ON THE ESSENTIAL LOGICAL STRUCTURE OF INTER-UNIVERSAL TEICHMÜLLER THEORY IN TERMS OF LOGICAL AND “∧”/LOGICAL OR “∨” RELATIONS: REPORT ON THE OCCASION OF THE PUBLICATION OF THE FOUR MAIN PAPERS ON INTER-UNIVERSAL TEICHMÜLLER THEORY

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Introduction

In the present paper, we give a detailed exposition of the essential logical structure of inter-universal Teichmüller theory in terms of elementary Boolean operators such as logical AND “∧” and logical OR “∨”. One important topic in this exposition is the issue of “redundant copies”, i.e., the issue of how the arbitrary identification of copies of isomorphic mathematical objects that appear in the various constructions of inter-universal Teichmüller theory impacts — and indeed invalidates — the essential logical structure of inter-universal Teichmüller theory. This issue continues, at least at the time of writing of the present paper, to be a focal point of fundamental misunderstandings and entirely unnecessary confusion concerning inter-universal Teichmüller theory in certain sectors of the mathematical community [cf. the discussion of Examples 2.4.5, 2.4.7].

We begin, in §1, by reporting on various non-mathematical aspects of the situation surrounding inter-universal Teichmüller theory, such as the issue of “redundant copies”. Perhaps the most central portion of this discussion of non-mathematical aspects of the situation surrounding inter-universal Teichmüller theory concerns the long-term, historical importance of producing detailed, explicit, mathematically substantive, and readily accessible written documentation of the essential logical structure of the issues under debate [cf. §1.5]. Such written documentation of the essential logical structure of the issues under debate is especially important in situations such as the situation that has arisen surrounding inter-universal Teichmüller theory, in which the proliferation of logically unrelated fabricated versions of the theory has led to fundamental misunderstandings and entirely unnecessary confusion concerning inter-universal Teichmüller theory in certain sectors of the mathematical community that are deeply detrimental to the operational normalcy of the field of mathematics [cf. §1.3, §1.8, §1.10, §1.11].

We then proceed, in §2, to discuss elementary aspects of the mathematics surrounding the essential logical structure of inter-universal Teichmüller theory. Our discussion of these elementary aspects, which concerns mathematics at the advanced undergraduate or beginning graduate level and does not require any advanced knowledge of anabelian geometry or inter-universal Teichmüller theory, focuses on the close relationship between

- integration and differentiation on — i.e., so to speak, the “differential geometry” of — the real line [cf. §2.1, §2.2, as well as Example 2.4.2],

- the geometry of adjacent closed intervals of the real line and the loops that arise by identifying various closed subspaces of such closed intervals [cf. §2.3; Example 2.4.1], and

- Boolean operators such as logical AND “∧” and logical OR “∨” [cf. §2.4].
One important unifying theme that relates these seemingly disparate topics is the theme of “carry operations”, which appear in the various arithmetic, geometric, and Boolean-logical situations discussed in §2 [cf. Example 2.4.6]. On the other hand, from the point of view of arithmetic geometry, the discussion of the projective line as a gluing of ring schemes along a multiplicative group scheme given in Example 2.4.7 yields a remarkably elementary qualitative model/analog of the essential logical structure surrounding the gluing given by the Θ-link in inter-universal Teichmüller theory.

The preparatory topics of §2 lead naturally to the detailed exposition of the essential logical structure of inter-universal Teichmüller theory given in §3. From a strictly rigorous point of view, this exposition assumes a substantial level of knowledge and understanding of the technicalities of inter-universal Teichmüller theory [which are surveyed, for instance, in [Alien]], although the essential mathematical content of most of the issues discussed may in fact be understood at the level of the elementary considerations discussed in §2. The essential logical structure of inter-universal Teichmüller theory may be represented symbolically as follows:

\[
A \land B = A \land (B_1 \oplus B_2 \oplus \ldots)
\]

\[
\Rightarrow A \land (B_1 \oplus B_2 \oplus \ldots \oplus B'_1 \oplus B'_2 \oplus \ldots)
\]

\[
\Rightarrow A \land (B_1 \oplus B_2 \oplus \ldots \oplus B'_1 \oplus B'_2 \oplus \ldots \oplus B''_1 \oplus B''_2 \oplus \ldots)
\]

\[\vdots\]

[cf. the discussion of (\land(\lor)-Chn) in §3.10]. [Here, “\lor” denotes the Boolean operator exclusive-OR, i.e., “XOR”.] Indeed, §3 is devoted, for the most part, to giving a detailed exposition of various aspects of this symbolic representation, such as the following:

- the logical AND “\land’s” in the above display may be understood as corresponding to the Θ-link of inter-universal Teichmüller theory and are closely related to the multiplicative structures of the rings that appear in the domain and codomain of the Θ-link;
- the logical XOR “\lor’s” in the above display may be understood as corresponding to various indeterminacies that arise mainly from the log-Kummer-correspondence, i.e., from sequences of iterates of the log-link of inter-universal Teichmüller theory, which may be thought of as a device for constructing additive log-shells.

This appearance of logical AND “\land’s” and logical XOR “\lor’s” is of interest in that it is reminiscent of the well-known description of the “carry-addition” operation on Teichmüller representatives of the truncated Witt ring \(\mathbb{Z}/4\mathbb{Z}\) in terms of Boolean addition “\lor” and Boolean multiplication “\land” in the field \(\mathbb{F}_2\) and may be understood as a sort of “Boolean intertwining” that mirrors, in a remarkable fashion, the “arithmetic intertwining” between addition and multiplication in number fields and local fields which is, in some sense, the main object of study in inter-universal Teichmüller theory [cf. the discussion of Example 2.4.6, (iii); the discussion surrounding (TrHrc) in §3.10].
Section 1: General summary for non-specialists

We begin with an overall summary of non-mathematical aspects of the situation surrounding [IUTchI-IV], which may be of interest to both non-mathematicians and mathematicians. We also refer to [FsADT], [FKvid], [FsDss], [FsPio] for a discussion of various aspects of this situation from slightly different points of view.

§ 1.1. Publication of [IUTchI-IV]

The four main papers [IUTchI-IV] on inter-universal Teichmüller theory (IUT) were accepted for publication in the Publications of the Research Institute for Mathematical Sciences (PRIMS) on February 5, 2020. This was announced at an online video news conference held at Kyoto University on April 3, 2020. The four papers were subsequently published in several special volumes of PRIMS, a leading international journal in the field of mathematics with a distinguished history dating back over half a century.

The refereeing for these Special Volumes was overseen by an Editorial Board for the Special Volumes chaired by Professors Masaki Kashiwara and Akio Tamagawa. [Needless to say, as the author of these four papers, I was completely excluded from the activities of this Editorial Board for the Special Volumes.] Professor Kashiwara, a professor emeritus at RIMS, Kyoto University, is a global leader in the fields of algebraic analysis and representation theory. Professor Tamagawa, currently a professor at RIMS, Kyoto University, is a leading pioneer in the field of anabelian geometry and related research in arithmetic geometry. Here, it should be noted that, to a substantial extent,

inter-universal Teichmüller theory arose as an extension/application — developed by the author in the highly mathematically stimulating environment at RIMS, Kyoto University, over the course of roughly two decades [i.e., 1992 - 2012] — of precisely the sort of anabelian geometry that was pioneered by Tamagawa.

It is for this reason that PRIMS stood out among mathematics journals worldwide as the most appropriate — i.e., in the sense of being by far the most [and indeed perhaps the only truly] technically qualified — journal for the task of refereeing and publishing the four papers [IUTchI-IV] on inter-universal Teichmüller theory.

Both Professors Kashiwara and Tamagawa have an outstandingly high international reputation, built up over distinguished careers that span several decades. It is entirely inconceivable that any refereeing process overseen by these mathematicians might be conducted relative to anything less than the highest mathematical standards, free of any inappropriate non-mathematical considerations. In an article in the Asahi Shimbun [a major Japanese newspaper] published shortly after the announcement of April 3, 2020, Professor Tamagawa is quoted as saying that he has

“100 percent confidence in the refereeing”

that was done for the four papers [IUTchI-IV].

In another article in the Asahi Shimbun [also published shortly after the announcement of April 3, 2020], Professors Shigefumi Mori, a professor emeritus at
RIMS, Kyoto University, and Nobushige Kurokawa, a professor emeritus at the Tokyo Institute of Technology, express their expectations about the possibility of applying inter-universal Teichmüller theory to other unsolved problems in number theory.

In particular, the results proven in these four papers [IUTchI-IV] may now be quoted in the mathematical literature as results proven in papers that have been published in a leading international journal in the field of mathematics after undergoing an exceptionally thorough [seven and a half year long] refereeing process.

§1.2. Redundancy assertions of the “redundant copies school” (RCS)

Unfortunately, it has been brought to my attention that, despite the developments discussed in §1.1, fundamental misunderstandings concerning the mathematical content of inter-universal Teichmüller theory persist in certain sectors of the mathematical community. These misunderstandings center around a certain oversimplification — which is patently flawed, i.e., leads to an immediate contradiction — of inter-universal Teichmüller theory. This oversimplified version of inter-universal Teichmüller theory is based on assertions of redundancy concerning various multiple copies of certain mathematical objects that appear in inter-universal Teichmüller theory. In the present paper, I shall refer to the school of thought [i.e., in the sense of a “collection of closely interrelated ideas”] constituted by these assertions as

the “RCS”, i.e., “redundant copies school [of thought]”.

One fundamental reason for the use of this term “RCS” [i.e., “redundant copies school [of thought]”] in the present paper, as opposed to proper names of mathematicians, is to emphasize the importance of concentrating on mathematical content, as opposed to non-mathematical — i.e., such as social, political, or psychological — aspects or interpretations of the situation.

Thus, in a word, the central assertions of the RCS may be summarized as follows:

Various multiple copies of certain mathematical objects in inter-universal Teichmüller theory are redundant and hence may be identified with one another. On the other hand, once one makes such identifications, one obtains an immediate contradiction.

In the present paper, I shall refer to redundancy in the sense of the assertions of the RCS as “RCS-redundancy”, to the identifications of RCS-redundant copies that appear in the assertions of the RCS as “RCS-identifications”, and to the oversimplified version of inter-universal Teichmüller theory obtained by implementing the RCS-identifications as “RCS-IUT”.

As discussed in [Rpt2018] [cf., especially, [Rpt2018], §18], there is absolutely no doubt that

RCS-IUT is indeed a meaningless and absurd theory that leads immediately to a contradiction.
A more technical discussion of this contradiction, in the language of inter-universal Teichmüller theory, is given in §3.1 below, while a digested version in more elementary language of the technical discussion of §3 may be found in Examples 2.4.5, 2.4.7 below.

Rather, the fundamental misunderstandings underlying the RCS lie in the assertions of RCS-redundancy. The usual sense of the word “redundant” suggests that there should be some sort of equivalence, or close logical relationship, between the original version of the theory [i.e., IUT] and the theory obtained [i.e., RCS-IUT] by implementing the RCS-identifications of RCS-redundant objects. In fact, however,

implementing the RCS-identifications of RCS-redundant objects radically alters/invalidates the essential logical structure of IUT

in such a fundamental way that it seems entirely unrealistic to verify any sort of “close logical relationship” between IUT and RCS-IUT.

A more technical discussion of the three main types of RCS-redundancy/RCS-identification — which we refer to as “(RC-FrÉt), “(RC-log), and “(RC-Θ)” — is given, in the language of inter-universal Teichmüller theory, in §3.2, §3.3, §3.4, below. In fact, however, the essential mathematical content of these three main types of RCS-redundancy/RCS-identification is entirely elementary and lies well within the framework of undergraduate-level mathematics. A discussion of this essentially elementary mathematical content is given in §2.3, §2.4 below [cf., especially, Examples 2.4.5, 2.4.7].

One important consequence of the technical considerations discussed in §3 below is the following:

from the point of view of the logical relationships between various assertions of the RCS, the most fundamental type of RCS-redundancy is (RC-Θ).

That is to say, (RC-Θ) may be understood as the logical cornerstone of the various assertions of the RCS.

§1.3. Qualitative assessment of assertions of the RCS

As discussed in detail in §3.4 below [cf. also §2.3, §2.4],

implementing the logical cornerstone RCS-identification of (RC-Θ) completely invalidates the crucial logical AND “∧” property satisfied by the Θ-link — a property that underlies the entire logical structure of inter-universal Teichmüller theory.

In particular, understanding the issue of how the RCS treats this fundamental conflict between the RCS-identification of (RC-Θ) and the crucial ∧-property of the Θ-link is central to the issue of assessing the assertions of the RCS.

In March 2018, discussions were held at RIMS with two adherents of the RCS concerning, in particular, (RC-Θ) [cf. [Rpt2018], [Dsc2018]]. Subsequent to these discussions, after a few e-mail exchanges, these two adherents of the RCS informed
me via e-mail in August 2018 — in response to an e-mail that I sent to them in which
I stated that I was prepared to continue discussing inter-universal Teichmüller the-
ory with them, but that I had gotten the impression that they were not interested
in continuing these discussions — that indeed they were not interested in continuing
these discussions concerning inter-universal Teichmüller theory. In the same e-mail,
I also stated that perhaps it might be more productive to continue these discussions
of inter-universal Teichmüller theory via different participants [i.e., via “represent-
tatives” of the two sides] and encouraged them to suggest possible candidates for
doing this, but they never responded to this portion of my e-mail. [Incidentally,
it should be understood that I have no objection to making these e-mail messages
public, but will refrain from doing so in the absence of explicit permission from the
two recipients of the e-mails.]

Since March 2018, I have spent a tremendous amount of time discussing the
fundamental “(RC-Θ) vs. ∧-property” conflict mentioned above with quite
a number of mathematicians. Moreover, over the past two years, many mathemati-
cians [including myself!] with whom I have been in contact have devoted a quite
substantial amount of time and effort to analyzing and discussing certain 10pp.
manuscripts written by adherents of the RCS — indeed to such an extent that by
now, many of us can cite numerous key passages in these manuscripts by memory.
More recently, one mathematician with whom I have been in contact has made a
quite intensive study of the mathematical content of recent blog posts by adherents
of the RCS.

Despite all of these efforts, the only justification for the logical cornerstone
RCS-identification of (RC-Θ) that we [i.e., I myself, together with the many
mathematicians with whom I have discussed these issues] could find either in oral
explanations during the discussions of March 2018 or in subsequent written records
produced by adherents of the RCS [i.e., such as the 10pp. manuscripts referred to
above or various blog posts] were statements of the form

“I don’t see why not”.

[I continue to find it utterly bizarre that such justifications of the assertions of
the RCS appear to be taken seriously by some professional mathematicians.] In
particular, we were unable to find any detailed mathematical discussion by adherents
of the RCS of the fundamental “(RC-Θ) vs. ∧-property” conflict mentioned
above. That is to say, in summary,

the mathematical justification for the “redundancy” asserted in the
logical cornerstone assertion (RC-Θ) of the RCS remains a complete
mystery to myself, as well as to all of the mathematicians that I have
consulted concerning this issue

[cf. the discussion of Examples 2.4.5, 2.4.7]. Put another way, the response of all
of the mathematicians with whom I have had technically meaningful discussions
concerning the assertions of the RCS was completely uniform and unanimous, i.e.,
to the effect that these assertions of the RCS were obviously completely math-
ematically inaccurate/absurd, and that they had no idea why adherents of the
RCS continued to make such manifestly absurd assertions. In particular, it should
be emphasized that
I continue to search for a professional mathematician [say, in the field of arithmetic geometry] who feels that he/she understands the mathematical content of the assertions of the RCS and is willing to discuss this mathematical content with me or other mathematicians with whom I am in contact [cf. the text at the beginning of [Dsc2018]]. It is worth noting that this situation also constitutes a serious violation of article (6.) of the subsection entitled “Responsibilities of authors” of the Code of Practice of the European Mathematical Society (cf. [EMSCOP]).

In this context, one important observation that should be kept in mind is the following [cf. the discussion of [Rpt2018], §18]:

(UndIg) There is a fundamental difference between

(UndIg1) criticism of a mathematical theory that is based on a solid, technically accurate understanding of the content and logical structure of the theory and

(UndIg2) criticism of a mathematical theory that is based on a fundamental ignorance of the content and logical structure of the theory.

An elementary classical example of this sort of difference is discussed in §2.1 below.

In the case of the RCS, the lack of any thorough mathematical discussion of the fundamental “(RC-Θ) vs. ∧-property” conflict mentioned above in the various oral/written explanations set forth by adherents of the RCS demonstrates, in a definitive way, that none of the adherents of the RCS has a solid, technically accurate understanding of the logical structure of inter-universal Teichmüller theory in its original form, i.e., in particular, of the central role played in this logical structure by the “∧-property” of the Θ-link. Put another way, the only logically consistent explanation of this state of affairs is that the theory “RCS-IUT” that adherents of the RCS have in mind, i.e., the theory that is the object of their criticism, is simply a completely different — and logically unrelated — theory from the theory constituted by inter-universal Teichmüller theory in its original form.

Finally, it should be mentioned that although some people have asserted parallels between the assertions of the RCS and the fundamental error in the first version of Wiles’s proof of the Modularity Conjecture in the mid-1990’s, this analogy is entirely inappropriate for numerous reasons. Indeed, as is well-known, nothing even remotely close to the phenomena discussed thus far in the present §1.3 occurred in the case of the error in the first version of Wiles’s proof. The fact that there was indeed a fatal error in the first version of Wiles’s proof was never disputed in any way by any of the parties involved; the only issue that arose was the issue of whether or not the proof could be fixed. By contrast, no essential errors have been found in inter-universal Teichmüller theory, since the four preprints
[IUTchI-IV] on inter-universal Teichmüller theory were released in August 2012. That is to say, in a word, the assertions of the RCS are nothing more than meaningless, superficial misunderstandings of inter-universal Teichmüller theory on the part of people who are clearly not operating on the basis of a solid, technically accurate understanding of the mathematical content and essential logical structure of inter-universal Teichmüller theory.

§1.4. The importance of extensive, long-term interaction

In general, the transmission of mathematical ideas between individuals who share a sufficient stock of common mathematical culture may be achieved in a relatively efficient way and in a relatively brief amount of time. Typical examples of this sort of situation in the context of interaction between professional mathematicians include

- one-hour mathematical lectures,
- week-long mathematical lecture series, and
- informal mathematical discussions for several days to a week.

In the context of mathematical education, typical examples include

- written or oral mathematical examinations and
- mathematics competitions.

The successful operation of each of these examples relies, in an essential way, on a common framework of mathematical culture that is shared by the various participants in the activity under consideration.

On the other hand, in the case of a fundamentally new area of research, such as inter-universal Teichmüller theory, which evolved out of research over the past quarter of a century concerning absolute anabelian geometry, certain types of categories arising from arithmetic geometry, and certain arithmetic aspects of theta functions, the collection of mathematicians who share such a sufficient stock of common mathematical culture tends to be relatively small in number. In particular, for most mathematicians — even many arithmetic geometers or anabelian geometers — short-term interaction of the sort that occurs in the various typical examples mentioned above is far from sufficient to achieve an effective transmission of mathematical ideas. That is to say, no matter how mathematically talented the participants in such platforms of interaction may be, it takes time for the participants to

- analyze and sort out numerous mutual misunderstandings,
- develop effective techniques of communication that can transcend such misunderstandings, and
- digest and absorb new ideas and modes of thought.

Depending on the mathematical content under consideration, as well as on the mathematical talent, mathematical background, and time constraints of the participants, this painstaking process of analysis/development/digestion/absorption may require
patiently sustained efforts to continue constructive, orderly mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] over a period of months or even years

to reach fruition. Indeed, my experience in exposing the ideas of inter-universal Teichmüller theory to numerous mathematicians over the past decade suggests strongly that, in the case of inter-universal Teichmüller theory, it is difficult to expedite this process to the extent that it can be satisfactorily achieved in less than half a year or so.

In particular, in the case of inter-universal Teichmüller theory, a week-long session of discussions such as the discussions held at RIMS in March 2018 with two adherents of the RCS [cf. [Rpt2018], [Dsc2018]] is far from sufficient. This is something that I emphasized, both orally during these discussions and in e-mails to these two adherents of the RCS during the summer of 2018 subsequent to these discussions.

§1.5. The historical significance of detailed, explicit, accessible records

As was discussed in §1.3, I continue to search for a professional mathematician [say, in the field of arithmetic geometry] who purports to understand the mathematical justification for the RCS-redundancy asserted in the logical cornerstone assertion (RC-Θ) — i.e., in particular, who has confronted the mathematical content of the fundamental “(RC-Θ) vs. ∧-property” conflict mentioned in §1.3 — and who is prepared to discuss this mathematical content with me or other mathematicians with whom I am in contact. Of course, a detailed, explicit, mathematically substantive, and readily accessible written exposition — i.e., as an alternative to direct mathematical discussions [via e-mail, online video discussions, or face-to-face meetings] — of the mathematical justification for the logical cornerstone assertion (RC-Θ) would also be quite welcome [cf. the discussion of [Rpt2014], (7)]. Moreover, in this context, it should be emphasized that such a detailed, explicit, mathematically substantive, and readily accessible written exposition would be of great value not only for professional mathematicians and graduate students who are involved with inter-universal Teichmüller theory at the present time, but also for scholars in the [perhaps distant!] future.

In general, it cannot be overemphasized that maintaining such detailed, explicit, mathematically substantive, and readily accessible written records is of fundamental importance to the development of mathematics.

Indeed, as was discussed in the final portion of [Rpt2018], §3, from a historical point of view, it is only by maintaining such written records that the field of mathematics can avoid the sort of well-known and well-documented confusion that lasted for so many centuries concerning “Fermat’s Last Theorem”. Moreover, it is fascinating to re-examine, from the point of view of a modern observer, the intense debates that occurred, during the time of Galileo, concerning the theory of heliocentrism or, during the time of Einstein, concerning the theory of relativity. Again, it cannot be overemphasized that
such historical re-examinations are technically possible precisely because of the existence of detailed, explicit, mathematically substantive, and readily accessible written expositions of the logical structure underlying the various central assertions that arose in the debate.

In this context, it should be noted that, from a historical point of view, one pattern that typically underlies the formidable deadlocks that tend to occur in such debates is the point of view, on the part of parties opposed to a newly developed theory, that

\[(CmSn)\] it is a “matter of course” or “common sense” — i.e., in the language of the above discussion, a matter that is so profoundly self-evident that any “decent, reasonable observer” would undoubtedly find detailed, explicit, mathematically substantive, and readily accessible written expositions of its logical structure to be entirely unnecessary — that the issues under consideration can be completely resolved within some existing, familiar framework of thought without the introduction of the newly developed theory, which is regarded as deeply disturbing and unlikely to be of use in any substantive mathematical sense.

In fact, however,

\[(OvDlk)\] ultimately, the only meaningful technical tool that humanity can apply to develop the cultural infrastructure necessary to overcome such deadlocks is precisely the production of detailed, explicit, mathematically substantive, and readily accessible written expositions of the logical structure underlying points of view that are regarded by certain parties as being a “matter of common sense”

[cf. the discussion of [EMSCOP] in §1.3; the discussion of (UndIg) in §1.3; the discussion of §1.10 below; §2.1 below; the discussion of (FxRng), (FxEuc), (FxFld) in §3.1 below].

§1.6. The importance of further dissemination

One fundamental and frequently discussed theme in the further development of inter-universal Teichmüller theory is the issue of increasing the number of professional mathematicians who have a solid, technically accurate understanding of the details of inter-universal Teichmüller theory. Indeed, this issue is in some sense the central topic of [Rpt2013], [Rpt2014]. As discussed in §1.4, in order to achieve such a solid, technically accurate understanding of the theory, it is necessary to devote a substantial amount of time and effort over a period of roughly half a year to two or three years, depending on various factors. It also requires the participation of professional mathematicians or graduate students who are

- sufficiently familiar with numerous more classical theories in arithmetic geometry [cf. the discussion of [Alien], §4.1, (ii); [Alien], §4.4, (ii)],
- sufficiently well motivated and enthusiastic about studying inter-universal Teichmüller theory, and
- sufficiently mathematically talented, and who have a
sufficient amount of time to devote to studying the theory.

As a result of quite substantial dissemination efforts not only on my part, but also on the part of many other mathematicians, the number of professional mathematicians who have achieved a sufficiently detailed understanding of inter-universal Teichmüller theory to make independent, well informed, definitive statements concerning the theory that may be confirmed by existing experts on the theory [cf. also the discussion of §1.7 below] is roughly on the order of 10. It is worth noting that although this collection of mathematicians is centered around RIMS, Kyoto University, it includes mathematicians of many nationalities and of age ranging from around 30 to around 60. One recent example demonstrated quite dramatically that it is quite possible to achieve a solid mathematical understanding of inter-universal Teichmüller theory as a graduate student by studying on one’s own, outside of Japan, and with essentially zero contact with RIMS, except for a very brief period of a few months at the final stage of the student’s study of inter-universal Teichmüller theory.

Finally, we observe, in the context of the discussion [cf. §1.3, §1.4, §1.5] of the assertions of the RCS, that another point that should be emphasized is that it is also of fundamental importance to

increase the number of professional mathematicians [say, in the field of arithmetic geometry] who have a solid technical understanding of the mathematical content of the assertions of the RCS, and who are prepared to discuss this mathematical content with members of the “IUT community”

[i.e., with mathematicians who are substantially involved in mathematical research and/or dissemination activities concerning inter-universal Teichmüller theory]. Here, we note in passing that such a solid technical understanding of the mathematical content of the assertions of the RCS is by no means “equivalent” to expressions of support for the RCS on the basis of non-mathematical — i.e., such as social, political, or psychological — reasons. In this context, it should also be emphasized and understood [cf. the discussion of [Rpt2014], (7)] that both

· producing detailed, explicit, mathematically substantive, and readily accessible written expositions of the mathematical justification of assertions of the RCS [such as (RC-Θ)], i.e., as discussed in §1.5, and

· increasing the number of professional mathematicians [say, in the field of arithmetic geometry] who have a solid technical understanding of the mathematical content of the assertions of the RCS, and who are prepared to discuss this mathematical content with members of the IUT community

are in the interest not only of the IUT community, but of the RCS as well. Moreover,

the process of attaining a solid, technically accurate understanding of the precise logical relationship between RCS-IUT and IUT, i.e., as exposed, for instance, in the present paper, can serve as a valuable pedagogical tool

[cf. the discussion of [Rpt2018], §17] for mathematicians currently in the process of studying inter-universal Teichmüller theory.
§1.7. The notion of an “expert”

One topic that sometimes arises in the context of discussions of dissemination of inter-universal Teichmüller theory [i.e., as in §1.6], is the following issue:

What is the definition of, or criterion for, being an “expert” on inter-universal Teichmüller theory?

In a word, it is very difficult to give a brief, definitive answer, e.g., in the form of a straightforward, easily applicable criterion, to this question. On the other hand, in this context, it should also be pointed out that the difficulties that arise in the case of inter-universal Teichmüller theory are, in fact, not so qualitatively different from the difficulties that arise in answering the analogous question for mathematical theories other than inter-universal Teichmüller theory. These difficulties arise throughout the daily life of professional mathematicians in numerous contexts, such as the following:

(Ev1) preparing suitable exercises or examination problems to educate and evaluate students,
(Ev2) evaluating junior mathematicians,
(Ev3) refereeing/evaluating mathematical papers for journals.

From my point of view, as the author of [IUTchI-IV], one fundamental criterion that I always keep in mind — not only the in case of [IUTchI-IV], but also in the case of other papers that I have written, as well as when I am involved in the various types of evaluation procedures (Ev1) ∼ (Ev3) discussed above — is the issue of

the extent to which the level of understanding of the mathematician in question enables the mathematician to “stand on his/her own two feet” with regard to various assertions concerning the theory, on the basis of independent, logical reasoning, without needing to be “propped up” or corrected by me or other known experts in the theory.

I often refer to this criterion as the criterion of autonomy of understanding. Of course, from a strictly rigorous point of view, this criterion is, in some sense, not so “well-defined” and, in many contexts, difficult to apply in a straightforward fashion. On the other hand, in the past, various mathematicians involved with inter-universal Teichmüller theory have demonstrated such an autonomous level of understanding in the following ways:

(Atm1) the ability to detect various minor errors/oversights in [IUTchI-III];
(Atm2) the ability to propose new, insightful ways of thinking about various aspects of inter-universal Teichmüller theory;
(Atm3) the ability to propose ways of modifying inter-universal Teichmüller theory so as to yield stronger or more efficient versions of the theory;
(Atm4) the ability to produce technically accurate oral or written expositions of inter-universal Teichmüller theory;
(Atm5) the ability to supervise or direct new mathematicians — i.e., by training/educating professional mathematicians or graduate students with regard to inter-universal Teichmüller theory — who, in due time, demonstrate various of the four types of ability (Atm1) ∼ (Atm4) discussed above.
Of course, just as in the case of other mathematical theories, different experts demonstrate their expertise in different ways. That is to say, experts in inter-universal Teichmüller theory often demonstrate their expertise with respect to some of these five types of ability (Atm1) \(\sim\) (Atm5), but not others.

In this context, it should be pointed out that one aspect of inter-universal Teichmüller theory that is currently still under development is the analogue for inter-universal Teichmüller theory of (Ev1), i.e., preparing suitable exercises for mathematicians currently in the process of studying inter-universal Teichmüller theory. This point of view may be seen in the discussion in the final portion of the Introduction to [Alien], as well as in the discussion of “valuable pedagogical tools” in [Rpt2018], §17. Indeed, many of the technical issues discussed in [Rpt2018], §15, may easily be reformulated as “exercises” or, alternatively, as “examination problems” for evaluating the level of understanding of mathematicians in the process of studying inter-universal Teichmüller theory.

§1.8. Fabricated versions spawn fabricated dramas

As discussed in §1.6, §1.7, by now there is a substantial number of mathematicians who have attained a thorough, accurate, and autonomous understanding of inter-universal Teichmüller theory. In each of the cases of such mathematicians that I have observed thus far, such an understanding of the theory was achieved essentially by means of a thorough study of the original papers [IUTchI-IV], followed by a period of constructive discussions with one or more existing experts that typically lasted roughly from two to six months to sort out and resolve various “bugs” in the mathematician’s understanding of the theory that arose when the mathematician studied the original papers on his/her own [cf. the discussion of §1.4].

On the other hand, there is also a growing collection of mathematicians who have a somewhat inaccurate and incomplete — and indeed often quite superficial — understanding of certain aspects of the theory. This in and of itself is not problematic — that is to say, so long as the mathematician in question maintains an appropriate level of self-awareness of the inaccurate and incomplete nature of his/her level of understanding of the theory — and indeed is a phenomenon that often occurs as abstract mathematical theories are disseminated.

Unfortunately, however, a certain portion of this collection of mathematicians [i.e., whose understanding of the theory is inaccurate and incomplete] have exhibited a tendency to assert/justify the validity of their inaccurate and incomplete understanding of the theory by means of “reformulations” or “simplifications” of the theory, which are in fact substantively different from and have no directly logical relationship to [e.g., are by no means “equivalent” to!] the original theory.

Indeed, the version, referred to in the present paper as “RCS-IUT” [cf. §1.2], that arises from implementing the assertions of the RCS appears to be the most famous of these fabricated versions of inter-universal Teichmüller theory [cf. also the discussion of Example 2.4.5 below for a more detailed discussion of various closely related variants of RCS-IUT]. On the other hand, other, less famous fabricated
versions of inter-universal Teichmüller theory have also come to my attention in recent years.

Here, before proceeding, we note that, in general, reformulations or simplifications of a mathematical theory are not necessarily problematic, i.e., so long as they are indeed based on a thorough and accurate understanding of the original theory and, moreover, can be shown to have a direct logical relationship to the original theory.

The authoring of fabricated versions of inter-universal Teichmüller theory appears to be motivated, to a substantial extent, by a deep desire to recast inter-universal Teichmüller theory in a “simplified” form that is much closer to the sort of mathematics with which the author of the fabricated version is already familiar/feels comfortable. On the other hand, this phenomenon of producing fabricated versions also appears to have been

substantially fueled by numerous grotesquely distorted mass media reports and comments on the English-language internet that blithely paint inter-universal Teichmüller theory as a sort of cult religion, fanatical political movement, mystical philosophy, or vague sketch/proposal for a mathematical theory.

Moreover, another unfortunate tendency, of which RCS-IUT is perhaps the most egregious example, is for fabricated versions of inter-universal Teichmüller theory to

spawn lurid social/political dramas revolving around the content of the fabricated version, which in fact have essentially nothing to do with the content of inter-universal Teichmüller theory.

Such lurid dramas then spawn further grotesquely distorted mass media reports and comments on the English-language internet, which then reinforce and enhance the social/political status of the fabricated version. Here, it should be emphasized that such vicious spirals have little [or nothing] to do with substantive mathematical content and indeed serve only to mass-produce unnecessary confusion that is entirely counterproductive, from the point of the view of charting a sound, sustainable course in the future development of the field of mathematics [cf. the discussion of [Alien], §4.4, (iv)].

In fact, of course, inter-universal Teichmüller theory is neither a religion, nor a political movement, nor a mystical philosophy, nor a vague sketch/proposal for a mathematical theory. Rather, it should be emphasized that

inter-universal Teichmüller theory is a rigorously formulated mathematical theory that has been verified countless times by quite a number of mathematicians, has undergone an exceptionally thorough seven and a half year long refereeing process, and was subsequently published in a leading international journal in the field of mathematics.

[cf. the discussion of §1.1]. In particular, in order to avoid the sort of vicious spirals referred to above, it is of the utmost importance

to concentrate, in discussions of inter-universal Teichmüller theory, on substantive mathematical content, as opposed to non-mathematical
such as social, political, or psychological — aspects or interpretations of the situation.

As discussed in §1.2, this is the main reason for the use of the term “RCS” in the present paper.

§1.9. Geographical vs. mathematical proximity

Historically, mathematical interaction between professional mathematicians relied on physical meetings or the exchange of hardcopy documents. Increasingly, however, advances in information technology have made it possible for mathematical interaction between professional mathematicians to be conducted electronically, by means of e-mail or online video communication. Of course, this does not imply that physical meetings or the exchange of hardcopy documents — especially in situations where physical meetings or the exchange of hardcopy documents do indeed function in a meaningful way, from the point of view of those involved — should necessarily be eschewed.

On the other hand, physical meetings between participants who live in distant regions requires travel. Moreover, travel, depending on the situations of the participants, can be a highly taxing enterprise. Indeed, travel, as well as lodging accommodations, typically requires the expenditure of a quite substantial amount of money, as well as physical and mental effort on the part of those involved. This effort can easily climb to unmanageable [i.e., from the point of view of certain of the participants] proportions, especially when substantial cultural — i.e., either in mathematical or in non-mathematical culture, or in both — differences are involved. The current situation involving the COVID-19 pandemic adds yet another dimension to the reckoning, from the point of view of the participants, of the physical and mental effort that must be expended in order to travel. As a result,

when, from the point of view of at least one of the key participants, the amount of effort, time, and/or money that must be expended to travel clearly exceeds, by a substantial margin — i.e., “≫” — the gain [i.e., relative to various mathematical or non-mathematical criteria of the key participant in question] that appears likely to be obtained from the travel under consideration, it is highly probable that the travel under consideration will end up simply not taking place.

One “classical” example of this phenomenon “≫” is the relative scarcity of professional mathematicians in Europe or North America who travel to Japan frequently [e.g., at least once a year] or for substantial periods to time.

I have, at various times in my career, been somewhat surprised by assertions on the part of some mathematicians to the effect that travel should somehow be forced on mathematicians, i.e., to the effect that some sort of coercion may somehow “override” the fundamental inequality “≫” that exists as a result of the circumstances in which a mathematician finds him/herself in. In my experience, although this sort of coercion to travel may result in some sort of superficial influence in the very short term, it can never succeed in the long term. That is to say, the fundamental circumstances that give rise to the fundamental inequality “≫” can never be altered by means of such coercive measures to travel [cf. the discussion of [Rpt2014], (8)].
In this context, I was most impressed by the following two concrete examples, which came to my attention recently. In describing these examples, I have often used somewhat indirect expressions, in order to protect the privacy of the people involved.

**Example 1.9.1: The insufficiency of geographical proximity.** This example concerns the results obtained in a paper written in the fall of 2019 by a graduate student (St1) from country (Ct1). This student (St1) showed his paper to a prominent senior researcher (Pf1) at a university in country (Ct2) in a certain area of number theory. The education and career of this researcher (Pf1) was conducted entirely at universities in countries (Ct2), (Ct3), and (Ct4). This researcher (Pf1) informed (St1) of his very positive evaluation of the originality of the results obtained in the paper by (St1). Another prominent senior researcher (Pf2) in a certain area of number theory was informed by (Pf1) of the paper by (St1). This researcher (Pf2), who works at a university in country (Ct2) in close physical proximity to (Pf1), also took a generally positive position with regard to the paper by (St1).

On the other hand, several months subsequent to this interaction between (Pf1) and (St1), a junior researcher (Pf3), who is originally from country (Ct5), but currently works at a university in country (Ct2) in close physical proximity to (Pf1) and (Pf2), informed student (St1) [via e-mail contacts between (Pf3) and (St1)’s advisor] that

the results of the paper by (St1) are in fact “well-known” and essentially contained in papers published in the 1990’s by (Pf4), a prominent senior researcher in country (Ct6).

[To be more precise, in fact the results of the paper by (St1) are not entirely contained in the papers by (Pf4) in the sense that the paper by (St1) contains certain numerically explicit estimates that are not contained in the papers by (Pf4).] Country (Ct6) is in close physical proximity to country (Ct3), and in fact, one of the research advisors of researcher (Pf4), when (Pf4) was a graduate student, was a prominent researcher (Pf5) who is originally from country (Ct7), but has pursued his career as a mathematician mainly in countries (Ct3) and (Ct6). Here, it should be pointed out that (Pf1), (Pf2), and (Pf4) are very close in age, and that (Pf1) received his undergraduate education in country (Ct3) at one of the universities that played in prominent role in the career of (Pf5). The paper by student (St1) concerns mathematics that has been studied extensively by — and indeed forms one of the central themes of the research of — both (Pf1) and (Pf4), but from very different points of view, using very different techniques, since (Pf1) and (Pf4) work in substantially different areas of number theory. On the other hand, at no time during the initial several months of interaction between (Pf1), (Pf2), and (St1) was the work of (Pf4) mentioned. That is to say, (Pf1) and (Pf2) discussed the results obtained in the paper by (St1) in a way that can only be explained by the hypothesis that

(Pf1) and (Pf2) were, at the time, entirely unaware of the very close relationship between the results obtained in the paper by (St1) and the papers in the 1990’s by (Pf4).

— i.e., despite the numerous opportunities afforded by close physical proximity, as well as proximity of age, for substantial interaction between (Pf1) and
(Pf4). The paper by student (St1) is currently submitted for publication to a certain mathematical journal. Student (St1) recently received a referee’s report for his paper, which apparently [i.e., judging from the comments made in the referee’s report] was written by a mathematician working in an area of number theory close to (Pf1). This referee’s report also makes no mention of the papers in the 1990’s by (Pf4) and the fact that the results obtained in the paper by (St1) appear, with the exception of certain numerically explicit estimates, to be essentially contained in these papers of (Pf4). Finally, it should be mentioned that each official language of each of these countries (Ct1), (Ct2), (Ct3), (Ct4), (Ct5), (Ct6), (Ct7) belongs to the European branch of the Indo-European family of languages, and that at least six of the ten pairs of countries in the list (Ct2), (Ct3), (Ct4), (Ct6), (Ct7) share a common official language [i.e., with the other country in the pair].

**Example 1.9.2:** The remarkable potency of mathematical proximity. This example concerns the study of inter-universal Teichmüller theory by a graduate student (St2), who is originally from country (Ct8), but was enrolled in the doctoral program in mathematics at a university in country (Ct9) under the supervision of a senior faculty member (Pf6), who is originally from country (Ct10). This graduate student (St2) began his study of inter-universal Teichmüller theory as a graduate student and continued his study during his years as a graduate student with essentially no mathematical contact with any researchers who are significantly involved with inter-universal Teichmüller theory, except for his advisor (Pf6) and one mid-career researcher (Pf7) from country (Ct11). Here, we remark that the official language of each of these countries (Ct8), (Ct9), (Ct10), (Ct11) belongs to the European branch of the Indo-European family of languages. In particular, with the exception of a few very brief e-mail exchanges with me and a brief two-week long stay at RIMS in 2016 to participate in a workshop on IUT, this student (St2) had essentially no mathematical contact, prior to the fall of 2019, with any researchers at Kyoto University who are involved with inter-universal Teichmüller theory. Even in these circumstances, this student was able not only to achieve a very technically sound understanding of inter-universal Teichmüller theory on his own, by reading [IUTchI-IV] and making use of various resources, activities, and contacts within country (Ct9), but also to succeed, as a graduate student, in making highly nontrivial original research contributions to a certain mild generalization of inter-universal Teichmüller theory, as well as to certain related aspects of anabelian geometry.

My first [i.e., with the exception of a few very brief e-mail exchanges prior to this] mathematical contact with this student (St2) was in the fall of 2019. Although this student (St2) initially had some technical questions concerning aspects of inter-universal Teichmüller theory that he was unable to understand on his own, after a few relatively brief discussions in person with me, he was able to find answers to these technical questions in a relatively short period of time [roughly a month or two] without much trouble.

§1.10. Mathematical intellectual property rights

The socio-political dynamics generated by the proliferation of logically unrelated fabricated versions of inter-universal Teichmüller theory — of which
RCS-IUT is perhaps the most frequently cited [cf. the discussion of §1.2, as well as Examples 2.4.5, 2.4.7 below] — and further fueled by

- grotesquely distorted mass media coverage and internet comments [cf. the discussion of §1.8], as well as by

- the conspicuous absence of detailed, explicit, mathematically substantive, and readily accessible written expositions of the logical structure underlying the various central assertions of the proponents of such socio-political dynamics [cf. the discussion of §1.5],

have had the effect of deeply disrupting the normal process of absorption of inter-universal Teichmüller theory by the worldwide mathematical community. Left unchecked, this state of affairs threatens to pave the way for a field — i.e., the field of mathematics — governed by socio-political dynamics [cf. the discussion of (CmSn) in §1.5], rather than by mathematical content.

From a historical point of view, various forms of institutional and conceptual infrastructure — such as the notions of

- a modern judiciary system;
- universal, inalienable human rights;
- the rule of law;
- due process of law; and
- burden of proof

— were gradually developed, precisely with the goal of averting the outbreak of the sort of socio-political dynamics that were viewed as detrimental to society. In this context, it is interesting to note the central role played, for instance, in courts of law, by the practice of producing
detailed, explicit, logically substantive, and readily accessible written documentation of the logical structure underlying the various central assertions of the parties involved.

This situation is very much reminiscent of the situation in mathematics discussed in §1.5 [cf. also the discussion of RCS-IUT in §1.3!], i.e., where we observe that it is not even possible to analyze or debate, in any sort of meaningfully definitive way, mathematical assertions — such as, for instance, the historically famous assertion of Fermat to the effect that he had a proof of “Fermat’s Last Theorem”, but did not write it down — in the absence of such written documentation of the logical structure of the issues under consideration.

From the point of view of the above discussion, it seems natural, in the case of mathematics, to introduce, especially in the context of issues such as the one discussed above involving logically unrelated fabricated versions of inter-universal Teichmüller theory, the notion of mathematical intellectual property rights [i.e., “MIPRs”]. As the name suggests, this notion is, in some sense, modeled on the conventional notion of intellectual property rights associated, for instance, with trademarks or brand names of corporations. In the case of this conventional notion, intellectual property rights may be understood as a tool for protecting the “reliability” or “creditworthiness” of trademarks or brand names of a corporation from the sort of severe injury to such trademarks or brand names that may ensue
from the proliferation of shoddy third-party imitations of products produced by the corporation. Here, we observe that this “severe injury” often revolves around the creation of severe obstacles to the execution of activities that play a central role in the operational normalcy of the corporation.

Unlike this conventional notion, MIPRs should be understood as being associated — not to corporations or individuals for some finite period of time, but rather — to mathematical notions and theories and, moreover, are of unlimited duration. The purpose of MIPRs may be understood as the protection of the “creditworthiness” of such a mathematical notion or theory from the severe injury to the operational normalcy of mathematical progress related to notion/theory that ensues from the proliferation of logically unrelated fabricated “fake” versions of the notion/theory.

Before proceeding, we pause to consider one relatively elementary example of this notion of MIPRs.

**Example 1.10.1: The Pythagorean Theorem.**

(i) Recall the Pythagorean Theorem concerning the length of the hypotenuse of a right triangle in the Euclidean plane. Thus, if $0 < x \leq y < z \in \mathbb{R}$ are the lengths of the sides of a right triangle in the Euclidean plane, then the Pythagorean Theorem states that

$$x^2 + y^2 = z^2.$$  

Various versions of this result apparently may be found not only in the writings of ancient Greece and Rome, but also in Babylonian, ancient Indian, and ancient Chinese documents. Well-known “elementary proofs” of this result may be obtained, for instance, by computing, in various equivalent ways, the area of suitable planar regions covered by right triangles or squares that are closely related to the given right triangle. On the other hand, such “elementary proofs” typically do not address the fundamental issue of how to define such notions as length, angle, and rotation, i.e., which are necessary in order to understand the precise content of the statement of the Pythagorean Theorem. Here, we observe that if, for instance, one tries to define the notion of the length of a line segment in Euclidean space in the conventional way, then the Pythagorean Theorem reduces, in effect, to a meaningless tautology! Moreover, although the notions of length and angle may be defined once one has defined the notion of a rotation, it is by no means clear how to give a natural definition of the notion of a rotation. For instance, one may attempt to define the notion of a rotation of Euclidean space as an element of the group generated by well-known matrices involving sines and cosines, but it is by no means clear that such a definition is “natural” or the “right definition” in some meaningful sense. Thus, in summary,

it is by no means clear that such “elementary proofs” may be regarded as genuine rigorous proofs in the sense of modern mathematics.

(ii) From a modern point of view, a natural, precise definition of the fundamental notion of a rotation of Euclidean space [from which, as observed in (i), natural definitions of the notions of the notions of length and angle may be easily
derived] may be given by thinking in terms of invariant tensor forms associated to compact subgroups of [the topological groups determined by] various general linear groups. From this modern point of view, the precise form of the Pythagorean Theorem [i.e., “$x^2 + y^2 = z^2$”] — and, in particular, the significance of the “2” in the exponent! — may be traced back to the theory of Brauer groups and the closely related local class field theory of the archimedean field “$\mathbb{R}$” of real numbers, i.e., in short, to various fundamental properties of the arithmetic of the topological field “$\mathbb{R}$”.

(iii) Considering the situation discussed in (i), (ii), it is by no means clear, in any sort of a priori or naive sense, just why the Pythagorean Theorem should take the precise form “$x^2 + y^2 = z^2$”. Indeed, since this precise form of the Pythagorean Theorem continues to appear utterly mysterious even to numerous modern-day high school students — i.e., who grow up immersed in an environment replete with countless cultural links to modern mathematics, science, and technology! — it seems reasonable to assume that it should have appeared all the more mysterious to the individuals who populated the various ancient civilizations mentioned in (i). In particular, it is by no means unnatural to consider the possibility that assertions similar to the following assertions [stated relative to the notation introduced in (i)] might have been made by some hypothetical individual at some time in human history:

(\text{Pyth1}) “I don’t understand why the relation in the Pythagorean Theorem is of the form ‘$x^2 + y^2 = z^2$’, rather than ‘$x^2 \cdot y^2 = z^2$’.”

(\text{Pyth2}) “I would like to investigate, in the context of the Pythagorean Theorem, whether or not the relation ‘$x^2 \cdot y^2 = z^2$’ holds.”

(\text{Pyth3}) “I investigated, in the context of the Pythagorean Theorem, whether or not the relation ‘$x^2 \cdot y^2 = z^2$’ holds and discovered that there exist examples that show that this relation does not in fact hold in general.”

(\text{Pyth4}) “The Pythagorean Theorem is false for the following reason: The Pythagorean Theorem states that ‘$x^2 \cdot y^2 = z^2$’, but there exist counterexamples that show that this relation does not hold in general.”

(iv) From the point of view of the discussion given above of MIPRs, the assertions (\text{Pyth1}), (\text{Pyth2}), (\text{Pyth3}) do not constitute a violation of the MIPRs of the Pythagorean Theorem, but rather are precisely the sorts of assertions/comments that occur naturally in normal, sound research and educational activities in mathematics. By contrast,

(\text{Pyth4}) may be regarded as a classical example of a violation of the MIPRs of the Pythagorean Theorem.

It is not difficult to imagine the deeply detrimental effects on the development of mathematics throughout history that would have occurred if violations of MIPRs similar to (\text{Pyth4}) regarding the Pythagorean Theorem arose and were left unchecked.
(v) From a historical point of view, it appears, in light of the discussion of (i), (ii), (iii), (iv), to be in some sense a sort of miracle that the Pythagorean Theorem was “discovered” in and, moreover, survived throughout the duration of numerous ancient civilizations, i.e., despite the fact that the ancient world was primarily dominated by dictatorial, authoritarian political regimes with little regard for such modern notions as a judiciary system [in the modern sense], inalienable human rights, the rule of law, due process of law, burden of proof, and so on. Of course, to a certain extent, this situation may be understood as a consequence of the fact that the Pythagorean Theorem is closely related to the task of direct measurement of lengths of various easily accessed [i.e., even in the ancient world!] physical objects. From this point of view of “direct measurement”, the “Pythagorean Theorem”, as understood in various ancient civilizations, should perhaps be regarded [cf. the discussion of (i), (ii)] not so much as a result in mathematics [in the modern sense of the term], but rather as a principally empirically substantiated result in physics. Nevertheless, even when viewed from this point of view, it still seems like something of a miracle that this result survived throughout the duration of numerous ancient civilizations, unaffected by numerous meaningless misunderstandings of the sort discussed in (iii), (iv), especially considering that similarly meaningless misunderstandings of the Pythagorean Theorem continue to plague modern-day high school students!

Prior to the introduction above of the notion of MIPRs, this notion does not appear to have played an important role, at least in any sort of explicit sense, in discussions or analyses of the development of mathematics. On the other hand, Example 1.10.1 [cf., especially, Example 1.10.1, (iv)!) shows how, even at a purely implicit level,

this notion of MIPRs has in fact played a fundamentally important role in the development of mathematics throughout history.

Of course, in the case of violations of the MIPRs of a mathematical notion or theory, conventional courts or judiciary systems are simply not equipped to play a meaningful role in dealing with such violations, since this would require an extensive technical knowledge and understanding, on the part of the judges or lawyers involved, of the mathematics under consideration. Indeed, it is useful to recall in this context that

- traditionally, any detrimental effects arising from such violations of the MIPRs of a mathematical notion or theory were typically averted in the field of mathematics by means of the refereeing systems of various well established mathematical journals;

- from the point of view of such traditional refereeing systems of well established mathematical journals, the burden of proof regarding the correctness of any novel assertions concerning existing mathematical notions or theories — such as, for instance, any assertions concerning some sort of logical relationship between a modified version of a theory and the original version of the theory — lies [not with the author of the original version of the theory (!), but rather] with the author of the manuscript containing the novel assertions.
On the other hand, as discussed in §1.8 [cf. also the discussion of §1.3], in the case of the quite egregious MIPRs violations constituted by logically unrelated fabricated versions of inter-universal Teichmüller theory, numerous mass media reports and internet comments released by individuals who are clearly not operating on the basis of a solid, technically accurate understanding of the mathematics involved are regarded, in certain sectors of the mathematical community, as carrying much more weight than an exceptionally thorough refereeing process in a well-established mathematical journal by experts on the mathematics under consideration. This state of affairs is deeply regrettable and should be regarded as a cause for alarm. Perhaps in the long term, new forms of institutional or conceptual infrastructure may be developed for averting the deeply detrimental effects of this sort of situation. At the time of writing, however, it appears that

the only meaningful technical tool currently available to humanity for dealing with this sort of situation lies in the production of detailed, explicit, mathematically substantive, and readily accessible written expositions of the logical structure underlying the assertions of the various parties involved [cf. the discussion surrounding (OvDlk) in §1.5],

i.e., even when such assertions are purported to be a “matter of course” or “common sense”, that is to say, a matter that is so profoundly self-evident that any “decent, reasonable observer” would undoubtedly find such written documentations of logical structure to be entirely unnecessary [cf. the discussion surrounding (CmSn) in §1.5]. From a historical point of view, such written documentations of logical structure can then serve as a valuable transgenerational or transcultural common core

of scholarly activity — a point of view that is reminiscent of the logical relator AND “∧”, which forms a central theme of the present paper.

§1.11. Social mirroring of mathematical logical structure

Discussions, on the part of some observers, concerning the situation surrounding inter-universal Teichmüller theory are often dominated by various mutually exclusive and socially divisive/antagonistic dichotomies [cf. also the related discussion of §1.8], i.e., such as the following:

(ExcDch) Is it the case that adherents of the RCS should be regarded as mathematically correct/reliable/reasonable mathematicians, while mathematicians associated with inter-universal Teichmüller theory should not, OR is it the other way around?

Here, the “OR” is typically understood as an “XOR”, i.e., exclusive-or. That is to say, such questions/dichotomies are typically understood as issues for which it can never be the case that an “AND” relation between the two possible alternatives under consideration holds. In discussions of mutually exclusive dichotomies such as (ExcDch), mathematicians associated with inter-universal Teichmüller theory, as well as the actual mathematical content of inter-universal Teichmüller theory, are often treated as completely and essentially disjoint entities, within the international mathematical community, from the mathematicians and mathematical research associated with the RCS.
In this context, it is interesting to note that this sort of mutually exclusive dichotomy is very much reminiscent of the essential logical structure of RCS-IUT, which, as discussed in the latter portion of Example 2.4.5 below, may be understood as being essentially logically equivalent to OR-IUT, as well as to XOR-IUT, i.e., to logically unrelated fabricated versions of inter-universal Teichmüller theory in which the crucial logical AND “∧” relation satisfied by the Θ-link of inter-universal Teichmüller theory is replaced by a logical OR “∨” relation or, alternatively, by a logical XOR “˙∨” relation.

In fact, however,

(MthCnn) although it is indeed the case that the international mathematical community, as well as the mathematical content of the research performed by the international mathematical community, does not consist [in the language of classical algebraic geometry] of a single irreducible component, it is nevertheless undeniably connected.

In the case of inter-universal Teichmüller theory, this abundant inter-connectivity may be explicitly witnessed in the following aspects of the theory:

(IntCnn1) The mathematical content of various aspects of inter-universal Teichmüller theory is closely related to various classical theories such as the following:

- the invariance of heights of abelian varieties under isogeny [cf. the discussion of [Alien], §2.3, §2.4, as well as the discussion of §3.5 below];
- the classical proof in characteristic zero of the geometric version of the Szpiro inequality via the Kodaira-Spencer morphism, phrased in terms of the theory of crystals [cf. the discussion of [Alien], §3.1, (v), as well as the discussion of §3.5, §3.10 below];
- Bogomolov's proof over the complex numbers of the geometric version of the Szpiro inequality [cf. the discussion of [Alien], §3.10, (vi)];
- classical complex Teichmüller theory [cf. the discussion of Example 3.3.1 in §3.3 below];
- the classical theory of the Jacobi identity for the theta function [cf. the discussion of Example 3.3.2 in §3.3 below];
- the classical theory of the computation of the Gaussian integral via polar coordinates [cf. [Alien], §3.8].

We refer to [Alien], §4, for a more detailed discussion of such relationships between inter-universal Teichmüller theory and various classical mathematical theories.

(IntCnn2) In the context of the assertions of the RCS, it is important to recall [cf. the discussion of (MthVI) in Example 2.4.5, (viii), below] that [perhaps somewhat surprisingly!]

in fact there is in some sense no disagreement among any of the parties involved with regard to the mathematical validity of the central mathematical assertions of the RCS.
— i.e., so long as one deletes the arbitrary label “inter-universal Teichmüller theory” imposed by adherents of the RCS on the logically unrelated fabricated versions [i.e., RCS-IUT/OR-IUT/XOR-IUT] of inter-universal Teichmüller theory that typically appear in discussions of adherents of the RCS [cf. the discussion of §1.2, §1.3, §1.8, §1.10].

The situation discussed in (InnCnn2) is of particular interest in the context of the present paper since the essential logical structure of this situation discussed in (InnCnn2) — i.e.,

(CmMth) of a common mathematical understanding of the mathematical validity of the various assertions under discussion, so long as one keeps track of the distinct labels “inter-universal Teichmüller theory” and “RCS-IUT/OR-IUT/XOR-IUT” — is remarkably similar to the essential logical structure of the situation surrounding the central theme of the present paper, namely, the crucial logical AND “∧” property of the Θ-link in inter-universal Teichmüller theory — cf. the discussion of §2.4, §3.4 below. At a more elementary level, (CmMth) may be understood as being qualitatively essentially the same phenomenon as the phenomenon discussed in Example 1.10.1, (iii), i.e., the distinction between (Pyth3) [where the distinct label “Pythagorean Theorem” is treated properly] and (Pyth4) [where the distinct label “Pythagorean Theorem” is not treated properly].

Section 2: Elementary mathematical aspects of “redundant copies”

The essence of the central mathematical assertions of the RCS revolves, perhaps somewhat remarkably, around quite elementary considerations that lie well within the framework of undergraduate-level mathematics. Before examining, in §3, the assertions of the RCS in the technical terminology of inter-universal Teichmüller theory, we pause to give a detailed exposition of these elementary considerations.

§2.1. The history of limits and integration

The classical notion of integration [e.g., for continuous real-valued functions on the real line], as well as the more fundamental, but closely related notion of a limit, have a long history, dating back [at least] to the 17-th century. Initially, these notions did not have rigorous definitions, i.e., were not “well-defined”, in the sense understood by mathematicians today. The lack of such rigorous definitions frequently led, up until around the end of the 19-th century, to “contradictions” or “paradoxes” in mathematical work — such as Grandi’s series

$$\sum_{n=0}^{\infty} (-1)^n$$

— concerning integrals or limits.

Ultimately, of course, the theory of limits and integrals evolved, especially during the period starting from around the mid-19-th century and lasting until around the early 20-th century, to the extent that such “contradictions/paradoxes”
could be resolved in a definitive way. This process of evolution involved, for instance, in the case of integration, first the introduction of the Riemann integral and later the introduction of the Lebesgue integral, which made it possible to integrate functions — such as, for instance, the indicator function on the real line of the subset of rational numbers — whose Riemann integral is not well-defined.

Here, it should be noted that at various key points during this evolution of the notions of limits and integration, the central “contradictions/paradoxes” that, at times, led to substantial criticism and confusion arose from a solid, technically accurate understanding of the content and logical structure of the assertions — such as, for instance, various possible approaches to computing the value of Grandi’s series — at the center of these “contradictions/paradoxes”. It is precisely for this reason that such criticism and confusion ultimately lead to substantive refinements in the theory that were sufficient to resolve the original “contradictions/paradoxes” in a definitive way.

Such constructive episodes in the history of mathematics — which may be studied by scholars today precisely because of the existence of detailed, explicit, mathematically substantive, and readily accessible written records! [cf. the discussion of §1.5] — stand in stark contrast to [cf. the discussion of (UndIg) in §1.3] criticism of a mathematical theory that is based on a fundamental ignorance of the content and logical structure of the theory, such as the following “false contradiction” in the theory of integration, which may be observed in some students who are still in an initial stage with regard to their study of the notion of integration.

**Example 2.1.1: False contradiction in the theory of integration.** Consider the following computation of the definite integral of a real-valued function on the real line

\[
\int_0^1 x^n \, dx = \frac{1}{n+1}
\]

for \( n \) a positive integer. Suppose that one takes the [drastically oversimplified and manifestly absurd, from the point of view of any observer who has an accurate understanding of the theory of integration!] point of view that the most general possible interpretation of the equation of the above display is one in which the following three symbols

\[
\int_0^1, \quad x, \quad dx
\]

are allowed to be arbitrary positive real numbers \( a, b, c \). Here, we note that in the case of “\( dx \)”, such a substitution “\( dx \mapsto c \)” could be “justified” by quoting conventional “\( \varepsilon-\delta \)” treatments of the theory of limits and integrals, in which infinitesimals — i.e., such as “\( dx \)” or “\( \varepsilon \)” — are allowed to be arbitrary positive real numbers, which are regarded as being “arbitrarily small”, and observing that any positive real number \( c \) is indeed much smaller than “most other positive real numbers” [such as \( 1000 \cdot c \), etc.]. On the other hand, by substituting the values \( n = 1, 2, 3 \), one obtains relations

\[
abc = 1, \quad ab^2 c = \frac{1}{2}, \quad ab^3 c = \frac{1}{3}.
\]

The first two of these relations imply that \( b = \frac{1}{2} \) [so \( b^2 = \frac{1}{4} \)], while the first and third relations imply that \( b^2 = \frac{3}{4} \neq \frac{1}{4} \) — a “contradiction”!
§2.2. Derivatives and integrals

In the context of the historical discussion of integration in §2.1, it is interesting to recall the fundamental theorem of calculus, i.e., the result to the effect that, roughly speaking, the operations of integration and differentiation of functions [i.e., real-valued functions on the real line satisfying suitable conditions] are inverse to one another. Thus, from a certain point of view,

the “essential information” contained in a function may be understood as being “essentially equivalent” to the “essential information” contained in the derivative of the function

— that is to say, since one may always go back and forth at will between a function and its derivative by integrating and then differentiating. This point of view might then tempt some observers to conclude that

any mathematical proof that relies, in an essential way, on consideration of the derivative of a function must be fundamentally flawed since any information that might possibly be extracted from the derivative of the function should already be available [cf. the “essential equivalence” discussed above] from the function prior to passing to the derivative, i.e., in “contradiction” to the essential dependence of the proof on passing to the derivative.

Alternatively, this point of view may be summarized in the following way:

the “essential equivalence” discussed above implies that any usage, in a mathematical proof, of the derivative of a function is necessarily inherently redundant in nature.

In fact, of course, such “pseudo-mathematical reasoning” is itself fundamentally flawed. Two examples of well-known proofs in arithmetic geometry that depend, in an essential way, on passing to the derivative will be discussed in the final portion of §3.2 below. These examples are in fact closely related to the mathematics that inspired inter-universal Teichmüller theory [cf. the discussion in the final portion of §3.2 below]. One central aspect of the situations discussed in §3.2 below is the exploitation of properties of [various more abstract analogues of] the derivative of a function that differ, in a very substantive, qualitative way, from the properties of the original function. One important example of this sort of situation is the validity/invalidity of various symmetry properties. This phenomenon may be observed in the following elementary example.

Example 2.2.1: Symmetry properties of derivatives. The real-valued function

$f(x) = x$

on the real line is not invariant [i.e., not symmetric] with respect to translations by an arbitrary constant $c \in \mathbb{R}$. That is to say, in general, it is not the case that $f(x + c) = f(x)$. On the other hand, the derivative

$f'(x) = 1$
of this function is manifestly invariant/symmetric with respect to such translations.

§2.3. Line segments vs. loops

By comparison to the examples given in §2.1, §2.2, the following elementary geometric examples are much more closely technically related to the assertions of the RCS concerning inter-universal Teichmüller theory.

Example 2.3.1: Endpoints of an oriented line segment.

(i) Write
\[ I \overset{\text{def}}{=} [0,1] \subset \mathbb{R} \]
for the closed unit interval [i.e., the set of nonnegative real numbers \( \leq 1 \)] in the real line \( \mathbb{R} \). Thus, \( I \) is equipped with a natural topology [i.e., induced by the topology of \( \mathbb{R} \)], hence can be regarded as a topological space, indeed more specifically, as a one-dimensional topological manifold with boundary that is equipped with a natural orientation [i.e., induced by the usual orientation of \( \mathbb{R} \)]. Write
\[ \alpha \overset{\text{def}}{=} \{0\}, \quad \beta \overset{\text{def}}{=} \{1\} \]
for the topological spaces [consisting of a single point!] determined by the two endpoints of \( I \). Thus, \( \alpha \) and \( \beta \) are isomorphic as topological spaces. In certain situations that occur in category theory, it is often customary to replace a given category by a full subcategory called a skeleton, which is equivalent to the given category, but also satisfies the property that any two isomorphic objects in the skeleton are equal. This point of view of working with skeletal categories [i.e., categories which are their own skeletons] is motivated by the idea that nonequal isomorphic objects are “redundant”. Of course, there are indeed various situations in which nonequal isomorphic objects are redundant in the sense that working with skeletal categories, as opposed to arbitrary categories, does not result in any substantive difference in the mathematics under consideration.

(ii) On the other hand, if, in the present discussion of \( I \), \( \alpha \), \( \beta \) — which one may visualize as follows

\[ \begin{array}{c}
\overset{\rightarrow}{I} \\
\overset{\rightarrow}{\alpha} \quad \beta
\end{array} \]

— one identifies \( \alpha \) and \( \beta \), then one obtains a new topological space, that is to say, more specifically, an oriented one-dimensional topological manifold [whose orientation is induced by the orientation of \( I \)]
\[ L \overset{\text{def}}{=} I/\langle \alpha \sim \beta \rangle \]
that is homeomorphic to the unit circle, i.e., may be visualized as a loop. Write \( \gamma_L \subseteq L \) for the image of \( \alpha \subseteq I \), or, equivalently, \( \beta \subseteq I \), in \( L \). As is well-known from elementary topology, the topological space \( L \) is structurally/qualitatively very
different from the topological space \( I \). For instance, whereas \( I \) has a trivial fundamental group, \( L \) has a nontrivial fundamental group [isomorphic to the additive group of integers \( \mathbb{Z} \)]. In particular,

it is by no means the case that the fact that \( \alpha \) and \( \beta \) are isomorphic as topological spaces implies a sort of "redundancy" to the effect that any mathematical argument involving \( I \) [cf. the above observation concerning fundamental groups!] is entirely equivalent to a corresponding mathematical argument in which \( \alpha \) and \( \beta \) are identified, i.e., in which "\( I \)" is replaced by "\( L \)."

(iii) In this context, we observe that the [one-dimensional oriented topological manifold with boundary] \( I \) does not admit any symmetries that switch \( \alpha \) and \( \beta \). Moreover, even if one passes to the quotient \( I \to L \), the [one-dimensional oriented topological manifold] \( L \) does not admit any symmetries that reverse the orientation of \( L \).

**Example 2.3.2:** Gluing of adjacent oriented line segments.

(i) A similar elementary geometric situation to the situation discussed in Example 2.3.1, but which is technically a bit more similar to the situation that arises in inter-universal Teichmüller theory may be given as follows. We begin with two distinct copies \( ^\dagger I \), \( ^\ddagger I \) of \( I \). Thus, \( ^\dagger I \) has endpoints \( ^\dagger \alpha \), \( ^\dagger \beta \) [i.e., corresponding respectively to the endpoints \( \alpha \), \( \beta \) of \( I \)]; similarly, \( ^\ddagger I \) has endpoints \( ^\ddagger \alpha \), \( ^\ddagger \beta \). We then proceed to form a new topological space \( J \) by gluing \( ^\dagger I \) to \( ^\ddagger I \) via the unique isomorphism of topological spaces \( ^\dagger \beta \sim ^\ddagger \alpha \). Thus, \( ^\dagger \beta \) and \( ^\ddagger \alpha \) are identified in \( J \). Let us write \( \gamma_J \) for the one-pointed topological space obtained by identifying \( ^\dagger \beta \) and \( ^\ddagger \alpha \). Thus, \( J \) may be visualized as follows:

![Diagram](https://via.placeholder.com/150)

(ii) Observe that the gluing operation that gave rise to \( J \) is such that we may regard \( ^\dagger I \) and \( ^\ddagger I \) as subspaces \( ^\dagger I \subseteq J \), \( ^\ddagger I \subseteq J \) of \( J \). Since each of these subspaces \( ^\dagger I \), \( ^\ddagger I \) of \( J \) is naturally isomorphic to \( I \), one may take the point of view, as in the discussion of Example 2.3.1, that these two subspaces are "redundant" and hence should be identified with one another [say, via the natural isomorphisms of \( ^\dagger I \), \( ^\ddagger I \) with \( I \)] to form a new topological space

\[
\mathbb{M} \overset{\text{def}}{=} J / ( ^\dagger I \sim ^\ddagger I )
\]

— where we observe that the natural isomorphisms of \( ^\dagger I \), \( ^\ddagger I \) with \( I \) determine a natural isomorphism of topological spaces \( \mathbb{M} \sim L = I / ( \alpha \sim \beta ) \), with the loop \( L \) considered in Example 2.3.1. Write \( \gamma_M \subseteq \mathbb{M} \) for the image of \( \gamma_J \subseteq J \) in \( \mathbb{M} \). Thus, the natural isomorphism \( \mathbb{M} \sim L \) maps \( \gamma_M \) isomorphically onto \( \gamma_L \). On the other hand, just as in the situation discussed in Example 2.3.1,
it is by no means the case that the fact that \( \uparrow I \) and \( \downarrow I \) are [in fact, naturally] isomorphic as topological spaces implies a sort of "redundancy" to the effect that any mathematical argument involving \( J \) is entirely equivalent to a corresponding mathematical argument in which \( \uparrow I \) and \( \downarrow I \) are identified [say, via the natural isomorphisms of \( \uparrow I, \downarrow I \) with \( I \)], i.e., in which "\( J \)" is replaced by "\( \mathbb{M} \)".

Indeed, for instance, one verifies immediately, just as in the situation of Example 2.3.1, that the fundamental groups of \( J \) and \( \mathbb{M} \) are not isomorphic. That is to say, just as in the situation of Example 2.3.1, the topological space \( J \) is structurally/qualitatively very different from the topological space \( \mathbb{M} \).

\[2.4. \text{ Logical AND "\( \land \)" vs. logical OR "\( \lor \)"} \]

The essential mathematical content of the elementary geometric examples discussed in §2.3 may be reformulated in terms of the symbolic logical relators AND "\( \land \)" and OR "\( \lor \)". This reformulation renders the elementary geometric examples of §2.3 in a form that is even more directly technically related to various central aspects of the assertions of the RCS concerning inter-universal Teichmüller theory.

\[\text{Example 2.4.1: \"\( \land \)" vs. \"\( \lor \)" for adjacent oriented line segments.}\]

(i) Recall the situation discussed in Example 2.3.2. Thus, \( J \supseteq \uparrow I \supseteq \downarrow \beta = \gamma_J = \uparrow \alpha \subseteq \downarrow I \subseteq J \), i.e.,

(AOL1) the following condition holds:

\[
\left( \gamma_J = \uparrow \beta \subseteq \downarrow I \right) \land \left( \gamma_J = \uparrow \alpha \subseteq \downarrow I \right).
\]

On the other hand, if one identifies \( \uparrow I, \downarrow I \), then one obtains a topological space \( \mathbb{M} \tilde{\to} \mathbb{L} \), i.e., a loop. Here, "\( \tilde{\to} \)" denotes the natural isomorphism discussed in Example 2.3.2, (ii). Now suppose that we are given a connected subspace

\[\gamma_I \subseteq I \]

whose image in the quotient \( I \to \mathbb{L} = I/\langle \alpha \sim \beta \rangle \) coincides with \( \gamma_L \subseteq \mathbb{L} \), i.e., with the image of \( \gamma_I \subseteq J \) via the composite of the quotient \( J \to \mathbb{M} = J/\langle \uparrow I \sim \downarrow I \rangle \) with the natural isomorphism \( \mathbb{M} \tilde{\to} \mathbb{L} \). Then observe that

(AOL2) the following condition holds: \( \gamma_I \in \{\alpha, \beta\} \), i.e.,

\[
\left( \gamma_I = \beta \subseteq I \right) \lor \left( \gamma_I = \alpha \subseteq I \right).
\]

Of course, (AOL3) the essential mathematical content discussed in this condition (AOL2) may be formally described as a condition involving the AND relator "\( \land \)"

\[
\left( \beta \in \{\alpha, \beta\} \right) \land \left( \alpha \in \{\alpha, \beta\} \right).
\]
But the essential mathematical content of the OR relator “∨” statement in (AOL2) remains unchanged.

(ii) On the other hand, [unlike the case with γ_J!]

(AOL4) the following condition does not hold:

\[ (γ_i = β ⊆ I) \land (γ_i = α ⊆ I). \]

That is to say, in summary, the operation of identifying \( †I, ‡I \) — e.g., on the grounds of “redundancy” [cf. the discussion of Example 2.3.2] — has the effect of passing from a situation in which

the AND relator “∧” holds [cf. (AOL1)]

to a situation in which

the OR relator “∨” holds [cf. (AOL2), (AOL3)], but

the AND relator “∧” does not hold [cf. (AOL4)]!

(iii) It turns out that this phenomenon — i.e., of an identification of “redundant copies” leading to a passage from the validity of an “∧” relation to the validity of an “∨” relation coupled with the invalidity of an “∧” relation — forms a very precise model of the situation that arises in the assertions of the RCS concerning inter-universal Teichmüller theory [cf. the discussion of §3.2, §3.4 below].

**Example 2.4.2: Differentials on oriented line segments.**

(i) In the situation of Example 2.4.1, one way to understand the gap between (AOL1) and (AOL4) — i.e., the central issue of whether the AND relator “∧” holds or does not hold — is to think in terms of the restriction to \( I ⊆ R \) of the coordinate function “\( x \)” of Example 2.2.1. Indeed,

(AOD1) one may interpret (AOL4) as the statement that the coordinate functions “\( x \)” on the two copies \( †I, ‡I \) that constitute \( J \) do not glue together to form a single, well-defined \( R \)-valued function on \( J \) [that is to say, since it is not clear whether the value of such a function on \( γ_J ⊆ J \) should be 0 or 1, i.e., such a function is not well-defined on \( γ_J ⊆ J \)];

(AOD2) on the other hand, (AOL1) may be interpreted as the statement that such a function [i.e., obtained by gluing together the coordinate functions “\( x \)” on the two copies \( †I, ‡I \) that constitute \( J \)] can indeed be defined if one regards its values as being [not in a single copy of \( R \), but rather] in the set \( †R, ‡R \) obtained by gluing together two distinct copies \( †R, ‡R \) of \( R \) by identifying \( †1 ∈ †R \) with \( ‡0 ∈ ‡R \).

(ii) On the other hand, if, instead of considering the coordinate function “\( x \)”, one considers the differential “\( dx \)” associated to this coordinate function [cf. the discussion of Example 2.2.1], then one observes immediately that
(AOD3) the differentials \( \mathrm{d}x \) on the two copies \( \uparrow \mathbb{I}, \downarrow \mathbb{I} \) that constitute \( \mathbb{J} \) do indeed glue together to form a single, well-defined differential on \( \mathbb{J} \) that, moreover, is compatible with the quotient \( \mathbb{J} \to \mathbb{M} = \mathbb{J}/(\uparrow \mathbb{I} \sim \downarrow \mathbb{I}) \) in the sense that, as is easily verified, it arises as the pull-back, via this quotient map \( \mathbb{J} \to \mathbb{M} \), of a [smooth] differential on the [smooth manifold constituted by] loop \( \mathbb{M} \).

Note, moreover, that the gluings and compatibility of (AOD3) may be achieved without considering functions or differentials valued in some sort of complicated [i.e., by comparison to \( \mathbb{R} \)] set such as \( \uparrow \downarrow \mathbb{R} \).

(iii) It turns out [cf. the discussion of Example 2.4.1, (iii)] that the phenomenon discussed in (AOD3) is closely related to the situation that arises in inter-universal Teichmüller theory [cf. the discussion of §3.2 below].

**Example 2.4.3:** Representation via subgroup indices of \( \land \) vs. \( \lor \).

(i) Let \( A \) be an abelian group and \( B_1, B_2 \subseteq A \) subgroups of \( A \) such that \( B_1 \cap B_2 \) has finite index in \( B_1 \) and \( B_2 \). Then one may define a positive rational number, which we call the index of \( B_2 \) relative to \( B_1 \),

\[
[B_1 : B_2] \overset{\text{def}}{=} [B_1 : B_1 \cap B_2]/[B_2 : B_1 \cap B_2] \in \mathbb{Q}_{>0}.
\]

Thus, \([B_1 : B_2] \cdot [B_2 : B_1] = 1\); when \( B_2 \subseteq B_1 \), this notion of index coincides with the usual notion of the index of \( B_2 \) in \( B_1 \).

(ii) Let \( n \) be a positive integer \( \geq 2 \). Consider the diagram of group homomorphisms

\[
G_1 \overset{n}{\longrightarrow} G_2 \overset{n}{\longrightarrow} G_3
\]

— where, for \( i = 1, 2, 3 \), \( G_i \) denotes a copy of [the additive group of rational integers] \( \mathbb{Z} \), and the arrows are given by multiplication by \( n \). For \( i = 1, 2, 3 \), write \( G_i^\mathbb{Q} \overset{\text{def}}{=} G_i \otimes \mathbb{Z} \mathbb{Q} \) for the tensor product of \( G_i \) over \( \mathbb{Z} \) with \( \mathbb{Q} \). Then observe that this diagram induces a diagram of group isomorphisms

\[
G_1^\mathbb{Q} \overset{\sim}{\longrightarrow} G_2^\mathbb{Q} \overset{\sim}{\longrightarrow} G_3^\mathbb{Q}
\]

— i.e., in which the arrows are isomorphisms. Let us use these isomorphisms to identify the groups \( G_i^\mathbb{Q} \), for \( i = 1, 2, 3 \), and denote the resulting group by \( G_n^\mathbb{Q} \).

(iii) Observe that the first diagram of (ii) is structurally reminiscent of the object \( \mathbb{J} \) discussed in Examples 2.3.2, 2.4.1, 2.4.2, i.e., if one regards

- the first arrow of the first diagram of (ii) as corresponding to \( \uparrow \mathbb{I} \),
- the second arrow of the first diagram of (ii) as corresponding to \( \downarrow \mathbb{I} \), and
- \( G_1, G_2, \) and \( G_3 \) as corresponding to \( \uparrow \alpha, \downarrow \alpha, \) and \( \downarrow \beta \), respectively.

Here, we observe that \( G_2 \) appears simultaneously as the codomain of the arrow \( G_1 \overset{n}{\longrightarrow} G_2 \) AND [cf. (AOL1)!] as the domain of the arrow \( G_2 \overset{n}{\longrightarrow} G_3 \). Moreover, we may consider indices of \( G_1, G_2, \) and \( G_3 \) as subgroups of \( G_2^\mathbb{Q} \)

\[
[G_2 : G_1] = [G_1 : G_2]^{-1} = n; \quad [G_3 : G_2] = [G_2 : G_3]^{-1} = n; \quad [G_3 : G_1] = [G_1 : G_3]^{-1} = n^2
\]
in a **consistent** fashion, i.e., in a fashion that does *not* give rise to any **contradictions**.

(iv) On the other hand, suppose that we *delete* the “distinct labels” $G_1, G_2, G_3$ from the copies of $\mathbb{Z}$ considered in the first diagram of (ii). This yields a diagram

$$\mathbb{Z} \xrightarrow{n \cdot} \mathbb{Z} \xrightarrow{n} \mathbb{Z}$$

in which the *second arrow* may be regarded as a *copy* of the *first arrow*. This situation might motivate some observers to conclude that these two arrows are “**redundant**” and hence should be identified with one another — cf. the discussion of the quotient $\mathbb{J} \rightarrow \mathbb{M}$ in Example 2.3.2, (ii) — to form a diagram

$$n \cdot M$$

consisting of a *single copy* of $\mathbb{Z}$ and the *endomorphism* of this single copy of $\mathbb{Z}$ given by *multiplication* by $n$. At first glance, this operation of *identification* may appear to give rise to various “**contradictions**” in the computation of the *index*, i.e., such as

$$1 = [G_1 : G_1] = [\mathbb{Z} : \mathbb{Z}] = [G_2 : G_1] = n \geq 2$$

and so on. In fact, however, if one takes into account the *OR* relator “∨” [but not the *AND* relator “∧”!] relations that one obtains upon executing the identification operation in question [cf. (AOL2), (AOL4)!], then one concludes that [after executing the identification operation in question!] each of the indices $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, *may only be computed up to multiplication by an integral power of* $n$, i.e., that

each index $[G_i : G_j]$, for $i, j \in \{1, 2, 3\}$, is only well-defined as “**some indeterminate element**” of $n\mathbb{Z} \defeq \{nm \mid m \in \mathbb{Z}\} \subseteq \mathbb{Q}_{>0}$.

In particular, in fact there is **no contradiction**.

**Example 2.4.4**: Logical “∧/∨” vs. “narrative ∧/∨”. Consider the following argument concerning a *natural number* $x \in \{1, 3\}$, i.e., a natural number for which it holds that $(x = 1) \lor (x = 3)$:

(Nar1) Suppose that $x = 3$. Then it follows that $x = 3 > 2$. That is to say, we *conclude* that $x > 2$.

(Nar2) Since $(x = 1) \lor (x = 3)$, we may consider the case $x = 1$. Then, by applying the conclusion of (Nar1), we conclude that $1 = x > 2$, i.e., that $1 > 2$ — a **contradiction**!

Of course, this argument is *completely fallacious*! On the other hand, it yields a readily understood concrete example of the *absurdity* that arises when, as is in effect done in (Nar2), *logical OR “∨” is confused with logical AND “∧”! In various contexts, this sort of confusion can arise from the *ambiguity of various narrative expressions* that appear in the discussion of a mathematical argument. This sort of ambiguity can lead to a situation in which
a “narrative AND ∧” — i.e., the fact that in a particular narrative exposition of an argument, one performs both the task of considering the case “x = 3” [cf. (Nar1)] and the task of considering the case “x = 1” [cf. (Nar2)] — is mistakenly construed as a logical AND “∧”.

In a similar vein, one may consider situations in which the roles played by “∧” and “∨” are reversed, i.e., in which a “narrative OR ∨” — i.e., the fact that in a particular narrative exposition of an argument, one’s attention is concentrated either on the task of considering one situation or on the task of considering another situation — is mistakenly construed as a logical OR “∨”. Indeed, it appears that one fundamental cause, in the context of the essential logical structure of inter-universal Teichmüller theory [cf. the discussion of Example 2.4.5 below!], of the confusion on the part of some mathematicians between logical AND “∧” and logical OR “∨” lies precisely in this sort of confusion between “narrative ∧/∨” and logical “∧/∨”.

Example 2.4.5: Numerical representation of “∧” vs. “∨”.

(i) A slightly more sophisticated numerical representation of the difference between “∧” and “∨” — which in fact mirrors the essential logical structure of inter-universal Teichmüller theory in a very direct fashion — may be given as follows [cf. [Alien], Example 3.11.4]. Indeed, the essential logical flow of inter-universal Teichmüller theory may be summarized as follows:

- one starts with from the definition of an object called the Θ-link;
- one then constructs a complicated apparatus that is referred to as the multiradial representation of the Θ-pilot [cf. [IUTchIII], Theorem 3.11];
- finally, one derives a final numerical estimate [cf. [IUTchIII], Corollary 3.12] in an essentially straightforward fashion from the multiradial representation of the Θ-pilot.

(ii) An elementary model of this essential logical flow may be given by means of real numbers $A, B \in \mathbb{R}_{>0}$ and $\epsilon, N \in \mathbb{R}$ such that $0 \leq \epsilon \leq 1$ in the following way:

- Θ-link:
  $$\left( N \overset{\text{def}}{=} -2B \right) \land \left( N \overset{\text{def}}{=} -A \right);$$

- multiradial representation of the Θ-pilot:
  $$\left( N = -2A + \epsilon \right) \land \left( N = -A \right);$$

- final numerical estimate:
  $$-2A + \epsilon = -A,$$
  hence $A = \epsilon$, i.e., $A \leq 1$.

Thus, the definition of the Θ-link and the construction of the multiradial representation of the Θ-pilot are meaningful/nontrivial precisely on account of the validity
of the **AND relator** “∧”, which is rendered possible, in the definition of the Θ-link, precisely by allowing the real numbers \(A, B\) to be [a priori] **distinct** real numbers — cf. (AOL1) vs. (AOL4), where we think in terms of the correspondences

\[
\begin{align*}
B & \leftrightarrow \uparrow \mathbb{I}, \quad N \leftrightarrow \gamma_\mathbb{I}, \quad A \leftrightarrow \downarrow \mathbb{I} \\
-2B & \leftrightarrow \uparrow \beta, \quad -A \leftrightarrow \uparrow \alpha, \quad -2A \leftrightarrow \downarrow \beta.
\end{align*}
\]

The passage from the **multiradial representation of the Θ-pilot** to the **final numerical estimate** is then immediate/straightforward/logically transparent.

(iii) By contrast, if, in the elementary numerical model of (ii), one replaces “∧” by “∨”, then our elementary numerical model of the logical structure of inter-universal Teichmüller theory takes the following form:

- **“∨” version of Θ-link:**

\[
\left( N \overset{\text{def}}{=} -2B \right) \lor \left( N \overset{\text{def}}{=} -A \right) \quad \text{[cf.} \left( N \overset{\text{def}}{=} -2A \right) \lor \left( N \overset{\text{def}}{=} -A \right)\text{]};
\]

- **“∨” version of multiradial representation of the Θ-pilot:**

\[
\left( N = -2A + \epsilon \right) \lor \left( N = -A \right);
\]

- **final numerical estimate:**

\[-2A + \epsilon = -A, \text{ hence } A = \epsilon, \text{ i.e., } A \leq 1.\]

That is to say, the use of **distinct** real numbers \(A, B\) in the definition of the “∨” version of Θ-link seems entirely superfluous [cf. (AOL2), relative to the correspondences discussed in (ii)]. This motivates one to **identify** \(A\) and \(B\) — i.e., to suppose “for the sake of simplicity” that \(A = B\) — which then has the effect of rendering the definition of the original “∧” version of the Θ-link invalid/self-contradictory [cf. (AOL4), relative to the correspondences discussed in (ii)]. Once one identifies \(A\) and \(B\), i.e., once one supposes “for the sake of simplicity” that \(A = B\), the passage from the “∨” version of Θ-link to the resulting “∨” version of the multiradial representation of the Θ-pilot then seems entirely meaningless/devoid of any interesting content. The passage from the resulting meaningless “∨” version of the multiradial representation of the Θ-pilot to the final numerical estimate then seems abrupt/mysterious/entirely unjustified, i.e., put another way, looks as if one **erroneously replaced** the “∨” in the meaningless “∨” version of the multiradial representation of the Θ-pilot by an “∧” **without any mathematical justification whatsoever**.

It is precisely this **pernicious chain of misunderstandings** emanating from the “redundancy” assertions of the RCS that has given rise to a substantial amount of unnecessary confusion concerning inter-universal Teichmüller theory.
(iv) Before proceeding, we observe that the sort of confusion discussed in (iii) between “∧” and “∨” can occur as the result of any of the following phenomena:

(AOC1) a confusion between “narrative ∧/∨” and logical “∧/∨”, as discussed in Example 2.4.4;

(AOC2) thinking in terms of the “fake ∧” of (AOL3), i.e., which, though formulated as a logical AND “∧” relation, is in fact, substantively speaking, a logical OR “∨” relation;

(AOC3) the symptom (Syp2) discussed in §3.6 below, i.e., a desire to see the “proof” of some sort of commutative diagram or “compatibility property” to the effect that taking log-volumes of pilot objects in the domain and codomain of the Θ-link yields the same real number;

(AOC4) a fundamental misunderstanding — which is often closely intertwined with the symptom (Syp2) discussed in (AOC3) — of the meaning of the crucial closed loop of §3.10, (Stp7), (Stp8), below [cf. §3.10, (Stp7), (Stp8), as well as the following discussion].

(v) Let us refer to the “∧” version of inter-universal Teichmüller theory discussed in (ii) — i.e., the original version of inter-universal Teichmüller theory, in which one interprets the Θ-link as a logical AND “∧” relation — as AND-IUT. Thus,

\[ \text{AND-IUT} = \text{IUT} \]

is the original version of inter-universal Teichmüller theory.

Let us refer to the “∨” version of inter-universal Teichmüller theory discussed in (iii) — i.e., the version of inter-universal Teichmüller theory that arises if one [mistakenly!] interprets the Θ-link as a logical OR “∨” relation — as OR-IUT. As discussed in (iii), in OR-IUT, one is motivated to implement the RCS-identifications of RCS-redundant copies of objects — i.e., in the language of (iii), to “identify A and B” — and hence to conclude that OR-IUT \(\Rightarrow\) RCS-IUT, where we recall that “RCS-IUT” refers to the version of inter-universal Teichmüller theory obtained by implementing the RCS-identifications of RCS-redundant copies of objects [cf. the discussion of §1.2]. On the other hand, it is not difficult to see that in RCS-IUT, one is forced to work with a (NeuORInd) indeterminacy [cf. the discussion at the end of §3.4 below, as well as the discussion of (ΘORInd) in §3.11 below], i.e., to interpret the Θ-link as a logical XOR “˙∨” relation [that is to say, a logical OR “∨” relation such that the corresponding logical AND “∧” relation cannot hold — cf. the discussion of (iii)]. In particular, we conclude that RCS-IUT \(\Rightarrow\) XOR-IUT \(\Rightarrow\) OR-IUT [where the second “ \(\Rightarrow\) ” is a consequence of well-known general properties of Boolean operators], i.e., in summary:

(XOR/RCS) we have equivalences XOR-IUT \(\iff\) OR-IUT \(\iff\) RCS-IUT.

In the following, I shall refer to the school of thought [i.e., in the sense of a “collection of closely interrelated ideas”] surrounding OR-IUT as ORS, i.e., the “OR school [of thought]”, and to the school of thought surrounding XOR-IUT as XORS, i.e., the “XOR school [of thought]”. Thus, XORS = ORS = RCS.

(vi) On the other hand, one may also consider yet another version of inter-universal Teichmüller theory, also motivated by the discussion of (iii), which we
refer to as EssOR-IUT, i.e., “essentially OR IUT”. This is the version of inter-universal Teichmüller theory in which one accepts, at the level of formal definitions, the logical AND “∧” version of the Θ-link as in (ii), i.e., without identifying A and B, but [for some unexplained reason!] one then arbitrarily shifts, when considering the multiradial representation of the Θ-pilot, to the logical OR “∨” interpretation of the multiradial representation of the Θ-pilot, i.e., as in (iii). That is to say, as the name “EssOR-IUT” suggests, the fundamental logical AND “∧” property of the Θ-link is never actually used in any sort of meaningful way in EssOR-IUT. In particular,

(EssOR/RCS) although, at a purely formal level, EssOR-IUT rejects RCS-IUT, the essential logical structure of EssOR-IUT still nevertheless gives rise to the abrupt/mysterious/entirely unjustified transition discussed in (iii) to the final numerical estimate.

It appears that the “arbitrarily shift” referred to above is often precipitated by the various phenomena discussed in (iv) [cf., especially, (AOC3), (AOC4)]. In the following, I shall refer to the school of thought [i.e., in the sense of a “collection of closely interrelated ideas”] surrounding EssOR-IUT as EssORS, i.e., the “essentially OR school [of thought]”.

(vii) In general, at the level of formalities of Boolean operators, “∧ ⇒ ∨”, but “∨ ̸⇒ ∧”. In particular, in the context of the transition to the final numerical estimate of inter-universal Teichmüller theory,

(∨ ̸⇒ ∧) it appears entirely hopeless/unrealistic to pass from the “∨” version of the multiradial representation of the Θ-pilot to the “∧” version of the multiradial representation of the Θ-pilot.

This is precisely the “abrupt/mysterious/entirely unjustified” transition [to the final numerical estimate] discussed in (iii).

(viii) The discussion of (v), (vi), (vii) may be summarized as follows [cf. also the discussion of §1.2, §1.3]:

- The fundamental misunderstanding on the part of adherents of the RCS = ORS = XORS to the effect that OR-IUT or XOR-IUT is indeed the content of AND-IUT = IUT leads to the mistaken interpretation of the assertion (XOR/RCS) as an equivalence between AND-IUT = IUT and RCS-IUT.

- The fundamental misunderstanding on the part of adherents of the RCS = ORS = XORS to the effect that OR-IUT or XOR-IUT is indeed the content of AND-IUT = IUT leads to the mistaken interpretation of the assertion (∨ ̸⇒ ∧) as a logical flaw in AND-IUT = IUT.

- The fundamental misunderstanding on the part of adherents of the EssORS to the effect that EssOR-IUT is indeed the content of AND-IUT = IUT leads either to the mistaken interpretation of the assertion (∨ ̸⇒ ∧) as a logical flaw in AND-IUT = IUT or to the mistaken interpretation of the assertion (∨ ̸⇒ ∧) as an indication the existence of some sort of infinitely complicated and mysterious argument — i.e., for concluding that “∨ ⇒ ∧”! — in inter-universal Teichmüller theory that requires years of concerted
effort to understand. Thus, as the descriptive “essential” suggests, there is in fact, from the point of view of the essential logical structure under consideration, very little difference between EssORS and XORS = ORS = RCS or between EssOR-IUT and XOR-IUT ⇔ OR-IUT ⇔ RCS-IUT.

- In fact, the **correct interpretation** of the assertion (∨ ̸⇒ ∧) consists of the conclusion that neither XOR-IUT ⇔ OR-IUT ⇔ RCS-IUT nor EssOR-IUT has any direct logical relationship to AND-IUT = IUT.

Here, we observe that the above analysis is in some sense remarkable in that it makes explicit the fact that, if one **forgets** the arbitrary label “inter-universal Teichmüller theory” placed on XOR-IUT or OR-IUT or EssOR-IUT by adherents of the RCS = ORS = XORS or the EssORS, then [perhaps somewhat surprisingly!] (MthVI) there is in fact **no disagreement** among any of the parties involved with regard to the **mathematical validity** of the mathematical assertions (XOR/RCS) and (∨ ̸⇒ ∧).

Indeed, this state of affairs may be understood as in some sense highlighting the essentially social/political/psychological, i.e., in summary, non-mathematical nature of the entirely unnecessary confusion that has arisen concerning inter-universal Teichmüller theory [cf. the discussion of §1.8, §1.11]. The above observations are summarized in the **“dictionary of assertions”** given below.

<table>
<thead>
<tr>
<th>Assertions of various schools of thought</th>
<th>Actual mathematical content</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCS = ORS = XORS: “IUT ⇔ RCS-IUT”</td>
<td>“XOR-IUT ⇔ OR-IUT ⇔ RCS-IUT”</td>
</tr>
<tr>
<td>RCS = ORS = XORS: “IUT is logically flawed.”</td>
<td>“∨ ̸⇒ ∧”, which implies that “(AND-IUT =) IUT ̸= RCS-IUT”, “(AND-IUT =) IUT ̸= OR-IUT”, “(AND-IUT =) IUT ̸= XOR-IUT”</td>
</tr>
<tr>
<td>EssORS: either “IUT is logically flawed.” or “The logical structure of IUT is infinitely complicated/mysterious.”</td>
<td>“∨ ̸⇒ ∧”, which implies that “(AND-IUT =) IUT ̸= EssOR-IUT”</td>
</tr>
</tbody>
</table>
Example 2.4.6: Carry operations in arithmetic, geometry, and Boolean logic.

(i) Observe that if, in the situation of Example 2.4.3, (ii), one focuses one’s attention on the subset $D_{4-i} \subseteq G_i$ in the copy of $\mathbb{Z}$ denoted by $G_i$, where $i = 1, 2, 3$, corresponding to $\{0, 1, 2, \ldots, n-1\} \subseteq \mathbb{Z}$, then the situation considered in Example 2.4.3, (ii), closely resembles the situation that arises in elementary arithmetic computations — such as addition and multiplication — involving base $n$ expansions of natural numbers. That is to say, one may think of

- $D_1$ as the first digit, i.e., when $n = 10$, the “ones digit”,
- $D_2$ as the second digit, i.e., when $n = 10$, the “tens digit”, and
- $D_3$ as the third digit, i.e., when $n = 10$, the “hundreds digit”

of such an expansion. When performing such elementary arithmetic computations — such as addition and multiplication — involving base $n$ expansions of natural numbers, recall that it is of fundamental importance to take into account the various carry operations that occur. In particular, we observe that

the use of distinct labels for distinct digits plays a fundamental role in elementary arithmetic computations involving base $n$ expansions of natural numbers

— cf. the distinct labels “$G_1$”, “$G_2$”, “$G_3$” in the discussion of Example 2.4.3, (ii), (iii). This situation is reminiscent of the important role played by the distinct labels “$A$”, “$B$” in the “$\Theta$-link” of Example 2.4.5, (ii). Note, moreover, that

deletion/confusion of these distinct labels for distinct digits has the effect of completely invalidating, at least in the usual “strict sense”, elementary arithmetic computations involving base $n$ expansions of natural numbers

— cf. the situation considered in Example 2.4.3, (iv). On the other hand, if one restricts one’s attention to a specific computational algorithm [involving, say, addition and multiplication operations], then in fact it is often possible — i.e., depending on the content of the specific computational algorithm under consideration — to obtain

estimates to the effect that applying the algorithm either with or without the use of distinct labels for distinct digits in the base $n$ expansions of the natural numbers involved in fact yields the same result, up to some explicitly bounded discrepancy.

[For instance, when $n = 10$, any algorithm that only involves addition and multiplication operations yields the same result modulo $9 (= 10 - 1)$, regardless of whether or not one uses distinct labels for distinct digits in decimal expansions of natural numbers.] Such estimates are reminiscent of the “multiradial representation” of Example 2.4.5, (ii).

(ii) Observe that the discussion of

- adjacent oriented line segments “$\mathbb{I}$”, “$\mathbb{I}$” and
- oriented loops “$\mathbb{L}$”, “$\mathbb{M}$”
in Examples 2.3.1, 2.3.2, 2.4.1, 2.4.2 [cf. also the discussion of Examples 2.4.3, (iii); 2.4.5, (ii)] may be regarded as a sort of limiting case of the discussion of base $n$ expansions of natural numbers in (i) above, i.e., if one

- considers the real numbers obtained by dividing the natural numbers $\leq 2n$ in the discussion of (i) above by $n$ and then
- passes to the limit $n \to +\infty$.

That is to say, in summary,

the adjacency of the oriented line segments "$I$", "$I$" may be understood as a sort of continuous, geometric representation of the carry operation that appears in elementary arithmetic computations involving base $n$ expansions of natural numbers.

(iii) From the point of view of discussions of the logical structure of mathematical arguments represented in terms of Boolean operators such as logical AND "$\land$" and logical OR "$\lor$", it is of interest to consider the discussion of (i) above in the binary case, i.e., the case $n = 2$. We begin our discussion of the binary case by recalling the following well-known facts:

- multiplication in the field $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the Boolean operator AND "$\land$";
- addition in the field $\mathbb{F}_2 = \{0, 1\}$ may be regarded as corresponding to the Boolean operator XOR [i.e., "exclusive OR"], which we denote by "$\lor$";
- "carry-addition" in the truncated ring of Witt vectors $\mathbb{F}_2 \times \mathbb{F}_2$ — i.e., addition of two elements of $\mathbb{F}_2 \cong \{0\} \times \mathbb{F}_2 \subseteq \mathbb{F}_2 \times \mathbb{F}_2$, regarded as Teichmüller representatives in the truncated ring of Witt vectors $\mathbb{Z}/4\mathbb{Z}$, that is to say, an addition operation in which one allows for the carry operation [cf. the discussion of (i)!] to the first factor of $\mathbb{F}_2 \times \mathbb{F}_2$ — may be regarded as corresponding to an operator that we shall refer to as the "COR", or "carry-OR", operator and denote by "$\hat{\lor}$"; thus, we have

$$\hat{\lor} = (\land, \hat{\lor})$$

[so $0 \hat{\lor} 0 = (0, 0)$; $1 \hat{\lor} 0 = 0 \hat{\lor} 1 = (0, 1)$; $1 \hat{\lor} 1 = (1, 0)$].

These well-known facts case may be summarized as follows:

$$(\hat{\lor} = \land \hat{\lor})$$

Conventional mixed-characteristic/"carry" addition in $\mathbb{Z}$ considered modulo 4 — i.e., "$\hat{\lor}$" may be described in terms of the "splitting" of the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ determined by the "Teichmüller representatives" $0, 1 \in \mathbb{Z}$ via the equation

$$\hat{\lor} = (\land, \hat{\lor})$$

— i.e., which exhibits "$\hat{\lor}$" as an operation obtained by "stacking" multiplication "$\land$" in $\mathbb{F}_2$ on top of addition "$\lor$" in $\mathbb{F}_2$. Here, we note that this splitting via Teichmüller representatives $0, 1 \in \mathbb{Z}$ is compatible
with the multiplicative structures in \( \mathbb{Z} \) and \( \mathbb{F}_2 \), but not with the additive structures in \( \mathbb{Z} \) and \( \mathbb{F}_2 \). Put another way, one may think of the ring structures of \( \mathbb{Z} \) and \( \mathbb{F}_2 \) as structures that share a common multiplicative structure [cf. “\(^\wedge\)”], but do not share a common additive structure [cf. “\(^\vee\)”].

These observations will be of fundamental importance in the theory developed in §3 [cf., especially, the discussion at the beginning of §3.10].

(iv) Some readers may object to the comparisons and analogies between inter-universal Teichmüller theory and the “mathematics of carry operations” discussed in (i), (ii), and (iii) as being inappropriate on the grounds that this mathematics of carry operations is much too “trivial” to be of any substantive interest. On the other hand, we observe that the mathematics of carry operations discussed in (i), (ii), and (iii) is intimately intertwined with numerous important developments in the history of mathematics. With regard to the content of (i), we recall that the use of place-value decimal numerals, i.e., that make use of notation for zero, appears to date back to Indian texts and inscriptions from the 7-th to 9-th centuries AD. Such numerals also reached the Arab world during this period, but this Hindu-Arabic numeral system apparently only became widely used in Europe during the late middle ages, between the 13-th and 15-th centuries. In this context, one noteworthy development was the book Liber Abaci published by the Italian mathematician Fibonacci in 1202, which promoted the use of the Hindu-Arabic numeral system in Europe. Here, it is useful to recall that, by comparison to earlier numeral systems, such as the Greco-Roman and Babylonian systems, place-value decimal numerals not only facilitate elementary arithmetic computations — i.e., via the systematic use of carry operations, as discussed in (i) — but also make it possible to express all — hence, in particular, infinitely many — natural numbers by means of finitely many symbols — i.e., unlike earlier numerical systems, in which only finitely many natural numbers could be expressed using finitely many symbols. This revolutionary importance of the development of place-value decimal numerals in India was recognized, for instance, in writings of the 18-th century French mathematician Laplace. In this context, it is also of interest to observe that the discussion in (ii) of the interpretation of the discussion of (i) in terms of line segments is reminiscent of the discussion of the “Euclidean algorithm” in Euclid’s Elements, in which numbers are often represented as lengths of line segments. Finally, we recall that the Boolean aspects discussed in (iii) played an important role in the [well-known!] development of modern digital computers in the 20-th century.

The gluings of adjacent line segments discussed in Examples 2.3.2, 2.4.1, 2.4.2 may in some sense be regarded as a sort of optimized elementary geometric/combinatorial representation of the essential logical “\(^\wedge/\vee\)” structure surrounding a gluing in a fashion that is qualitatively entirely structurally similar to the gluings that occur in inter-universal Teichmüller theory, which will be discussed in more detail in §3 below. The somewhat more numerical/arithemetic situations discussed in Examples
The projective line as a gluing of ring schemes along a multiplicative group scheme. In the following discussion, we take \( k \) to be a field and \( q \in k \) to be an element such that \( q^3 \neq q \) [i.e., \( q \notin \{0, 1, -1\} \)]. Write \( k^\times \overset{\text{def}}{=} k \setminus \{0\} \), \( \mathbb{A}^1 \) for the affine line \( \text{Spec}(k[T]) \) over \( k \), \( \mathbb{G}_m \) for the open subscheme \( \text{Spec}(k[T, T^{-1}]) \) of \( \mathbb{A}^1 \) obtained by removing the origin. Thus, the standard coordinate \( T \) on \( \mathbb{A}^1 \), \( \mathbb{G}_m \) determines natural bijections \( \mathbb{A}^1(k) \to k \), \( \mathbb{G}_m(k) \to k^\times \) of the respective sets of \( k \)-rational points of \( \mathbb{A}^1 \), \( \mathbb{G}_m \) with corresponding subsets of \( k \). Also, we recall that \( \mathbb{A}^1 \) is equipped with a well-known natural structure of ring scheme over \( k \), while \( \mathbb{G}_m \) is equipped with a well-known natural structure of [multiplicative] group scheme over \( k \).

(i) Write \( \dagger \mathbb{A}^1 \), \( \ddagger \mathbb{A}^1 \) for the \( k \)-ring schemes given by copies of \( \mathbb{A}^1 \) equipped with the respective labels “\( \dagger \)” , “\( \ddagger \)” . We regard \( \dagger \mathbb{A}^1 \) as being further equipped with the \( k \)-rational point \( \dagger q^{-1} \in \dagger \mathbb{A}^1(k) \) (\( \sim \) \( k \)) corresponding to the multiplicative inverse of the element \( q \in k \) and \( \ddagger \mathbb{A}^1 \) as being further equipped with the \( k \)-rational point \( \ddagger q \in \ddagger \mathbb{A}^1(k) \) (\( \sim \) \( k \)) corresponding to the element \( q \in k \). Similarly, we write \( \dagger \mathbb{G}_m \), \( \ddagger \mathbb{G}_m \) for the [multiplicative] \( k \)-group schemes given by copies of \( \mathbb{G}_m \), equipped with the respective labels “\( \dagger \)” , “\( \ddagger \)” . Thus, \( \dagger q^{-1} \in \dagger \mathbb{G}_m(k) \) (\( \subseteq \dagger \mathbb{A}^1(k) \)), \( \ddagger q \in \ddagger \mathbb{G}_m(k) \) (\( \subseteq \ddagger \mathbb{A}^1(k) \)).

(ii) Relative to the notation of (i), we observe that

(ii-a) there exists a unique isomorphism of \( k \)-ring schemes \( \dagger \mathbb{A}^1 \sim \ddagger \mathbb{A}^1 \), but that

(ii-b) the pairs \( (\dagger \mathbb{A}^1, \dagger q^{-1}) \) and \( (\ddagger \mathbb{A}^1, \ddagger q) \) are not isomorphic, i.e., as pairs consisting of a \( k \)-ring scheme equipped with a \( k \)-rational point [cf. our assumption that \( q^3 \neq q \)].

By contrast,

(ii-c) there exists a unique isomorphism of \( \mathbb{G}_m \) pairs \( (\dagger \mathbb{G}_m, \dagger q^{-1}) \to (\ddagger \mathbb{G}_m, \ddagger q) \), i.e., of pairs consisting of a [multiplicative] \( k \)-group scheme equipped with a \( k \)-rational point.

Here, we observe that the isomorphism \( (\dagger \mathbb{G}_m, \dagger q^{-1}) \to (\ddagger \mathbb{G}_m, \ddagger q) \) of (ii-c) does not extend [cf. (ii-b)!] to an isomorphism \( (\dagger \mathbb{A}^1, \dagger q^{-1}) \to (\ddagger \mathbb{A}^1, \ddagger q) \). In particular,

(ii-d) the isomorphism of [multiplicative] \( k \)-group schemes \( \dagger \mathbb{G}_m \sim \ddagger \mathbb{G}_m \) is not compatible with the \( k \)-ring scheme structures of \( \dagger \mathbb{A}^1 \) \( (\supset \dagger \mathbb{G}_m) \), \( \ddagger \mathbb{A}^1 \) \( (\supset \ddagger \mathbb{G}_m) \).

Next, we observe that
(ii-e) the standard construction of the projective line may be regarded as the result of gluing $(\overset{\dagger}{\mathbb{A}}^1, \overset{\dagger}{q}^{-1})$ to $(\overset{\dagger}{\mathbb{A}}^1, \overset{\dagger}{q})$ along the isomorphism

$$ (\overset{\dagger}{\mathbb{G}_m}, \overset{\dagger}{q}^{-1}) \sim (\overset{\dagger}{\mathbb{G}_m}, \overset{\dagger}{q}) $$

of (ii-c); thus, relative to this gluing, $\overset{\dagger}{\mathbb{G}_m} \sim \overset{\dagger}{\mathbb{G}_m}$ may be regarded simultaneously as an open subscheme of $\overset{\dagger}{\mathbb{A}}^1$ AND [cf. “∧”!] as an open subscheme of $\overset{\dagger}{\mathbb{A}}^1$.

In particular, (ii-d), (ii-e) may be summarized as follows:

the standard construction of the projective line may be regarded as a gluing of two ring schemes along an isomorphism of multiplicative group schemes that is not compatible with the ring scheme structures on either side of the gluing.

Moreover, we note that, relative to this gluing,

(ii-f) the notion of a regular function on $\overset{\dagger}{\mathbb{A}}^1$ cannot be expressed directly in terms of the notion of a regular function on $\overset{\dagger}{\mathbb{A}}^1$, whereas

(ii-g) the notion of a rational function on $\overset{\dagger}{\mathbb{A}}^1$ can be expressed directly in terms of — i.e., in essence, coincides, relative to the above gluing, with — the notion of a rational function on $\overset{\dagger}{\mathbb{A}}^1$.

Finally, we observe that

(ii-h) if, in the gluing of (ii-e), one arbitrarily deletes the distinct labels “†”, “‡”, then the resulting “gluing without labels” amounts to a gluing of a single copy of $\mathbb{A}^1$ to itself that maps the standard coordinate “$T$” on $\mathbb{A}^1$ [regarded, say, as a rational function on $\mathbb{A}^1$] to $T^{-1}$; that is to say, such a “gluing without labels” results in a contradiction [i.e., since $T \neq T^{-1}$!], unless one passes to some sort of quotient of $\mathbb{A}^1$ — which amounts, from a foundational/logical point of view, to the introduction of some sort of indeterminacy, i.e., to the consideration of some sort of collection of possibilities [cf. “∨”!].

(iii) The discussion of the projective line in (ii) is truly remarkable in that it completely parallels — i.e., relative to the correspondence

$$ “-1” \leftrightarrow “j^2” $$

between the exponent “$-1$” in the discussion of (ii) and the exponents “$j^2$”, where $j$ ranges from 1 to $l^*$, in the discussion of §3.4 — numerous aspects of the $\Theta$-link of inter-universal Teichmüller theory, which we shall discuss in more detail in §3 [cf., especially, §3.4]. Indeed,

(iii-a) the isomorphism of (ii-a) may be understood as corresponding to the fact that the $(\Theta^{\pm \text{ell}\mathbb{NF}^{-}})$Hodge theaters on either side of the $\Theta$-link in inter-universal Teichmüller theory are isomorphic, while

(iii-b) the observation of (ii-b) may be understood as corresponding to the fact that there is no isomorphism of $(\Theta^{\pm \text{ell}\mathbb{NF}^{-}})$Hodge theaters as in (iii-a) that maps the $\Theta$-pilot in the domain of the $\Theta$-link [which corresponds
to \( ^{1} q^{-1} \) to the \( q \)-pilot in the codomain of the \( \Theta \)-link [which corresponds to \( ^{1} q \)].

On the other hand,

(iii-c) the isomorphism of (ii-c) may be understood as corresponding to the full poly-isomorphism of [multiplicative!] \( F^{\oplus^\times^\mu} \)-prime-strips that constitutes the \( \Theta \)-link, while

(iii-d) the observation of (ii-d) may be understood as corresponding to the fact that the full poly-isomorphism of (iii-c) is not compatible with the ring structures determined by the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theaters on either side of the \( \Theta \)-link, i.e., in particular, does not arise from a poly-isomorphism between these \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theaters on either side of the \( \Theta \)-link [cf. (iii-b)].

Next, we observe that

(iii-e) the gluing of (ii-e) may be understood as corresponding to the gluing constituted by the \( \Theta \)-link between the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theaters on either side of the \( \Theta \)-link, i.e., a gluing along [multiplicative!] \( F^{\oplus^\times^\mu} \)-prime-strips that is not compatible with the ring structures in the domain and codomain of the \( \Theta \)-link, but which allows one to obtain a single \( F^{\oplus^\times^\mu} \)-prime-strip, up to isomorphism, that may be interpreted simultaneously as the \( F^{\oplus^\times^\mu} \)-prime-strip arising from the \( \Theta \)-pilot in the domain of the \( \Theta \)-link AND [cf. \( ^{\wedge} \)!] as the \( F^{\oplus^\times^\mu} \)-prime-strip arising from the \( q \)-pilot in the codomain of the \( \Theta \)-link.

Here, we recall that this crucial logical AND \( ^{\wedge} \) property of the \( \Theta \)-link is the central theme of the present paper [cf. the discussion of Examples 2.4.1, 2.4.2, 2.4.3, 2.4.4, 2.4.5, 2.4.6!]. Next, we observe that

(iii-f) the observation of (ii-f) may be understood as corresponding to the fact that, at least from an \textit{a priori} point of view, there is no natural way to express the \( \Theta \)-pilot of the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theater in the domain of the \( \Theta \)-link, relative to the gluing of (iii-e), in terms of the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theater in the codomain of the \( \Theta \)-link, while

(iii-g) the observation of (ii-g) may be understood as corresponding to the simultaneous holomorphic expressibility (SHE) property of the multipartiradial representation of the \( \Theta \)-pilot [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)], which allows one to express the \( \Theta \)-pilot of the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theater in the domain of the \( \Theta \)-link, relative to the gluing of (iii-e), in terms of the \( (\Theta^{\pm \text{ell}} \text{NF}-) \) Hodge theater in the codomain of the \( \Theta \)-link.

Finally, we note that

(iii-h) the \textit{“gluing without labels”} discussed in (ii-h) may be understood as corresponding to the oversimplified version \textit{“RCS-IUT”} of inter-universal Teichmüller theory obtained by implementing the RCS-identifications of RCS-redundant copies of objects [cf. the discussion of §1.2, Example 2.4.5], which leads to an immediate contradiction, unless one introduces some sort of quotient/indeterminacy, i.e., which amounts to the consideration of some sort of collection of possibilities [cf. \( ^{\vee} \)!].
In particular, relative to this remarkably close structural resemblance between
the gluing that appears in the standard construction of the projective line and
the gluing constituted by the Θ-link of inter-universal Teichmüller theory, the
central assertion

“IUT ⇔ RCS-IUT”

of the RCS [cf. the discussion of Example 2.4.5] may be understood as correspond-
ing to the assertion of an “obvious equivalence” [cf. the discussion of §1.3] between

- the projective line, on the one hand, and
- the affine line regarded up to some sort of identification of the standard
  coordinate “T” on the affine line with its inverse, on the other.

Section 3: The logical structure of inter-universal Teichmüller theory

In the present §3, we give a detailed exposition of the essential logical struc-
ture of inter-universal Teichmüller theory, with a special focus on issues related to
RCS-redundancy. From a strictly rigorous point of view, this exposition assumes a
substantial level of knowledge and understanding of the technical details of inter-
universal Teichmüller theory [which are surveyed, for instance, in [Alien]]. On the
other hand, in a certain qualitative sense, the discussion of the present §3 may in
fact be understood, at a relatively elementary level, via the analogies that we dis-
cuss with the topics covered in §2. Indeed, in this context, it should be emphasized
that, despite the relatively novel nature of the set-up of inter-universal Teichmüller
theory,

the essential mathematical content that lies at the heart of all of the issues
covered in the present §3 concerns entirely well-known mathematics at the
advanced undergraduate or beginning graduate level [i.e., the topics covered
in §2].

§3.1. One-dimensionality via identification of RCS-redundant copies

Inter-universal Teichmüller theory concerns the explicit description of the rela-
tionship between various possible intertwinnings — namely,

the “Θ”- and “q-” intertwinnings

— between the two underlying combinatorial/arithmetic dimensions of a
ring [cf., e.g., [Alien], §2.11; [Alien], §3.11, (v), as well as the discussion of §3.9
below]. There are many different ways of thinking about these two underlying
combinatorial/arithmetic dimensions of a ring; one way to understand these two
dimensions is to think of them as corresponding, respectively, to the unit group
and value group of the various local fields that appear as completions of a number
field at one of its valuations.

In more technical language, this sort of decomposition into unit groups and
value groups may be seen in the \( F_{\Theta}^{\ast \times \mu} \)-prime-strips that appear in the Θ-link
of inter-universal Teichmüller theory. Thus, if one thinks in terms of such \( F_{\Theta}^{\ast \times \mu} \)-
prime-strips, then inter-universal Teichmüller theory may be summarized as follows:
(2-Dim) The main content of inter-universal Teichmüller theory is an explicit description, up to certain relatively mild indeterminacies, of the $\Theta$-intertwining on the $[\text{two-dimensional}] \mathcal{F}^\dagger \rtimes^\mu$-prime-strips that appear in the $\Theta$-link in terms of the $\varphi$-intertwining on these $\mathcal{F}^\dagger \rtimes^\mu$-prime-strips by means of the log-link and various types of Kummer theory that are used to relate Frobenius-like and étale-like structures.

In particular, the essential mathematical content of inter-universal Teichmüller theory concerns an a priori variable relationship between the two underlying combinatorial/arithmetic dimensions of a ring.

Put another way, if one arbitrarily “crushes” these two dimensions into a single dimension — i.e., in more technical language, assumes that

(1-Dim) there exists a consistent choice of a fixed relationship between these two dimensions of (2-Dim), so that these two dimensions may, in effect, be regarded as a single dimension

— then one immediately obtains a superficial contradiction [cf. the discussion of Example 3.1.1, (i-b), (ii-b), below]. Indeed, this is one of the central assertions of the RCS [cf. the discussion following Example 3.1.1]. This is not a “new” observation, but rather, in some sense, the starting point of inter-universal Teichmüller theory, i.e., the initial motivation for regarding the relationship between the two underlying combinatorial/arithmetic dimensions of a ring as being variable, rather than fixed.

**Example 3.1.1:** Elementary models of gluings and intertwinings. In the following, we shall write $V$ for the topological group $\mathbb{R}_{>0}$. Let $x, y \in V$ be [not necessarily distinct!] elements of $V$ and $Y \subseteq V$ a nonempty subset of $V$. Let $V^\dagger, V^\ddagger$ be two distinct labeled copies of $V$, which we think of as corresponding to the positive portions of the real and imaginary axes in the complex plane.

(i) Let $\dagger V, \ddagger V$ be two not necessarily distinct copies of $V$. We shall write $\dagger y \in \dagger V, \ddagger x \in \ddagger V$ for the respective elements determined by $y, x \in V$.

(i-a) Suppose that $\dagger V, \ddagger V$ are distinct copies of $V$. Write $W$ for the topological space obtained by gluing $\dagger V, \ddagger V$ along the homeomorphic subspaces $\{\dagger y\} \subseteq \dagger V, \{\ddagger x\} \subseteq \ddagger V$. Then observe that this construction of $W$ is well-defined and free of any internal contradictions. Moreover, the existence of $W$ does not imply any nontrivial conclusions concerning $x$ and $y$.

Note the sharp contrast between the situation discussed in (i-a) and the following situation:

(i-b) Suppose that $\dagger V, \ddagger V$ are in fact the same copy of $V$, i.e., $^*V \overset{\text{def}}{=} \dagger V = \ddagger V$.

Consider the assertion that

the topological space $^*V$ is obtained by gluing $\dagger V, \ddagger V$ along the homeomorphic subspaces $\{\dagger y\} \subseteq \dagger V, \{\ddagger x\} \subseteq \ddagger V$.

Then observe that this assertion concerning $^*V$ is well-defined and free of internal contradictions only in the case where $x = y$. That is to say, the existence of a topological space $^*V$ as described in the above assertion
implies the nontrivial conclusion that \( x = y \), or, equivalently, a “contradiction” to the assertion that \( x \neq y \).

One may also consider the following variant of (i-b):

(i-c) One replaces \( \{y\} \subseteq \uparrow V \) in (i-b) by the nonempty subset \( \uparrow Y \subseteq \uparrow V \) [i.e., determined by \( Y \subseteq V \)], where one thinks of this subset as a set of “possible \( y \)'s”. The resulting “assertion” then becomes a corresponding collection of assertions related by logical OR “\( \lor \)'s”, and the final nontrivial conclusion is that \( x \in Y \).

(ii) The elementary models presented in (i) may be interpreted as essentially equivalent representations of various models of “holomorphic structures” [cf. the discussion below of (InfH), as well as Examples 3.3.1, 3.3.2] — i.e., in the terminology of the discussion preceding the present Example 3.1.1, “intertwinings” — between the “real” and “imaginary” dimensions \( V^r, V^im \). Here, we think of “holomorphic structures”/“intertwinings” as being defined by assignments

\[
V^r \ni 1^r \quad \mapsto \quad ? \in V^im
\]

[where \( 1^r \subseteq V^r \) denotes the element determined by \( 1 \in V \)], corresponding to “counterclockwise rotations by 90 degrees”, or, alternatively, “multiplication by \( \sqrt{-1} \)”. Indeed, let \( \uparrow V^r, \downarrow V^r \) be two not necessarily distinct copies of \( V^r \), and \( \uparrow V^im, \downarrow V^im \) two not necessarily distinct copies of \( V^im \). We shall write \( \uparrow y^im \in \uparrow V^im, \downarrow y^im \in \downarrow V^im \) for the respective elements determined by \( y, x \in V \). Then the discussion of (i-a) may be translated into a discussion concerning intertwinings by arguing as follows:

(ii-a) Suppose that \( \uparrow V^r, \downarrow V^r \) are distinct copies of \( V^r \), \( \uparrow V^im, \downarrow V^im \) are distinct copies of \( V^im \). Here, we think of \( \uparrow V^r, \downarrow V^r \) as being equipped with the intertwining given by taking “\( ? \)” to be \( \uparrow y^im \in \uparrow V^im \); we think of \( \uparrow y^im \in \uparrow V^im, \downarrow y^im \in \downarrow V^im \). Then one applies (i-a), relative to the correspondences \( \uparrow y^im \mapsto \downarrow y^im \). This yields a gluing as in (i-a) that is well-defined and free of any internal contradictions. Moreover, the existence of such a gluing does not imply any nontrivial conclusions concerning \( x \) and \( y \).

In a similar vein:

(ii-b) Suppose that \( \uparrow V^r, \downarrow V^r \) are in fact the same copy of \( V^r \), i.e., \( *V^r \) and \( \uparrow V^im, \downarrow V^im \) are in fact the same copy of \( V^im \), i.e., \( *V^im \) is the intertwining on \( \uparrow V^r \), and \( \downarrow V^im \) with the intertwining on \( \uparrow V^r \), \( \downarrow V^im \) — that is well-defined and free of internal contradictions only in the case where \( x = y \). That is to say, the existence of such a gluing implies the nontrivial conclusion that \( x = y \), or, equivalently, a “contradiction” to the assertion that \( x \neq y \).

(ii-c) One replaces \( \{\uparrow y^im\} \subseteq \uparrow V^im \) in (ii-b) [cf. also the notation of (ii-a)] by the nonempty subset \( \uparrow Y^im \subseteq \uparrow V^im \) [i.e., the subset determined by \( Y \subseteq V \)],
where one thinks of this subset as a set of “possible y’s”. The resulting “assertion” then becomes a corresponding collection of assertions related by logical OR “∨’s”, and the final nontrivial conclusion is that \( x \in Y \).

(iii) Relative to the analogy with inter-universal Teichmüller theory, we have correspondences with objects that appear in the elementary models of (ii) as follows:

\[
\begin{align*}
V^{rl} & \leftrightarrow \text{the value group portion of an } \mathcal{F}^{\text{ell}+\times \mu}\text{-prime-strip;} \\
V^{im} & \leftrightarrow \text{the unit group portion of an } \mathcal{F}^{\text{ell}+\times \mu}\text{-prime-strip;} \\
\dagger/\ddagger & \leftrightarrow (\Theta^{\pm\text{ell}}\text{NF-})\text{Hodge theaters in the domain/codomain of the } \Theta\text{-link;} \\
\text{intertwinings involving “y”} & \leftrightarrow \Theta\text{-intertwinings;} \\
\text{intertwinings involving “x”} & \leftrightarrow q\text{-intertwinings;}
\end{align*}
\]

(cf. the discussion at the beginning of §3.4]. Here, we note that from the point of view of intertwinings, the unit group portion corresponding to “\( V^{inn} \)” must be understood as being log-shifted by −1, relative to the value group portion corresponding to “\( V^{rl} \)” [cf. the discussion below of (InF)], as well as Examples 3.3.1, 3.3.2]. That is to say, if the value group portion corresponding to “\( V^{rl} \)” is located at the coordinate \((n, m)\) of the log-theta-lattice, then the unit group portion corresponding to “\( V^{inn} \)” must be understood as being located at the coordinate \((n, m−1)\) of the log-theta-lattice. In particular, the unit group and value group portions corresponding to a pair “\((V^{rl}, V^{im})\)” belong to different \( \mathcal{F}^{\text{ell}+\times \mu} \)-prime-strips. From the point of view of the discussion of (1-Dim), the “consistent choice of a fixed relationship” corresponds to the coincidence of intertwinings in (ii-b), while the resulting “superficial contradiction” corresponds to the “contradiction” discussed in (ii-b). On the other hand, the “explicit description”/“variable relationship” of (2-Dim), which leads naturally to a numerical estimate/inequality concerning log-volumes [cf. Example 2.4.5, (ii)], corresponds to the situation involving various possibilities discussed in (ii-c), which leads to the nontrivial conclusion “\( x \in Y \)” [cf. the discussion of “closed loops” in (Stp7), (Stp8) of §3.10 below; the discussion of (DltLb) in §3.11 below; the discussion of [IUTchIII], Remark 3.12.2, (ii)].

(iv) Finally, we observe in passing that the fixed intertwining of (ii-b) [cf. also the discussion of (ii-b) in (iii), as well as the discussion of (FxRng), (FxEuc), (FxFld) below] may be regarded as being analogous to the well-known classical holomorphic approach to the theory of moduli of [one-dimensional] complex tori, that is to say, in which one works with a copy of the upper half-plane “\( \mathbb{H} \)” with a fixed holomorphic structure and thinks of the moduli of complex tori as a “variation of period matrices” [i.e., the holomorphic parameter “\( z \in \mathbb{H} \)”], which may be taken, in the notation of (ii-b), to be “\( ix \)” or “\( iy \)”]. By contrast, the situation involving the set \( \dagger Y^{im} \subseteq \dagger V^{im} \) discussed in (ii-c) may be regarded as analogous to the [real analytic] Teichmüller approach to the theory of moduli of complex tori [cf. the discussion of Example 3.3.1], i.e., in which the holomorphic structure is subject to Teichmüller dilations [corresponding to various elements in the set \( \dagger Y^{im} \)], relative to the fixed “real analytic” pair given by \( \dagger V^{rl}, \dagger V^{im} \).

One central assertion of the RCS [which appears, for instance, in certain 10pp. manuscripts written by adherents of the RCS] is to the effect that the existence, as
in (1-Dim), of a consistent choice of a fixed relationship between the two dimensions of (2-Dim) may be derived as a consequence — i.e., in more succinct notation,

\[(\text{RC-FrÉt}), (\text{RC-log}), (\text{RC-Θ}) \implies \text{ ”(1-Dim) ”}\]

— of certain “redundant copies assertions”, as follows:

(RC-FrÉt) the Frobenius-like and étale-like versions of objects in inter-universal Teichmüller theory are “redundant”, i.e., may be identified with one another without affecting the essential logical structure of the theory;

(RC-log) the \((\Theta^\pm_{\text{ell}}\text{NF-})\text{Hodge theaters}\) on either side of the log-link in inter-universal Teichmüller theory are “redundant”, i.e., may be identified with one another without affecting the essential logical structure of the theory;

(RC-Θ) the \((\Theta^\pm_{\text{ell}}\text{NF-})\text{Hodge theaters}\) on either side of the \(\Theta\)-link in inter-universal Teichmüller theory are “redundant”, i.e., may be identified with one another without affecting the essential logical structure of the theory.

In the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4], we discuss in more detail the falsity of each of these “RCS-redundancy” assertions [i.e., (RC-FrÉt), (RC-log), (RC-Θ)].

Here, it should be noted that this falsity of (RC-FrÉt), (RC-log), (RC-Θ) is by no means a difficult or subtle issue, but rather a sort of matter of “belaboring the intuitively obvious” from the point of view of mathematicians who are thoroughly familiar with inter-universal Teichmüller theory. Nevertheless, as discussed in [Rpt2018], §17, it is a pedagogically meaningful exercise to write out and discuss the details surrounding this sort of issue. Moreover, as discussed in §1.5 of the present paper, it is desirable from a historical point of view to produce detailed, explicit, and readily accessible written expositions concerning this sort of issue.

This state of affairs prompts the following question:

Why do adherents of the RCS continue to insist on asserting the validity of these assertions (RC-FrÉt), (RC-log), (RC-Θ)?

Any sort of complete, definitive answer to this question lies beyond the scope of the present paper. On the other hand, it seems natural to conjecture that one fundamental motivation for these assertions of RCS-redundancy may be found in the fact that

(FxRng) many arithmetic geometers have only experienced working in situations where all schemes — or, alternatively, rings — that appear in a theory are regarded as belonging to a single category that is fixed throughout the theory, hence are related to another via ring homomorphisms, i.e., in such a way that the ring structure of the various rings involved is always respected [cf. the discussion of §1.5, as well as the discussion of §3.8 below].
It is not difficult to imagine that the heuristics and intuition that result from years [or decades!] of immersive experience in such mathematical situations could create a mindset that is fertile ground for the RCS-redundancy assertions that will be discussed in detail in the remainder of the present §3 [cf., especially, §3.2, §3.3, §3.4].

Finally, we observe that this situation is, in certain respects, reminiscent of the situation that occurred in the late 19-th century with regard to such novel [i.e., at the time] notions as the notion of an abstract manifold or an abstract Riemann surface. That is to say,

(FxEuc) from the point of view of anyone for whom it is a “matter of course” or “common sense” that all geometry must take place within some fixed, static ambient Euclidean space — such as, for instance, the complex plane — such abstract geometric notions as the notion of an abstract manifold or abstract Riemann surface might come across as deeply disturbing and unlikely to be of use in any substantive mathematical sense [cf. the discussion of §1.5; the discussion of [IUTchI], §12].

In this context, it is of interest — especially from a historical point of view — to recall that, in some sense, the most fundamental classical example of such an abstract geometry is the Riemann surface that arises by applying the technique of analytic continuation to the complex logarithm, i.e., which may be regarded as a sort of distant ancestor [cf. the discussion of [IUTchI], Remark 5.1.4; [Alien], §3.3, (vi)] of the log-link of inter-universal Teichmüller theory. Another [in fact closely related!] fundamental classical example of such an abstract geometry is the hyperbolic geometry of the upper half-plane, which may also be regarded as a sort of distant ancestor of numerous aspects of inter-universal Teichmüller theory [cf. (InfH) and Example 3.3.2 in §3.3 below, as well as the discussion of [IUTchI], Remark 6.12.3, (iii); [IUTchIII], Remark 2.3.3, (ix), (x)].

Another historically important instance of this sort of situation may be seen in the introduction, in the early 19-th century, of Galois groups — i.e., of [finite] automorphism groups of abstract fields — as a tool for investigating the roots of polynomial equations. That is to say,

(FxFld) until the advent of Galois groups/abstract fields, the issue of investigating the roots of polynomial equations was always regarded — again as a “matter of course” or “common sense” — as an issue of investigating various “exotic numbers” inside some fixed, static ambient field such as the field of complex numbers; moreover, from this more classical “common sense” point of view, the idea of working with automorphisms of abstract fields — i.e., fields that are not constrained [since such constraints would rule out the existence of nontrivial automorphisms!] to be treated as subsets of some fixed, static ambient field — might come across as deeply disturbing and unlikely to be of use in any substantive mathematical sense [cf. the discussion of §1.5].

On the other hand, from the point of view of inter-universal Teichmüller theory, this radical transition

roots as concrete numbers $\rightsquigarrow$ Galois groups/abstract fields
that occurred in the early 19-th century may be regarded as a sort of distant ancestor of the transition

Galois groups/abstract fields $\rightsquigarrow$ abstract groups/anabelian algorithms

that occurs in inter-universal Teichmüller theory [cf. the discussion at the beginning of §3.2 below; the discussion of §3.8 below; the discussion of the final portion of [Alien], §4.4, (i)].

§3.2. RCS-redundancy of Frobenius-like/étale-like versions of objects

We begin by recalling that $(\Theta^{\pm\ell\text{ell}}\text{NF})$-Hodge theaters — i.e., lattice points in the log-theta-lattice — give rise to both Frobenius-like and étale-like objects. Whereas the datum of a Frobenius-like object depends, a priori, on the coordinates $(n,m)$ of the $(\Theta^{\pm\ell\text{ell}}\text{NF})$-Hodge theater from which it arises, étale-like objects satisfy various [horizontal/vertical] coricity properties to the effect that they map isomorphically to corresponding objects in a vertically [in the case of vertical coricity] or horizontally [in the case of horizontal coricity] neighboring $(\Theta^{\pm\text{ell}}\text{NF})$-Hodge theater of the log-theta-lattice [cf., e.g., the discussion of [Alien], §2.7, (ii), (iii), (iv); [Alien], §2.8, $2^{\text{Fr/ét}}$; [Alien], §3.3, (ii), (vi), (vii); [Rpt2018], §15]. Here, we recall that étale-like objects correspond, for the most part, to

arithmetic fundamental groups — such as, for instance, the étale fundamental group $\pi_1(X)$ of a hyperbolic curve $X$ over a number field or mixed characteristic local field

— or, more generally, to objects that may be reconstructed from such arithmetic fundamental groups, so long as the object is regarded as being equipped with auxiliary data consisting of the arithmetic fundamental group from which it was reconstructed, together with the reconstruction algorithm that was applied to reconstruct the object. Here, we recall that, in this context, it is of fundamental importance that these arithmetic fundamental groups be treated simply as abstract topological groups [cf. the discussion of §3.8 below for more details]. Étale-like objects also satisfy a

crucial symmetry property with respect to permutation of adjacent vertical lines of the log-theta-lattice

[cf. [Alien], §3.2; the discussion surrounding Fig. 3.12 in [Alien], §3.6, (i)]. That is to say, in summary,

the crucial coricity/symmetry properties satisfied by étale-like objects — which are, in essence, a formal consequence of treating the arithmetic fundamental groups that appear as abstract topological groups [cf. the discussion of §3.8 below for more details] — play a central role in the multiradial algorithms of inter-universal Teichmüller theory [i.e., [IUTcHIII], Theorem 3.11] and are not satisfied by Frobenius-like objects

— cf., e.g., the discussion of [Alien], §2.7, (iii); [Alien], §3.1, (iii); [Alien], Example 3.2.2; [Rpt2018], §15, (LbΘ), (Lbθ), (LbMn), (EtFr), (EtΘ), (Etθ), (Etθ), (Etθ), (EtMn).
On the other hand, once one implements the RCS-identifications discussed in (RC-log), (RC-Θ), there is, in effect, “only one” \((Θ^{±\ell}\text{NF})\)-Hodge theater in the log-theta-lattice, so all issues of determining relationships between corresponding objects in \((Θ^{±\ell}\text{NF})\)-Hodge theaters at distinct coordinates \(\left(n, m\right)\) of the log-theta-lattice appear, at first glance, to have been “trivially resolved”. Put another way,

once one implements the RCS-identifications of (RC-log), (RC-Θ), even Frobenius-like objects appear, at first glance, to satisfy all possible coricity/symmetry properties, i.e., at a more symbolic level,

\[(\text{RC-log}), \ (\text{RC-Θ}) \quad \Rightarrow \quad \text{(RC-FrÉt)}\].

In particular, the assertions of the RCS discussed in §3.1 and the present §3.2 may be summarized, at a symbolic level, as follows:

\[(\text{RC-log}), \ (\text{RC-Θ}) \quad \Rightarrow \quad \text{(RC-FrÉt)}, \ (\text{RC-log}), \ (\text{RC-Θ}) \quad \Rightarrow \quad \text{(1-Dim)}\].

In fact, however, the RCS-identifications of (RC-log), (RC-Θ) do not resolve such issues [i.e., of relating corresponding objects in \((Θ^{±\ell}\text{NF})\)-Hodge theaters at distinct coordinates \(\left(n, m\right)\) of the log-theta-lattice] at all [cf. the discussion of symmetries in Example 2.3.1, (iii)!], but rather merely have the effect of translating/reformulating such issues of relating corresponding objects in \((Θ^{±\ell}\text{NF})\)-Hodge theaters at distinct coordinates \(\left(n, m\right)\) of the log-theta-lattice into issues of tracking the effect on objects in \((Θ^{±\ell}\text{NF})\)-Hodge theaters as one moves along the paths constituted by various composites of Θ- and log-links.

On the other hand, at a purely formal level,

the discussion given above of the falsity of (RC-FrÉt) — i.e., as a consequence of the crucial coricity/symmetry properties discussed above — is, in some sense, predicated on the falsity of (RC-log), (RC-Θ).

This falsity of (RC-log), (RC-Θ) will be discussed in detail in §3.3, §3.4, below.

In this context, it is useful to observe that the situation surrounding the Θ-link and (RC-Θ), (RC-FrÉt) (respectively, the log-link and (RC-log), (RC-FrÉt)) is structurally reminiscent of the object \(\mathcal{J}\) discussed in Examples 2.3.2, 2.4.1, 2.4.2 [cf. also the correspondences discussed in Example 2.4.5, (ii); the discussion of [IUTchIII], Remark 1.2.2, (vi), (vii)], i.e., if one regards \(\text{StR1}\) the domain of the Θ- (respectively, log-) link as corresponding to \(\uparrow\mathbb{I}\),

\(\text{StR2}\) the codomain of the Θ- (respectively, log-) link as corresponding to \(\downarrow\mathbb{I}\),

\(\text{StR3}\) the gluing data — i.e., a certain \(\mathcal{F}^{\uparrow\beta} \times \mu\)-prime-strip (respectively, \(\mathcal{F}\)-prime-strip) — that arises from the domain \((Θ^{±\ell}\text{NF})\)-Hodge theater of the Θ- (respectively, log-) link as corresponding to \(\uparrow\beta = γ_{\uparrow}\),

\(\text{StR4}\) the gluing data — i.e., a certain \(\mathcal{F}^{\downarrow\alpha} \times \mu\)-prime-strip (respectively, \(\mathcal{F}\)-prime-strip) — that arises from the codomain \((Θ^{±\ell}\text{NF})\)-Hodge theater of the Θ- (respectively, log-) link as corresponding to \(γ_{\downarrow} = \downarrow\alpha\),

\(\text{StR5}\) the étale-like objects that are coric with respect to the Θ- (respectively,
log-) link as corresponding to the glued differential discussed in (AOD3), and

(StR6) the RCS-identification of (RC-Θ) (respectively, (RC-log)) as corresponding to the operation of passing to the quotient

\[ J \rightarrow M = J/(I \sim \mathcal{J}) \leadsto L = I/(\alpha \sim \beta). \]

This strong structural similarity will play an important role in the discussion of §3.3, §3.4, below.

Finally, we observe that the portion, i.e., (StR5), of this strong structural similarity involving the glued differential discussed in (AOD3) is particularly of interest in the context of the discussion of [Alien], §2. That is to say, as discussed in the first paragraph of [Alien], §2.6, étale-like objects in inter-universal Teichmüller theory play an analogous role to the role played by tangent bundles/sheaves of differentials in the

- special case of the invariance of the height under isogenies between abelian varieties [due to Faltings] discussed in [Alien], §2.3, §2.4 [cf. also [Rpt2018], §16, (DiIsm)], as well as in the

- discussion of differentiation of $p$-adic liftings of the Frobenius morphism given in [Alien], §2.5.

The efficacy of the technique of considering induced maps on differentials in the various examples discussed in [Alien], §2.3, §2.4, §2.5, is also notable in the context of the discussion of the fundamental theorem of calculus in §2.2, as well as in the context of (RC-FrÉt) and (StR5).

§3.3. RCS-redundant copies in the domain/codomain of the log-link

The Θ-link of inter-universal Teichmüller theory is defined, in the style of classical complex Teichmüller theory [cf. Example 3.3.1 below; [IUTchI], Remark 3.9.3], as a deformation of the ring structure in a $(\Theta^{\pm\text{ell}}\text{NF})$-Hodge theater that depends, in an essential way, on the splitting into unit groups and value groups of the various localizations of the number field involved. On the other hand, the log-link of inter-universal Teichmüller theory [i.e., in essence, the $p$-adic logarithm at primes of the number field of residue characteristic $p$] has the effect of juggling/rotating these unit groups and value groups, e.g., by mapping units to non-units [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. In particular, there is no natural way to relate the two Θ-links [i.e., the two horizontal arrows in the following diagram] that emanate from the domain and codomain of the log-link [i.e., the left-hand vertical arrow in the following diagram]

\[ \bullet \xrightarrow{\Theta} \bullet \]

\[ \uparrow \text{log} \]

\[ \bullet \xrightarrow{\Theta} \bullet \]
— that is to say, there is no natural candidate for “??” [i.e., such as, for instance, an isomorphism or the log-link between the two bullets “•” on the right-hand side of the diagram] that makes the diagram commute. Indeed, it is an easy exercise, in the style of [Rpt2018], §15, (LbΘ), (Lblog), (LbMn), to show that neither of these candidates for “??” [i.e., an isomorphism or the log-link] yields a commutative diagram.

Thus, in summary, any identification of the domain and codomain of the log-link [cf. (RC-log)!] yields a situation in which the local splittings into unit groups and value groups of the resulting identified “•’s” are no longer well-defined. In particular,

any such identification of the domain and codomain of the log-link [cf. (RC-log)!] yields a situation in which the Θ-link is not well-defined

— i.e., a situation in which the apparatus of inter-universal Teichmüller theory completely ceases to function — cf. the discussion of the definition of the Θ-link in the latter half of [Alien], §3.3, (ii). This discussion may be summarized, at a symbolic level, as follows:

\[
\text{definition of the Θ-link} \implies \text{falsity of (RC-log)}. 
\]

Next, we observe [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)] that the non-existence of a solution for “??” in the above diagram [i.e., that makes the diagram commute] amounts, at a structural level, to essentially the same phenomenon as the incompatibility of the dilations that appear in classical complex Teichmüller theory with multiplication by non-real roots of unity [cf. Example 3.3.1 below]. Write \(\mathbb{R}, \mathbb{C}\), respectively, for the topological fields of real and complex numbers. Then as observed in the discussion of the latter half of [Alien], §3.3, (ii) [cf., especially, the discussion surrounding [Alien], Fig. 3.6]:

(InFH) this structural similarity is consistent with the analogy discussed in loc. cit. between

\[ \begin{array}{c}
\cdot \text{the “infinite } H \text{” portion of the log-theta-lattice consisting of the two vertical lines [i.e., of log-links] on either side of a horizontal arrow [i.e., a Θ-link] of the log-theta-lattice and} \\
\cdot \text{the elementary theory surrounding the bijection} \\
\end{array} \]

\[ \mathbb{C}^\times \backslash GL_2^+ (\mathbb{R}) / \mathbb{C}^\times \cong [0, 1) \]

\[ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \mapsto \frac{\lambda - 1}{\lambda + 1} \]

— where \(\lambda \in \mathbb{R}_{>1}\); \(GL_2^+ (\mathbb{R})\) denotes the group of 2 \times 2 real matrices of positive determinant; \(\mathbb{C}^\times\) denotes the multiplicative group of \(\mathbb{C}\), which we regard as a subgroup of \(GL_2^+ (\mathbb{R})\) via the assignment \(a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\), for \(a, b \in \mathbb{R}\) such that \((a, b) \neq (0, 0)\);
the domain of the bijection is the set of double cosets.

That is to say,

- the dilation $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ — cf. the dilations that appear in classical complex Teichmüller theory, i.e., as reviewed in Example 3.3.1 below — corresponds to the $\Theta$-link portion of an “infinite H” [cf. Example 3.3.2, (iii), below], while

- the two copies of the group of toral rotations “$\mathbb{C}^\times$” [e.g., by roots of unity in $\mathbb{C}^\times$] on either side of “$GL^+_2(\mathbb{R})$” — which may be thought of as a representation of the holomorphic structures in the domain and codomain of the dilation $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ [cf. the discussion of Example 3.3.1 below] — correspond, respectively, to the two vertical lines of $\log$-links in the “infinite H” on either side of the $\Theta$-link [cf. the discussion of Example 3.3.2, (iv), below].

**Example 3.3.1**: Classical complex Teichmüller theory. Let $\lambda \in \mathbb{R}_{>1}$. Recall the most fundamental deformation of complex structure in classical complex Teichmüller theory

$$\Lambda : \mathbb{C} \to \mathbb{C}$$

$$\mathbb{C} \ni z = x + iy \mapsto \zeta = \xi + i\eta \overset{\text{def}}{=} \lambda \cdot x + iy \in \mathbb{C}$$

— where $x, y \in \mathbb{R}$. Let $n \geq 2$ be an integer, $\omega$ a primitive $n$-th root of unity. Write $(\omega \in) \, \mu_n \subseteq \mathbb{C}$ for the group of $n$-th roots of unity. Then observe that

if $n \geq 3$, then there does not exist $\omega' \in \mu_n$ such that $\Lambda(\omega \cdot z) = \omega' \cdot \Lambda(z)$ for all $z \in \mathbb{C}$.

[Indeed, this observation follows immediately from the fact that if $n \geq 3$, then $\omega \notin \mathbb{R}$.] That is to say, in words,
A is not compatible with multiplication by \( \mu_n \) unless \( n = 2 \) [in which case \( \omega = -1 \)].

This incompatibility with “indeterminacies” arising from multiplication by \( \mu_n \), for \( n \geq 3 \), may be understood as one fundamental reason for the special role played by square differentials [i.e., as opposed to \( n \)-th power differentials, for \( n \geq 3 \)] in classical complex Teichmüller theory [cf. the discussion of [IUTchI], Remark 3.9.3, (iii), (iv)].

**Example 3.3.2:** The Jacobi identity for the classical theta function.

(i) Write \( z = x + iy \) for the standard coordinate on the upper half-plane \( \mathcal{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \). Recall the theta function on \( \mathcal{H} \)

\[
\Theta(q) \overset{\text{def}}{=} \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}.
\]

— where we write \( q \overset{\text{def}}{=} e^{2\pi iz} \). Restricting to the imaginary axis [i.e., \( x = 0 \)] yields a function

\[
\theta(t) \overset{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}.
\]

— where we write \( t \overset{\text{def}}{=} y \).

(ii) Next, let us observe that

\[
t \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})
\]

maps \( z \mapsto -z^{-1} \), hence \( iy \mapsto iy^{-1} \), i.e., \( t \mapsto t^{-1} \), while, for \( \lambda \in \mathbb{R}_{\geq 1} \),

\[
\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL^+(\mathbb{R})
\]

maps \( z \mapsto \lambda \cdot z \), hence \( iy \mapsto i\lambda \cdot y \), i.e., \( t \mapsto \lambda \cdot t \).

(iii) Next, we observe the following:

· As \( t \to +\infty \), the terms in the series for \( \theta(t) \) are rapidly decreasing, and \( \theta(t) \to +0 \). In particular, the series for \( \theta(t) \) is relatively easy to compute.

· As \( t \to +0 \), the terms in the series for \( \theta(t) \) decrease very slowly, and \( \theta(t) \to +\infty \). In particular, the series for \( \theta(t) \) is very difficult to compute.

Thus, in summary, the “flow/dilation” \( \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \) along the imaginary axis may be regarded as a sort of “link”, in the context of the theta function \( \theta(t) \), between small values [i.e., \( \theta(t) \to +0 \) as \( t \to +\infty \)] and large values [i.e., \( \theta(t) \to +iy \) as \( t \to +0 \)]. That is to say, this flow/dilation along the imaginary axis behaves in a way that
is strongly reminiscent of the $\Theta$-link of inter-universal Teichmüller theory [cf. the discussion of (InfH)].

(iv) The Jacobi identity for the theta function $\theta(t)$

$$\theta(t) = t^{-\frac{1}{2}} \cdot \theta(t^{-1})$$

allows one to analyze the behavior of $\theta(t)$ as $t \to +0$, which is very difficult to compute [cf. (iii)], in terms of the behavior of $\theta(t)$ as $t \to +\infty$, which is relatively easy to compute [cf. (iii)] — cf. the discussion of the Jacobi identity in [Pano], §3, §4; [Alien], §4.1, (i). Observe that this identity may be understood as a sort of invariance with respect to $\nu$ [cf. (ii)], up to a certain easily computed factor [i.e., $t^{-\frac{1}{2}}$]. Note that $\nu$ “juggles”, or “rotates/permutes”, the two dimensions of $\mathbb{R}^2$. This aspect of $\nu$ is strongly reminiscent of the log-link of inter-universal Teichmüller theory, which “juggles”, or “rotates/permutes”, the two underlying dimensions of the ring structures in a vertical column of the log-theta-lattice [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. By contrast, we note that the theta function $\theta(t)$ does not satisfy any interesting properties of invariance with respect to the dilations $(\lambda^0_0 1)$.

Finally, we recall that, in any vertical line of log-links in the log-theta-lattice,

- the discrepancy between the [holomorphic] Frobenius-like copies of objects on either side of a log-link [cf. (RC-log)], as well as
- the discrepancy between [holomorphic] Frobenius-like copies of objects and [holomorphic] étale-like copies of objects [cf. (RC-FrÉt)],

may be understood as the extent to which the diagram of arrows that constitutes the log-Kummer-correspondence associated to this vertical line of log-links fails to commute.

This failure to commute may be estimated by means of the indeterminacy (Ind3), i.e., by interpreting this failure to commute as a sort of “upper semicommutativity”. This indeterminacy (Ind3) is highly nontrivial and, in particular, gives rise to the inequality that appears in the final computation of log-volumes in inter-universal Teichmüller theory [cf. [IUTchIV], Corollary 3.12]. In this context, it is important to recall that the theory surrounding this indeterminacy (Ind3) depends, in an essential way, on the absolute anabelian geometry of [AbsTopIII], §1, i.e., which allows one to reconstruct a hyperbolic curve $X$ over a number field or mixed characteristic local field from the abstract profinite group determined by the étale fundamental group $\pi_1(X)$ of the curve. That is to say, in summary, this absolute anabelian geometry allows one to show that

the discrepancies between the various [holomorphic] Frobenius-like and [holomorphic] étale-like copies of objects in a vertical line of log-links [cf. (RC-log), (RC-FrÉt)] in the log-theta-lattice are “bounded by” the relatively mild indeterminacy (Ind3).

On the other hand, this absolute anabelian geometry most certainly does not imply that these discrepancies are trivial/non-existent, i.e., as asserted in (RC-log), (RC-FrÉt) — cf. the discussion of the falsity of (RC-log), (RC-FrÉt) in §3.2 and the present §3.3.
§3.4. RCS-redundant copies in the domain/codomain of the $\Theta$-link

The $\Theta$-link of inter-universal Teichmüller theory

\[ \bullet \xrightarrow{\Theta} \bullet \]

is defined as a gluing between the $(\Theta^\pm \text{ell} \text{NF})$-Hodge theater “$\bullet$” in the domain of the arrow and the $(\Theta^\pm \text{ell} \text{NF})$-Hodge theater “$\bullet$” in the codomain of the arrow along $\mathcal{F}^\pm \times \mu^\ast$-prime-strips “$\ast$” that arise from the $\Theta$-pilot object “$q$-plt” in the domain and the $q$-pilot object “$q$-plt” in the codomain. Here, it is important to note that this gluing is obtained by regarding these $\mathcal{F}^\pm \times \mu^\ast$-prime-strips “$\ast$” as being known only up to isomorphism. This point of view, i.e., of regarding these $\mathcal{F}^\pm \times \mu^\ast$-prime-strips “$\ast$” as being known only up to isomorphism, is implemented formally by taking the gluing to be the full poly-isomorphism — i.e., the set of all isomorphisms — between the $\mathcal{F}^\pm \times \mu^\ast$-prime-strips arising from the domain and codomain of the $\Theta$-link. Here, we recall that

- $q$-plt essentially amounts to the arithmetic line bundle determined by [the ideal generated by] some $2l$-th root $q$ of the $q$-parameters at the valuations $\in \mathcal{V}^{\text{bad}}$, while

- $\Theta$-plt essentially amounts to the collection of arithmetic line bundles determined by [the ideals generated by] the collection \{$q^{j^2}$\}, as $j$ ranges over the integers $1, \ldots, l^* \overset{\text{def}}{=} \frac{l - 1}{2}$ [where $l$ is the prime number that appears in the initial $\Theta$-data under consideration].

Also, we recall that each $(\Theta^\pm \text{ell} \text{NF})$-Hodge theater “$\bullet$” gives rise to an associated model “$\mathcal{R}\text{ing}$” of the ring/scheme theory surrounding the elliptic curve under consideration. In the following discussion, we shall write

- $\dagger$ for the “$\bullet$” in the domain of the $\Theta$-link,

- $\ddagger$ for the “$\bullet$” in the codomain of the $\Theta$-link,

- $\Box$ for an arbitrary element of the set consisting of “$\dagger$”, “$\ddagger$”, and the “empty symbol” [i.e., no symbol at all],

- $\Box \Theta$-plt $\in \Box \mathcal{R}\text{ing}$ for the $\Theta$-pilot arising from the collection “$\{\Box q^{j^2}\}$” that appears in the model of ring/scheme theory associated to $\Box \bullet$, and

- $\Box q$-plt $\in \Box \mathcal{R}\text{ing}$ for the $q$-pilot arising from the “$\Box q$” that appears in the model of ring/scheme theory associated to $\Box \bullet$.

Finally, we recall that since, for $j \neq 1$, the valuation [at each valuation $\in \mathcal{V}^{\text{bad}}$] of $q^{j^2}$ differs from that of $q$, the arithmetic degrees of the line bundles constituted by $\Box q$-plt and $\Theta$-plt differ.

Thus, at a more formal level, the above description of the gluing that constitutes the $\Theta$-link may be summarized as follows:

\[ \dagger \mathcal{R}\text{ing} \ni \dagger \Theta$-plt $\leftarrow: \ast :\rightarrow \ddagger q$-plt $\in \ddagger \mathcal{R}\text{ing} \]

\[ \mathcal{R}\text{ing} \ni q$-plt $\neq \Theta$-plt $\in \mathcal{R}\text{ing} \]
[where “←:” and “:→” denote the assignments that constitute the *gluing* discussed above].

In this context, we note the following **fundamental observation**, which underlies the entire logical structure of inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 3.12.2, (ctw), (ftw); [Alien], §3.11, (iv)]:

(AOΘ1) the following condition **holds**:

\[
(*) :\rightarrow \Theta\text{-plt} \in \text{↑Ring} \land (*) :\rightarrow q\text{-plt} \in \text{‡Ring}.
\]

By contrast, if one simply **deletes** the distinct labels “↑”, “‡” [cf. (RC-Θ)!], then

(AOΘ2) the following condition **holds**:

\[
(*) :\rightarrow \Theta\text{-plt} \in \text{Ring} \lor (*) :\rightarrow q\text{-plt} \in \text{Ring}.
\]

Of course,

(AOΘ3) the essential mathematical content discussed in this condition (AOΘ2) may be **formally** described as a condition involving the *AND relator* “∧”:

\[
(\text{q-plt} \in \{\text{q-plt}, \Theta\text{-plt}\}) \land (\Theta\text{-plt} \in \{\text{q-plt}, \Theta\text{-plt}\}).
\]

On the other hand, precisely as a **consequence** of the fact [discussed above] that \(\text{Ring} \ni \text{q-plt} \neq \Theta\text{-plt} \in \text{Ring}\),

(AOΘ4) the following condition does **not** hold:

\[
(*) :\rightarrow \Theta\text{-plt} \in \text{Ring} \land (*) :\rightarrow q\text{-plt} \in \text{Ring}.
\]

That is to say, the operation of **identifying** ↑, ↓ [hence also ↑Ring, ↓Ring] — e.g., on the grounds of “**redundancy**” [i.e., as asserted in (RC-Θ)!] — by **deleting** the distinct labels “↑”, “‡” has the effect of passing from a situation in which

*the AND relator* “∧” *holds* [cf. (AOΘ1)]

to a situation in which

*the OR relator* “∨” *holds* [cf. (AOΘ2), (AOΘ3)], but

*the AND relator* “∧” *does not hold* [cf. (AOΘ4)]!

In particular, relative to the *correspondences*

\[
\begin{align*}
\text{↑•}, \text{↑Ring} & \leftrightarrow \text{↑I}; \quad * \leftrightarrow \gamma_3; \quad \text{↑•}, \text{↑Ring} & \leftrightarrow \text{↑I} \\
\text{↑Θ-plt} & \leftrightarrow \text{↑β}; \quad \text{↑q-plt} & \leftrightarrow \text{↑α}
\end{align*}
\]
[cf. the correspondences (StR1) ∼ (StR6) discussed in §3.2; the correspondences discussed in Example 2.4.5, (ii); the discussion of [Alien], §3.11, (iv)], one obtains very precise structural resemblances

\[(AO\Theta1) \leftrightarrow (AOL1),\]
\[(AO\Theta2) \leftrightarrow (AOL2),\]
\[(AO\Theta3) \leftrightarrow (AOL3),\]
\[(AO\Theta4) \leftrightarrow (AOL4)\]

with the situation discussed in Example 2.4.1, (i), (ii). Thus, in summary,

the falsity of (RC-Θ) may be understood as a consequence of the falsity [cf. (AOΘ4)] of the crucial AND relator “∧” in the absence of distinct labels, in stark contrast to the truth [cf. (AOΘ1)] of the crucial AND relator “∧” as an essentially tautological consequence of the use of the distinct labels “†”, “‡”.

In the context of the central role played in the logical structure of inter-universal Teichmüller theory by the validity of (AOΘ1), it is important to note [cf. the discussion of (AOΘ4)]!

(NoRng) there does not exist an isomorphism of ring structures \(\dagger\text{Ring} \rightsquigarrow \dagger\text{Ring}\) that induces, on value groups of corresponding local rings, the desired assignment \(\{\dagger q^2\} \mapsto \dagger q\) [i.e., that appears in the Θ-link].

On the other hand, if, instead of considering the full ring structures of \(\dagger\text{Ring}, \dagger\text{Ring}\), one considers [cf. the discussion of [Rpt2018], §6]

- certain suitable subquotients — i.e., in the notation of [Alien], §3.3, (vii), \((a^q), (a^q), \mathcal{O}_k\) — of the underlying multiplicative monoids of corresponding local fields, as well as
- the absolute Galois groups — i.e., in the notation of [Alien], §3.3, (vii), \((a^q), (a^q), G_k\) — associated to corresponding local rings, regarded as abstract topological groups [that is to say, not as Galois groups, or equivalently/alternatively, as groups of field automorphisms! — cf. the discussion of §3.8 below],

then one obtains structures — i.e., the structures that constitute the \(\mathcal{F}^{\dagger \times \mu}\)-prime-strips that appear in the Θ-link — that are simultaneously associated [as “underlying structures”] to both \(\dagger\text{Ring}\) and \(\dagger\text{Ring}\) via isomorphisms [i.e., of certain suitable multiplicative monoids equipped with actions by certain suitable abstract topological groups] that restrict, on the subquotient monoids that correspond to the respective value groups, to the desired assignment \(\{\dagger q^2\} \mapsto \dagger q\). It is this crucial simultaneity that yields, as a tautological consequence, the validity of the AND relator “∧” in (AOΘ1).

Working, as in the discussion above, with multiplicative monoids equipped with actions by abstract topological groups, necessarily gives rise to certain indeterminacies, called (Ind1), (Ind2), that play an important role in inter-universal Teichmüller theory. Certain aspects of these indeterminacies (Ind1), (Ind2) will be discussed in more detail in §3.5 below. In this context, we recall that one central
assertion of the RCS [cf. the discussion of (SSInd), (SSIId) in [Rpt2018], §7, §10] is to the effect that

(NeuRng) these indeterminacies (Ind1), (Ind2) may be eliminated, without affecting the essential logical structure of inter-universal Teichmüller theory, by taking the multiplicative monoids and abstract topological groups that appear in the $\mathcal{F}_{\mu}^{\times \mu}$-prime-strips of the above discussion to be equipped with rigidifications by regarding them as arising from some fixed "neutral" ring structure $\Box \text{Ring}$.

On the other hand, as discussed in (NoRng) above, there does not exist any ring structure that is compatible [i.e., in the sense discussed in (NoRng)], with the desired assignment \(\{\dagger q^2\} \mapsto \ddagger q\). That is to say, in summary,

(NeuroInd) working with such a fixed "neutral" ring structure $\Box \text{Ring}$ as in (NeuRng) means either that

(NeuroInd1) there is no relationship between "*" and $\Box \Theta$-plt $\Box \text{Ring}$ [cf. the situation discussed in [Rpt2018], §10, (SSIId)], or that

(NeuroInd2) the relationship between "*" and $\Box \Theta$-plt $\Box \text{Ring}$ is always necessarily subject to an indeterminacy [cf. (AOΘ2), (AOΘ3)!]

\[ (* : \rightarrow \Box \Theta \text{-plt} \in \Box \text{Ring}) \lor (* : \rightarrow \Box \Theta \text{-plt} \in \Box \text{Ring}) \].

Here, we observe that whichever of these "options"/"indeterminacies" that appear in (NeuroInd) [i.e., (NeuroInd1), (NeuroInd2)] one chooses to adopt, one is forced to contend with an indeterminacy that is, in some sense, much more drastic than the relatively mild indeterminacies (Ind1), (Ind2) whose elimination formed the original motivation for the introduction of $\Box \text{Ring}$!

Finally, we observe that this much more drastic indeterminacy (NeuroInd) means [cf. the discussion of Example 2.4.4!] that throughout any argument, one must always take the position that the only possible relationship between "*" and $\Box \Theta$-plt, $\Box \Theta$-plt is one in which

(PltRel) "*" maps either to $\Box \Theta$-plt or — i.e., "∨"! — to $\Box \Theta$-plt, but not both!

Since $\dagger \text{Ring}$ may be thought of as a ring structure in which "*" tautologically maps to $\dagger \Theta$-plt, while $\Box \text{Ring}$ may be thought of as a ring structure in which "*" tautologically maps to $\dagger \Theta$-plt, one may rephrase the above observation as the observation that one must always take the position that the only possible relationship between $\Box \Theta$-plt, on the one hand, and $\dagger \Theta$-plt, $\dagger \text{Ring}$, on the other, is one in which

(RngRel) the ring structure $\Box \text{Ring}$ is identified either with the ring structure $\dagger \text{Ring}$ or — i.e., "∨"! — with the ring structure $\dagger \Theta$-plt, but not both!

At this point, let us recall [cf., e.g., the discussion of §3.5, §3.11, below; [Rpt2018], §9, (GIUT), (OCR)] that

inter-universal Teichmüller theory requires, in an essential way, the use of the log-links, hence, in particular, [in order to define the power series of
the various $p$-adic logarithm functions that constitute these $\log$-links! the 
ring structures $\mathcal{Q}^{\text{ring}}$, $\mathcal{Q}^{\text{ring}}$ on both sides — i.e., “∧”! — of the Θ-link

[cf. the discussion surrounding (InfH) of the two vertical lines of $\log$-links in the “infinite H” on either side of the Θ-link]. In particular, we conclude formally that

it is impossible to implement the arguments of inter-universal Teichmüller theory once this sort of much more drastic indeterminacy (NeuORInd) has been imposed.

§3.5. Gluings, indeterminacies, and pilot discrepancy

As discussed in §3.4, the Θ-link involves a gluing

$$\{\mathcal{Q}^{\text{ring}}_j \} \mapsto \mathcal{Q}^{\text{ring}}$$

that identifies $\mathcal{Q}^{\text{ring}}$ [i.e., $2l$-th roots of the $q$-parameters at primes of multiplicative reduction of the [copy belonging to $\mathcal{Q}^{\text{ring}}$ of the] elliptic curve under consideration] with elements, i.e., the $\mathcal{Q}^{\text{ring}}_j$’s, which, when $j \neq 1$, have different valuations from the valuation of $\mathcal{Q}^{\text{ring}}$.

On the other hand, in inter-universal Teichmüller theory, by applying the multiradial representation [IUTchIII], Theorem 3.11, which involves various indeterminacies (Ind1), (Ind2), (Ind3), and then forming [cf. [IUTchIII], Corollary 3.12, and its proof] the holomorphic hull of the union of possible images of the Θ-pilot in this multiradial representation,

(ΘGl) one may treat both sides of the Θ-link gluing of the above display as belonging to a single ring theory without disturbing [cf. the crucial AND relator “∧” property discussed in §3.4!] the gluing.

Alternative ways to understand the essential mathematical content of (ΘGl) include the following:

(NonInf) One may think of (ΘGl) as a statement concerning the mutually non-interference or simultaneous executability of the Kummer theories surrounding the $q$-pilot and Θ-pilot relative to the gluing of abstract $\mathcal{F}^{\text{ring}}_\times \times_\mu$-prime-strips constituted by the Θ-link, i.e., when the Kummer theory surrounding the $q$-pilot is held fixed, and one allows the Kummer theory surrounding the Θ-pilot to be subject to various indeterminacies.

(Cohab) One may think of (ΘGl) as a statement concerning the “cohabitation”, or “coexistence”, of the $q$-pilot and Θ-pilot — relative to the gluing of abstract $\mathcal{F}^{\text{ring}}_\times \times_\mu$-prime-strips constituted by the Θ-link — within the common container obtained by applying the multiradial representation of the Θ-pilot, forming the holomorphic hull [relative to the holomorphic structure [i.e., (Θ±elNF)-Hodge theater] that gave rise to the $q$-pilot under consideration], and finally taking log-volumes.

In this context, it is important to recall that this sort of phenomenon — i.e.,
of computations of global degrees/heights of elliptic curves in situations where a certain “confusion”, up to suitable indeterminacies, is allowed between \( q \)-parameters of the elliptic curves and certain large positive powers of these \( q \)-parameters [i.e., as in \((\Theta Gl)\)]

— may be seen in various classical examples such as

- the proof by Faltings of the invariance of heights of abelian varieties under isogeny [cf. the discussion of [Alien], §2.3, §2.4],

- the classical proof in characteristic zero of the geometric version of the Szpiro inequality via the Kodaira-Spencer morphism, phrased in terms of the theory of crystals [cf. the discussion of [Alien], §3.1, (v)], and

- Bogomolov’s proof over the complex numbers of the geometric version of the Szpiro inequality [cf. the discussion of [Alien], §3.10, (vi)]

— cf. also the discussion of [Rpt2018], §16. Moreover, in the case of crystals, we observe that, relative to the notation introduced in Example 2.4.5, (v), (vi), we have correspondences as follows:

**CrAND** The logical AND “\( \land \)” that appears in the multiradial representation of the \( \Theta \)-pilot in IUT (= AND-IUT) may be understood as being analogous to the fact that crystals, i.e., “\( \land \)-crystals”, may be thought of as objects [on infinitesimal neighborhoods of the diagonal inside products of two copies of the scheme under consideration] that may be simultaneously interpreted, up to isomorphism, as pull-backs via one projection morphism \( \land \) [cf. “\( \land \)!”] as pull-backs via the other projection morphism [cf. the discussion of \((\land(\lor)\text{-Chn1})\) in §3.10 below; the discussion of [Alien], §3.11, (iv), (2nd), concerning the interpretation of the discussion of crystals in [Alien], §3.1, (v), (3KS), in terms of the logical relator “\( \land \)”].

**CrOR** Thus, from the point of view of the analogy discussed in (CrAND), the logical OR “\( \lor \)” that appears throughout OR-IUT may be understood as corresponding to working with “\( \lor \)-crystals” [i.e., as opposed to crystals \( (= \land\text{-crystals}) \)] , that is to say, with objects [on infinitesimal neighborhoods of the diagonal inside products of two copies of the scheme under consideration] that may be interpreted, up to isomorphism, as pull-backs via one projection morphism \( \lor \) [cf. “\( \lor \)!”] as pull-backs via the other projection morphism. Here, we observe that this defining “\( \lor \)” condition of an \( \lor \)-crystal is essentially vacuous since one may obtain \( \lor \)-crystals from arbitrary objects on the scheme under consideration simply by pulling back such an object to the infinitesimal neighborhood of the diagonal under consideration via one of the two projection morphisms.

**CrRCS** In a similar vein, from the point of view of the analogies discussed in (CrAND) and (CrOR), RCS-IUT may be understood as corresponding to the modified version of the usual theory of crystals obtained by replacing the infinitesimal neighborhoods of the diagonal inside products of two copies of the scheme under consideration [i.e., that appear in the usual theory of crystals!] by the diagonal itself. Such a replacement clearly renders the usual theory of crystals trivial/meaningless, in a fashion that is essentially very similar to the triviality of \( \lor \)-crystals discussed in (CrOR). Finally, we observe that this similarity between the
modified versions of the usual theory of crystals discussed in (CrOR) and the present (CrRCS) is entirely analogous to the equivalence OR-IUT $\iff$ RCS-IUT observed in Example 2.4.5, (v), (XOR/RCS).

Unfortunately, however, the situation summarized above in (ΘGl) has resulted in certain frequently voiced misunderstandings by some mathematicians. One such frequently voiced misunderstanding is to the effect that

(CnfInd1+2) the situation summarized in (ΘGl) may be explained as a consequence of a “confusion” between $q$-parameters and large positive powers of these $q$-parameters that results from the indeterminacies (Ind1), (Ind2).

In fact, however, as discussed in Example 3.5.1, (iii), below,

at least in the case of $q$-parameters of sufficiently small valuation [i.e., sufficiently large positive order, in the sense of loc. cit.], such a “confusion” [i.e., between $q$-parameters and large positive powers of these $q$-parameters] can never occur as a consequence of (Ind1), (Ind2), i.e., both of which amount to automorphisms of the [underlying topological module of the] log-shells involved

[cf. also the discussion of (ΘInd) in [Rpt2018], §11]. In this context, we note that this misunderstanding (CnfInd1+2) appears to be caused in many cases, at least in part, by a more general misunderstanding concerning the operation of passage to underlying structures [cf. Example 3.5.2 below]. A more detailed discussion of the operation of passage to underlying structures may be found in §3.9 below.

As discussed in [Rpt2018], §11, the “confusion” summarized in (ΘGl) occurs in inter-universal Teichmüller theory as a consequence not only of the local indeterminacies (Ind1), (Ind2), (Ind3), but also of the constraints imposed by the global realified Frobenioid portions of the $\mathcal{F}^\rightarrow\times\mu$-prime-strips that appear in the Θ-link. In this context, it is of particular importance to observe that

(CnfInd3) the indeterminacy (Ind3), which constrains one to restrict one’s attention to upper bounds [i.e., but not lower bounds!] on the log-volume that is the subject of the computation of [IUTchIII], Corollary 3.12, already by itself — i.e., without considering (Ind1), (Ind2), or global realified Frobenioids! [cf. the discussion of (Ind3>1+2) in §3.11 below] — is sufficient to account for the possibility of a “confusion” of the sort summarized in (ΘGl) [i.e., between $q$-parameters and large positive powers of these $q$-parameters].

Indeed, the indeterminacy (Ind3) is defined in precisely such a way as to identify the ideals generated by arbitrary positive powers of the $q$-parameters.

**Example 3.5.1:** Bounded nature of log-shell automorphism indeterminacies. Write $\mathbb{Z}_p$ for the ring of $p$-adic integers, for some prime number $p$; $\mathbb{Q}_p$ for the field of fractions of $\mathbb{Z}_p$.

(i) Let $M$ be a finitely generated free $\mathbb{Z}_p$-module, which, in the following discussion, we shall think of as being embedded in $M_{\mathbb{Q}_p} \overset{\text{def}}{=} M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

\[ \alpha : M \xrightarrow{\sim} M \]
an automorphism of the $\mathbb{Z}_p$-module $M$. For $n \in \mathbb{Z}$, write

$$\mathcal{U}(M,n) \overset{\text{def}}{=} \{ x \in M_{\mathbb{Q}_p} \mid x \in p^n \cdot M, \ x \not\in p^{n+1} \cdot M \} \subseteq M_{\mathbb{Q}_p}.$$

Then observe that $\alpha$ induces a bijection

$$\mathcal{U}(M,n) \overset{\sim}{\to} \mathcal{U}(M,n)$$

for every $n \in \mathbb{Z}$.

(ii) In the notation of (i), suppose, for simplicity, that $p$ is odd. Let $K$ be a finite field extension of $\mathbb{Q}_p$. Write $\mathcal{O}_K \subseteq K$ for the ring of integers of $K$; $\mathcal{O}_K^\times \subseteq \mathcal{O}_K$ for the group of units of $\mathcal{O}_K$; $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for the maximal ideal of $K$; $\mathcal{I}_K \subseteq K$ for the log-shell associated to $K$ [cf., e.g., the discussion of [IUTchIII], Remark 1.2.2, (i)], i.e., the result of multiplying by $p^{-1}$ the image log $\log_p(\mathcal{O}_K^\times)$ of $\mathcal{O}_K^\times$ by the $p$-adic logarithm $\log_p(-)$. Thus,

$$\mathcal{O}_K \subseteq \mathcal{I}_K \subseteq p^{-c} \cdot \mathcal{O}_K$$

for some nonnegative integer of $c$ that depends only on the isomorphism class of the field $K$ [cf. [IUTchIV], Proposition 1.2, (i)]. In particular, there exists a positive integer $s$ that depends only on the isomorphism class of the field $K$ such that for any automorphism

$$\phi : \mathcal{I}_K \overset{\sim}{\to} \mathcal{I}_K$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ and any $n \in \mathbb{Z}$, it holds that

$$\phi(\mathcal{U}(\mathcal{O}_K,n)) \subseteq \bigcup_{i=-s}^{s} \mathcal{U}(\mathcal{O}_K,n+i)$$

[where $i$ ranges over the integers between $-s$ and $s$].

(iii) In the situation of (ii), we define the order of a nonzero element $x \in K$ to be the unique $n \in \mathbb{Z}$ such that $x \in \mathfrak{m}^n$, $x \not\in \mathfrak{m}^{n+1}$. One thus concludes from the final portion of the discussion of (ii) that there exists a positive integer $t$ that depends only on the isomorphism class of the field $K$ such that for any automorphism

$$\phi : \mathcal{I}_K \overset{\sim}{\to} \mathcal{I}_K$$

of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ and any nonzero element $q \in \mathcal{O}_K$ [i.e., such as the $q$-parameter of a Tate curve over $K$!], the absolute value of the difference between the orders of $q$ and $\phi(q)$ is $\leq t$, i.e., in words,

automorphisms of the $\mathbb{Z}_p$-module $\mathcal{I}_K$ only give rise to bounded discrepancies in the orders of nonzero elements of $\mathcal{O}_K$.

Example 3.5.2: Examples of gluings. Distinct auxiliary structures on some common [i.e., "∧"!] underlying structure may be thought of as gluings of the distinct auxiliary structures along the common underlying structure. Here, we observe
that, in general, *distinct auxiliary structures on a common underlying structure are not* necessarily mapped to one another by some *automorphism of the common underlying structure*. Concrete examples of these generalities may be found in quite substantial abundance throughout arithmetic geometry and include, in particular, the examples (i), (ii), (iii), (iv), (v) given below, as well as the elementary Examples 2.3.2, 2.4.1, 2.4.2, 2.4.3, 2.4.7, 3.3.1 discussed in §2.3, §2.4, §3.3 [cf. also the discussion of [Rpt2018], §11]. In passing, we observe that

these examples may also be understood as interesting examples of the sort of **gluing/logical AND** “∧” relation that appears in the Θ-/log-links of inter-universal Teichmüller theory, i.e., examples of situations that are qualitatively similar to the Θ-/log-links of inter-universal Teichmüller theory in the sense that they involve *distinct auxiliary structures* that are glued together along some *common auxiliary structure* [cf. the discussion of (StR1) ∼ (StR6) in §3.2; the discussion of §3.4; the portion of the present §3.5 preceding Example 3.5.1].  

(i) The group structures of the **finite abelian groups** \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}/4\mathbb{Z} \) are not mapped to one another by any *isomorphism of sets*, despite the fact that the underlying sets of these two groups are indeed isomorphic to one another. This example is also of interest in light of the discussion of truncated Witt vectors in Example 2.4.6, (iii).  

(ii) The **scheme structures** of non-isomorphic algebraic curves over a common algebraically closed field are not mapped to one another by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of such algebraic curves over a common algebraically closed field are indeed isomorphic to one another.  

(iii) The **holomorphic structures** of non-isomorphic compact Riemann surfaces of the same genus are not mapped to one another by any *isomorphism of topological spaces*, despite the fact that the underlying topological spaces of such Riemann surfaces are indeed isomorphic to one another.  

(iv) The **field structures** of non-isomorphic mixed-characteristic local fields [which, by local class field theory, may be regarded as [the formal union with “{0}” of] some suitable subquotient of their respective absolute Galois groups] are not, in general, mapped to one another by any *isomorphism of profinite groups* between the respective absolute Galois groups [cf., e.g., [Ymgt], §2, Theorem, for an example of this phenomenon].  

(v) In the notation of Example 3.5.1, (i), let \( X \) be a *proper smooth curve of genus \( \geq 2 \) over \( \mathbb{F}_p \) def = \( \mathbb{Z}_p/p\mathbb{Z}_p \). Thus, \( X \) may be thought of as an “underlying structure” associated to any *lifting of \( X \) to \( \mathbb{Z}_p \)*, i.e., any flat \( \mathbb{Z}_p \)-scheme \( Y \) equipped an isomorphism of \( \mathbb{F}_p \)-schemes \( Y \times_{\mathbb{Z}_p} \mathbb{F}_p \to X \). Then observe that *non-isomorphic liftings* of \( X \) to \( \mathbb{Z}_p \) are not, in general, mapped to one another by any *automorphism of the \( \mathbb{F}_p \)-scheme \( X \). [Indeed, this is particularly easy to see if one chooses \( X \) such that \( X \) does not admit any nontrivial automorphisms.] Finally, we note that this example may be regarded as a sort of *\( p \)-adic analogue* of the example of (iii).
§3.6. Chains of logical AND relations

From the point of view of the simple qualitative model of inter-universal Teichmüller theory given in Example 2.4.5, the discussion of §3.4 concerns the AND relator “∧” in the “Θ-link” portion of Example 2.4.5, (ii). On the other hand, strictly speaking, this portion of inter-universal Teichmüller theory only concerns the initial definition of the Θ-link. That is to say, the bulk of the theory developed in [IUTchIII] concerns, from the point of view of the simple qualitative model of inter-universal Teichmüller theory given in Example 2.4.5, (ii), the preservation of the AND relator “∧” as one passes from

- the “Θ-link” portion of Example 2.4.5, (ii), to
- the “multiradial representation” portion of Example 2.4.5, (ii).

By contrast, the passage from the “multiradial representation” portion of Example 2.4.5, (ii), to the “final numerical estimate” portion of Example 2.4.5, (ii) — i.e., which corresponds to the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 — is [cf. the discussion of the final portion of Example 2.4.5, (ii)!] relatively straightforward [cf. the discussion of §3.10, §3.11, below].

At this point, it is perhaps of interest to consider “typical symptoms” of mathematicians who are operating under fundamental misunderstandings concerning the essential logical structure of inter-universal Teichmüller theory. Such typical symptoms, which are in fact closely related to one another, include the following:

(Syp1) a sense of unjustified and acutely harsh abruptness in the passage from [IUTchIII], Theorem 3.11, to [IUTchIII], Corollary 3.12 [cf. the discussion of the final portions of Example 2.4.5, (ii), (iii)!];

(Syp2) a desire to see the “proof” of some sort of commutative diagram or “compatibility property” to the effect that taking log-volumes of pilot objects in the domain and codomain of the Θ-link yields the same real number [a property which, in fact, can never be proved since it is false! — cf. the discussion of §3.5];

(Syp3) a desire to see the inequality of the final numerical estimate obtained as the result of concatenating some chain of intermediate inequalities, i.e., as is often done in proofs in real/complex/functional analysis or analytic number theory.

Here, it should be noted that (Syp2) and (Syp3) often occur as approaches to mitigating the “harsh abruptness” of (Syp1).

With regard to (Syp3), it should be emphasized that it is entirely unrealistic to attempt to obtain the inequality of the final numerical estimate as the result of concatenating some chain of intermediate inequalities since this is simply not the way in which the logical structure of inter-universal Teichmüller theory is organized. That is to say, in a word, the logical structure of inter-universal Teichmüller theory does not proceed by concatenating some sort of chain of intermediate inequalities. Rather,

(∧-Chn) the logical structure of inter-universal Teichmüller theory proceeds by observing a chain of AND relations “∧”
[cf. the discussion of [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (c.iw), (f.iw); [Alien], §3.11, (iv), (v)]. As observed in Example 2.4.5, (ii), (iii), once one follows this chain of AND relations “∧” up to and including the multiradial representation of the Θ-pilot [i.e., [IUTchIII], Theorem 3.11], the passage to the final numerical estimate [i.e., [IUTchIII], Corollary 3.12] is relatively straightforward [i.e., as one might expect, from the use of the word “corollary”!].

One essentially formal consequence of (∧-Chn) is the following: Since the definition of the Θ-link, the construction of the multiradial representation of the Θ-pilot, and the ultimate passage to the final numerical estimate consist of a finite number of steps, one natural and effective way to analyze/diagnose [cf. the discussion of §1.4!] the precise content of misunderstandings of inter-universal Teichmüller theory is to determine

\((∧\text{-Dgn})\) precisely where in the finite sequence of steps that appear is the first step at which the person feels that the preservation of the crucial AND relator “∧” is no longer clear.

In some sense, the starting point of the various AND relations “∧” that appear in the multiradial algorithm of [IUTchIII], Theorem 3.11, is the observation that

\((∧\text{-Input})\) the input data for this multiradial algorithm consists solely of an abstract \(F^{\bullet\bullet}×\mu\text{-prime-strip}\); moreover, this multiradial algorithm is functorial with respect to arbitrary isomorphisms between \(F^{\bullet\bullet}×\mu\text{-prime-strips}\) [cf. [IUTchIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i)]. This property \((∧\text{-Input})\) means that the multiradial algorithm may be applied to any \(F^{\bullet\bullet}×\mu\text{-prime-strip}\) that appears, or alternatively/equivalently, that any \(F^{\bullet\bullet}×\mu\text{-prime-strip}\) may serve as the gluing data [cf. the “γ3” in the analogies discussed in §3.2, (StR3), (StR4), as well as Example 2.4.5, (ii)!] between a given situation [i.e., such as the \((Θ^{\pm\ell\ell}\text{-NF})\text{-Hodge theater in the codomain of the Θ-link!}\)] and the content of the multiradial algorithm.

On the other hand, in order to conclude that the multiradial algorithm yields output data satisfying suitable AND relations “∧”, it is necessary also to examine in detail the content of this output data, i.e., in particular, in the context of the central IPL and SHE properties discussed in [IUTchIII], Remark 3.11.1, (iii), as well as the chain of (sub)quotients aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)]. In a word, the essential “principle” that is applied throughout the various steps of the multiradial algorithm in order to derive new AND relations “∧” from old AND relations “∧” is the following “principle of extension of indeterminacies”:

\((\text{ExtInd})\) If \(A, B,\) and \(C\) are propositions, then it holds [that \(B \Rightarrow B \lor C\) and hence] that

\[A \land B \Rightarrow A \land (B \lor C).\]

One important tool that is frequently used in inter-universal Teichmüller theory in a fashion that is closely related to (ExtInd) is the notion of a poly-morphism [cf. the discussion of §3.7 below for more details].

In the context of (ExtInd), it is interesting to note that, from the point of view of the discussion of §3.4,
the “∨” that appears in the conclusion — i.e., $A \wedge (B \lor C)$ — of (ExtInd) may be understood as amounting to essentially the same phenomenon as the “∨” that appears in (NeuORInd2) [e.g., by taking “C” to be $A$].

That is to say, instead of generating AND relations “∧” tautologically by means of the introduction of distinct labels [i.e., as in (AOΘ1)] — i.e., by introducing a new distinct label for “C” so as to conclude a tautological relation

$$A \wedge B \wedge C$$

— (ExtInd) allows one to generate new AND relations “∧” while avoiding the introduction of new distinct labels. As discussed in §3.4, this point of view [i.e., of avoiding the introduction of new distinct labels] leads inevitably to OR relations “∨”, i.e., as in (NeuORInd2) or as in the conclusion “$A \wedge (B \lor C)$” of (ExtInd). As discussed above, the reason that one wishes to avoid the introduction of new distinct labels when applying (ExtInd) is precisely that

(sQLTL) one wishes to apply (ExtInd) to form “(sub)quotients/splittings” of the log-theta-lattice [cf. the title of [IUTchIII]], i.e., to project the vertical line on the left-hand side of the infinite “H” portion of the log-theta-lattice onto the vertical line on the right-hand side of this infinite “H” by somehow achieving some sort of “crushing together” of distinct coordinates [i.e., “(n, m)”, where $n, m \in \mathbb{Z}$] of the log-theta-lattice

[cf. the discussion of §3.11 below; [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (e^{it}), (f^{it}); [Alien], §3.11, (iv), (v)).

At this point, it is of interest to note that there are, in some sense, two ways in which (ExtInd) is applied during the execution of the various steps of the multiradial algorithm [cf. the discussion of §3.10, §3.11, below, for more details]:

(ExtInd1) operations that consist of simply adding more possibilities/indeterminacies [which corresponds to passing from $B$ to $B \lor C$] within some fixed container;

(ExtInd2) operations in which one identifies [i.e., “crushes together”, by passing from $B$ to $B \lor C$] objects with distinct labels, at the cost of passing to a situation in which the object is regarded as being only known up to isomorphism.

Typical examples of (ExtInd1) include the upper semi-continuity of (Ind3), as well as the passage to holomorphic hulls. Typically, such applications of (ExtInd1) play an important role in establishing various symmetry or invariance properties such as multiradiality. This sort of establishment of various symmetry or invariance properties by means of (ExtInd1) then allows one to apply label crushing operations as in (ExtInd2). Put another way,

- (ExtInd1) may be understood as a sort of operation whose purpose is to prepare suitable descent data, while

- (ExtInd2) may be thought of as a sort of actual descent operation, i.e., from data that depends on the specification of a member of some collection of distinct labels to data that is independent of such a label specification.
We refer to the discussion of §3.8 below for more details on foundational aspects of (ExtInd2) and to the discussion of §3.9 below for more details concerning the notion of “descent”. Typical examples of (ExtInd2) in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

- identifying “Π_v”’s [where v ∈ V] at different vertical coordinates [i.e., “(n,m)” and “(n,m’),” for n, m, m’ ∈ ℤ] of the log-theta-lattice, which results in a “Π_v regarded up to isomorphism” that is labeled by a new label “(n, ◦)”;
- identifying “G_v”’s [where v ∈ V] at different horizontal or vertical coordinates [i.e., “(n,m)” and “(n’, m’),” for n, n’, m, m’ ∈ ℤ] of the log-theta-lattice, which results in a “G_v regarded up to isomorphism” that is labeled by a new label “(◦, ◦)”;
- identifying the F ⊩ ▶ ×μ-prime-strips in the Θ-link that arise from the Θ- and q-pilot objects in distinct (Θ±ellNF-)Hodge theaters [i.e., the (Θ±ellNF-)Hodge theaters in the domain and codomain of the Θ-link] by working with these F ⊩ ▶ ×μ-prime-strips up to isomorphism.

In some sense, the most nontrivial instances of the application of (ExtInd) in the context of the multiradial algorithm occur in relation to the log-Kummer correspondence [i.e., in the vertical line on the left-hand side of the infinite “H”] and closely related operations of Galois evaluation [cf. the discussion of §3.11 below]. The Kummer theories that appear in this log-Kummer-correspondence — i.e., Kummer theories for

- multiplicative monoids of nonzero elements of rings of integers in mixed-characteristic local fields,
- mono-theta environments/theta monoids, and
- pseudo-monoids of κ-coric functions

— involve the construction of various [Kummer] isomorphisms between

- Frobenius-like data and
- corresponding data constructed via anabelian algorithms from étale-like objects.

The output of such algorithms typically involves constructing the “corresponding data” as one possibility among many. Here, we note that either of these Frobenius-like/étale-like versions of “corresponding data” is — unlike, for instance, the data that constitutes an F ⊩ ▶ ×μ-prime-strip! — sufficiently robust that it completely determines [even when only regarded up to isomorphism!] the [usual] embedding of the Θ-pilot.

That is to say, taken as a whole, the multiradial algorithm — and, especially, the portion of the multiradial algorithm that involves the log-Kummer correspondence and closely related operations of Galois evaluation — plays the role of exhibiting the Frobenius-like Θ-pilot as one possibility within a collection of possibilities constructed via anabelian algorithms from étale-like data.
Thus, in this situation, one obtains the crucial preservation of the AND relation “∧” by applying (ExtInd) twice, i.e., by applying

- (ExtInd1) to the enlargement of the collection of possibilities under consideration and
- (ExtInd2) to the Kummer isomorphisms involved, when one passes from Frobenius-like object labels “\((n,m)\)” [where \(n,m \in \mathbb{Z}\)] to étale-like object labels “\((n,\circ)\)” [where \(n \in \mathbb{Z}\)].

This is precisely what is meant by the chain of (sub)quotients aspect of the SHE property [cf. [IUTchIII], Remark 3.11.1, (iii); [IUTchIII], Remark 3.9.5, (viii), (ix)] discussed above [cf. also the discussion of §3.10, §3.11, below].

§3.7. Poly-morphisms and logical AND relations

Poly-morphisms — i.e., sets of morphisms between objects — appear throughout inter-universal Teichmüller theory as a tool for facilitating

the explicit enumeration of a collection of possibilities.

Composable ordered pairs of poly-morphisms [i.e., pairs for which the domain of the first member in the pair coincides with the codomain of the second member in the pair] may be composed by considering the set of morphisms obtained by composing the morphisms that belong to the sets of morphisms that constitute the given pair of poly-morphisms. Such compositions of poly-morphisms allow one to keep track — in a precise and explicit fashion — of collections of possibilities under consideration.

From the point of view of chains of AND relations “∧”, as discussed in §3.6,

the collections of possibilities enumerated by poly-morphisms are to be understood as being related to one another via OR relations “∨”.

That is to say, poly-morphisms may be thought of as a sort of indeterminacy, which is used in inter-universal Teichmüller theory to produce structures that satisfy various symmetry or invariance properties, hence yield suitable descent data [cf. the discussion of (ExtInd1) in §3.6; the discussion of §3.9 below].

Thus, for instance, in the case of the full poly-isomorphism that constitutes the Θ-link, one may understand the fundamental AND relation “∧” of (AOΘ1) — which, for simplicity, we denote by

\[ A \land B \]

[where \(A\) and \(B\) correspond, respectively, in the notation of the discussion of §3.4, to “\(* : \dagger q\text{-plt} \in \dagger \mathbb{Z}\text{Ring}\)” and “\(* : \dagger \Theta\text{-plt} \in \dagger \mathbb{Z}\text{Ring}\)”] — may be understood as a relation “\(A \land (B_1 \lor B_2 \lor \ldots)\)”, i.e., a relation to the effect that

if one fixes the \(q\)-pilot \(\dagger q\text{-plt}\), then this \(q\)-pilot is glued, via the Θ-link, to the \(\Theta\)-pilot \(\dagger \Theta\text{-plt}\) by means of one isomorphism [of the full poly-isomorphism that constitutes the Θ-link] or another isomorphism, or yet another isomorphism, etc.
[Here, the various possible gluings that constitute \( B \) are denoted by \( B_1, B_2, \ldots \). In particular, as discussed in \((\wedge\text{-Chn})\), if one starts with the \( \Theta \)-link and then considers various subsequent logical AND relations “\( \wedge \)” that arise — for instance, by considering various composites of poly-morphisms! — by applying \((\text{ExtInd})\), then

\[(\wedge(\vee)\text{-Chn})\] the essential logical structure of inter-universal Teichmüller theory, as discussed in \((\wedge\text{-Chn})\), may be understood as follows:

\[
A \wedge B = A \wedge (B_1 \vee B_2 \vee \ldots )
\]

\[
\implies A \wedge (B_1 \vee B_2 \vee \ldots \vee B'_1 \vee B'_2 \vee \ldots )
\]

\[
\implies A \wedge (B_1 \vee B_2 \vee \ldots \vee B'_1 \vee B'_2 \vee \ldots \vee B''_1 \vee B''_2 \vee \ldots )
\]

Finally, we recall that various “classical examples” of the notion of a poly-morphism include

- the collection of maps between topological spaces that constitutes a homotopy class, or stable homotopy class, of maps;
- the collection of morphisms between complexes that constitutes a morphism of the associated derived category;
- the collection of morphisms obtained by considering some sort of orbit by some sort of group action on the domain or codomain of a given morphism

[cf. the discussion of [Rpt2018], §13, (PMEx1), (PMEx2), (PMQut)]. Also, in this context, it is useful to recall [cf. the discussion of [Alien], §4.1, (iv)] that

- gluings via poly-morphisms are closely related to the sorts of gluings that occur in the construction of algebraic stacks [i.e., algebraic stacks which are not algebraic spaces].

§3.8. Inter-universality and logical AND relations

One fundamental aspect of inter-universal Teichmüller theory lies in the consideration of distinct universes that arise naturally when one considers Galois categories — i.e., étale fundamental groups — associated to various schemes. Here, it is important to note that, when phrased in this way,

this fundamental aspect of inter-universal Teichmüller theory is, at least from the point of view of mathematical foundations, no different from the situation that arises in [SGA1].

On the other hand, the fundamental difference between the situation considered in [SGA1] and the situations considered in inter-universal Teichmüller theory lies in the fact that, whereas in [SGA1], the various distinct schemes that appear are related to one another by means of morphisms of schemes or rings,

the various distinct schemes that appear in inter-universal Teichmüller theory are related to one another, in general, by means of relations — such as the log- and \( \Theta \)-links — that are non-ring/scheme-theoretic in
nature, i.e., in the sense that they do not arise from morphisms of schemes or rings.

In general, when considering relations between distinct mathematical objects, it is of fundamental importance to specify those mathematical structures that are common — i.e., in the terminology of inter-universal Teichmüller theory, coric — to the various distinct mathematical objects under consideration. Here, we observe that

this notion of being “common”/“coric” to the various distinct mathematical objects under consideration constitutes, when formulated at a formal, symbolic level, a logical AND relation “∧”.

— cf. the discussion of §3.4, §3.5, §3.6, §3.7.

Thus, in the situations considered in [SGA1], the ring/scheme structures of the various distinct schemes that appear are coric and hence allow one to relate the universes/Galois categories/étale fundamental groups associated to these distinct schemes in a way that makes use of the common ring/scheme structures between these schemes. At a concrete level, this means that

in the situations considered in [SGA1], étale fundamental groups may be related to one in such a way that the only indeterminacies that occur are inner automorphism indeterminacies.

Moreover, these inner automorphism indeterminacies are by no means superfluous — cf. the discussion of Examples 3.8.1, 3.8.2 below.

Example 3.8.1: Inevitability of inner automorphism indeterminacies. The unavoidability of inner automorphism indeterminacies may be understood in very elementary terms, as follows.

(i) Let \( k \) be a perfect field; \( \overline{k} \) an algebraic closure of \( k \); \( N \subseteq G_k \defeq \text{Gal}(\overline{k}/k) \) a normal closed subgroup of \( G_k \); \( \sigma \in G_k \) such that the automorphism \( \iota_\sigma : N \xrightarrow{\sim} N \) of \( N \) given by conjugating by \( \sigma \) is not inner. [One verifies immediately that, for instance, if \( k \) is a number field or a mixed-characteristic local field, then such \( N, \sigma \) do indeed exist.] Write \( k_N \subseteq \overline{k} \) for the subfield of \( N \)-invariants of \( \overline{k} \), \( G_{k_N} \defeq N \subseteq G_k \), \( Q_N \defeq G_k/G_{k_N} \). Then observe that this situation yields an example of a situation in which one may verify directly that

the functoriality of the étale fundamental group only holds if one allows for inner automorphism indeterminacies in the definition of the étale fundamental group.

Indeed, let us first observe that the “basepoints” of \( k \) and \( k_N \) determined by \( \overline{k} \) allows us to regard \( G_k \) and \( G_{k_N} \), respectively, as the étale fundamental groups of \( k \) and \( k_N \). Thus, if one assumes that the functoriality of the étale fundamental group holds even in the absence of inner automorphism indeterminacies, then the commutative diagram of schemes

\[
\begin{array}{ccc}
\text{Spec}(k_N) & \xrightarrow{\sigma} & \text{Spec}(k_N) \\
\downarrow & & \downarrow \\
\text{Spec}(k) & & \text{Spec}(k)
\end{array}
\]

would have to be commutative.
where the diagonal morphisms are the natural morphisms] induces a commutative diagram of profinite groups

\[
\begin{array}{ccc}
G_{k_N} & \overset{\iota_{\sigma}}{\rightarrow} & G_{k_N} \\
\downarrow & & \downarrow \\
G_k & & \\
\end{array}
\]

which [since the natural inclusion \( N = G_{k_N} \hookrightarrow G_k \) is injective!] implies that \( \iota_{\sigma} \) is the identity automorphism, in contradiction to our assumption concerning \( \sigma \!).

(ii) The phenomenon discussed in (i) may be understood as a consequence of the fact that, whereas \( \text{Spec}(k) \) is coric in the commutative diagram of schemes that appears in (i) [i.e., in the sense that this diagram does indeed commute!], \( \text{Spec}(\overline{k}) \) is not coric in the diagram of schemes

\[
\begin{array}{ccc}
\text{Spec}(\overline{k}) & & \\
\downarrow & & \downarrow \\
\text{Spec}(k_N) & \overset{\sigma}{\rightarrow} & \text{Spec}(k_N) \\
\downarrow & & \downarrow \\
\text{Spec}(k) & & \\
\end{array}
\]

[where the diagonal morphisms are the natural morphisms], i.e., in the sense that the upper portion of this diagram does not commute!

(iii) Finally, we consider the natural exact sequence

\[
1 \rightarrow G_{k_N} \rightarrow G_k \rightarrow Q_N \rightarrow 1
\]

of profinite groups. Then observe that the inner automorphisms indeterminacies of \( G_k \) [cf. the discussion of (i), (ii)!] induce outer automorphism indeterminacies of \( G_{k_N} \) that will not, in general, be inner. That is to say,

if one considers \( G_{k_N} \) in the context of this natural exact sequence, then one must in fact consider \( G_{k_N} \) [not only up to inner automorphism indeterminacies, i.e., as discussed in (i), (ii), but also] up to certain outer automorphism indeterminacies.

Relative to the point of view of the discussion of (ii), these outer automorphism indeterminacies may be understood as a consequence of the fact that, in the context of the field extensions \( k \subseteq k_N \subseteq \overline{k} \) and the automorphisms of these field extensions induced by elements of \( G_k \),

the field \( k \) is coric, whereas the field \( k_N \) is not coric

— i.e., in the context of these field extensions and automorphisms of field extensions, the relationship of \( k \) to the various field extensions that appear is constant and fixed, whereas the relationship of \( k_N \) to the various field extensions that appear is
variable, i.e., subject to indeterminacies arising from the action of elements of $G_k$.

**Example 3.8.2:** Inter-universality and the structure of $(Θ^{±\text{ell}}\text{NF})$-Hodge theaters. In the following discussion of $(Θ^{±\text{ell}}\text{NF})$-Hodge theaters, we fix a collection of initial $Θ$-data

$$(\overline{F}/F, X_F, l, \mathbb{C}_K, \mathbb{Y}, \mathbb{Y}_{\text{mod}}, \mathfrak{e})$$

as in [IUTchI], Definition 3.1, and apply the notational conventions of [IUTchI], Definition 3.1. In particular, we recall that $E \overset{\text{def}}{=} E_F$ is an elliptic curve over the number field $F$; $\overline{F}$ is an algebraic closure of $F$; $l$ is a prime number; $K \subseteq \overline{F}$ is the extension field of $F$ determined by the composite of the fields of definition of the closed points of the finite group scheme $E[l] \subseteq E$ of $l$-torsion points of $E$. For simplicity, we assume that $l > 5$.

(i) We begin by recalling the following:

(i-a) The point of view of classical Galois theory with regard to constructing finite Galois extensions of fields may be summarized, in the case of the Galois extension $K/F$, as follows:

- one starts with a base field $F$;
- one then constructs a finite field extension $K$ of $F$ that is saturated with respect to Galois conjugation over $F$.

Thus, relative to this classical point of view, one is constrained to viewing the situation from the point of view of the base field $F$. This constraint obliges one to always take into account the entirety of Galois conjugates [over $F$] of objects associated to $K$.

The point of view of (i-a) is fundamentally incompatible with the main goal of the construction of $(Θ^{±\text{ell}}\text{NF})$-Hodge theaters in [IUTchI], namely, the simulation of a global multiplicative subspace of $E[l]$ [cf. the discussion of global multiplicative subspaces in [IUTchI], §I1; [Alien], §2.3, §2.4; [Alien], §3.3, (iv)], together with a global canonical generator, up to $\pm 1$, of the quotient of $E[l]$ by the global multiplicative subspace [cf. the discussion of global canonical generators in [IUTchI], §I1; [Alien], §3.3, (iv)]. In some sense, the technical starting point of the “simulation of a global multiplicative subspace” implemented in [IUTchI] may be summarized as follows:

(i-b) The “simulation of a global multiplicative subspace” given in [IUTchI] is achieved by, in some sense, reversing the flow of the classical construction reviewed in (i-a), i.e., by

- viewing the situation [not from the point of view of the base field $F$, but rather] from the point of view of the hyperbolic orbicurve

$$\mathbb{C}_K$$

— which may be thought of as data that amounts to $K$, together with a fixed choice of a quotient “$Q$” [cf. [IUTchI], Definition
regarding the base field $F_{\text{mod}}$ — or, at the level of hyperbolic orbicurves, $C_{F_{\text{mod}}}$ [cf. [IUTchI], Remark 3.1.7, (i)] — as a finite étale quotient of $K$ [or, at the level of hyperbolic orbicurves, $C_K$], i.e., which amounts to thinking in terms of [compatible] finite étale quotients

$$\text{Spec}(K) \to \text{Spec}(F_{\text{mod}}), \quad C_K \to C_{F_{\text{mod}}}$$

— which are regarded as objects constructed from $\text{Spec}(K)$, $C_K$.

This approach allows one to concentrate on a fixed [simulated global multiplicative] subspace and hence [unlike the situation discussed in (i-a)!] to exclude the various nontrivial Galois conjugates over $F$ of this fixed simulated global multiplicative subspace.

The approach of (i-b) has numerous important technical consequences [to be discussed in (ii), (iii), (iv), below].

(ii) From the point of view of étale-like objects [i.e., arithmetic fundamental groups], constructing a quotient $C_K \to C_{F_{\text{mod}}}$ as in (i-b) corresponds to constructing a profinite group “$\Pi_{C_{F_{\text{mod}}}}$” from the profinite group $\Pi_{C_K}$ that contains $\Pi_{C_K}$ as an open subgroup. In light of the well-known slimness of $\Pi_{C_{F_{\text{mod}}}}$ [cf., e.g., [AbsTopI], Proposition 2.3, (ii)], such a construction of “$\Pi_{C_{F_{\text{mod}}}}$” amounts to the construction of a finite group of outer automorphisms of some open subgroup of $\Pi_{C_K}$. This finite group may be thought of as a finite quotient group $\Pi_{C_{F_{\text{mod}}}} \to \Gamma_{\text{mod}}$ of $\Pi_{C_{F_{\text{mod}}}}$. If we think of the absolute Galois group $G_{F_{\text{mod}}}$ of the number field $F_{\text{mod}}$ as a quotient $\Pi_{C_{F_{\text{mod}}}} \to G_{F_{\text{mod}}}$ of $\Pi_{C_{F_{\text{mod}}}}$, then this finite quotient group $\Gamma_{\text{mod}}$ determines a finite quotient group $G_{F_{\text{mod}}} \to \Gamma_{\text{Gal}}$ of $G_{F_{\text{mod}}}$. Here, we recall from the construction of [IUTchI], Example 4.3, (i), that $\Gamma_{\text{Gal}}$ has a natural subquotient that may be identified with $F^*_1 = F^*_1/\{\pm 1\}$, i.e., which corresponds to the multiplicative $F^*_1$-symmetry of the $(\Theta^{\pm\ell}\text{NF})$-Hodge theater. In particular, $\Gamma_{\text{Gal}}$, hence also $\Gamma_{\text{mod}}$, is a finite group of order $> 2$, which implies, by well-known properties of absolute Galois groups of number fields [cf., e.g., [NSW], Theorem 12.1.7] that

$$\text{(NoSpl)} \text{ The surjection } G_{F_{\text{mod}}} \to \Gamma_{\text{Gal}}_{\text{mod}} \text{ of profinite groups does not admit a splitting.}$$

Here, we note that [in light of the well-known slimness of $G_{F_{\text{mod}}}$ — cf., e.g., [AbsTopI], Theorem 1.7, (iii)] this nonexistence of a splitting may be reformulated as the assertion that the natural outer action of $\Gamma_{\text{Gal}}_{\text{mod}}$ on the kernel $\text{Ker}(G_{F_{\text{mod}}} \to \Gamma_{\text{Gal}}_{\text{mod}})$ does not admit a lifting to an action of $\Gamma_{\text{Gal}}_{\text{mod}}$ on $\text{Ker}(G_{F_{\text{mod}}} \to \Gamma_{\text{Gal}}_{\text{mod}})$, i.e., to an action that is free of inner automorphism indeterminacies. In particular, it follows [a fortiori!] that the natural outer action of $\Gamma_{\text{mod}}$ on $\text{Ker}(\Pi_{C_{F_{\text{mod}}}} \to \Gamma_{\text{mod}})$ does not admit a lifting to an action of $\Gamma_{\text{mod}}$ on $\text{Ker}(\Pi_{C_{F_{\text{mod}}}} \to \Gamma_{\text{mod}})$, i.e., to an action that is free of inner automorphism indeterminacies. That is to say, in summary, the inner automorphism indeterminacies in these natural outer actions are essential and unavoidable.

(iii) The existence of the inner automorphism indeterminacies discussed in (ii) implies, in particular, that the permutations of prime-strips in the multiplicative
symmetry portion of a \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater induced by the \(\mathbb{F}_l^*\)-symmetries of the \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater necessarily give rise to inner automorphism indeterminacies in the isomorphisms between the copies of local absolute Galois groups \(G_v\) [where \(v \in \mathbb{V}^{\text{non}}\)] that appear in prime-strips with distinct labels \(\in \mathbb{F}_l^*\) [cf. [IUTchI], Remark 4.5.1, (iii); [IUTchII], Remark 2.5.2, (iii); [IUTchII], Remarks 4.7.2, 4.7.6; [Alien], §3.6, (iii)]. Put another way, (NoSyn) there is no well-defined synchronization between these copies of \(G_v\) that appear in prime-strips at distinct labels \(\in \mathbb{F}_l^*\) free of inner automorphism — i.e., conjugacy — indeterminacies.

In this context, we recall that such a conjugate synchronization is of fundamental importance in inter-universal Teichmüller theory since it is necessary in order to construct the data that appears in the unit group portion of the \(\mathbb{F}_l^*\times\mu\)-prime-strip that appears in the domain of the \(\Theta\)-link, i.e., data that is required to be free of any dependence on the distinct labels \(\in \mathbb{F}_l^*\). Such a conjugate synchronization is achieved by applying the \(\mathbb{F}_l^*\times\pm\)-symmetries [where we recall that \(\mathbb{F}_l^*\times\pm \overset{\text{def}}{=} \mathbb{F}_l\times\{\pm 1\}\), i.e., relative to the natural action of \(\{\pm 1\}\) on \(\mathbb{F}_l\) in the additive symmetry portion of the \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater under consideration [cf. [IUTchII], Corollary 3.5, (i); [IUTchII], Remark 3.5.2, (ii); [IUTchII], Remark 4.5.3, (i); [IUTchIII], Theorem 1.5, (iii); [IUTchIII], Remark 1.5.1, (i); [Alien], §3.6, (ii)]. Here, we observe that in order to achieve this conjugate synchronization via the \(\mathbb{F}_l^*\times\pm\)-symmetry of the various copies of \(G_v\) that appear in prime-strips with distinct labels, it is of fundamental importance to keep these copies of \(G_v\) isolated from the absolute Galois groups of number fields that appear in the discussion of (ii) [i.e., since, as observed in (ii), it is precisely the intrinsic structure of these global absolute Galois groups that gives rise to the unwanted inner automorphism/conjugacy indeterminacies!]. This local-global isolation requirement — i.e., in effect, the requirement that (LGIsl) these copies of the local absolute Galois group \(G_v\) be regarded not as subgroups of some global absolute Galois group, but rather as coric objects that are treated as being independent of any sort of embedding into a global absolute Galois group [cf. [IUTchII], Remark 4.7.6; [Alien], §3.6, (iii)] — will have important consequences, as we shall see in the discussion of (iv) below.

(iv) As discussed in (iii), the issue (SymIsl) of isolating the \(\mathbb{F}_l^*\times\pm\)-symmetry from the \(\mathbb{F}_l^*\)-symmetry in order to achieve conjugate synchronization is one important reason for imposing the local-global isolation requirement (LGIsl) in the context of the construction of \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theaters. In fact, however, this property (LGIsl) is fundamental to the entire structure of a \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater [cf., [IUTchII], Fig. 6.5; [Alien], Fig. 3.8]. That is to say, the issue (SymIsl) may be thought of as being reflected in the gluing along certain collections of prime-strips between the additive and multiplicative symmetry portions of the \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater [cf., [IUTchII], Fig. 6.5; [Alien], Fig. 3.8]. In fact, however, (SctNF) even within the multiplicative symmetry portion of a \((\Theta^{\pm\text{ell}}\text{NF}^-)\)Hodge theater, the goal of simulating a global canonical generator requires
one to treat the various prime-strips that appear in the multiplicative symmetry portion of the \( (\Theta^{\pm}NF-) \) Hodge theater as “sections”, in some suitable sense, of the finite étale quotient \( \text{Spec}(K) \to \text{Spec}(F_{\text{mod}}) \)

— a point of view that is fundamentally incompatible with the prime decomposition trees of the number fields \( K, F_{\text{mod}} \), hence again requires one to impose (LGIsl) [cf. [IUTchI], Remarks 4.3.1, 4.3.2; [Alien], §3.3, (iv)]. On the other hand, let us recall that the ring structure of the nonarchimedean local field that gives rise to \( G_v \) cannot be reconstructed from the abstract topological group \( G_v \) [cf. [NSW], the Closing Remark preceding Theorem 12.2.7; [AbsTopIII], §13; [Alien], Example 2.12.3, (i)]. In particular, once one imposes (LGIsl), the crucial reconstruction of the ring structures of the nonarchimedean local fields that give rise to the various copies of \( G_v \) — where we recall that such ring structures play a fundamental and indispensable role in the definition of the log-link! — can only be conducted if one applies the absolute anabelian algorithms of [AbsTopIII], §1, locally at each \( v \in V_{\text{non}} \) [not to \( G_v \), but rather] to \( \Pi_v \), i.e., one must always regard each coric copy of \( G_v \) as a “certain quotient” of a corresponding coric copy of \( \Pi_v \). Indeed, from a historical point of view [cf. the discussion of [IUTchI], Remark 4.3.2],

it was precisely these local-global isolation aspects — i.e., surrounding (LGIsl), as discussed in (iii) and the present (iv) — of the structure of \( (\Theta^{\pm}NF-) \) Hodge theaters that motivated the author to develop the absolute anabelian algorithms of [AbsTopIII], §1, in the first place!

Unlike the situations considered in [SGA1] [cf. the discussion of Example 3.8.1], in which the ring/scheme structures of the various distinct schemes that appear are coric, the ring structures of the rings that appear on either side of the log- and \( \Theta \)-links of inter-universal Teichmüller theory — i.e., such as number fields or completions of number fields at various valuations — are not coric with respect to the respective links. This leads one naturally to consider weaker structures [cf. the discussion of Example 3.8.2, (iii), (iv)] such as

- sets equipped with a topology and a continuous action of a topological group, in the case of the log-link, or
- realified Frobenioids or topological monoids equipped with a continuous action of a topological group, in the case of the \( \Theta \)-link,

which are indeed coric with respect to the respective links. Indeed, it is precisely this sort of consideration — i.e., of weaker coric structures to relate the universes/Galois categories/étale fundamental groups associated to ring/scheme structures on opposite sides of the links under consideration [cf. the discussion preceding Example 3.8.1] — that gave rise to the term “inter-universal”.

Here, we note that it is of fundamental importance that these topological groups [which typically in fact arise as Galois groups or arithmetic fundamental groups of schemes] be treated as abstract topological groups, rather than as Galois groups or arithmetic fundamental groups [cf. the discussion at the beginning of §3.2; the discussion of Example 3.8.2, (iii), (iv)]. That is to say, to treat these topological groups as Galois groups or arithmetic fundamental groups requires the use of the ring/scheme structures involved, i.e., the use of structures which are
not available since they are not common/coric to the rings/schemes that appear on opposite sides of the log-/Θ-link [cf. the discussion of [Alien], §2.10; IUTchIII], Remarks 1.1.2, 1.2.4, 1.2.5; IUTchIV], Remarks 3.6.1, 3.6.2, 3.6.3]. In this context, it is also of fundamental importance to observe that it is precisely because these topological groups must be treated as abstract topological groups that anabelian results play a central role in inter-universal Teichmüller theory.

One consequence of the constraint [discussed above] that one must typically work, in inter-universal Teichmüller theory, with structures that are substantially weaker than ring structures is the necessity, in inter-universal Teichmüller theory, of allowing for various indeterminacies, such as (Ind1), (Ind2), (Ind3), that are somewhat more involved than the relatively simple inner automorphism indeterminacies that occur in [SGA1]. Here, we recall that from the discussion of (\∧(\lor)-Chn) in §3.7 that

it is precisely the numerous indeterminacies that arise in inter-universal Teichmüller theory that give rise to the numerous logical OR relations “\lor” in the display of (\∧(\lor)-Chn).

On the other hand, once one takes such indeterminacies into account, i.e.,

once one consents to work with various objects “up to certain suitable indeterminacies” — e.g., by means of poly-morphisms, as discussed in §3.7 — it is natural to identify, by applying (ExtInd2) [as discussed in §3.6], objects that are related to one another by means of collections of isomorphisms [i.e., poly-isomorphisms] that are uniquely determined up to suitable indeterminacies.

Here, we observe that this sort of (ExtInd2) identification that occurs repeatedly in inter-universal Teichmüller theory [cf. the discussion of §3.6] may at first glance appear somewhat novel. In fact, however, from the point of view of mathematical foundations — i.e., just as in the discussion of inter-universality given above! — this sort of (ExtInd2) identification is qualitatively very similar to numerous classical constructions such as the following:

(AlgCl) the notion of an algebraic closure of a field [cf. the discussion of Example 3.8.1], which is not constrained to be a specific set constructed from the field;

(DrInv) various categorical constructions such as direct and inverse limits [i.e., such as fiber products of schemes] that are defined by means of some sort of universal property, and which are not constrained to be specific sets even when the given direct or inverse systems are specified set-theoretically;

(HomRs) various constructions of (co)homology modules in homological algebra that depend on the use of resolutions that satisfy certain abstract properties, but which are not constrained in a strict set-theoretic sense even if the original objects resolved by such resolutions are specified set-theoretically.

That is to say, in each of the classical constructions, the “output object” is, strictly speaking, from the point of view of mathematical foundations, not well-defined as a particular set, but rather as a collection of sets [where we note that, typically,
this “collection” is not a set] that are related to one another — and hence, in common practice, identified with one another, in the fashion of (ExtInd2)! — via unique [modulo, say, some sort of well-defined indeterminacy] isomorphisms by means of some sort of “universal” property.

In this context, it is also important to note that, from a foundational point of view, the sort of “(sub)quotient” obtained by applying (ExtInd2) [cf. the discussion of “(sub)quotients” in (sQLTL) and indeed throughout §3.6] must be regarded, a priori, as a formal (sub)quotient, i.e., as some sort of diagram of arrows. That is to say, at least from an a priori point of view,

any explicit construction of a “naive set-theoretic (sub)quotient” necessarily requires the use of some sort of set-theoretic enumeration of each of the individual [set-theoretic] objects that are identified, up to isomorphism, via an application of (ExtInd2). On the other hand, as is well-known, typically such set-theoretic enumerations — which often reduce, roughly speaking, to consideration of the “set of all sets”! — lead immediately to a contradiction.

Indeed, it is precisely this aspect of the constructions of inter-universal Teichmüller theory that motivated the author to include the discussion of species in [IUTchIV], §3.

Finally, we recall [cf. also the discussion of §3.10 below] that

it is only in the final portion of inter-universal Teichmüller theory, i.e., once one obtains a formal (sub)quotient that forms a “closed loop”, that one may pass from this formal (sub)quotient to a “coarse/set-theoretic (sub)quotient” by taking the log-volume [cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (ix); Step (x) of the proof of [IUTchIII], Corollary 3.12].

§3.9. Passage and descent to underlying structures

One fundamental aspect of inter-universal Teichmüller theory lies in the use of numerous functorial algorithms that consist of the construction

\[ \text{input data} \xrightarrow{\sim} \text{output data} \]

of certain output data associated to given input data. When one applies such functorial algorithms, there are two ways in which the output data may be treated [cf. [Alien], §2.7, (iii); the discussion of “post-anabelian structures” in [IUTchII], Remark 1.11.3, (iii), (v); [IUTchIII], Remark 1.2.2, (vii)]:

One may consider the output data independently of the given input data and functorial algorithms used to construct the output data. In this case, the output data may be regarded as a sort of “underlying structure” associated to the input data.

One may consider the output data as data equipped with the additional structure constituted by the input data, together with the functorial algorithm that gave rise to the output data by applying the algorithm to the input data.
Typical examples of this phenomenon in inter-universal Teichmüller theory are the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

\( \text{(sQGOut)} \) **Functorial algorithms that associate to} \ \Pi_v \ [\text{where} \ v \in \mathcal{V}^{\text{non}}] \ \text{some subquotient group of} \ \Pi_v, \ \text{such as, for instance, the quotient} \ \Pi_v \to G_v; \n\]

In this sort of situation, treatment of the output data [i.e., subquotient group of \( \Pi_v \)] according to (InOut) is indicated by a “(\( \Pi_v \))” following the notation for the particular subquotient under consideration; by contrast, treatment of the output data [i.e., subquotient group of \( \Pi_v \)] according to (UdOut) is indicated by the omission of this “(\( \Pi_v \))”.

\( \text{(MnOut)} \) **Functorial algorithms that associate to} \ \Pi_v \ [\text{where} \ v \in \mathcal{V}^{\text{non}}] \ \text{some sort of abelian monoid equipped with a continuous action by} \ \Pi_v, \ \text{such as, for instance,} \ [\text{data isomorphic to}] \ \text{various subquotient monoids [i.e., “}\bigcirc\bigcirc\text{”}, \ \text{“}\bigcirc\times\bigcirc\text{”}, \ \text{“}\bigcirc\times\mu\text{”, etc.] of the multiplicative monoid} \ F_v^\times: \n\]

In this sort of situation, treatment of the output data [i.e., monoid equipped with an action by \( \Pi_v \)] according to (InOut) is indicated by a “(\( \Pi_v \))” following the notation for the particular monoid equipped with an action by \( \Pi_v \) under consideration; by contrast, treatment of the output data [i.e., monoid equipped with an action by \( \Pi_v \)] according to (UdOut) is indicated by the omission of this “(\( \Pi_v \))”.

\( \text{(PSOut)} \) **Functorial algorithms that associate some sort of prime-strip to some sort of input data:** In this sort of situation, treatment of the output data [i.e., some sort of prime-strip] according to (InOut) is indicated by a “(−)” [where “−” is the given input data] following the notation for the particular prime-strip under consideration; by contrast, treatment of the output data [i.e., some sort of prime-strip] according to (UdOut) is indicated by the omission of this “(−)”.

Perhaps the most central example of (PSOut) in inter-universal Teichmüller theory is the notion of the “\( q-/\Theta\)-intertwinings” on an \( F^{\Phi\times\mu}_\pi\)-prime-strip [cf. the discussion of [Alien], §3.11, (v); [IUTchIII], Remark 3.9.5, (viii), (ix); [IUTchIII], Remark 3.12.2, (ii)]:

\( \text{(ItwOut)} \) This terminology refers to the treatment of the \( F^{\Phi\times\mu}_\pi\)-prime-strip according to (InOut), relative to the functorial algorithm for constructing the \( q\)-pilot \( F^{\Phi\times\mu}_\pi\)-prime-strip [in the case of the “\( q\)-intertwining”] or the \( \Theta\)-pilot \( F^{\Phi\times\mu}_\pi\)-prime-strip [in the case of the “\( \Theta\)-intertwining”] from some \( \Theta^{\pm\text{ell}}\) \( NF\)- or \( D\cdot\Theta^{\pm\text{ell}}\) \( NF\)-Hodge theater.

In any situation in which one considers a construction from the point of view of (UdOut) — that is to say, as a construction that produces “underlying data” [i.e., “output data”] from “original data” [i.e., “input data”]

\[
\begin{array}{c|c|c}
\text{input data} & \sim & \text{output data} \\
\hline
\text{original data} & \text{underlying data} \\
\end{array}
\]

— it is natural to consider the issue of descent to [a functorial algorithm in] the underlying data of a functorial algorithm in the original data. Here, we say that
a functorial algorithm $\Phi$ in the original data descends to a functorial algorithm $\Psi$ in the underlying data if there exists a functorial isomorphism

$$\Phi \sim \Psi|_{\text{original data}}$$

between $\Phi$ and the restriction of $\Psi$, i.e., relative to the given construction $\text{original data} \rightsquigarrow \text{underlying data}$.

That is to say, roughly speaking, to say that the functorial algorithm $\Phi$ in the original data descends to the underlying data means, in essence, that although the construction constituted by $\Phi$ depends, a priori, on the “finer” original data, in fact, up to natural isomorphism, it only depends on the “coarser” underlying data.

One elementary example of the phenomenon of descent may be seen in the situation discussed in (HomRs) in §3.8:

(HmDsc) The various constructions of (co)homology modules in homological algebra are, strictly speaking, constructions that require as input data not just some given module [whose (co)homology is computed by the construction], but also some sort of resolution of the given module that satisfies certain properties. In fact, however, such constructions of (co)homology modules typically descend, up to unique isomorphism, to constructions whose input data consists solely of the given module.

Another illustrative elementary example of the phenomenon of descent is the following:

**Example 3.9.1: Categories of open subschemes.** Let $X$ be a scheme, $T$ a topological space. Write

- $|X|$ for the underlying topological space of $X$,
- $\text{Open}(X)$ for the category of open subschemes of $X$ and open immersions over $X$,
- $\text{Open}(T)$ for the category of open subsets of $T$ and open immersions over $T$.

Then the functorial algorithm

$$X \mapsto \text{Open}(X)$$

— defined, say, on the category of schemes and morphisms of schemes — is easily verified to descend, relative to the construction $X \rightsquigarrow |X|$, to the functorial algorithm

$$T \mapsto \text{Open}(T)$$

— defined, say, on the category of topological spaces and continuous maps of topological spaces. That is to say, one verifies immediately that there is a natural functorial isomorphism

$$\text{Open}(X) \sim \text{Open}(|X|)$$

[i.e., in this case, following the conventions employed in inter-universal Teichmüller theory, a natural functorial isomorphism class of equivalences of categories — cf. the discussion of “Monoids and Categories” in [IUTchI], §0].
On the other hand, perhaps the most fundamental example, in the context of inter-universal Teichmüller theory, of this phenomenon of descent is the following [cf. the notational conventions of [IUTchI], Definition 3.1, (e), (f)]:

(MnDsc) The topological multiplicative monoid determined by the topological ring given by [the union with \{0\} of] \(\mathcal{O}^\circ(\Pi_X)\) [cf. [Alien], Example 2.12.3, (iii)] — that is to say, a construction that, a priori, from the point of view of [AbsTopIII], Theorem 1.9; [AbsTopIII], Corollary 1.10, is a functorial algorithm in the topological group

\[\Pi_X\]

[i.e., “\(\Pi_u\)” from the point of view (sQGOOut)] — in fact descends [cf. the discussion at the beginning of [Alien], §2.12; the discussion of [Alien], Example 2.12.3, (i)], relative to passage to the underlying quotient group discussed in (SQGOOut), to a functorial algorithm in the topological group

\[G_k\]

[i.e., “\(G_v\)” from the point of view (sQGOOut)].

Finally, we remark that often, in inter-universal Teichmüller theory, the output data of the functorial algorithm \(\Phi\) of the above discussion is regarded “stack-theoretically”. That is to say, the output data is not a single “set-theoretic object”, but rather a collection [which is not necessarily a set!] of set-theoretic objects linked by uniquely determined poly-isomorphisms of some sort. Typically, this sort of situation arises when one applies \((\text{ExtInd}2)\) — cf. the discussion of \((\text{NSsQ})\) in §3.8. The most central example of this phenomenon in inter-universal Teichmüller theory is the multiradial algorithm — and, especially, the portion of the multiradial algorithm that involves the log-Kummer correspondence and closely related operations of Galois evaluation — which plays the role of

exhibiting the Frobenius-like \(\Theta\)-pilot as one possibility within a collection of possibilities constructed via anabelian algorithms from étale-like data

[cf. the discussion at the end of §3.6, as well as the discussion of §3.10, §3.11, below]. That is to say, the log-Kummer correspondence and closely related operations of Galois evaluation exhibit the Frobenius-like \(\Theta\)-pilot as one possibility within a collection of possibilities constructed via anabelian algorithms from étale-like data not in a set-theoretic sense [i.e., one possibility/element contained in a set of possibilities], but rather in a “stack-theoretic sense”, in accordance with various applications of (ExtInd2) [cf. the discussion at the end of §3.6], i.e., as

one possibility, up to isomorphism, within some [not necessarily set-theoretic!] collection of possibilities.

As discussed in (LVsQ) in §3.8, one arrives at a set-theoretic situation — i.e., one possibility/element contained in a set of possibilities — only after one obtains a “closed loop”, which allows one to pass to a “coarse/set-theoretic (sub)quotient” by taking the log-volume.
§3.10. Detailed description of the chain of logical AND relations

We begin the present §3.10 with the following well-known and, in some sense, essentially tautological observation: Just as every form of data — i.e., ranging from text files and webpages to audiovisual data — that can be processed by a computer can, ultimately, be expressed as a [perhaps very long!] chain of “0’s” and “1’s”, the well-known functional completeness, in the sense of propositional calculus, of the collection of Boolean operators consisting of logical AND “∧”, logical OR “∨”, and negation “¬” motivates the point of view that one can, in principle, express the essential logical structure of any mathematical argument or theory in terms of elementary logical relations, i.e., such as logical AND “∧”, logical OR “∨”, and negation “¬”.

Indeed, it is precisely this point of view that formed the central motivation and conceptual starting point of the exposition given in the present paper.

From the point of view of the correspondence with the terminology and modes of expression that actually appear in [IUTchI-III] and [Alien], the representation given in the present paper of the essential logical structure of inter-universal Teichmüller theory in terms of elementary logical relations, i.e., such as logical AND “∧” and logical OR “∨”, may be understood as follows:

- **Logical AND “∧”** corresponds to such terms as
  - simultaneous execution and
  - gluing

  [cf. [IUTchIII], Remark 3.11.1, (ii); [IUTchIII], Remark 3.12.2, (ii), (c^tw), (f^tw); the final portion of [Alien], §3.7, (i); [Alien], §3.11, (iv)].

- **Logical OR “∨”** corresponds to such terms as
  - indeterminacies,
  - poly-morphisms, and
  - projection/(sub)quotient/splitting

  [cf. §3.7; the title of [IUTchIII]; [IUTchIII], Remark 3.9.5, (xiii), (ix); [Alien], §3.11, (v); [Alien], §4.1, (iv)].

Recall the essential logical structure of inter-universal Teichmüller theory summarized in \((∧(∨)-\text{Chn})\)

\[
A \land B = A \land (B_1 \lor B_2 \lor \ldots) \\
\implies A \land (B_1 \lor B_2 \lor \ldots \lor B'_1 \lor B'_2 \lor \ldots) \\
\implies A \land (B_1 \lor B_2 \lor \ldots \lor B'_1 \lor B'_2 \lor \ldots \lor B''_1 \lor B''_2 \lor \ldots) \\
\vdots
\]

[cf. the discussion of §3.6, §3.7]. Observe that if the description of the various “possibilities” related via “∨’s” in the above displays is suitably formulated, i.e., without superfluous overlaps, then in fact these logical OR “∨’s” may be understood as logical XOR “˙∨’s”, i.e., we conclude the following:
(∧(\dot{\vee})-Chn) The essential logical structure of inter-universal Teichmüller theory may be summarized as follows:

\[ A \land B = A \land (B_1 \dot{\vee} B_2 \dot{\vee} \ldots) \]
\[ \Rightarrow A \land (B_1 \dot{\vee} B_2 \dot{\vee} \ldots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ldots) \]
\[ \Rightarrow A \land (B_1 \dot{\vee} B_2 \dot{\vee} \ldots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \ldots \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \ldots) \]
\[ \vdots \]

Here, we observe the following:

(∧(\dot{\vee})-Chn1) The “∧’s” in the above display
- arise from the Θ-link, which may be thought of as a relationship between certain portions of the multiplicative structures of the ring structures arising from the \((\Theta^{\text{ell}} \text{NF})\text{-Hodge theaters}\) in the domain and codomain of the Θ-link that are common [cf. “∧”!] to these ring structures.

This situation is reminiscent of
- the fact that from the point of view of Boolean algebras, “∧” corresponds to the multiplicative structure of the field \(\mathbb{F}_2\), which may be regarded, via the splitting determined by Teichmüller representatives, as a multiplicative structure that is common [cf. “∧”!] to \(\mathbb{Z}\) and \(\mathbb{F}_2\) [cf. Example 2.4.6, (iii)], as well as of
- the discussion of [Alien], §3.11, (iv), (2\text{and}), concerning the interpretation of the discussion of crystals in [Alien], §3.1, (v), (3\text{KS}), in terms of the logical relation “∧”, i.e., as objects that may be simultaneously interpreted, up to isomorphism, as pull-backs via one projection morphism and [cf. “∧”!] as pull-backs via the other projection morphism.

(∧(\dot{\vee})-Chn2) The “\dot{\vee}’s” in the above display may be understood as corresponding to
- various indeterminacies that arise mainly from the log-Kummer-correspondence, i.e., from sequences of iterates of the log-link, which may be thought of as a device for constructing additive log-shells. The additive structures of the ring structures arising from the \((\Theta^{\text{ell}} \text{NF})\text{-Hodge theaters}\) in the domain and codomain of the Θ-link are structures which, unlike the corresponding multiplicative structures, are not common [cf. “\dot{\vee}”!] to these ring structures in the domain and codomain of the Θ-link.
This situation is reminiscent of

- the fact that from the point of view of Boolean algebras, “∨” corresponds to the additive structure of the field \( \mathbb{F}_2 \), which is an additive structure that is not shared [cf. “∨”!], relative to the splitting determined by Teichmüller representatives, by \( \mathbb{Z} \) and \( \mathbb{F}_2 \) [cf. Example 2.4.6, (iii)], as well as of

- the discussion of [Alien], §3.11, (iv), (2\textsuperscript{and}), concerning the interpretation of the discussion of crystals in [Alien], §3.1, (v), (3\textsuperscript{KS}) in terms of the logical relator “∧”, i.e., where we recall that the two pull-backs of the rank one Hodge subbundle [cf. [Alien], §3.1, (v), (5\textsuperscript{KS}); the discussion of Hodge structures in [IUTchI], §12] do not, in general, coincide [cf. “∨”!], but rather differ by an additive “deformation discrepancy”, namely, the Kodaira-Spencer morphism.

\((∧(∨)-\text{Chn}3)\) Taken together, \((∧(∨)-\text{Chn}1)\) and \((∧(∨)-\text{Chn}2)\) may be understood as expressing the fact that the “∨’s” and “∧’s” of the above display correspond, respectively, to the two underlying combinatorial dimensions — i.e., addition and multiplication — of a ring or, alternatively, to the two-dimensional nature of the log-theta-lattice [cf. the discussion of [IUTchIII], Remark 3.12.2, (i); the latter portion of [Alien], §3.3, (ii)]. Thus, these two dimensions may be understood, alternatively, as corresponding to

- the arithmetically intertwined Θ-link and log-link of inter-universal Teichmüller theory, which give rise to the multiradial representation up to suitable indeterminacies [cf. “∧(∨)”!] of the Θ-pilot;

- the description given in Example 2.4.6, (iii), of the carry-addition operation on the truncated ring of Witt vectors \( \mathbb{Z}/4\mathbb{Z} \) in terms of “∧” and “∨” [cf. “∧(∨)”!];

- the filtered crystal discussed in [Alien], §3.1, (v), (5\textsuperscript{KS}), where one may think of the filtration [i.e., rank one Hodge subbundle] as “being well-defined up to indeterminacies” [cf. “∧(∨)”!], i.e., up to a “discrepancy”, which is given by the Kodaira-Spencer morphism.

\((∧(∨)-\text{Chn}4)\) The two dimensions discussed in \((∧(∨)-\text{Chn}3)\) may be understood as corresponding to the two dimensions — i.e.,

- the successive iterates of the Frobenius morphism in positive characteristic and
- successive extensions to mixed characteristic
— of a ring of Witt vectors [cf. the discussion of the latter portion of [Alien], §3.3, (ii)]. This relationship to the two dimensions of a ring of Witt vectors is entirely consistent with the way in which truncated rings of Witt vectors occur in the discussion of Example 2.4.6, (iii), i.e., with the expression

\[ \tilde{\land} = (\land, \tilde{\lor}) \]

of mixed characteristic “carry-addition” as a sort of “intertwining” between addition and multiplication in the field \( \mathbb{F}_2 \) obtained by “stacking” multiplication “\( \land \)” on top of addition “\( \lor \)”.

Moreover, in this context, we note that the various correspondences observed in \((\land(\tilde{\land})\text{-Chn3})\) and \((\land(\tilde{\land})\text{-Chn4})\) are particularly fascinating in that they assert that the “arithmetic intertwining” in a ring between addition and multiplication — i.e., the mathematical structure which is in some sense the main object of study in inter-universal Teichmüller theory — may be elucidated by means of a theory [i.e., inter-universal Teichmüller theory!] whose essential logical structure, when written symbolically in terms of Boolean operators such as “\( \land \)” and “\( \lor \)”, amounts precisely to the description [cf. the discussion of \((\tilde{\land} = \land, \tilde{\lor})\) in Example 2.4.6, (iii)] of the “Boolean intertwining” that appears in Boolean carry-addition “\( \tilde{\lor} \)” between Boolean addition “\( \lor \)” and Boolean multiplication “\( \land \)”. Put another way, it is as if

\[ \text{(TrHrc)} \quad \text{the arithmetic intertwining which is the main object of study in inter-universal Teichmüller theory somehow “induces”/“is reflected in” a sort of “structural carry operation”, or “trans-hierarchical similitude”, to the Boolean intertwining that constitutes the essential logical structure of the theory [i.e., inter-universal Teichmüller theory] that is used to describe it:} \]

\[ \text{arithmetic intertwining} \rightsquigarrow \text{Boolean intertwining!} \]

Finally, we observe that it also interesting to note that the essential mechanism underlying the Kummer theory of theta functions — which plays a central role in inter-universal Teichmüller theory, i.e., in inducing the trans-hierarchical similitude discussed in (TrHrc) — namely, the correspondence

\[ \text{Kummer theory of theta functions} \iff \text{one valuation/cusp} \]
\[ \text{Kummer theory of algebraic rational functions} \iff \text{multiple valuations/cusps} \]

discussed in [IUTchIII], Remark 2.3.3, (viii), (ix), may itself be thought of as a sort of trans-hierarchical similitude between number fields/local fields and function fields over number fields/local fields.

At a more technical level, the essential logical structure of inter-universal Teichmüller theory summarized symbolically in \((\land(\tilde{\land})\text{-Chn})\) may be understood as consisting of the following steps:

\[ \text{(Stp1) log-Kummer-correspondence and Galois evaluation: This step consists of} \]
exhibiting the Frobenius-like Θ-pilot at the lattice point (0, 0) of the log-theta-lattice — i.e., the data that gives rise to the $F^n \times \mu$-prime-strip that appears in the domain of the Θ-link — as one possibility within a collection of possibilities [cf. (ExtInd1)!] constructed via anabelian algorithms from holomorphic [relative to the 0-column] étale-like data.

In this context, it is perhaps worth mentioning that it is a logical tautology that the content of the above display may, equivalently, be phrased as follows: this step consists of

the negation “¬” of the assertion of the nonexistence of the Frobenius-like Θ-pilot at the lattice point (0, 0) of the log-theta-lattice — i.e., the data that gives rise to the $F^n \times \mu$-prime-strip that appears in the domain of the Θ-link — within the collection of possibilities constructed via certain anabelian algorithms from holomorphic [relative to the 0-column] étale-like data

[cf. also the discussion of (RcnLb) below]. At the level of labels of lattice points of the log-theta-lattice, this step corresponds to the descent operation

$$(0, 0) \rightsquigarrow (0, \circ)$$

[cf. the discussion at the end of §3.6; the discussion at the end of §3.9; [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), (iii)]. Finally, we recall that this step already involves the introduction of the (Ind3) indeterminacy, which may be understood as a sort of coarse approximation [cf. (ExtInd1), as well as the discussion of (logORInd) in §3.11 below] of the complicated apparatus constituted by the log-Kummer-correspondence and Galois evaluation.

(Stp2) **Introduction of (Ind1):** This step consists of observing that

the anabelian construction algorithms of (Stp1) in fact descend to — i.e., are equivalent to algorithms that only require as input data the weaker data constituted by [cf. the discussion of “descent” in §3.9] — the associated mono-analytic étale-like data, i.e., in the notation of (sQGOut), the “$G$’s”.

At the level of labels of lattice points of the log-theta-lattice, this step corresponds to the descent operation

$$(0, \circ) \rightsquigarrow (0, \circ)^+$$

[cf. [IUTchIII], Remark 3.9.5, (viii), (sQ1), (sQ2); [IUTchIII], Theorem 3.11, (i), as well as the references to [IUTchIII], Theorem 3.11, (i), in [IUTchIII], Theorem 3.11, (iii)]. Finally, we recall that this step involves the introduction of the (Ind1) indeterminacy, which [very mildly!] — cf. the discussion of (Ind3>1+2) in §3.11 below] increases the collection of possibilities under consideration [cf. (ExtInd1)].

(Stp3) **Introduction of (Ind2):** This step consists of observing that
the anabelian construction algorithms of (Stp2) in fact descendent to — i.e., are equivalent to algorithms that only require as input data the weaker data constituted by [cf. the discussion of “descent” in §3.9] — the associated mono-analytic Frobenius-like data, i.e., in the notation of (sQGOut) and (MnOut), the “$G_\mu \curvearrowright \mathcal{O}_F^\times$’s”.

[That is to say, one constructs log-shells, for instance, as submonoids of “$\mathcal{O}_F^\times$”, as opposed to subquotients of “$G_\mu$”.] At the level of labels of lattice points of the log-theta-lattice, this step corresponds to the descent operation

$$(0,0)^+ \quad \rightsquigarrow \quad (0,0)^+$$

— i.e., corresponding to the full poly-isomorphism of $\mathcal{F}_+^\times \times^\mu$-prime-strips constituted by the $\Theta$-link — this descent operation means that the algorithm under consideration may be regarded as an algorithm whose input data is the mono-analytic Frobenius-like data $(1,0)^+$ arising from the codomain of the $\Theta$-link. This step involves the introduction of the (Ind2) indeterminacy, which [very mildly! — cf. the discussion of (Ind3>1+2) in §3.11 below] increases, at least from an a priori point of view, the collection of possibilities under consideration [cf. (ExtInd1)]. Finally, we recall that this step plays the important role of

isolating the log-link indeterminacies in the domain [i.e., the (Ind3) indeterminacy of (Stp1)] and the codomain [i.e., the log-shift adjustment discussed in (Stp7) below] of the $\Theta$-link from one another

[cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob7-2); [Alien], §3.6, (iv)]. Here, we recall [cf. the discussion of the final portion of [Alien], §3.3, (ii)] that these $\log$-link indeterminacies on either side of the $\Theta$-link may be understood, in the context of the discussion of (InfH) in §3.3, as corresponding to the copies “$\mathbb{C}_\times$” on either side of the double coset space “$\mathbb{C}_\times \setminus GL_2^+ \mathbb{R} / \mathbb{C}_\times$”.]

(Stp4) Passage to the holomorphic hull: The passage from the collection of possible regions that appear in the output data of (Stp3) to the collection of regions contained in the holomorphic hull — relative to the 1-column of the log-theta-lattice — of the union of possible regions of the output data of (Stp3) [cf. [IUTchIII], Remark 3.9.5, (vi); [IUTchIII], Remark 3.9.5, (vii), (Ob5); [IUTchIII], Remark 3.9.5, (viii), (sQ3)] is a simple, straightforward application of (ExtInd1), that is to say, of increasing the set of possibilities [i.e., of “$\vee$’s”]. The purpose of this step, together with
(Stp5) Passage to hull-approximants: This step consists of passing from the collection of arbitrary regions contained the holomorphic hull of (Stp4) to hull-approximants, i.e., regions that have the same global log-volume as the original “arbitrary regions”, but which correspond to arithmetic vector bundles [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob1), (Ob2)]. This operation does not affect the logical “∧/∨” structure of the algorithm since this operation of passing to hull-approximants does

not affect the collection of possible value group portions —
i.e., “$\mathcal{F}^{\mu\Rightarrow\mu}$-prime-strips” — of $\mathcal{F}^{\mu\Rightarrow\mu}$-prime-strips determined by forming the log-volume of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp6) Passage to a suitable positive rational tensor power of the determinant: This step consists of passing from the regions corresponding to arithmetic vector bundles obtained in (Stp4), (Stp5) to a suitable tensor power root of a tensor power of the determinant arithmetic line bundle of such an arithmetic vector bundle [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob3), (Ob4); [IUTchIII], Remark 3.9.5, (viii), (sQ3)]. Just as in the case of (Stp5), this operation does not affect the logical “∧/∨” structure of the algorithm since this operation of passing to a suitable positive rational tensor power of the determinant does

not affect the collection of possible value group portions —
i.e., “$\mathcal{F}^{\mu\Rightarrow\mu}$-prime-strips” — of $\mathcal{F}^{\mu\Rightarrow\mu}$-prime-strips determined by forming the log-volume of these regions

[cf. the discussion of [IUTchIII], Remark 2.4.2; the discussion of (IPL) in [IUTchIII], Remark 3.11.1, (iii)].

(Stp7) Log-shift adjustment: The arithmetic line bundles that appear in (Stp6) occur with respect to the arithmetic holomorphic structure — i.e., in effect, ring structure — at the label $(1,1)$ of the log-theta-lattice, i.e., at a label vertically shifted by $+1$ relative to the label $(1,0)$ that forms the codomain of the $\Theta$-link [cf. the discussion of [IUTchIII], Remark 3.9.5, (vii), (Ob8); [IUTchIII], Remark 3.9.5, (viii), (sQ4)]. That is to say, by applying the algorithm discussed in (Stp1) $\sim$ (Stp6) at each lattice point $(1,m)$ [where $m \in \mathbb{Z}$] of the 1-column of the log-theta-lattice, we obtain algorithms with input data at $(1,m)$ and output data at $(1,m+1)$ — cf. the diagonal arrows of the diagram shown below. Thus, the totality of all of these diagonal arrows may be thought of a sort of endomorphism of the 1-column of the log-theta-lattice, i.e., an algorithm whose input data is the 1-column of the log-theta-lattice, and whose output data lies in the same 1-column of the log-theta-lattice. In particular, we obtain a closed
loop [cf. the discussion of Example 3.1.1, (iii); the discussion below of (DstMp), (FxGl), (NoCmplss), (Englf); the discussion of [IUTchIII], Remark 3.9.5, (ix); [Alien], §3.11, (v)]. Put another way, in the language of the discussion surrounding (DltLb) in §3.11 below, we obtain a situation that simulates — via the introduction of suitable indeterminacies [cf. the discussion of (Stp8) below] — a situation in which the distinct labels on the domain and codomain of the Θ-link have been eliminated.

1-column 1-column
: :
: :
• •
| log | log |
| log | log |
| log | log |
: :
: :

(Stp8) Passage to log-volumes: The closed loop of (Stp7) [cf. also the discussion of Example 3.1.1, (iii); the discussion below of (DstMp), (FxGl), (NoCmplss), (Englf)] implies that the crucial logical AND “∧” relation carefully maintained throughout the execution of (Stp1) ∼ (Stp7) yields, upon taking the log-volume, a logical AND “∧” relationship between the original q-pilot input $F^{[*]>$-prime-strip and a certain algorithmically constructed collection of possible output $F^{[*]$-prime-strips within the same container, i.e., some copy of the real numbers “$\mathbb{R}$” [cf. [IUTchIII], Remark 3.9.5, (vii), (Ob9); [IUTchIII], Remark 3.9.5, (viii), (sQ5); [IUTchIII], Remark 3.9.5, (ix); the discussion of Substeps (xi-d), (xi-e) of the proof of [IUTchIII], Corollary 3.12; the discussion of [IUTchIII], Remark 3.12.2, (ii); [Alien], §3.11, (v)]. The inequality in the statement of [IUTchIII], Corollary 3.12, then follows as a formal consequence [cf. the discussion of Substeps (xi-f), (xi-g) of the proof of [IUTchIII], Corollary 3.12]. Here, we observe that the various indeterminacies [such as (Ind1), (Ind2), (Ind3)] that arise in the course of applying (Stp1) ∼ (Stp7) may be thought of as a sort of monodromy associated to the closed loop of (Stp7) [cf. also the discussion below of (DstMp), (FxGl), (NoCmplss), (Englf)]. Moreover, in this context, we note that
the diagram of arrows “↗” from the 1-column to the 1-column in (Stp7) admits symmetries

\[(1, m) \mapsto (1, m + 1)\]

[where \(m \in \mathbb{Z}\)] that are compatible with all of the arrows in the diagram, as well as with the various arrows of the \(\log\)-Kummer-correspondence in the 1-column. These symmetries allow one to synchronize the various “monodromy indeterminacies” associated to each “↗” [i.e., to each application of (Stp1) \(\sim\) (Stp6)], so that one may think in terms of a single, synchronized collection of “monodromy indeterminacies” associated to the totality of “↗’s” in (Stp7).

Before proceeding, we pause to examine the meaning of the term “closed loop” in (Stp7), (Stp8), which is sometimes a source of confusion. The intended meaning of this term is that the sequence of mathematics objects on which the series of operations in (Stp1) \(\sim\) (Stp6) [cf. also [IUTchIII], Fig. I.8] are performed forms a closed loop in the sense that the ultimate output data lies in the same container [i.e., up to a log-shift in the 1-column, as discussed in (Stp7)] as the input data.

On the other hand, at the level of the actual mathematical objects that one is working with, the term “closed loop” has the potential to result in certain fundamental misunderstandings, since it may be [mistakenly!] interpreted as suggesting that

\[\text{(DstMp) one is considering two distinct mappings of abstract prime-strips to } [\Theta, q-]\text{-pilot prime-strips.}\]

Once one takes this point of view (DstMp), there is inevitably an issue of diagram commutativity, i.e., the issue discussed in §3.6, (Syp2), that one must contend with. As discussed in Example 2.4.5, (v), (vi), (vii), (viii), this point of view (DstMp) corresponds to EssOR-IUT [i.e., “essentially OR IUT”], which, as the name suggests, is essentially [thought not precisely!] equivalent to OR-IUT, and in particular, constitutes a fundamental misunderstanding of the logical structure of inter-universal Teichmüller theory.

Indeed, the chain of AND relations “∧” discussed in §3.6, as well as the present §3.10, which lies at the heart of the essential logical structure of inter-universal Teichmüller theory, consists precisely of

\[\text{(FxGl) fixing the Frobenius-like } q\text{-pilot at the lattice point } (1, 0), \text{ as well as the gluing [i.e., “∧”!] via the } \Theta\text{-link of this } q\text{-pilot at } (1, 0) \text{ to the Frobenius-like } \Theta\text{-pilot at the lattice point } (0, 0) \text{ [cf. [IUTchIII], Remark 3.12.2, (ii), (eitw), (eitw), (fitw)].} \]

One then proceeds to add to this \(\Theta\)-pilot at the lattice point \((0, 0)\) more and more possibilities/indeterminacies [i.e., “∨”, or, alternatively, “∨”!] in order to obtain data that descends to the same label [i.e., up to a log-shift in the 1-column, as discussed in (Stp7)] as the \(q\)-pilot at \((1, 0)\). That is to say,

\[\text{(NoCmplss) there is never any issue of compatibility between two distinct mappings of abstract prime-strips, as in (DstMp).}\]
From a pictorial point of view, at the level of mathematical objects, one is working in (Stp1) \sim (Stp8) — not with a “closed loop” (!), but rather — a single fixed line segment

\[ \bullet = = \bigtriangleup = = \bullet \]

corresponding to the gluing [i.e., “\(\wedge\)!”] of (FxGl) [so the “\(\bullet\)’s” on the left and right correspond, respectively, to the \(\Theta\)-pilot at \((0,0)\) and the \(q\)-pilot at \((1,0)\)], which is then subjected to subsequent “fuzzifications” [i.e., “\(\lor\)”, or alternatively, “\(\check{\lor}\)!”] of the \(\Theta\)-pilot at \((0,0)\) [denoted in the following display by the notation “\((\ldots)\)”], which may be thought of as representing a “fuzzy disc” that contains the “\(\bullet\)” on the left

\[
\begin{align*}
\bullet & = = \bigtriangleup = = \bullet \\
\sim & \quad ( \lor \bullet = ) = \bigtriangleup = = \bullet \\
\sim & \quad ( \lor \lor \bullet = = ) \bigtriangleup = = \bullet \\
\sim & \quad ( \lor \lor \lor \bullet = = \bigtriangleup ) = = \bullet \\
\sim & \quad ( \lor \lor \lor \lor \bullet = = \bigtriangleup = = ) \bullet \\
\sim & \quad ( \lor \lor \lor \lor \lor \bullet = = \bigtriangleup = = = ) \bullet \\
\end{align*}
\]

that terminate in a situation [cf. the final line of the above display] in which

(Englf) the fuzzifications engulf the \(q\)-pilot at \((1,0)\), i.e., a situation in which the distinct labels may be eliminated and nontrivial consequences may be obtained [cf. (Stp7), (Stp8), as well as the discussion surrounding (DltLb) in §3.11 below].

Thus, in summary, throughout the series of fuzzification operations constituted by (Stp1) \sim (Stp8), the line segment representing the gluing [i.e., “\(\wedge\)!”] of (FxGl) remains fixed, so [cf. (NoCmpIss), (Englf)] there is never any issue of compatibility between two distinct mappings of abstract prime-strips, as in (DstMp).

§3.11. The central importance of the \(\log\)-Kummer correspondence

In the context of the discussion of §3.10, it is important to recall that, whereas (Stp2) \sim (Stp8) are technically trivial in the sense that they concern operations that are very elementary and only require a few lines to describe, the \(\log\)-Kummer correspondence and Galois evaluation operations that comprise (Stp1) depend on the highly nontrivial theory of [EtTh] and [AbsTopIII]. Moreover, the technical description of these operations that comprise (Stp1) occupies the bulk of [IUTchI-III]. The central importance of (Stp1) may also be seen in the subordinate nature of (Ind1), (Ind2) [which occur in (Stp2), (Stp3)] relative to (Ind3) [which occurs in (Stp1)], i.e., in the sense that
(Ind3 $> 1+2$) once one constructs the output of the multiradial representation of the $\Theta$-pilot [cf. [IUTchIII], Theorem 3.11, (ii)] via tensor-packets of log-shells in such a way that each local portion of this output is stable with respect to the indeterminacy (Ind3), these local portions of the output are automatically “essentially stable” [i.e., stable up to discrepancies at the valuations $\in \mathbb{Y}_{\text{bad}}$ that affect the resulting log-volumes only up to very small/essentially negligible order] with respect to the indeterminacies (Ind1), (Ind2) [cf. [IUTchIII], Theorem 3.11, (i)].

Finally, we observe that this property (Ind3 $> 1+2$) is strongly reminiscent of the discussion of (CnfInd1+2) and (CnfInd3) in §3.5.

One way to understand the content of the operations of (Stp1) is as follows. These operations may be regarded as a sort of

(logORInd) saturation of the Frobenius-like $\Theta$-pilot at the lattice point $(0,0)$ of the log-theta-lattice — i.e., which is linked, via the $\Theta$-link, to the Frobenius-like $q$-pilot at $(0,1)$ — with respect to all of the possibilities that occur in the 0-column of the log-theta-lattice, i.e., all of the possibilities that arise from a possible confusion between the domain and codomain of the log-links in the 0-column [cf. the description of (Stp1)].

In this sense, the content of (Stp1) is formally reminiscent of the “(NeuORInd)” that appeared in the discussion of §3.4, i.e., which may be understood as a sort of

(ΘORInd) saturation of the Frobenius-like $\Theta$-pilot at the lattice point $(0,0)$ of the log-theta-lattice with respect to all of the possibilities — i.e., “$\Theta$-plt”, “$q$-plt” [cf. (NeuORInd2)] — that arise from a possible confusion between the domain and codomain of the $\Theta$-link joining the lattice points $(0,0)$ and $(0,1)$.

[In this context, we note that the logical OR “$\lor$’s” that appear in (logORInd), (ΘORInd) may in fact be understood as logical XOR “$\dot{\lor}$’s” — cf. the discussion surrounding $(\land(\lor)$-Chn) in §3.10.] On the other hand, whereas, as observed in the discussion at the end of §3.4, (ΘORInd) yields a meaningless/useless situation that does not give rise to any interesting mathematical consequences, (logORInd), by contrast, is a highly potent technical device that forms the technical core of inter-universal Teichmüller theory.

Before preceding, we observe that, in this context, it is interesting to note that both of these “saturation operations” (logORInd) and (ΘORInd) are in some sense qualitatively similar to the label crushing operation (ExtInd2).

Indeed, (ExtInd2) consists, roughly speaking, of regarding mathematical objects of a certain type up to isomorphism, i.e., of saturating within an isomorphism class of mathematical objects of a certain type [cf. the discussion of (ExtInd2) in §3.8, as well as the discussion of (DltLb) below].

The stark contrast between the potency of (logORInd) and the utterly meaningless nature of (ΘORInd) is highly reminiscent of the central role played, in Example 3.3.2, (iv), by invariance with respect to

$$\iota = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^\times \subseteq GL_2^+(\mathbb{R})$$
[where we recall from (InfH) that $C \times \log$ corresponds to the log-link], which lies in stark contrast to the utterly meaningless nature of considering invariance with respect to dilations $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in GL_2^+(\mathbb{R})$ [where we recall from (InfH) that such dilations correspond to the $\Theta$-link].

One way to witness the potency of (logORInd) is as follows. Recall that the $\Theta$-link, by definition [cf. [IUTchIII], Definition 3.8, (ii)], consists of

- a dilation applied to the local value group portions of the ring structures in its domain and codomain, coupled with
- a full poly-isomorphism — which preserves log-volumes, hence is non-dilating! — between the local $\mathcal{O}^\times \mu$'s, i.e., the local unit group portions, of these ring structures.

By contrast, the log-links in the 0-column of the log-theta-lattice have the effect of "juggling/rotating/permuting" the local value group portions and local unit group portions of the ring structures that appear in this 0-column [cf., e.g., the discussion of [Alien], Example 2.12.3, (v)]. From this point of view, the tautologically vertically coric — i.e., invariant with respect to the application of the log-link! — nature of the output data of (logORInd) is already somewhat "shocking" in nature. That is to say, the tautologically vertically coric nature of this output data of (logORInd) suggests that

- (Di/NDi) this output data already exhibits some sort of equivalence, up to perhaps some sort of mild discrepancy, between the dilated and non-dilated portions of the $\Theta$-link.

Such an equivalence already strongly suggests that some sort of bound on heights should follow as a formal consequence, i.e., in the style of the classical argument that implies the isogeny invariance of heights of elliptic curves [cf. the discussion of [Alien], §2.3, §2.4].

Finally, we conclude by emphasizing that, in inter-universal Teichmüller theory,

- (DltLb) ultimately one does want to find some way in which to delete/eliminate the distinct labels on the $\Theta$- and $q$-pilot objects [i.e., "$\Theta$-plt" and "$q$-plt"] in the domain and codomain of the $\Theta$-link

[cf. the discussion of Example 3.1.1, (iii); the discussion of (AOL4), (AO$\Theta$4) in §3.4; (Stp7), (Stp8) in §3.10], that is to say, not via the naive, simple-minded approach of (\ThetaORInd) [i.e., (NeuORInd2) in the discussion of §3.4], but rather via the indirect approach of applying descent operations

$$
\begin{align*}
(0, 0) & \xrightarrow{(Stp1)} (0, \circ) \xrightarrow{(Stp2)} (0, \circ)^+ \xrightarrow{(Stp3)} (0, 0)^+ \xrightarrow{(Stp3)} (1, 0)^+ \\
\end{align*}
$$

as discussed in (Stp1) $\sim$ (Stp8) [cf., especially, (Stp7), (Stp8)] of §3.10, i.e., an approach that centers around (logORInd). This approach is based on the various anabelian reconstruction algorithms discussed in (Stp1) $\sim$ (Stp3), which allow one to exhibit the Frobenius-like $\Theta$-pilot object at $(0, 0)$ as one possibility among some broader collection of possibilities that arise from the introduction of various types of indeterminacy. In this context, we observe [cf. the discussion of (ExtInd2), (NNsQ)] at the end of §3.9 that since such anabelian reconstruction algorithms only reconstruct various types of mathematical objects [i.e.,
monoids/pseudo-monoids/mono-theta environments, etc.] not “set-theoretically on the nose” [i.e., not in the sense of strict set-theoretic equality], but rather up to [a typically essentially unique, if one allows for suitable indeterminacies] isomorphism, it is not immediately clear

(RcnLb) in what sense such anabelian reconstruction algorithms yield a reconstruction of the crucial labels — i.e., such as “(0, 0)” — that underlie the crucial logical AND “∧” structure discussed in §3.4 [cf., especially, (AOL1), (A0Θ1)].

The point here is that indeed such anabelian reconstruction algorithms are not capable of reconstructing such labels “set-theoretically on the nose”.

On the other hand, in this context, it is important to recall the essential substantive content of the various labels involved:

(HolFrLb) (0, 0): The holomorphic Frobenius-like data labeled by (0, 0) consists of various monoids/pseudo-monoids/mono-theta environments, etc., regarded as abstract monoids/pseudo-monoids/mono-theta environments, etc., i.e., as objects that are not equipped with the auxiliary data of how they might have been reconstructed via anabelian algorithms from holomorphic étale-like data labeled (0, 0) [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. In particular, such monoids/pseudo-monoids/mono-theta environments, etc., are not invariant with respect to the “juggling/rotating/permuting” of local value group portions and local unit group portions effected by the log-links in the 0-column of the log-theta-lattice, but rather correspond to a temporary cessation [cf. the label (0, 0) as opposed to the label (0, ∘)] of this operation of juggling/rotation/permutation.

(MnAlyLb) (0, 0): The mono-analytic Frobenius-like data labeled by (0, 0) ⊢ consists of the \( F^{\Theta \_\pi \_\mu \_\pi} \)-prime-strip determined by the Frobenius-like Θ-pilot at (0, 0), regarded as an abstract \( F^{\Theta \_\pi \_\mu \_\pi} \)-prime-strip [cf. the discussion of (UdOut), (InOut), (PSOut), (ItwOut) in §3.9]. Thus, the transition of labels

\[
(0, 0) \rightsquigarrow (0, 0)^{\dagger}
\]

consists of an operation of forgetting some sort of auxiliary structure [cf. the discussion of (UdOut) in §3.9]. Here, we recall that this construction of the \( F^{\Theta \_\pi \_\mu \_\pi} \)-prime-strip determined by the Frobenius-like Θ-pilot at (0, 0) is technically possible precisely because of the “temporary cessation” discussed above [cf. the discussion of the definition of the Θ-link in [Alien], §3.3, (ii), as well as in §3.3 of the present paper].

Thus, the nontrivial substantive content of the anabelian reconstruction algorithms of (Stp1) \( \sim \) (Stp3) — and hence of the descent operations

\[
(0, 0) \rightsquigarrow (\text{Stp1}) \rightsquigarrow (\text{Stp3}) \rightsquigarrow (0, 0)^{\dagger}
\]

that result from these anabelian reconstruction algorithms — consists of statements to the effect that

(FrgInv) the operation of forgetting discussed in (MnAlyLb) can in fact, if one allows for suitable indeterminacies, be inverted.
It is precisely this invertibility (FrgInv), up to suitable indeterminacies, of the operation of forgetting discussed in (MnAlgLb), together with the fact that

(GluDt) the only data appearing in the reconstruction algorithms [i.e., in the 0-column] that is glued [cf. the discussion of [IUTcIII], Remark 3.11.1, (ii); the final portion of [Alien], §3.7, (i)] to data in the 1-column is the \( \mathcal{F}^{\uparrow \bigcirc \times \mu} \)-prime-strip labeled \((0,0)\)^{\circ},

that ensures that the descent operations discussed above do indeed preserve the crucial logical AND “\( \land \)" relations discussed in §3.4, §3.6, §3.7, §3.10, i.e., even though the reconstruction algorithms underlying these descent operations do not yield reconstructions of the various labels “\((0,0)\)”, etc., “set-theoretically on the nose”.

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