On surjective homomorphisms from a configuration space group to a surface group

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Foundations and Perspectives of Anabelian Geometry

2021/07/02

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§1 Introduction: geometric homomorphism

K: field of characteristic zero

X: hyperbolic curve$/K$ of type (g, r)

In the rest of this talk, suppose: K: algebraically closed unless otherwise specified.

$$X_n := \{(x_1, \ldots, x_n) \in X \times_K \cdots \times_K X \mid x_i \neq x_j \ (\forall i \neq j)\}$$

: n-th configuration space of X

Let l: prime, Σ: a set of prime numbers s.t.

$\Sigma = \{l\}$ or Σ contains all prime numbers.

Write $\Pi_n^\Sigma := \pi_1(X_n)^\Sigma$ (the maximal pro-Σ quotient).

We shall refer to (a profinite group isom to) Π_n^Σ (resp. Π_1^Σ) as a configuration space group (resp. surface group).

Definition (generalized projection)

$p : X_n \to X_m \ (0 \leq m \leq n)$

is a generalized projection morphism $\overset{\text{def}}{\Leftrightarrow}$

- If $(g, r) \notin \{(0,3), (1,1)\}$, then
  $p : X_n \to X_m$: projection morphism
- If $(g, r) = (0,3)$, then $X_n \cong (\mathcal{M}_{0,n+3})_K$
  $p : X_n \overset{\sim}{\to} (\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,m+3})_K \overset{\sim}{\to} X_m$
- If $(g, r) = (1,1)$, then $X_n \cong E_{n+1}/E \ \ (E := X^{\mathrm{cpt}})$
  $p : X_n \overset{\sim}{\to} E_{n+1}/E \to E_{m+1}/E \overset{\sim}{\to} X_m$

Definition (generalized fiber subgroup)

A (generalized) projection morphism induces $\Pi_n^\Sigma \twoheadrightarrow \Pi_m^\Sigma$.

$\ker(\Pi_n^\Sigma \twoheadrightarrow \Pi_m^\Sigma)$: (generalized) fiber subgroup

(of co-length m)

$\mathrm{GFS}_m(\Pi_n^\Sigma)$: the set of gen. fiber subgps of co-length m

$\mathrm{FS}_m(\Pi_n^\Sigma)$: the set of fiber subgps of co-length m

Note: $N \in \mathrm{GFS}_m(\Pi_n^\Sigma)$ is isomorphic to "Π_{n-m}^Σ" of a hyperbolic curve of type $(g, r+m)$.

Definition (exceptional morphism $(g = 0, 1)$)

An open immersion $X \hookrightarrow Y$ (Y: of type $(0, 3)$ or $(1, 1)$) determines $X_n \hookrightarrow Y_n$.

If $p : Y_n \to Y$ is a generalized projection which is not a projection, then we shall refer to the composite $X_n \hookrightarrow Y_n \xrightarrow{p} Y$ as an exceptional morphism.

Definition (exceptional subgroup)

An exceptional morphism induces $\Pi_n^\Sigma \twoheadrightarrow \pi_1(Y)^\Sigma$.

$\ker(\Pi_n^\Sigma \twoheadrightarrow \pi_1(Y)^\Sigma)$: exceptional subgroup

$\mathrm{ES}(\Pi_n^\Sigma)$: the set of excep. subgps

$(\mathrm{ES}(\Pi_n^\Sigma) := \emptyset$ if $g \geq 2)$

$$\mathrm{GFS}_1(\Pi_n^\Sigma) = \begin{cases} \mathrm{FS}_1(\Pi_n^\Sigma) & ((g,r) \notin \{(0,3),(1,1)\}) \\ \mathrm{FS}_1(\Pi_n^\Sigma) \cup \mathrm{ES}(\Pi_n^\Sigma) & ((g,r) \in \{(0,3),(1,1)\}) \end{cases}$$

Main Theorem (S.)

Let H: surface group and

$\varphi : \Pi_n^{\Sigma} \twoheadrightarrow H$: surjective homomorphism.

Then ${}^{\exists}N \in \mathrm{FS}_1(\Pi_n^{\Sigma}) \cup \mathrm{ES}(\Pi_n^{\Sigma})$ s.t. $N \subset \ker \varphi$.

(In other words, any surjective homomorphism from a configuration space group to a surface group factors through some "geometric" homomorphism.)

Remark

- The case H: not free of rank 2 is proved by Hoshi-Minamide-Mochizuki.
- The case $g \geq 2$ is essentially proved by Mochizuki-Tamagawa.
- An alternative proof of the case $g \geq 2$ for "topological fundamental groups" is given by L. Chen.

§2 Lie algebra

Definition (Lie algebra associated to X_n)

Write $\Pi_n^l(1) := \Pi_n^l (:= \Pi_n^{\{l\}})$,

$\Pi_n^l(2) := \ker(\Pi_n^l \twoheadrightarrow (\pi_1(X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}})^l)^{\mathrm{ab}})$,

$\Pi_n^l(m) := \langle \overline{[\Pi_n^l(m_1), \Pi_n^l(m_2)]} \mid m_1 + m_2 = m \rangle \ (m \geq 3)$,

$\mathrm{Gr}^m(\Pi_n^l) := \Pi_n^l(m)/\Pi_n^l(m+1)$,

$\mathrm{Gr}(\Pi_n^l) := \bigoplus_{m \geq 1} \mathrm{Gr}^m(\Pi_n^l)$.

Then $\mathrm{Gr}(\Pi_n^l)$: graded Lie algebra over $\mathbb{Z}_l$.

$\mathrm{Gr}(\Pi_n^l)$ has a presentation with generators

$X_i^{(k)}, Y_i^{(k)} \in \mathrm{Gr}^1(\Pi_n^l)$, $Z_j^{(k)}, W_h^{(k)} \in \mathrm{Gr}^2(\Pi_n^l)$

$(1 \leq i \leq g,\ 1 \leq j \leq r,\ 1 \leq k, h \leq n)$

and relations (R1–10):

$$\sum_{i=1}^{g}[X_i^{(k)}, Y_i^{(k)}] + \sum_{j=1}^{r} Z_j^{(k)} + \sum_{h=1}^{n} W_h^{(k)} = 0, \qquad \text{(R1)}$$

$$W_k^{(k)} = 0, \qquad \text{(R2)}$$

$$W_h^{(k)} = W_k^{(h)}, \qquad \text{(R3)}$$

$$[X_i^{(k)}, X_{i'}^{(k')}] = [Y_i^{(k)}, Y_{i'}^{(k')}] = 0 \quad (k \neq k'), \tag{R4}$$

$$[X_i^{(k)}, Y_{i'}^{(k')}] = 0 \quad (i \neq i', k \neq k'), \tag{R5}$$

$$[X_i^{(k)}, Y_i^{(k')}] = W_k^{(k')} \quad (k \neq k'), \tag{R6}$$

$$[X_i^{(k)}, Z_j^{(k')}] = [Y_i^{(k)}, Z_j^{(k')}] = 0 \quad (k \neq k'), \tag{R7}$$

$$[Z_j^{(k)}, Z_{j'}^{(k')}] = 0 \quad (j \neq j', k \neq k'), \tag{R8}$$

$$[X_i^{(k)}, W_h^{(k')}] = [Y_i^{(k)}, W_h^{(k')}] = [Z_j^{(k)}, W_h^{(k')}] = 0 \quad (k \notin \{k', h\}), \tag{R9}$$

$$[W_h^{(k)}, W_{h'}^{(k')}] = 0 \quad (\{k, h\} \cap \{k', h'\} = \emptyset). \tag{R10}$$

Example

- $\underline{n=1}$ (surface algebra)
  Generators: $X_1^{(1)} \cdots X_g^{(1)} \Big| Y_1^{(1)} \cdots Y_g^{(1)} \Big| Z_1^{(1)} \cdots Z_r^{(1)}$
  Relation: $\sum_{i=1}^{g} [X_i^{(1)}, Y_i^{(1)}] + \sum_{j=1}^{r} Z_j^{(1)} = 0$

(If $r > 0$, then $\mathrm{Gr}(\Pi_1^l)$: free of rank $2g + r - 1$.)

- $\underline{n = 2, g = 0}$ $(r \geq 3)$

  Generators: $\begin{array}{cccc|cc} Z_1^{(1)} & Z_2^{(1)} & \cdots & Z_r^{(1)} & 0 & W \\ Z_1^{(2)} & Z_2^{(2)} & \cdots & Z_r^{(2)} & W & 0 \end{array}$

  Relations: $\sum_{j=1}^{r} Z_j^{(k)} + W = 0 \ (k = 1, 2),$
  $[Z_j^{(1)}, Z_{j'}^{(2)}] = 0 \ (j \neq j')$

Definition (fiber ideal, exceptional ideal)

A proj. mor. $X_n \to X_m$ induces $\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l)$.

$\ker(\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l))$: fiber ideal (of co-length m)

$\mathrm{FI}_m(\mathrm{Gr}(\Pi_n^l))$: the set of fiber ideals of co-length m

An excep. mor. $X_n \to Y$ induces $\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\pi_1(Y)^l)$.

$\ker(\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\pi_1(Y)^l))$: exceptional ideal

$\mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$: the set of excep. ideals

Theorem A (S.)

Let $\mathfrak{h}$: surface algebra$/\mathbb{Z}_l$ and

$\varphi : \mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathfrak{h}$: surjective homomorphism$/\mathbb{Z}_l$.

Then ${}^\exists \mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}(\Pi_n^l)) \cup \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$ s.t. $\mathfrak{i} \subset \ker \varphi$.

lower central series ↓

If $g = 0$ or $r \leq 1$, then $\mathrm{Gr}(\Pi_n^l) \cong \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l)$

$\Rightarrow$ Main Theorem for $g = 0$ or $r \leq 1$

follows from Theorem A.

Sketch of the proof of Theorem A:

Lemma 1

Let $\mathfrak{h}$: surface algebra$/\mathbb{Z}_l$.

(i) $\mathfrak{h}$ is a free $\mathbb{Z}_l$-module.

(ii) $a, b \in \mathfrak{h}$, $[a, b] = 0$

$\Rightarrow$ a and b are linearly dependent over $\mathbb{Q}_l$.

(If "r"> 0, then, since $\mathfrak{h}$ is a free Lie algebra$/\mathbb{Z}_l$,

Lemma 1 is well-known.)

e.g. (the case $n = 2, g = 0$)

Recall $\mathrm{Gr}(\Pi_2^l)$ has the following presentation:

Generators: $\begin{array}{cccc|cc} Z_1^{(1)} & Z_2^{(1)} & \cdots & Z_r^{(1)} & 0 & W \\ Z_1^{(2)} & Z_2^{(2)} & \cdots & Z_r^{(2)} & W & 0 \end{array}$

Relations: $\sum_{j=1}^r Z_j^{(k)} + W = 0 \ (k = 1, 2),$

$[Z_j^{(1)}, Z_{j'}^{(2)}] = 0 \ (j \neq j')$

Write $z_j^{(k)} := \varphi(Z_j^{(k)})$, $w := \varphi(W)$.

We may assume: $\alpha := z_1^{(1)} \neq 0$.

$$[Z_j^{(1)}, Z_{j'}^{(2)}] = 0 \ (j \neq j')$$

$$\Rightarrow [\alpha, z_j^{(2)}] = \varphi([Z_1^{(1)}, Z_j^{(2)}]) = 0 \ (j \neq 1)$$

$$\overset{\text{Lem1}}{\Rightarrow} z_j^{(2)} \in \alpha\mathbb{Q}_l \ (j \neq 1)$$

$$\left(\begin{array}{cccc|cc} \alpha & z_2^{(1)} & \cdots & z_r^{(1)} & 0 & w \\ z_1^{(2)} & z_2^{(2)} & \cdots & z_r^{(2)} & w & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|cc} \alpha & z_2^{(1)} & \cdots & z_r^{(1)} & 0 & w \\ z_1^{(2)} & \boxed{\alpha} & \cdots & \boxed{\alpha} & w & 0 \end{array}\right)$$

Suppose: $z_j^{(1)} \in \alpha\mathbb{Q}_l \ (j \neq 1)$

$\sum_{j=1}^{r} Z_j^{(k)} + W = 0 \ (k = 1, 2)$

$\Rightarrow w \in \alpha\mathbb{Q}_l, \ z_1^{(2)} \in \alpha\mathbb{Q}_l$

Since $\mathrm{rank}_{\mathbb{Z}_l} \mathfrak{h}^{\mathrm{ab}} \geq 2$, we obtain a contradiction.

$$\begin{array}{cccc|cc} \alpha & \boxed{\alpha} & \cdots & \boxed{\alpha} & 0 & w \\ z_1^{(2)} & \boxed{\alpha} & \cdots & \boxed{\alpha} & w & 0 \end{array} \Rightarrow \begin{array}{cccc|cc} \alpha & \boxed{\alpha} & \cdots & \boxed{\alpha} & 0 & \boxed{\alpha} \\ \boxed{\alpha} & \boxed{\alpha} & \cdots & \boxed{\alpha} & \boxed{\alpha} & 0 \end{array}$$

We may assume: $\beta := z_2^{(1)} \notin \alpha\mathbb{Q}_l$.

By the above argument, $z_j^{(2)} \in \beta\mathbb{Q}_l \ (j \neq 2)$.

In particular, $z_j^{(2)} \in \alpha\mathbb{Q}_l \cap \beta\mathbb{Q}_l = \{0\} \ (j \neq 1, 2)$.

$$\left(\begin{array}{ccccc|cc} \alpha & \beta & z_3^{(1)} & \cdots & z_r^{(1)} & 0 & w \\ z_1^{(2)} & \boxed{\alpha} & \boxed{\alpha} & \cdots & \boxed{\alpha} & w & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc|cc} \alpha & \beta & z_3^{(1)} & \cdots & z_r^{(1)} & 0 & w \\ \boxed{\beta} & \boxed{\alpha} & 0 & \cdots & 0 & w & 0 \end{array}\right)$$

Put $z_2^{(2)} = a\alpha,\ z_1^{(2)} = b\beta$.

$\sum_{j=1}^{r} Z_j^{(k)} + W = 0\ (k = 1, 2)$

$\Rightarrow w = -a\alpha - b\beta,\ \sum_{j=3}^{r} z_j^{(k)} = (a-1)\alpha + (b-1)\beta$

$$\left.\begin{array}{ccccc} \alpha & \beta & z_3^{(1)} & \cdots & z_r^{(1)} \\ b\beta & a\alpha & 0 & \cdots & 0 \end{array}\right| \begin{array}{cc} 0 & w \\ w & 0 \end{array}$$

$$\Downarrow$$

$$\left.\begin{array}{ccccc} \alpha & \beta & z_3^{(1)} & \cdots & z_r^{(1)} \\ b\beta & a\alpha & 0 & \cdots & 0 \end{array}\right| \begin{array}{cc} 0 & -a\alpha - b\beta \\ -a\alpha - b\beta & 0 \end{array}$$

If $a \neq 0$, then

$[Z_j^{(1)}, Z_{j'}^{(2)}] = 0 \ (j \neq j')$ implies $z_j^{(1)} \in \alpha\mathbb{Q}_l \ (j \geq 3)$

$\sum_{j=3}^{r} z_j^{(k)} = (a-1)\alpha + (b-1)\beta$ implies $b = 1$

Similarly, $b \neq 0 \Rightarrow a = 1$

$\Rightarrow (a, b) = (0, 0), (1, 1)$

If $(a, b) = (1, 1)$, then

$[Z_j^{(1)}, Z_{j'}^{(2)}] = 0 \ (j \neq j')$

implies $z_j^{(1)} \in \alpha\mathbb{Q}_l \cap \beta\mathbb{Q}_l = \{0\} \ (j \neq 1, 2)$.

$$\left(\begin{array}{ccccc|cc} \alpha & \beta & z_3^{(1)} & \cdots & z_r^{(1)} & 0 & -\alpha-\beta \\ \beta & \alpha & 0 & \cdots & 0 & -\alpha-\beta & 0 \end{array}\right)$$

$$\Downarrow$$

$$\left(\begin{array}{ccccc|cc} \alpha & \beta & 0 & \cdots & 0 & 0 & -\alpha-\beta \\ \beta & \alpha & 0 & \cdots & 0 & -\alpha-\beta & 0 \end{array}\right)$$

Conclusion

$\mathrm{Gr}(\Pi_2^l) \twoheadrightarrow \mathfrak{h}$ is one of the following forms:

$\frac{\text{Type } 1}{\text{(Corr. to } {}^{\exists}\text{proj)}}$

$$\begin{array}{cccc|cc} * & * & \cdots & * & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{array}$$

$\frac{\text{Type } 2}{\text{(Corr. to } {}^{\exists}\text{excep)}}$

$$\begin{array}{ccccc|cc} \alpha & \beta & 0 & \cdots & 0 & 0 & -\alpha-\beta \\ \beta & \alpha & 0 & \cdots & 0 & -\alpha-\beta & 0 \end{array}$$

□

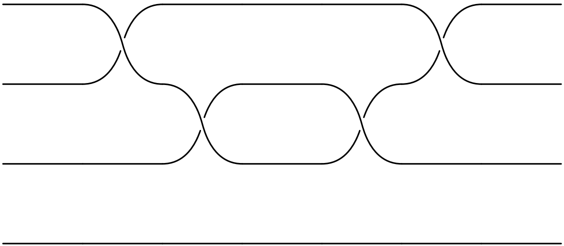
§3 Pure braids

a point of $X_n(/\mathbb{C})$ $\leftrightarrow$ ordered n distinct points of X

$g \in \pi_1^{\mathrm{top}}(X_n) =: \Pi_n$ $\leftrightarrow$ pure braid on n strands on X

$p_k : \Pi_n \twoheadrightarrow \Pi_{n-1}$ $\leftrightarrow$ forgetting the k-th strand

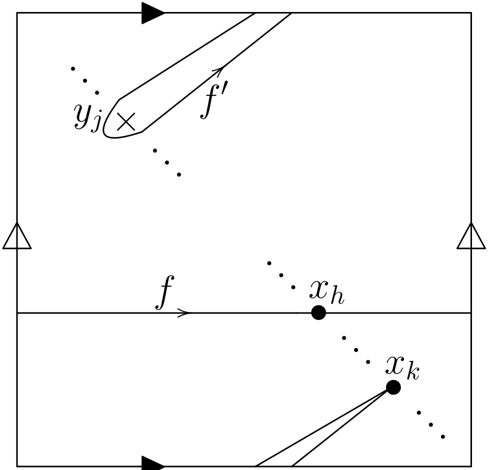
(Π_n^{Σ}: identified with the maximal pro-Σ completion of Π_n)



$$
\begin{aligned}
\ker p_k = \langle \alpha_1^{(k)}, \ldots, \alpha_g^{(k)}, \beta_1^{(k)}, \ldots, \beta_g^{(k)}, \\
\gamma_1^{(k)}, \ldots, \gamma_r^{(k)}, \delta_1^{(k)}, \ldots, \delta_n^{(k)} \mid \delta_k^{(k)} = 1, \\
\textstyle\prod_{i=1}^{g} [\alpha_i^{(k)}, \beta_i^{(k)}] \prod_{j=1}^{r} \gamma_j^{(k)} \prod_{h=1}^{n} \delta_h^{(k)} = 1 \rangle
\end{aligned}
$$

Lemma 2

If $k \neq h$ and $j \neq j'$, then $\alpha_i^{(h)}$ (resp. $\beta_i^{(h)}$, $\gamma_{j'}^{(h)}$) commutes with some conjugate of $\gamma_j^{(k)}$.



$(g = 1$, $(x_1, \ldots, x_n)$: basepoint, $y_1, \ldots, y_r$: cusps, f, f' represent some conjugate of $\alpha_1^{(h)}$, $\gamma_j^{(k)}$)

Theorem B (S.)

Suppose: $g > 0$.

H: pro-l surface group, $\varphi : \Pi_n^l \twoheadrightarrow H$,

$\varphi(\gamma_r^{(k)}) \neq 1$ for ${}^{\exists}k$

$\Rightarrow {}^{\forall}h \neq k \quad \ker p_h \subset \ker \varphi$

(Since $\langle \ker p_h | h \neq k \rangle \in \mathrm{FS}_1(\Pi_n^l)$, to prove

Main Theorem, we can reduce to the case $r \leq 1$.)

Proof of Theorem B:

Fact (Suppose: $g > 0$)

Write $A := \{\alpha_1^{(h)}, \ldots, \alpha_g^{(h)}, \beta_1^{(h)}, \ldots, \beta_g^{(h)}, \gamma_1^{(h)}, \ldots, \gamma_{r-1}^{(h)}\}$,
$J := \overline{\langle A \rangle} \subset \ker p_h (\subset \Pi_n^l)$.

Then $\mathrm{Im}(J \to \Pi_n^{l,\mathrm{ab}}) = \mathrm{Im}(\ker p_h \to \Pi_n^{l,\mathrm{ab}})$

By Lemma 2, ${}^\forall a \in A \ \ {}^\exists b_a \in H$ s.t.

$\varphi(a)$ commutes with $b_a\varphi(\gamma_r^{(k)})b_a^{-1}(\neq 1)$.

$\Rightarrow {}^\exists c_a \in \mathbb{Q}_l$ s.t. $\varphi(a) = (b_a\varphi(\gamma_r^{(k)})b_a^{-1})^{c_a}$

$\Rightarrow \varphi^{\mathrm{ab}}(\ker p_h) = \varphi^{\mathrm{ab}}(J) \subset \varphi^{\mathrm{ab}}(\gamma_r^{(k)})\mathbb{Q}_l$

$(\varphi^{\mathrm{ab}} : \Pi_n^l \overset{\varphi}{\twoheadrightarrow} H \twoheadrightarrow H^{\mathrm{ab}})$

In particular, $\varphi(\ker p_h) \subset H$: not open

Since any top. fin. gen. normal closed subgroup of H is open or trivial,

$\varphi(\ker p_h) = \{1\}$, i.e., $\ker p_h \subset \ker \varphi$. □

§4 Application

Definition (hyperbolic polycurve)

S: scheme, Z: scheme$/S$

Z/S: hyperbolic polycurve (of dim. n)

$\overset{\text{def}}{\Leftrightarrow}$ ${}^{\exists}$a sequence $Z = Z_{(n)} \to \cdots \to Z_{(1)} \to Z_{(0)} = S$

s.t. ${}^{\forall}i \;\; Z_{(i)}/Z_{(i-1)}$: relative hyperbolic curve

Example X_n/K is a hyperbolic polycurve.

(${}^{\exists}$a seq. $X_n \to X_{n-1} \to \cdots X_1 \to X_0 = \operatorname{Spec} K$

determined by (generalized) projections)

As an application of Main Theorem, we obtain a Grothendieck's anabelian conjecture-type result:

Theorem C (S.)

Suppose:

- K is generalized sub-l-adic (i.e., $K \hookrightarrow {}^{\exists}L/\operatorname{Frac}(W(\overline{\mathbb{F}_l}))$: fin. gen.).
- $g > 0$. (g: genus of X)

Let Z/K: hyperbolic polycurve over $\operatorname{Spec} K$.
Then the natural map

$$\operatorname{Isom}_K(X_n, Z) \to \operatorname{Isom}_{G_K}(\pi_1(X_n), \pi_1(Z))/\operatorname{Inn}(\pi_1(Z \times_K \overline{K}))$$

is bijective.

Sketch of the proof of Theorem C:

We can show:

Lemma 3 (S.)

Suppose: $g = 1$ and $r \geq 2$.

A: finite nonzero Σ-torsion Π_n^Σ-module, $N \in \mathrm{ES}(\Pi_n^\Sigma)$

$\Rightarrow H^n(N, A)$: infinite.

$$N := \ker(\underbrace{\pi_1(X_n \times_K \overline{K})}_{\Pi_n^{\mathrm{prof}}} \xrightarrow{\sim} \pi_1(Z \times_K \overline{K}) \twoheadrightarrow \pi_1(Z_{(1)} \times_K \overline{K}))$$

↑ arises from $\pi_1(X_n) \xrightarrow{\sim}_{G_K} \pi_1(Z)$

By Main Theorem, we can show:

$N \in \mathrm{FS}_1(\pi_1(X_n \times_K \overline{K})) \cup \mathrm{ES}(\pi_1(X_n \times_K \overline{K}))$.

Moreover, by Lemma 3,

$N \notin \mathrm{ES}(\pi_1(X_n \times_K \overline{K}))$ if $g = 1$ and $r \geq 2$.

Thus, $N \in \mathrm{GFS}_1(\pi_1(X_n \times_K \overline{K}))$.

Theorem C follows from induction and GC for hyperbolic curves/gen. sub-l-adic field (Mochizuki). □

Remark

(i) Grothendieck's anabelian conjecture for

・(Hoshi) hyp. polycurves of $\dim \leq 4$ (over a sub-l-adic field, i.e., a subfield of a fin. gen. ext. of $\mathbb{Q}_l$),

・(Schmidt-Stix) strongly hyperbolic Artin neighborhoods (over a fin. gen. ext. over $\mathbb{Q}$)

hold.

(ii)(S.) The isomorphism class of a hyp. polycurve (over a gen. sub-l-adic field K) is determined by $\pi_1(X) \twoheadrightarrow G_K$ up to finitely many possibilities.