DEVELOPMENTS OF ANABELIAN GEOMETRY OF CURVES OVER FINITE FIELDS

Akio Tamagawa

June 28, 2021

Abstract. This is a survey talk on anabelian geometry of curves over finite fields. It will cover various topics, from Uchida's theorem for function fields in 1970s to several recent developments.

Contents

§0. Introduction.

0.1. Fundamental groups.

S: a connected scheme
$\xi : \mathrm{Spec}(\Omega) \to S$: a geometric point ($\Omega$: a separably closed field)
$\Longrightarrow \pi_1(S) = \pi_1(S, \xi)$: a profinite group

F: a field
S: a geometrically connected F-scheme
$\Longrightarrow 1 \to \pi_1(S_{\overline{F}}) \to \pi_1(S) \stackrel{\mathrm{pr}}{\to} G_F \to 1$: exact
$G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F) = \pi_1(\mathrm{Spec}(F))$: the absolute Galois group of F
$\pi_1(S)$: called the arithmetic fundamental group
$\pi_1(S_{\overline{F}})$: called the geometric fundamental group

Quotients

$\mathfrak{Primes}$: the set of prime numbers

Γ: a profinite group

Γ^*: a characteristic quotient of Γ (referred to as (maximal) $*$ quotient), e.g.,

$$* = \begin{cases} \text{pro-}\Sigma \text{ [maximal pro-}\Sigma\text{ quotient] } (\Sigma \subset \mathfrak{Primes}), \\ \text{pro-}l = \text{pro-}\{l\} \ (l \in \mathfrak{Primes}), \\ \text{pro-}l' = \text{pro-}(\mathfrak{Primes} \setminus \{l\}) \ (l \in \mathfrak{Primes}), \\ \text{ab [abelianization, i.e. maximal abelian quotient]}, \\ \text{solv [maximal prosolvable quotient]}, \\ m\text{-solv [maximal } m\text{-step solvable quotient] } (m \geq 0), \\ \text{etc.} \end{cases}$$

$1 \to \overline{\Pi} \to \Pi \to G \to 1$: an exact sequence of profinite groups

$\Pi^{(*)} := \Pi / \operatorname{Ker}(\overline{\Pi} \twoheadrightarrow \overline{\Pi}^*)$

$\Longrightarrow \ 1 \to \overline{\Pi}^* \to \Pi^{(*)} \to G \to 1$: exact

Apply this to $1 \to \pi_1(S_{\overline{F}}) \to \pi_1(S) \to G_F \to 1$. Then

$\pi_1(S)^{(*)}$: called the maximal geometrically $*$ quotient of $\pi_1(S)$

0.2. Anabelian geometry (AG).

Grothendieck conjecture (GC): For an "anabelian scheme" S, (the isomorphism class of) S can be recovered group-theoretically from $\pi_1(S)$.

Mono-anabelian/bi-anabelian/weak bi-anabelian geometry
- Mono-AG: A purely group-theoretic algorithm for reconstructing (a scheme isomorphic to) S starting from $\pi_1(S)$ exists (or can be constructed).
- Bi-AG: For S_1, S_2, and an isomorphism $\pi_1(S_1) \xrightarrow{\sim} \pi_1(S_2)$, there exists an (a unique) isomorphism $S_1 \xrightarrow{\sim} S_2$ that induces the isomorphism $\pi_1(S_1) \xrightarrow{\sim} \pi_1(S_2)$ up to conjugacy. Namely, the natural map $\mathrm{Isom}(S_1, S_2) \to \mathrm{Isom}(\pi_1(S_1), \pi_1(S_2))/\mathrm{Inn}(\pi_1(S_2))$ is a bijection.
- Weak bi-AG: For S_1, S_2, if $\pi_1(S_1) \simeq \pi_1(S_2)$, then $S_1 \simeq S_2$.

In this talk, we ignore the difference between mono/bi-AG and write $\pi_1(S) \rightsquigarrow S$ for the mono/bi-anabelian results, while we write $\pi_1(S) \rightsquigarrow [S]$ for the weak bi-anabelian results.

Absolute/semi-absolute/relative anabelian geometry
- Absolute AG: $\pi_1(S) \rightsquigarrow S$ or $[S]$
- Semi-absolute AG: $(\pi_1(S), \pi_1(S_{\overline{F}})) \rightsquigarrow S$ or $[S]$
- Relative AG: F being fixed, $(\pi_1(S) \twoheadrightarrow G_F) \rightsquigarrow S$ or $[S]$

In this talk, we ignore the difference among absolute/semi-absolute/relative AG.

0.3. What are treated in this talk.

Contents (bis)

0.4. What are not treated in this talk.

- Number fields and integer rings (Neukirch, Ikeda, Iwasawa, Uchida, Hoshi, Ivanov, Saïdi, T, Shimizu, ...)
- Curves over algebraic closures of finite fields (Pop, Saïdi, Raynaud, T, Sarashina, Yang, ...)
- Curves over fields finitely generated over finite fields (Stix, Yamaguchi, ...)
- Curves over power series fields over finite fields (...)
- Curves over fields of characteristic 0 (Nakamura, T, Mochizuki, Hoshi, Tsujimura, Lepage, Porowski, Murotani, ...)
- Higher-dimensional varieties over finite fields (...)
- Function fields of several variables over finite fields (Bogomolov, Pop, Saïdi, T, ...)
- etc.

0.5. Notation/terminology.

From now on, we use the following notation/terminology:

- k: a finite field
- p: the characteristic of k
- q: the cardinality $|k|$ of k
- A curve: a scheme smooth, geometrically connected, separated and of dimension 1 over a field (except for "stable curve" in §3)
- S^{cl}: the set of closed points of a scheme S
- X: a curve over k
- X^{cpt}: the smooth compactification of X
- g: the genus of X^{cpt}
- r: the cardinality of $(X^{\mathrm{cpt}} \smallsetminus X)(\overline{k})$

(X: hyperbolic/affine/proper $\iff 2g-2+r>0/r>0/r=0$)

- $K = k(X)$: the function field of X
- $\mathrm{Sub}(\Gamma)$: the set of closed subgroups of a profinite group Γ
- $\mathrm{OSub}(\Gamma)$: the set of open subgroups of a profinite group Γ

§1. Birational AG (Uchida's theorem).

The following is the beginning of the history of AG of curves over finite fields (with the Neukirch-Uchida theorem for number fields as a pre-history).

Theorem [Uchida 1977]. $G_K \rightsquigarrow K$.

Outline of proof. Here, we may assume $X = X^{\mathrm{cpt}}$.

Step 1. Local theory and characterization of various invariants

1-1. Decomposition groups D_x ($x \in X^{\mathrm{cl}}$)

Show the separatedness, i.e. the injectivity of the map $X^{\mathrm{cl}} \twoheadrightarrow \mathrm{Dec}(G_K)/\mathrm{Inn}(G_K) \subset \mathrm{Sub}(G_K)/\mathrm{Inn}(G_K)$, $x \mapsto D_x$, and characterize the subset $\mathrm{Dec}(G_K) \subset \mathrm{Sub}(G_K)$ group-theoretically: For $D \in \mathrm{Sub}(G_K)$, $D \in \mathrm{Dec}(G_K) \iff D$ is a maximal element of $\{H \in \mathrm{Sub}(G_K) \mid \exists l \in \mathfrak{Primes}, \exists H_0 \in \mathrm{OSub}(H), \text{ s.t.} \forall H' \in \mathrm{OSub}(H_0),\ H^2(H', \mathbb{F}_l) \simeq \mathbb{F}_l\}$.

The proof of this step resorts to the local-global principle for Brauer groups.

1-2. The characteristic p

For $l \in \mathfrak{Primes}$, $l = p \iff \mathrm{cd}_l(G_K) = 1$

1-3. The cyclotomic character $\chi_{\mathrm{cycl}} : G_K \to (\hat{\mathbb{Z}}^{\text{pro-}p'})^\times$

For each $x \in X^{\mathrm{cl}}$, $\chi_{\mathrm{cycl}}|_{D_x}$ is the character associated to the conjugacy action of D_x on $\mathrm{Ker}(D_x \twoheadrightarrow D_x^{\mathrm{ab}})^{\text{ab, pro-}p'}$ ($\simeq \hat{\mathbb{Z}}^{\text{pro-}p'}$). Use this and Chebotarev: $G_K = \overline{\langle D_x \mid x \in X^{\mathrm{cl}} \rangle}$.

1-4. Inertia groups I_x, wild inertia groups I_x^{wild}, cardinality q_x of residue fields $k(x)$, and Frobenius elements Frob_x ($x \in X^{\mathrm{cl}}$)

$I_x = \mathrm{Ker}(\chi_{\mathrm{cycl}}|_{D_x})$, I_x^{wild} is a unique pro-p-Sylow subgroup of I_x, $q_x = |(D_x^{\mathrm{ab}})_{\mathrm{tor}}| + 1$, and $\mathrm{Frob}_x \in D_x/I_x$ is characterized by $\chi_{\mathrm{cycl}}(\mathrm{Frob}_x) = q_x \in (\hat{\mathbb{Z}}^{\text{pro-}p'})^\times$.

Step 2. Multiplicative groups

2-1. Local multiplicative groups $K_x^\times \supset O_x^\times \supset U_x \stackrel{\mathrm{def}}{=} \mathrm{Ker}(O_x^\times \twoheadrightarrow k(x)^\times)$ $(x \in X^{\mathrm{cl}})$

$K_x^\times$ is the inverse image of $\langle \mathrm{Frob}_x \rangle \subset D_x/I_x$ in D_x^{ab}, $O_x^\times = \mathrm{Im}(I_x \to D_x^{\mathrm{ab}})$, and $U_x = \mathrm{Im}(I_x^{\mathrm{wild}} \to D_x^{\mathrm{ab}})$ (local class field theory). Further, the natural map $\mathrm{ord}_x : K_x^\times \to \mathbb{Z}$ is characterized by $\mathrm{ord}_x(O_x^\times) = \{0\}$ and $\mathrm{ord}_x(\mathrm{Frob}_x) = 1$.

2-2. Global multiplicative group $K^\times$

$K^\times = \mathrm{Ker}((\prod'_{x \in X^{\mathrm{cl}}} K_x^\times) \to G_K^{\mathrm{ab}})$ (global class field theory). Further, for each $x \in X^{\mathrm{cl}}$, $\mathrm{ord}_x = \mathrm{ord}_x |_{K^\times}$, $\mathcal{O}_{X,x}^\times = K^\times \cap O_x^\times$, and $U_{X,x} (\stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathcal{O}_{X,x}^\times \twoheadrightarrow k(x)^\times)) = K^\times \cap U_x$.

Step 3. Additive structure on $K = K^\times \cup \{0\}$

Uchida's lemma.
$(K^\times, \cdot, X^{\mathrm{cl}}, (\mathrm{ord}_x)_{x \in X^{\mathrm{cl}}}, (U_{X,x})_{x \in X^{\mathrm{cl}}})$ *(for all constant field extensions of K)* $\rightsquigarrow (K, +)$

Proof.
- Additive structure on the constant field k: Consider minimal functions, i.e elements of $K \smallsetminus k$ with degree of poles minimal, and evaluate them at three points.
- Additive structure on the residue fields $k(x)$ $(x \in X^{\mathrm{cl}})$: Identify the residue field with the constant field (after a constant field extension).
- Additive structure on K: Use reductions. □ □

§2. AG.

Theorem [T 1997] (for $r > 0$) [Mochizuki 2007] (for $r = 0$).
(i) *If $r > 0$ or $2g - 2 + r > 0$, $\pi_1(X) \rightsquigarrow X$.*
(ii) *If $2g - 2 + r > 0$, $\pi_1^{\mathrm{tame}}(X) \rightsquigarrow X$.*

Outline of proof. For simplicity, we only treat (i).
Step 1. Local theory and characterization of various invariants
1-1. The quotient $\pi_1(X) \twoheadrightarrow G_k$ and the geometric fundamental groups $\pi_1(X_{\overline{k}})$
The p'-part $\pi_1(X) \twoheadrightarrow G_k^{\text{pro-}p'} (\simeq \hat{\mathbb{Z}}^{\text{pro-}p'})$ of the quotient $\pi_1(X) \twoheadrightarrow G_k (\simeq \hat{\mathbb{Z}})$ is identified with $\pi_1(X) \twoheadrightarrow \pi_1(X)^{\text{ab, pro-}p'}/(\text{torsion})$. For the p-part $\pi_1(X) \twoheadrightarrow G_k^{\text{pro-}p} (\simeq \mathbb{Z}_p)$, we resort to Iwasawa theory for ($\mathbb{Z}_p$-extensions of) function fields (details omitted). Further, $\pi_1(X_{\overline{k}}) = \mathrm{Ker}(\pi_1(X) \twoheadrightarrow G_k)$.
1-2. The characteristic p
For $l \in \mathfrak{Primes}$, $l = p \iff \pi_1(X_{\overline{k}})^{\text{ab, pro-}l'}$ is a free $\hat{\mathbb{Z}}^{\text{pro-}l'}$-module.
1-3. The invariant $\varepsilon \in \{0, 1\}$
Set $\varepsilon = 0$ (resp. 1) if $r > 0$ (resp. $r = 0$). Then $\varepsilon = 1 \iff \pi_1(X)$ is finitely generated.
1-4. The Frobenius element $\mathrm{Frob} \in G_k$
Set $M := \pi_1(X_{\overline{k}})^{\text{ab, pro-}p'}$. Then the character χ associated to the G_k-module $(M^{\wedge \max})^{\otimes 2}$ $(\simeq \hat{\mathbb{Z}}^{\text{pro-}p'})$ is $\chi_{\text{cycl}}^{2(g+r-\varepsilon)}$, where $\chi_{\text{cycl}} : G_k \to (\hat{\mathbb{Z}}^{\text{pro-}p'})^\times$ is the cyclotomic character. For $F \in G_k$, $F = \mathrm{Frob} \iff \chi(F) = \min(p^{\mathbb{Z}_{>0}} \cap \mathrm{Im}(\chi))$ $(= q^{2(g+r-\varepsilon)})$.

1-5. The cardinality q of k

Let A be the set of complex absolute values of eigenvalues of Frob acting on the free $\hat{\mathbb{Z}}^{\text{pro-}p'}$-module M. If $\varepsilon = 1$, then $A = \{q^{1/2}\}$. If $\varepsilon = 0$, then (possibly after replacing X by a suitable cover) $A = \{q^{1/2}, q\}$. This characterizes q.

1-6. Characterization of decomposition groups D_x $(x \in (X^{\text{cpt}})^{\text{cl}})$

First, assume $r = 0$. Show the separatedness, i.e. the injectivity of the map $X^{\text{cl}} \twoheadrightarrow \text{Dec}(\pi_1(X))/\operatorname{Inn}(\pi_1(X)) \subset \text{Sub}(\pi_1(X))/\operatorname{Inn}(G_K)$, $x \mapsto D_x$, and characterize the subset $\text{Dec}(\pi_1(X)) \subset \text{Sub}(\pi_1(X))$ group-theoretically: For $D \in \text{Sub}(\pi_1(X))$, $D \in \text{Dec}(\pi_1(X))$ $\iff$ D is a maximal element of
$\{Z \in \text{Sub}(\pi_1(X)) \mid Z \cap \pi_1(X_{\overline{k}}) = \{1\},\ \text{pr}(Z) \in \text{OSub}(G_k),\ \text{and } \forall' l \in \mathfrak{Primes},\ \forall H \in \text{OSub}(\pi_1(X)) \text{ containing } Z,\ 1 + q^{n_Z} - \text{tr}(\text{Frob}^{n_Z} \mid \overline{H}^{\text{ab, pro-}l}) \in \mathbb{Z}_{>0}\}$, where $n_Z = (G_k : \text{pr}(Z))$, $\overline{H} = H \cap \pi_1(X_{\overline{k}})$. The proof of this fact resorts to the Lefscetz trace formula for étale cohomology. For $r > 0$, we consider the compactification of the cover corresponding to the above H (details omitted).

1-7. Inertia groups I_x, wild inertia groups I_x^{wild}, cardinality q_x of residue fields $k(x)$, and Frobenius elements Frob_x $(x \in (X^{\text{cpt}})^{\text{cl}})$

For each $x \in (X^{\text{cpt}})^{\text{cl}}$, $I_x = D_x \cap \pi_1(X_{\overline{k}})$, I_x^{wild} is a unique pro-p-Sylow subgroup of I_x, $q_x = q^{(G_k:\text{pr}(D_x))}$, and $\text{Frob}_x \in D_x/I_x$ is characterized by $\text{pr}(\text{Frob}_x) = \text{Frob}^{(G_k:\text{pr}(D_x))}$.

Step 2. Multiplicative groups (for $r > 0$)

2-1. Local multiplicative groups $K_x^\times \supset O_x^\times \supset U_x$ $(x \in (X^{\text{cpt}})^{\text{cl}})$

For $x \in X^{\mathrm{cl}}$, $K_x^\times/O_x^\times = \langle \mathrm{Frob}_x \rangle \subset D_x$, and the natural map $\mathrm{ord}_x : K_x^\times/O_x^\times \to \mathbb{Z}$ is characterized by $\mathrm{ord}_x(\mathrm{Frob}_x) = 1$. For $x \in X^{\mathrm{cpt}} \smallsetminus X$, $K_x^\times$ is the inverse image of $\langle \mathrm{Frob}_x \rangle \subset D_x/I_x$ in D_x^{ab}, $O_x^\times = \mathrm{Im}(I_x \to D_x^{\mathrm{ab}})$, and $U_x = \mathrm{Im}(I_x^{\mathrm{wild}} \to D_x^{\mathrm{ab}})$ (local class field theory). Further, the natural map $\mathrm{ord}_x : K_x^\times \to \mathbb{Z}$ is characterized by $\mathrm{ord}_x(O_x^\times) = \{0\}$ and $\mathrm{ord}_x(\mathrm{Frob}_x) = 1$.

<u>2-2. Global multiplicative group $K^\times$</u>

$K^\times = \mathrm{Ker}((\prod'_{x \in (X^{\mathrm{cpt}})^{\mathrm{cl}}} W_x) \to G_K^{\mathrm{ab}})$, where $W_x = K_x^\times/O_x^\times$ $(x \in X^{\mathrm{cl}})$, $K_x^\times$ $(x \in X^{\mathrm{cpt}} \smallsetminus X)$ (global class field theory). Here, we have used $r > 0$. Further, $\mathrm{ord}_x = \mathrm{ord}_x |_{K^\times}$ for each $x \in (X^{\mathrm{cpt}})^{\mathrm{cl}}$ and $U_{X^{\mathrm{cpt}},x} = K^\times \cap U_x$ for each $x \in X^{\mathrm{cpt}} \smallsetminus X$ are recovered.

<u>Step 3. Additive structure on $K = K^\times \cup \{0\}$ (for $r > 0$)</u>

By replacing X with a suitable cover if necessary, we may assume $r \geq 3$. Then we may resort to the following strengthening of Uchida's lemma.

Lemma.
$(K^\times, \cdot, (X^{\mathrm{cpt}})^{\mathrm{cl}}, (\mathrm{ord}_x)_{x \in (X^{\mathrm{cpt}})^{\mathrm{cl}}}, (U_{X^{\mathrm{cpt}},x})_{x \in X^{\mathrm{cpt}} \smallsetminus X})$ *(for all constant field extensions of K)* $\rightsquigarrow (K, +)$

Step 4. Cuspidalizations (for $r = 0$)
Roughly speaking, the $r = 0$ case can be treated by reducing the problem to the $r > 0$ case (or, even to the function field case in §1). More precisely, Mochizuki's theory of cuspidalizations imply:

$(\pi_1(X), S \subset X^{\mathrm{cl}} = \mathrm{Dec}(\pi_1(X))/\mathrm{Inn}(\pi_1(X)), |S| < \infty) \rightsquigarrow \pi_1(X \setminus S)^{\text{c-ab}} \twoheadrightarrow \pi_1(X)$,

$(\pi_1(X), S \subset X^{\mathrm{cl}} = \mathrm{Dec}(\pi_1(X))/\mathrm{Inn}(\pi_1(X)), |S_{\overline{k}}| = 1,) \rightsquigarrow \pi_1(X \setminus S)^{\text{c-pro-}l} \twoheadrightarrow \pi_1(X)$ $(l \in \mathfrak{Primes} \setminus \{p\})$,

which are compatible in a certain sense. Here, setting $J_S = \mathrm{Ker}(\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X))$, $\pi_1(X \setminus S)^{\text{c-ab}} \stackrel{\text{def}}{=} \pi_1(X \setminus S)/\mathrm{Ker}(J_S \to J_S^{\text{ab, pro-}p'})$ and $\pi_1(X \setminus S)^{\text{c-pro-}l} \stackrel{\text{def}}{=} \pi_1(X \setminus S)/\mathrm{Ker}(J_S \to J_S^{\text{pro-}l})$ are called the maximal cuspidally abelian (pro-p') and maximal cuspidally pro-l quotients of $\pi_1(X \setminus S)$, respectively.

Note that $(\pi_1(X \setminus S)^{\text{c-ab}})^{\mathrm{ab}} = \pi_1^{\mathrm{tame}}(X \setminus S)^{\mathrm{ab}}$. The multiplicative group $K^\times$ (equipped with $(\mathrm{ord}_x)_{x \in X^{\mathrm{cl}}}$) is constructed from $\pi_1(X \setminus S)^{\text{c-ab}} \twoheadrightarrow \pi_1(X)$ (via Kummer theory), and the other data needed to apply (the strengthening of) Uchida's lemma are constructed by using $\pi_1(X \setminus S)^{\text{c-pro-}l} \twoheadrightarrow \pi_1(X)$. Now, (the strengthening of) Uchida's lemma finishes the proof. □

§3. Log AG.

Let $\mathrm{Spec}(k)^{\log}$ (or simply $k^{\log}$) be the log scheme whose underlying scheme is $\mathrm{Spec}(k)$ and whose log structure is (isomorphic to) the one associated to the chart $\mathbb{N} \to k$ given by the zero map. (Equivalently, the log structure is obtained by pulling back the log structure on $\mathrm{Spec}(W(k))$ given by the divisor $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(W(k))$.) Set $G_{k^{\log}} = \pi_1(\mathrm{Spec}(k)^{\log})$ (which is identified with $G^{\mathrm{tame}}_{\mathrm{Frac}(W(k))}$).

Let $X^{\log}$ be a proper stable log-curve over $k^{\log}$ such that X is not smooth over k.

Theorem [Mochizuki 1996]. $\pi_1(X^{\log})$ *(or, more precisely,* $\pi_1(X^{\log}) \twoheadrightarrow G_{k^{\log}}$*)* $\rightsquigarrow X^{\log}$

Outline of proof. Combinatorial-anabelian-geometric arguments + [T 1997].
Step 1. Show $\pi_1(X^{\log}) \rightsquigarrow$ the set I of irreducible components of X.
Step 2. Show $\pi_1(X^{\log}) \rightsquigarrow \pi_1^{\mathrm{tame}}(Y^{\mathrm{sm}})$ $(Y \in I)$.
Step 3. Show $\pi_1(X^{\log}) \rightsquigarrow$ the set N of nodes of X.
Step 4. Show $\pi_1(X^{\log}) \rightsquigarrow$ the dual graph of X (whose set of vertices is I and whose set of edges is N).
Step 5. Show $\pi_1(X^{\log}) \rightsquigarrow$ the log structure at each $y \in N$.
Step 6. End of proof. For each $Y \in I$, apply [T 1997] to $\pi_1^{\mathrm{tame}}(Y^{\mathrm{sm}})$ to recover Y^{sm}. Reconstruct $X^{\log}$ from $\{Y\}_{Y \in I}$ according to the recipe given by Steps 4 and 5. □

§4. Pro-Σ AG.

$\Sigma \subset \mathfrak{Primes}$. $\Sigma' \stackrel{\text{def}}{=} \mathfrak{Primes} \smallsetminus \Sigma$.
A: a semi-abelian variety over k.
- Σ is A-large $\iff$ the Σ'-adic representation $G_k \to \prod_{l \in \Sigma'} \mathrm{GL}(T_l(A))$ is not injective.
- Σ satisfies (ϵ_A) $\stackrel{\text{def}}{\iff}$ $\forall k'/k$, $[k' : k] < \infty$, $\exists k''/k'$, $[k'' : k'] < \infty$, s.t. $2|A(k'')\{\Sigma'\}| < |k''|$.

Lemma. *Assume* $\dim(A) > 0$. *Consider the following conditions:*
(i) Σ *is cofinite, i.e.* Σ' *is finite.*
(ii) Σ *is* A*-large.*
(iii) Σ *is* $(\mathbb{G}_m)_k$*-large and satisfies* (ϵ_A)*.*
(iv) Σ *is* $(\mathbb{G}_m)_k$*-large.*
(v) Σ *is infinite.*
Then (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v).

Theorem 1 [Saïdi-T 2009$_{1,2}$] (for $\Sigma = \mathfrak{Primes} \smallsetminus \{p\}$) [Saïdi-T 2017] (general). *Assume* $X = X^{\mathrm{cpt}}$ *and that* Σ *is* $(\mathbb{G}_m)_k$*-large and satisfies* (ϵ_{J_X})*. Then* $G_K^{(\text{pro-}\Sigma)} \rightsquigarrow K$*.*

Theorem 2 [Saïdi-T 2009$_1$] (for $\Sigma = \mathfrak{Primes} \smallsetminus \{p\}$) [Saïdi-T 2018] (general). *Assume* $\exists X'$ *a finite étale cover of* X *such that* $(X')^{\mathrm{cpt}}$ *is hyperbolic (i.e. of genus* ≥ 2*) and that* Σ *is* $J_{(X')^{\mathrm{cpt}}}$*-large. Then* $\pi_1(X)^{(\text{pro-}\Sigma)} \rightsquigarrow X$*.*

Outline of proof of Theorem 1.
Step 1. Local theory and characterization of various invariants
Similar to [Uchida 1977].
Step 2. Multiplicative groups
2-1. Local multiplicative groups
Similar to [Uchida 1977] (local class field theory). But we only get various local multiplicative groups with the unit group $O_x^\times$ replaced by $(O_x^\times)^{\text{pro-}\Sigma}$.
2-2. Global multiplicative groups
Similar to [Uchida 1977] (global class field theory). But we only get $(K^\times)^{(\Sigma)} \stackrel{\text{def}}{=} K^\times/(k^\times\{\Sigma'\})$ instead of $K^\times$. Here, we use the $(\mathbb{G}_m)_k$-largeness and (ϵ_{J_X}).
Step 3. Additive structure
As the constant field is not available fully, we cannot resort to Uchida's lemma. Instead, we apply the fundamental theorem of projective geometry to the infinite-dimensional projective space $K^\times/k^\times = (K^\times)^{(\Sigma)}/(\text{torsion})$ over k. For this, we regard $(K^\times)^{(\Sigma)}$ as the set of "pseudo-functions" with values in $(k(x)^\times)^\Sigma$ instead of $k(x)^\times$ ($x \in X^{\text{cl}}$). Via evaluations of pseudo-functions at points of X^{cl}, we recover lines in the projective space $K^\times/k^\times$. Here again, we use the $(\mathbb{G}_m)_k$-largeness and (ϵ_{J_X}). $\square$

Outline of proof of Theorem 2. For simplicity, we assume $X = X^{\mathrm{cpt}}$ and that Σ is J_X-large. (The general case can be reduced to this case.)
Step 1. Local theory and characterization of various invariants
Similar to [T 1997] (Lefschetz trace formula), but the problem is that the separatedness is not available fully. We define the set of exceptional points $E \subset X^{\mathrm{cl}}$ outside which the separatedness is available, and recover (the decomposition groups of) $X^{\mathrm{cl}} \smallsetminus E$. The J_X-largeness implies $k(E) \subsetneq \overline{k}$ and, in particular, $|X^{\mathrm{cl}} \smallsetminus E| = \infty$.
Step 2. Multiplicative groups
By using a variant of the theory of cuspidalizations with exceptional points, we reconstruct $\mathcal{O}_E^\times/(k^\times\{\Sigma'\})$ up to ambiguity coming from $J_X(k)\{\Sigma'\}$.
Step 3. Additive structure
Similar to the proof of Theorem 1, but there are two extra problems: the above problem of ambiguity coming from $J_X(k)\{\Sigma'\}$ and the problem that $\mathcal{O}_E^\times/k^\times$ itself is not a projective space but a mere subset of the projective space $(\mathcal{O}_E \smallsetminus \{0\})/k^\times$ By establishing a certain generalization of the fundamental theorem of projective geometry, we recover the additive structure. $\square$

§5. m-step solvable AG.
In [Uchida 1977] (resp. [T 1997]), the following prosolvable variant is also shown: $G_K^{\mathrm{solv}}(= G_K^{(\mathrm{solv})}) \rightsquigarrow K$ (resp. $\pi_1(X)^{\mathrm{solv}}(= \pi_1(X)^{(\mathrm{solv})})$ or $\pi_1^{\mathrm{tame}}(X)^{\mathrm{solv}}(= \pi_1^{\mathrm{tame}}(X)^{(\mathrm{solv})}) \rightsquigarrow X$). Here, we consider (finite-step) solvable variants.

Theorem 1 [Saïdi-T, in preparation].
(i) *Assume* $m \geq 2$. *Then* $G_K^{m\text{-solv}} \rightsquigarrow [K]$.
(ii) *Assume* $m \geq 2$. *Then* $G_K^{(m\text{-solv})} \rightsquigarrow K$.
(iii) *Assume* $m \geq 3$. *Then* $G_K^{m\text{-solv}} \rightsquigarrow K$.

Theorem 2 [Yamaguchi, in preparation]. *Assume* $2g-2+r > 0$, $r > 0$ *and* $m \geq 3$. *Then* $\pi_1^{\mathrm{tame}}(X)^{(m\text{-solv})} \rightsquigarrow X$.

Remark. [de Smit-Solomatin, preprint] shows that $G_K^{1\text{-solv}}(= G_K^{\mathrm{ab}}) \rightsquigarrow [K]$ does not hold in general.

Outline of proof of Theorem 1. We may assume $X = X^{\mathrm{cpt}}$.
Step 1. Local theory and characterization of various invariants
The main point is to establish local theory: $G_K^{\text{2-solv}} \rightsquigarrow X^{\mathrm{cl}} = \mathrm{Dec}(G_K^{\mathrm{ab}})$, by observing the structure of abelianizations of arithmetic and geometric fundamental groups of abelian covers of X. We also show: $G_K^{\text{2-solv}} \rightsquigarrow$ the cyclotomic character $\chi_{\mathrm{cycl}} : G_K^{\mathrm{ab}} \to (\hat{\mathbb{Z}}^{\text{pro-}p'})^{\times}$.
Step 2. Multiplicative groups
Similar to [Uchida 1977].
Step 3. Additive structure
For (i), we resort to [Cornelissen-de Smit-Li-Marcolli-Smit 2019] to recover the isomorphim class of K. For (ii)(iii), we resort to Uchida's lemma, similarly to [Uchida 1977]. □

Outline of proof of Theorem 2.
Similar to [T 1997]. One of the main points is to establish local theory: If $g \geq 1$, $\pi_1^{\mathrm{tame}}(X)^{(\text{2-solv})} \rightsquigarrow ((X^{\mathrm{cpt}})^{\mathrm{cl}} \twoheadrightarrow \mathrm{Dec}(\pi_1(X)^{\mathrm{ab}}))$, by using the Lefschetz trace formula. □

§6. Hom version.

K_1, K_2: function fields

- $\gamma \in \mathrm{Hom}(K_2, K_1)$ is separable $\overset{\text{def}}{\iff}$ $K_1/\gamma(K_2)$ is a separable extension.

- $\sigma \in \mathrm{Hom}(G_{K_1}, G_{K_2})$ is rigid $\overset{\text{def}}{\iff}$ σ is open and $\exists H_i \in \mathrm{OSub}(G_{K_i})$ for $i = 1, 2$, such that $\sigma(H_1) \subset H_2$ and that $\forall D_1 \in \mathrm{Dec}(H_1)$, $\sigma(D_1) \in \mathrm{Dec}(H_2)$

- $\sigma \in \mathrm{Hom}(G_{K_1}, G_{K_2})$ is well-behaved $\overset{\text{def}}{\iff}$ σ is open and $\forall D_1 \in \mathrm{Dec}(G_{K_1})$, $\exists D_2 \in \mathrm{Dec}(G_{K_2})$, s.t. $\sigma(D_1) \in \mathrm{OSub}(D_2)$ ($\implies$ $\phi : \mathrm{Dec}(G_{K_1}) \to \mathrm{Dec}(G_{K_2})$).

- $\sigma \in \mathrm{Hom}(G_{K_1}, G_{K_2})$ is proper $\overset{\text{def}}{\iff}$ σ is well-behaved, and the map $(\mathrm{Dec}(G_{K_1})/\,\mathrm{Inn}(G_{K_1})) \to (\mathrm{Dec}(G_{K_2})/\,\mathrm{Inn}(G_{K_2}))$ induced by ϕ has finite fibers.

- $\sigma \in \mathrm{Hom}(G_{K_1}, G_{K_2})$ is inertia-rigid $\overset{\text{def}}{\iff}$ σ is well-behaved and $\exists \tau : \hat{\mathbb{Z}}^{\text{pro-}p'}(1)_{K_1} \hookrightarrow \hat{\mathbb{Z}}^{\text{pro-}p'}(1)_{K_2}$, $\forall D_1 \in \mathrm{Dec}(G_{K_1})$, $\exists e = e(D_1) \in \mathbb{Z}_{>0}$, s.t. $I_1^{\text{tame}} \to I_2^{\text{tame}}$ is identified with $e\tau$, where $D_2 = \phi(D_1) \in \mathrm{Dec}(G_{K_2})$ and I_i^{tame} is the tame inertia subquotient of D_i for $i = 1, 2$.

Theorem [Saïdi-T 2011]. *The natural map* $\mathrm{Hom}(K_2, K_1) \to \mathrm{Hom}(G_{K_1}, G_{K_2})/\,\mathrm{Inn}(G_{K_2})$ *induces bijections*

$$\mathrm{Hom}(K_2, K_1)^{\text{separable}} \overset{\sim}{\to} \mathrm{Hom}(G_{K_1}, G_{K_2})^{\text{rigid}}/\,\mathrm{Inn}(G_{K_2}),$$
$$\mathrm{Hom}(K_2, K_1)^{\text{separable}} \overset{\sim}{\to} \mathrm{Hom}(G_{K_1}, G_{K_2})^{\text{proper, inertia-rigid}}/\,\mathrm{Inn}(G_{K_2}).$$

Outline of proof. Omit! □

References

[Cornelissen-de Smit-Li-Marcolli-Smit 2019] Cornelissen, G., de Smit, B., Li, X., Marcolli, M. and Smit, H., Characterization of global fields by Dirichlet L-series, Res. Number Theory 5 (2019), Art. 7, 15 pp.

[de Smit-Solomatin, preprint] de Smit, B., Solomatin, P., On abelianized absolute Galois group of global function fields, preprint, arXiv:1703.05729.

[Mochizuki 1996] Mochizuki, S., The profinite Grothendieck conjecture for closed hyperbolic curves over number fields, J. Math. Sci. Univ. Tokyo 3 (1996), 571–627.

[Mochizuki 2007] Mochizuki, S., Absolute anabelian cuspidalizations of proper hyperbolic curves, J. Math. Kyoto Univ. 47 (2007), 451–539.

[Saïdi-T 2009_1] Saïdi, M., Tamagawa, A., A prime-to-p version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields of characteristic $p > 0$, Publ. Res. Inst. Math. Sci. 45 (2009), 135–186.

[Saïdi-T 2009_2] Saïdi, M., Tamagawa, A., On the anabelian geometry of hyperbolic curves over finite fields, Algebraic number theory and related topics 2007, 67–89, RIMS Kôkyûroku Bessatsu, B12, Res. Inst. Math. Sci. (RIMS), 2009.

[Saïdi-T 2011] Saïdi, M. and Tamagawa, A., On the Hom-form of Grothendieck's birational anabelian conjecture in characteristic $p > 0$, Algebra and Number Theory, 5(2) (2011), 131–184.

[Saïdi-T 2017] Saïdi, M., and Tamagawa, A., A refined version of Grothendieck's birational anabelian conjecture for curves over finite fields, Advances in Mathematics 310 (2017) 610–662.

[Saïdi-T 2018] Saïdi, M., and Tamagawa, A., A refined version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields, J. Algebraic Geom. 27 (2018), 383–448.

[Saïdi-T, in preparation] Saïdi, M. and Tamagawa, A., The m-step solvable anabelian geometry of global function fields, in preparation.

[Sawada 2021] Sawada, K., Algorithmic approach to Uchida's theorem for ne-dimensional function fields over finite fields, Inter-universal Teichmüller theory summit 2016, 1–21, RIMS Kôkyûroku Bessatsu, B84, Res. Inst. Math. Sci. (RIMS), 2021.

[T 1997] Tamagawa, A., The Grothendieck conjecture for affine curves, Compositio Math. 109 (1997), 135–194.

[Uchida 1977] Uchida, K., Isomorphisms of Galois groups of algebraic function fields, Ann. of Math. (2) 106 (1977), 589–598.

[Yamaguchi, in preparation] Yamaguchi, N., The m-step solvable anabelian geometry for affine hyperbolic curves over finitely generated fields, in preparation.

Akio Tamagawa
Research Institute for Mathematical Sciences
Kyoto University
KYOTO 606-8502
Japan
tamagawa@kurims.kyoto-u.ac.jp