vdots \end{array}$$

[cf. the discussion of §3.6, §3.7]. Observe that if the description of the *various "possibilities"* related via "$\vee$'s" in the above displays is *suitably formulated*, i.e., without superfluous overlaps, then in fact these *logical OR "$\vee$'s"* may be understood as *logical XOR "$\dot{\vee}$'s"*, i.e., we conclude the following:

($\wedge(\dot{\vee})$-Chn) The *essential logical structure* of inter-universal Teichmüller theory may be summarized as follows:

$$\begin{array}{rcl} A \wedge B & = & A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots) \\ & \Longrightarrow & A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1' \ \dot{\vee}\ B_2' \ \dot{\vee}\ \dots) \\ & \Longrightarrow & A \wedge (B_1 \ \dot{\vee}\ B_2 \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1' \ \dot{\vee}\ B_2' \ \dot{\vee}\ \dots \ \dot{\vee}\ B_1'' \ \dot{\vee}\ B_2'' \ \dot{\vee}\ \dots) \\ & & \vdots \end{array}$$

Here, we observe the following:

($\wedge(\dot{\vee})$-Chn1) The "$\wedge$'s" in the above display

· arise from the **Θ-link**, which may be thought of as a relationship between certain portions of the **multiplicative** structures of the ring structures arising from the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters* in the domain and codomain of the Θ-*link* that are **common** [cf. "$\wedge$!"] to these ring structures.

This situation is reminiscent of

- the fact that from the point of view of **Boolean algebras**, "$\wedge$" corresponds to the **multiplicative** structure of the *field* $\mathbb{F}_2$, which may be regarded, via the splitting determined by Teichmüller representatives, as a multiplicative structure that is **common** [cf. "$\wedge$!"] to $\mathbb{Z}$ and $\mathbb{F}_2$ [cf. Example 2.4.6, (iii)], as well as of

- the discussion of [Alien], §3.11, (iv), (2^{and}), concerning the interpretation of the discussion of **crystals** in [Alien], §3.1, (v), (3^{KS}), in terms of the logical relator "$\wedge$", i.e., as objects that may be **simultaneously** interpreted, up to isomorphism, as pull-backs via one projection morphism **and** [cf. "$\wedge$!"] pull-backs via another projection morphism.

($\wedge(\dot{\vee})$-Chn2) The "$\dot{\vee}$'s" in the above display may be understood as corresponding to

- various *indeterminacies* that arise mainly from the **$\mathfrak{log}$-Kummer-correspondence**, i.e., from sequences of iterates of the **$\mathfrak{log}$-link**, which may be thought of as a device for constructing **additive** log-shells. The additive structures of the ring structures arising from the *($\Theta^{\pm \mathrm{ell}}$NF-)Hodge theaters* in the domain and codomain of the Θ-*link* are structures which, unlike the corresponding multiplicative structures, are **not common** [cf. "$\dot{\vee}$"!] to these ring structures in the domain and codomain of the Θ-*link*.

This situation is reminiscent of

- the fact that from the point of view of **Boolean algebras**, "$\dot{\vee}$" corresponds to the **additive** structure of the *field* $\mathbb{F}_2$, which is an additive structure that is **not shared** [cf. "$\dot{\vee}$"!], relative to the splitting determined by Teichmüller representatives, by $\mathbb{Z}$ and $\mathbb{F}_2$ [cf. Example 2.4.6, (iii)], as well as of

- the discussion of [Alien], §3.11, (iv), (2^{and}), concerning the interpretation of the discussion of *crystals* in [Alien], §3.1, (v), (3^{KS}) in terms of the logical relator "$\wedge$", i.e., where we recall that the two pull-backs of the **rank one Hodge subbundle** [cf. [Alien], §3.1, (v), (5^{KS}); the discussion of Hodge structures in [IUTchI], §I2] do **not**, in general, **coincide** [cf. "$\dot{\vee}$"!], but rather differ by an **additive** *"deformation discrepancy"*, namely, the **Kodaira-Spencer morphism**.

($\wedge(\dot{\vee})$-Chn3) Taken together, ($\wedge(\dot{\vee})$-Chn1) and ($\wedge(\dot{\vee})$-Chn2) may be understood as expressing the fact that the "$\dot{\vee}$'s" and "$\wedge$'s" of the above display correspond, respectively, to the **two underlying combinatorial dimensions** — i.e., **addition** and **multiplication** — of a ring or, alternatively, to the **two-dimensional** nature of the **log-theta-lattice** [cf. the discussion of [IUTchIII], Remark 3.12.2, (i); the latter portion of [Alien], §3.3, (ii)]. Thus, these two dimensions may be understood, alternatively, as corresponding to

- the **arithmetically intertwined Θ-link** and $\mathfrak{log}$**-link** of inter-universal Teichmüller theory, which give rise to the **multiradial representation** up to suitable **indeterminacies** [cf. "$\wedge(\dot{\vee})$"!] of the **Θ-pilot**;
- the description given in Example 2.4.6, (iii), of the **carry-addition** operation on the truncated ring of Witt vectors $\mathbb{Z}/4\mathbb{Z}$ in terms of "$\wedge$" and "$\dot{\vee}$" [cf. "$\wedge(\dot{\vee})$"!];
- the **filtered crystal** discussed in [Alien], §3.1, (v), (5^{KS}), where one may think of the *filtration* [i.e., rank one Hodge subbundle] as "being well-defined up to *indeterminacies*" [cf. "$\wedge(\dot{\vee})$"!], i.e., up to a *"discrepancy"*, which is given by the **Kodaira-Spencer morphism**.

($\wedge(\dot{\vee})$-Chn4) The two dimensions discussed in ($\wedge(\dot{\vee})$-Chn3) may be understood as corresponding to the two dimensions — i.e.,

- the **successive iterates** of the **Frobenius morphism** in **positive characteristic** and
- **successive extensions** to **mixed characteristic**

— of a ring of **Witt vectors** [cf. the discussion of the latter portion of [Alien], §3.3, (ii)]. This relationship to the two dimensions of a ring of **Witt vectors** is *entirely consistent* with the way in which truncated rings of Witt vectors occur in the discussion of Example 2.4.6, (iii), i.e., with the expression

$$\ddot{\vee} \ = \ (\wedge, \dot{\vee})$$

of *mixed characteristic* **"carry-addition"** as a sort of **"intertwining"** between addition and multiplication in the *field* $\mathbb{F}_2$ obtained by *"stacking"* **multiplication** *"$\wedge$" on top of* **addition** *"$\dot{\vee}$"*.

Moreover, in this context, we note that the various correspondences observed in ($\wedge(\dot{\vee})$-Chn3) and ($\wedge(\dot{\vee})$-Chn4) are particularly *fascinating* in that they assert that the **"arithmetic intertwining"** in a ring between **addition** and **multiplication** — i.e., the **mathematical structure** which is in some sense the main object of study in inter-universal Teichmüller theory — may be elucidated by means of a theory [i.e., inter-universal Teichmüller theory!] whose **essential logical structure**, when written symbolically in terms of **Boolean operators** such as "$\wedge$" and "$\dot{\vee}$", amounts precisely to the description [cf. the discussion of ($\ddot{\vee} = \wedge \, \dot{\vee}$) in Example 2.4.6, (iii)] of the **"Boolean intertwining"** that appears in Boolean carry-addition "$\ddot{\vee}$" between Boolean addition "$\dot{\vee}$" and Boolean multiplication "$\wedge$". Put another way, it is as if

(TrHrc) the *arithmetic intertwining* which is the main object of study in inter-universal Teichmüller theory somehow *"induces"/"is reflected in"* a sort of **"structural carry operation"**, or **"trans-hierarchical similitude"**, to the *Boolean intertwining* that constitutes the *essential logical structure* of the theory [i.e., inter-universal Teichmüller theory] that is used to describe it:

$$\textbf{arithmetic intertwining} \quad \rightsquigarrow \quad \textbf{Boolean intertwining}!$$

Finally, we observe that it also interesting to note that the *essential mechanism* underlying the **Kummer theory** of **theta functions** — which plays a central role in inter-universal Teichmüller theory, i.e., in inducing the *trans-hierarchical similude* discussed in (TrHrc) — namely, the correspondence

$$\begin{array}{rcl} \text{Kummer theory of } \textit{theta functions} & \longleftrightarrow & \textit{one} \text{ valuation/cusp} \\ \text{Kummer theory of } \textit{algebraic rational functions} & \longleftrightarrow & \textit{multiple} \text{ valuations/cusps} \end{array}$$

discussed in [IUTchIII], Remark 2.3.3, (viii), (ix), may itself be thought of as a sort of **trans-hierarchical similitude** between **number fields/local fields** and **function fields** over number fields/local fields.

At a more technical level, the *essential logical structure* of inter-universal Teichmüller theory summarized symbolically in ($\wedge(\dot{\vee})$-Chn) may be understood as consisting of the following steps:

(Stp1) **$\mathfrak{log}$-Kummer-correspondence and Galois evaluation**: This step consists of

> *exhibiting the* **Frobenius-like Θ-pilot** *at the lattice point* $(0, 0)$ *of the log-theta-lattice — i.e., the data that gives rise to the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip that appears in the* **domain** *of the* **Θ-link** *— as* **one possibility** *within a* **collection of possibilities** *constructed via anabelian algorithms from* **holomorphic** *[relative to the* 0*-column]* **étale-like data**.

In this context, it is perhaps worth mentioning that it is a *logical tautology* that the content of the above display may, equivalently, be phrased as follows: this step consists of

> *the* **negation "$\neg$"** *of the assertion of the* **nonexistence** *of the* **Frobenius-like Θ-pilot** *at the lattice point* $(0, 0)$ *of the log-theta-lattice — i.e., the data that gives rise to the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip that appears in the* **domain** *of the* **Θ-link** *— within the* **collection of possibilities** *constructed via certain anabelian algorithms from* **holomorphic** *[relative to the* 0*-column]* **étale-like data**

[cf. also the discussion of (RcnLb) below]. At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0,0) \quad \rightsquigarrow \quad (0,\circ)$$

[cf. the discussion at the end of §3.6; the discussion at the end of §3.9; [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (ii), (iii)]. Finally, we recall that this step already involves the introduction of the **(Ind3) indeterminacy**.

(Stp2) **Introduction of (Ind1)**: This step consists of observing that

> *the anabelian construction algorithms of (Stp1) in fact* **descend** *to — i.e., are* **equivalent** *to algorithms that only require as input data the* **weaker data** *constituted by [cf. the discussion of "descent" in §3.9] — the associated* **mono-analytic étale-like** *data, i.e., in the notation of (sQGOut), the "$G_{\underline{v}}$'s".*

At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0,\circ) \quad \rightsquigarrow \quad (0,\circ)^{\vdash}$$

[cf. [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), as well as the references to [IUTchIII], Theorem 3.11, (i), in [IUTchIII], Theorem 3.11, (iii)]. Finally, we recall that this step involves the introduction of the **(Ind1) indeterminacy**.

(Stp3) **Introduction of (Ind2)**: This step consists of observing that

> *the anabelian construction algorithms of (Stp2) in fact* **descend** *to — i.e., are* **equivalent** *to algorithms that only require as input data the* **weaker data** *constituted by [cf. the discussion of "descent" in §3.9] — the associated* **mono-analytic Frobenius-like** *data, i.e., in the notation of (MnOut), the "$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$'s".*

[That is to say, one constructs log-shells, for instance, as submonoids of "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$", as opposed to subquotients of "$G_{\underline{v}}$".] At the level of *labels* of lattice points of the log-theta-lattice, this step corresponds to the *descent operation*

$$(0,\circ)^{\vdash} \quad \rightsquigarrow \quad (0,0)^{\vdash}$$

[cf. [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), as well as the references to [IUTchIII], Theorem 3.11, (i), in [IUTchIII], Theorem 3.11, (iii)]. Since the **Θ-link** may be thought of as a sort of *equivalence of labels*

$$(0,0)^{\vdash} \quad \Longleftrightarrow \quad (1,0)^{\vdash}$$

— i.e., corresponding to the *full poly-isomorphism* of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips constituted by the Θ-link — this *descent operation* means that the algorithm under consideration may be regarded as an algorithm whose **input**

data is the **mono-analytic Frobenius-like** data $(1,0)^{\vdash}$ arising from the **codomain** of the Θ-link. Finally, we recall that this step involves the introduction of the **(Ind2) indeterminacy** and plays the important role of

> **isolating** the **$\mathfrak{log}$-link** indeterminacies in the **domain** [i.e., the (Ind3) indeterminacy of (Stp1)] and the **codomain** [i.e., the *log-shift adjustment* discussed in (Stp7) below] of the **Θ-link** from one another

[cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob7-2); [Alien], §3.6, (iv)]. Here, we recall [cf. the discussion of the final portion of [Alien], §3.3, (ii)] that these $\mathfrak{log}$-*link* indeterminacies on either side of the Θ-link may be understood, in the context of the discussion of (InfH) in §3.3, as corresponding to the copies "$\mathbb{C}^{\times}$" on either side of the double coset space "$\mathbb{C}^{\times}\backslash GL_2^{+}(\mathbb{R})/\mathbb{C}^{\times}$".

(Stp4) <u>**Passage to the holomorphic hull**</u>: The passage from the collection of *possible regions* that appear in the *output data* of (Stp3) to the collection of regions contained in the **holomorphic hull** — relative to the 1-*column* of the log-theta-lattice — of the union of possible regions of the output data of (Stp3) [cf. [IUTchIII], Remark 3.9.5, (vi); [IUTchIII], Remark 3.9.5, (vii), (Ob5); [IUTchIII], Remark 3.9.5, (viii), (sQ3)] is a simple, straighforward application of (ExtInd1), that is to say, of *increasing the set of possibilities* [i.e., of "$\dot{\vee}$'s"]. The purpose of this step, together with (Stp5) below, is to pass from *arbitrary regions* to regions corresponding to *arithmetic vector bundles* [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob1), (Ob2)].

(Stp5) <u>**Passage to hull-approximants**</u>: This step consists of passing from the collection of arbitrary regions contained the *holomorphic hull* of (Stp4) to **hull-approximants**, i.e., regions that have the *same global log-volume* as the original "arbitrary regions", but which correspond to *arithmetic vector bundles* [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob6); [IUTchIII], Remark 3.9.5, (viii), (sQ3)]. This operation does *not affect* the logical "$\wedge/\vee$" structure of the algorithm since this operation of passing to hull-approximants does

> **not affect** the collection of **possible value group portions** — i.e., *"$\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips"* — of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips determined by forming the *log-volume* of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp6) <u>**Passage to a suitable positive rational tensor power of the determinant**</u>: This step consists of passing from the [regions corresponding to] *arithmetic vector bundles* obtained in (Stp4), (Stp5) to a *suitable tensor power root* of a *tensor power* of the *determinant arithmetic line bundle* of such an arithmetic vector bundle [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob3), (Ob4); [IUTchIII], Remark 3.9.5, (viii), (sQ3)]. Just as in the case of (Stp5), this operation does *not affect* the logical "$\wedge/\vee$" structure of the algorithm since this operation of passing to a suitable positive rational tensor power of the determinant does

> **not affect** the collection of **possible value group portions** — i.e., *"$\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips"* — of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips determined by forming the *log-volume* of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp7) **Log-shift adjustment**: The *arithmetic line bundles* that appear in (Stp6) occur with respect to the arithmetic holomorphic structure — i.e., in effect, *ring structure* — at the label $(1,1)$ of the log-theta-lattice, i.e., at a label *vertically shifted* by $+1$ relative to the label $(1,0)$ that forms the codomain of the Θ-link [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob8); [IUTchIII], Remark 3.9.5, (viii), (sQ4)]. That is to say, by applying the algorithm discussed in (Stp1) $\sim$ (Stp6) at *each lattice point* $(1,m)$ [where $m \in \mathbb{Z}$] of the 1-column of the log-theta-lattice, we obtain algorithms with *input data* at $(1,m)$ and *output data* at $(1,m+1)$ — cf. the *diagonal arrows* of the diagram shown below. Thus, the **totality** of all of these *diagonal arrows* may be thought of a sort of **endomorphism** of the 1-column of the log-theta-lattice, i.e., an algorithm whose **input data** is the 1-column of the log-theta-lattice, and whose **output data** lies in the **same** 1-column of the log-theta-lattice. In particular, we obtain a **closed loop** [cf. the discussion of [IUTchIII], Remark 3.9.5, (ix); [Alien], §3.11, (v)]. Put another way, in the language of the discussion surrounding (DltLb) in §3.11 below, we obtain a situation that **simulates** — via the introduction of suitable **indeterminacies** [cf. the discussion of (Stp8) below] — a situation in which the **distinct labels** on the domain and codomain of the Θ-link have been **eliminated**.

$$
\begin{tikzcd}
\text{1-column} & \text{1-column} \\
\vdots & \vdots \\
\bullet & \bullet \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[ur] & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[ur] & \bullet \arrow[u, "\mathfrak{log}"'] \\
\bullet \arrow[u, "\mathfrak{log}"'] \arrow[ur] & \bullet \arrow[u, "\mathfrak{log}"'] \\
\vdots & \vdots
\end{tikzcd}
$$

(Stp8) **Passage to log-volumes**: The **closed loop** of (Stp7) implies that the crucial **logical AND "$\wedge$"** relation *carefully maintained throughout the execution of (Stp1) $\sim$ (Stp7)* yields, upon taking the **log-volume**, a

> **logical AND "$\wedge$"** relationship between the **original q-pilot input $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip** and a certain algorithmically constructed collection of **possible output $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips** within the **same container**, i.e., some copy of the real numbers "$\mathbb{R}$"

[cf. [IUTchIII], Remark 3.9.5, (vii), (Ob9); [IUTchIII], Remark 3.9.5, (viii), (sQ5); [IUTchIII], Remark 3.9.5, (ix); the discussion of Substeps (xi-d), (xi-e) of the proof of [IUTchIII], Corollary 3.12; the discussion of [IUTchIII], Remark 3.12.2, (ii); [Alien], §3.11, (v)]. The **inequality** in the statement of [IUTchIII], Corollary 3.12, then follows as a formal consequence [cf. the discussion of Substeps (xi-f), (xi-g) of the proof of [IUTchIII], Corollary 3.12]. Here, we observe that the various *indeterminacies* [such as (Ind1), (Ind2), (Ind3)] that arise in the course of applying (Stp1) $\sim$ (Stp7) may be thought of as a sort of *monodromy* associated to the *closed loop* of (Stp7). Moreover, in this context, we note that the *diagram* of arrows "$\nearrow$" from the 1-*column* to the 1-*column* in (Stp7) admits **symmetries**

$$(1, m) \mapsto (1, m + 1)$$

[where $m \in \mathbb{Z}$] that are *compatible* with all of the arrows in the *diagram*, as well as with the various arrows of the $\mathfrak{log}$-*Kummer-correspondence* in the 1-*column*. These *symmetries* allow one to **synchronize** the various **"monodromy indeterminacies"** associated to each "$\nearrow$" [i.e., to each application of (Stp1) $\sim$ (Stp6)], so that one may think in terms of a **single, synchronized** collection of *"monodromy indeterminacies"* associated to the *totality* of "$\nearrow$'s" in (Stp7).

§3.11. The central importance of the $\mathfrak{log}$-Kummer correspondence

In the context of the discussion of §3.10, it is important to recall that, whereas (Stp2) $\sim$ (Stp8) are *technically trivial* in the sense that they concern operations that are very elementary and only require a few lines to describe, the **$\mathfrak{log}$-Kummer correspondence** and **Galois evaluation** operations that comprise (Stp1) depend on the *highly nontrivial* theory of [EtTh] and [AbsTopIII]. Moreover, the technical description of these operations that comprise (Stp1) occupies the bulk of [IUTchI-III]. The central importance of (Stp1) may also be seen in the **subordinate nature** of (Ind1), (Ind2) [which occur in (Stp2), (Stp3)] relative to (Ind3) [which occurs in (Stp1)], i.e., in the sense that

(Ind3>1+2) once one constructs the *output* of the **multiradial representation** of the Θ-pilot [cf. [IUTchIII], Theorem 3.11, (ii)] via **tensor-packets of log-shells** in such a way that each local portion of this output is *stable* with respect to the indeterminacy (Ind3), these local portions of the output are *automatically stable* with respect to the indeterminacies (Ind1), (Ind2) [cf. [IUTchIII], Theorem 3.11, (i)].

Finally, we observe that this property (Ind3>1+2) is strongly reminiscent of the discussion of (CnfInd1+2) and (CnfInd3) in §3.5.

One way to understand the content of the operations of (Stp1) is as follows. These operations may be regarded as a sort of

($\mathfrak{log}$ORInd) **saturation** of the **Frobenius-like Θ-pilot** at the lattice point $(0, 0)$ of the log-theta-lattice — i.e., which is *linked*, via the Θ-link, to the **Frobenius-like *q*-pilot** at $(0, 1)$ — with respect to *all of the* **possibilities** that occur in the 0-column of the log-theta-lattice, i.e., all of the possibilities that arise from a *possible confusion* between the *domain* and *codomain* of the **$\mathfrak{log}$-links** in the 0-column [cf. the description of (Stp1)].

In this sense, the content of (Stp1) is *formally reminiscent* of the "(NeuORInd)" that appeared in the discussion of §3.4, i.e., which may be understood as a sort of

(ΘORInd) **saturation** of the **Frobenius-like Θ-pilot** at the lattice point $(0, 0)$ of the log-theta-lattice with respect to *all of the* **possibilities** — i.e., "Θ-$\mathfrak{plt}$", "$\mathfrak{q}$-$\mathfrak{plt}$" [cf. (NeuORInd2)] — that arise from a *possible confusion* between the *domain* and *codomain* of the **Θ-link** joining the lattice points $(0, 0)$ and $(0, 1)$.

[In this context, we note that the *logical OR "$\vee$'s"* that appear in ($\mathfrak{log}$ORInd), (ΘORInd) may in fact be understood as *logical XOR "$\dot{\vee}$'s"* — cf. the discussion surrounding ($\wedge$($\dot{\vee}$)-Chn) in §3.10.] On the other hand, whereas, as observed in the discussion at the end of §3.4, (ΘORInd) yields a *meaningless/useless* situation that does not give rise to any interesting mathematical consequences, ($\mathfrak{log}$ORInd), by contrast, is a **highly potent technical device** that forms the **technical core** of inter-universal Teichmüller theory.

The *stark contrast* between the **potency** of ($\mathfrak{log}$ORInd) and the *utterly meaningless* nature of (ΘORInd) is *highly reminiscent* of the **central role** played, in Example 3.3.2, (iv), by *invariance* with respect to

$$\iota \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL_2^+(\mathbb{R})$$

[where we recall from (InfH) that $\mathbb{C}^\times$ corresponds to the **$\mathfrak{log}$-link**!], which lies in stark contrast to the *utterly meaningless* nature of considering invariance with respect to *dilations* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ [where we recall from (InfH) that such dilations correspond to the Θ-*link*].

One way to witness the potency of ($\mathfrak{log}$ORInd) is as follows. Recall that the **Θ-link**, by definition [cf. [IUTchIII], Definition 3.8, (ii)], consists of

· a **dilation** applied to the *local value group* portions of the ring structures in its domain and codomain, coupled with

· a **full poly-isomorphism** — which *preserves log-volumes*, hence is **non-dilating**! — between the local "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s", i.e., the *local unit group* portions, of these ring structures.

By contrast, the **$\mathfrak{log}$-links** in the 0-column of the log-theta-lattice have the effect of *"juggling/rotating/permuting"* the *local value group* portions and *local unit group* portions of the ring structures that appear in this 0-column [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. From this point of view, the *tautologically vertically coric* — i.e., *invariant* with respect to the application of the $\mathfrak{log}$-*link*! — nature of the *output* data of ($\mathfrak{log}$ORInd) is already somewhat **"shocking"** in nature. That is to say, the tautologically vertically coric nature of this output data of ($\mathfrak{log}$ORInd) suggests that

(Di/NDi) this output data already exhibits some sort of **equivalence**, up to perhaps some sort of mild discrepancy, between the **dilated** and **non-dilated** portions of the Θ-link.

Such an *equivalence* already strongly suggests that some sort of **bound** on **heights** should follow as a *formal consequence*, i.e., in the style of the classical argument that implies the *isogeny invariance of heights of elliptic curves* [cf. the discussion of [Alien], §2.3, §2.4].

Finally, we conclude by emphasizing that, in inter-universal Teichmüller theory,

(DltLb) ultimately one *does* want to find some way in which to **delete/eliminate** the **distinct labels** on the Θ- and q-pilot objects [i.e., "Θ-$\mathfrak{plt}$" and "$\mathfrak{q}$-$\mathfrak{plt}$"] in the domain and codomain of the Θ-link

[cf. the discussion of (AOL4), (AOΘ4) in §3.4; (Stp7), (Stp8) in §3.10], that is to say, **not** via the *naive, simple-minded* approach of (ΘORInd) [i.e., (NeuORInd2) in the discussion of §3.4], but rather via the *indirect approach* of applying **descent operations**

$$(0,0) \quad \overset{\text{(Stp1)}}{\rightsquigarrow} \quad (0,\circ) \quad \overset{\text{(Stp2)}}{\rightsquigarrow} \quad (0,\circ)^{\vdash} \quad \overset{\text{(Stp3)}}{\rightsquigarrow} \quad (0,0)^{\vdash} \quad \overset{\text{(Stp3)}}{\Longleftrightarrow} \quad (1,0)^{\vdash}$$

as discussed in (Stp1) $\sim$ (Stp8) [cf., especially, (Stp7), (Stp8)] of §3.10, i.e., an approach that centers around ($\mathfrak{log}$ORInd). This approach is based on the various **anabelian reconstruction algorithms** discussed in (Stp1) $\sim$ (Stp3), which allow one to exhibit the **Frobenius-like Θ-pilot object** at $(0,0)$ as *one possibility* among *some broader collection of possibilities* that arise from the introduction of various types of **indeterminacy**. In this context, we observe [cf. the discussion of (ExtInd2), (NSsQ) at the end of §3.9] that since such anabelian reconstruction algorithms only reconstruct various types of mathematical objects [i.e., monoids/pseudo-monoids/mono-theta environments, etc.] *not "set-theoretically on the nose"*, so to speak, but rather *up to [a typically essentially unique, if one allows for suitable indeterminacies] isomorphism*, it is not immediately clear

(RcnLb) in what sense such anabelian reconstruction algorithms yield a **reconstruction** of the **crucial labels** — i.e., such as "$(0,0)$" — that underlie the **crucial logical AND "$\wedge$"** structure discussed in §3.4 [cf., especially, (AOL1), (AOΘ1)].

The point here is that indeed such anabelian reconstruction algorithms are *not capable* of reconstructing such labels *"set-theoretically on the nose"*.

On the other hand, in this context, it is important to recall the *essential substantive content* of the various **labels** involved:

(HolFrLb) $\underline{(0,0)}$: The **holomorphic Frobenius-like** data labeled by $(0,0)$ consists of various monoids/pseudo-monoids/mono-theta environments, etc., regarded as *abstract monoids/pseudo-monoids/mono-theta environments, etc.*, i.e., as objects that are *not* equipped with the auxiliary data of how they might have been reconstructed via anabelian algorithms from *holomorphic étale-like* data labeled $(0, \circ)$ [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. In particular, such monoids/pseudo-monoids/mono-theta environments, etc., are **not invariant** with respect to the *"juggling/rotating/permuting"* of *local value group* portions and *local unit group* portions effected by the **log-links** in the 0-column of the log-theta-lattice, but rather correspond to a **temporary cessation** [cf. the label $(0, 0)$ as opposed to the label $(0, \circ)$!] of this operation of *juggling/rotation/permutation*.

(MnAlyLb) $\underline{(0, 0)^{\vdash}}$: The **mono-analytic Frobenius-like** data labeled by $(0, 0)^{\vdash}$ consists of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* determined by the *Frobenius-like* Θ-*pilot* at $(0, 0)$, regarded as an *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. Thus, the transition of labels

$$(0, 0) \quad \rightsquigarrow \quad (0, 0)^{\vdash}$$

consists of an operation of **forgetting** some sort of auxiliary structure [cf. the discussion of (UdOut) in §3.9]. Here, we recall that this construction of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip determined by the Frobenius-like Θ-pilot at $(0, 0)$ is *technically possible* precisely because of the *"temporary cessation"* discussed above [cf. the discussion of the *definition of the* Θ-*link* in [Alien], §3.3, (ii), as well as in §3.3 of the present paper].

Thus, the *nontrivial substantive content* of the anabelian reconstruction algorithms of (Stp1) $\sim$ (Stp3) — and hence of the **descent operations**

$$(0, 0) \quad \overset{\text{(Stp1)} \sim \text{(Stp3)}}{\rightsquigarrow} \quad (0, 0)^{\vdash}$$

that result from these anabelian reconstruction algorithms — consists of statements to the effect that

(FrgInv) the operation of **forgetting** discussed in (MnAlyLb) can in fact, if one allows for suitable **indeterminacies**, be **inverted**.

It is precisely this **invertibility** (FrgInv), up to suitable *indeterminacies*, of the operation of **forgetting** discussed in (MnAlyLb), together with the fact that

(GluDt) the **only data** appearing in the reconstruction algorithms [i.e., in the 0-*column*] that is **glued** [cf. the discussion of [IUTchIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i)] to data in the 1-*column* is the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* labeled $(0, 0)^{\vdash}$,

that ensures that the *descent operations* discussed above do indeed **preserve** the crucial **logical AND "$\wedge$"** relations discussed in §3.4, §3.6, §3.7, §3.10, i.e., even though the reconstruction algorithms underlying these descent operations do *not* yield reconstructions of the various labels "$(0, 0)$", etc., "set-theoretically on the nose".

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