A Combinatorial Anabelian Result for Stable Log Curves over Log Points

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"Combinatorial Anabelian Geometry and Related Topics"

§1: Semi-graphs of Anabelioids of PSC-type

Σ: a nonempty set of prime numbers

Definition

k: an algebraically closed field of characteristic $\notin \Sigma$
$X^{\log}$: a stable log curve/an fs log scheme $S^{\log}$ w/ $S = \mathrm{Spec}(k)$

Write:
(X, D): the pointed stable curve/k associated to $X^{\log}$
$U_X \overset{\mathrm{def}}{=} X^{\mathrm{sm}} \cap (X \setminus D) \subseteq X$
$\mathbb{G}$: the dual semi-graph of $X^{\log}$
v: a vertex of $\mathbb{G}$
  C_v: the irreducible component of X corresponding to v
  $X_v \overset{\mathrm{def}}{=} C_v \cap U_X$
e: an open edge of $\mathbb{G}$ w/ the branch b that abuts to v
  C_e: the completion of X at the point corresponding to e ($\cong \mathrm{Spec}(k[[t]])$)
  $X_e \overset{\mathrm{def}}{=} C_e \setminus \{e\}$ ($\cong \mathrm{Spec}(k[[t]][1/t])$)
  $\iota_b \colon X_e \to X_v$: the natural morphism corresponding to b

e: a closed edge of $\mathbb{G}$ w/ the two distinct branches b_1, b_2 that abut to v_1, v_2, respectively (possibly $v_1 = v_2$)
  C_e: the completion of X at the node corresponding to e ($\cong \mathrm{Spec}(k[[t_1, t_2]]/(t_1 t_2))$)
  Fix an isomorphism

$$\begin{array}{ccccccc} X_{v_1} & \xleftarrow{\text{“}b_1\text{”}} & \text{“}\mathrm{Spec}(k[[t_1]][1/t_1])\text{”} & \overset{\sim}{\dashrightarrow} & \text{“}\mathrm{Spec}(k[[t_2]][1/t_2])\text{”} & \xrightarrow{\text{“}b_2\text{”}} & X_{v_2} \\ & & \text{immersion} \searrow & & \swarrow \text{immersion} & & \\ & & & C_e & & & \end{array}$$

  $X_e \overset{\mathrm{def}}{=}$ “$\mathrm{Spec}(k[[t_1]][1/t_1]) \overset{\text{fixed}}{\cong} \mathrm{Spec}(k[[t_2]][1/t_2])$”
  $\iota_{b_i} \colon X_e \hookrightarrow X_{v_i}$: the natural morphism corresponding to b_i

Definition, continued

k: an algebraically closed field of characteristic $\notin \Sigma$
$X^{\log}$: a stable log curve/an fs log scheme $S^{\log}$ w/ $S = \mathrm{Spec}(k)$

Write:
(X, D): the associated pointed stable curve/k
$\mathbb{G}$: the dual semi-graph of $X^{\log}$
v: a vertex of $\mathbb{G}$
$X_v \stackrel{\text{def}}{=} C_v \cap U_X$
e: an open edge of $\mathbb{G}$ w/ the branch b that abuts to v
$X_e \stackrel{\text{def}}{=} C_e \setminus \{e\}$ ($\cong \mathrm{Spec}(k[[t]][1/t])$)
$\iota_b \colon X_e \hookrightarrow X_v$: the closed immersion corresponding to b
e: a closed edge of $\mathbb{G}$ w/ the two distinct branches b_1, b_2
that abut to v_1, v_2, respectively (possibly $v_1 = v_2$)
$X_e \stackrel{\text{def}}{=}$ "$\mathrm{Spec}(k[[t_1]][1/t_1]) \stackrel{\text{fixed}}{\cong} \mathrm{Spec}(k[[t_2]][1/t_2])$"
$\iota_{b_i} \colon X_e \hookrightarrow X_{v_i}$: the closed immersion corresponding to b_i

Define a semi-graph $\mathcal{G}^{\Sigma}_{X^{\log}}$ of anabelioids as follows:

- the underlying semi-graph $\stackrel{\text{def}}{=} \mathbb{G}$
- the anabelioid $\mathcal{G}_v$ corresponding to a vertex $v \stackrel{\text{def}}{=} \Sigma\text{-Fét}(X_v)$
- the anabelioid $\mathcal{G}_e$ corresponding to an edge $e \stackrel{\text{def}}{=} \Sigma\text{-Fét}(X_e)$
- the morphism $b_* \colon \mathcal{G}_e \to \mathcal{G}_v$ that corr'g to the branch b of e abutting to v
$\stackrel{\text{def}}{=} \mathcal{G}_e \to \mathcal{G}_v$ obtained by pulling back by ι_b, i.e., $\iota_b^* \colon \Sigma\text{-Fét}(X_v) \to \Sigma\text{-Fét}(X_e)$

Definition

$\mathcal{G}$: a connected semi-graph of anabeliods

$\Rightarrow \mathcal{B}(\mathcal{G})$: the connected anabelioid of $(S_v, \phi_e)_{v\text{: a vertex, } e\text{: a closed edge}}$, where

- S_v: an object of the connected anabelioid $\mathcal{G}_v$
- $\phi_e : b_1^* S_{v_1} \xrightarrow{\sim} b_2^* S_{v_2}$: an isomorphism in the connected anabelioid $\mathcal{G}_e$
  (b_1, b_2 are the two distinct branches of e that abut to v_1, v_2, respectively)

Proposition 1.1

In the above situation:

- $\exists$a natural continuous isomorphism $\pi_1(\mathcal{B}(\mathcal{G}^{\Sigma}_{X^{\log}}))^{\Sigma} \xrightarrow{\sim} \pi_1^{\mathrm{adm}}(X, D)^{\Sigma}$
- $\exists$a natural $\pi_1(X^{\log})$-conjugacy class of continuous isomorphisms
  $\pi_1(\mathcal{B}(\mathcal{G}^{\Sigma}_{X^{\log}}))^{\Sigma} \xrightarrow{\sim} \mathrm{Ker}(\pi_1(X^{\log})^{\Sigma} \twoheadrightarrow \pi_1(S^{\log})^{\Sigma}) \cong \mathrm{Ker}(\pi_1(X^{\log}) \twoheadrightarrow \pi_1(S^{\log}))^{\Sigma}$

Definition

$\mathcal{G}$: a semi-graph of anabelioids

$\mathcal{G}$: <u>of (pro-Σ) PSC-type</u> $\overset{\mathrm{def}}{\Leftrightarrow} \exists(k, X^{\log})$ as above s.t. $\mathcal{G} \cong \mathcal{G}^{\Sigma}_{X^{\log}}$

Remark

graph of groups (cf., e.g., "Trees" by Serre)

semi-graph of <u>anabelioids</u> ass'd to $X^{\log} \overset{??}{\Leftrightarrow}$ semi-graph of <u>profinite groups</u> ass'd to $X^{\log}$

In order to define and study the notion of the semi-graph of <u>prof. gps</u> ass'd to $X^{\log}$,
one has to fix <u>basepoints</u> of all the components of $X^{\log}$ simultaneously.
On the other hand, there is no natural/consistent choice of such <u>basepoints</u> in general.

$\Rightarrow$ The notion of a semi-graph of <u>profinite groups</u> is <u>quite unnatural/unsuitable</u>
from the point of view of combinatorial anabelian geometry.

In the remainder of the present §1:
$\mathcal{G}$: a semi-graph of anabelioids of pro-Σ PSC-type
$\widetilde{\mathcal{G}} = \{\mathcal{G}^i \to \mathcal{G}\}_{i\in I}$: a universal pro-Σ covering,
i.e., a some cofinal, i.e., in $\mathcal{B}(\mathcal{G})$, collection of connected fét Σ-Galois coverings

Definition

$\Pi_{\mathcal{G}} \stackrel{\text{def}}{=} \varprojlim_{\widetilde{\mathcal{G}}\to\mathcal{H}^{\text{fin. Gal.}}\to\mathcal{G}} \mathrm{Aut}(\mathcal{H}/\mathcal{G})$: the PSC-fundamental group of $\mathcal{G}$ (w.r.t. $\widetilde{\mathcal{G}}$)

Proposition 1.2 [CbGC, Remark 1.1.3]

$\Pi_{\mathcal{G}}$: a nonabelian pro-Σ surface group
(follows from Prp 1.1)

Definition

a (Galois) $\Pi_{\mathcal{G}}$-covering $\stackrel{\text{def}}{\Leftrightarrow}$ a finite intermediate (Galois) covering of $\widetilde{\mathcal{G}} \to \mathcal{G}$

Remark

$\forall \Pi_{\mathcal{G}}$-covering has a natural structure of semi-graph of anabelioids of pro-Σ PSC-type.

Definition

$\mathrm{Vert}(\mathcal{G})$: the set of vertices of (the underlying semi-graph of) $\mathcal{G}$
$\mathrm{Cusp}(\mathcal{G})$: the set of open edges of (the underlying semi-graph of) $\mathcal{G}$
$\mathrm{Node}(\mathcal{G})$: the set of closed edges of (the underlying semi-graph of) $\mathcal{G}$
$\mathrm{Edge}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G}) \cup \mathrm{Node}(\mathcal{G})$
$\mathrm{VCN}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Vert}(\mathcal{G}) \cup \mathrm{Cusp}(\mathcal{G}) \cup \mathrm{Node}(\mathcal{G})$

$\square \in \{\mathrm{Vert}, \mathrm{Cusp}, \mathrm{Node}, \mathrm{Edge}, \mathrm{VCN}\} \Rightarrow \square(\widetilde{\mathcal{G}}) \overset{\mathrm{def}}{=} \varprojlim_{\widetilde{\mathcal{G}} \to \mathcal{H}^{\text{fin. Gal.}} \to \mathcal{G}} \square(\mathcal{H})$

Definition

$\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}}) \Rightarrow \Pi_{\widetilde{z}} \subseteq \Pi_{\mathcal{G}}$: the stablizer of $\widetilde{z}$ w.r.t. $\Pi_{\mathcal{G}} \curvearrowright \square(\widetilde{\mathcal{G}})$,
<u>VCN-subgroup</u> associated to $\widetilde{z}$

a <u>verticial</u> (resp. a <u>cuspidal</u>; a <u>nodal</u>; an <u>edge-like</u>) subgroup $\overset{\mathrm{def}}{\Leftrightarrow}$
a VCN-subgroup associated to $\in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (resp. $\mathrm{Cusp}(\widetilde{\mathcal{G}})$; $\mathrm{Node}(\widetilde{\mathcal{G}})$; $\mathrm{Edge}(\widetilde{\mathcal{G}})$)

Observe: $z \in \mathrm{VCN}(\mathcal{G})$ determines a $\Pi_{\mathcal{G}}$-conjugacy class of VCN-subgroup, i.e.,
by considering the $\Pi_{\mathcal{G}}$-conjugacy class of $\Pi_{\widetilde{z}}$ for some $\mathrm{VCN}(\widetilde{\mathcal{G}}) \ni \widetilde{z} \mapsto z$.
$\Rightarrow$ the notion of <u>"a VCN-subgp ass'd to $\in \mathrm{VCN}(\mathcal{G})$, well-defined up to conjugation"</u>

Definition

$\square \in \{\mathrm{Vert}, \mathrm{Cusp}, \mathrm{Node}, \mathrm{Edge}\}$
$\Pi_{\mathcal{G}}^{\text{ab-}\square} \subseteq \Pi_{\mathcal{G}}^{\mathrm{ab}}$: the subgp top'y gen'd by the images of the VCN-subgps ass'd to $\in \square(\widetilde{\mathcal{G}})$
$\Pi_{\mathcal{G}}^{\mathrm{ab}/\square} \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}}^{\mathrm{ab}} / \Pi_{\mathcal{G}}^{\text{ab-}\square}$
$\Rightarrow$

- $0 \to \Pi_{\mathcal{G}}^{\text{ab-}\square} \to \Pi_{\mathcal{G}}^{\mathrm{ab}} \to \Pi_{\mathcal{G}}^{\mathrm{ab}/\square} \to 0$
- $\Pi_{\mathcal{G}}^{\text{ab-Cusp}}, \Pi_{\mathcal{G}}^{\text{ab-Node}} \subseteq \Pi_{\mathcal{G}}^{\text{ab-Edge}} = \Pi_{\mathcal{G}}^{\text{ab-Cusp}} + \Pi_{\mathcal{G}}^{\text{ab-Node}} \subseteq \Pi_{\mathcal{G}}^{\text{ab-Vert}} \subseteq \Pi_{\mathcal{G}}^{\mathrm{ab}}$
- $\Pi_{\mathcal{G}}^{\mathrm{ab}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab/Cusp}}, \Pi_{\mathcal{G}}^{\mathrm{ab/Node}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab/Edge}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab/Vert}}$

Definition

$\mathcal{G}'$: a semi-graph of anabelioids of pro-Σ PSC-type
$\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{G}'}$: a continuous (outer) isomorphism

- α: graphic $\overset{\text{def}}{\Leftrightarrow}$
  $\exists \mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ that induces the (outer isomorphism det'd by the) isomorphism α
- $\square \in \{\text{verticial}, \text{cuspidal}, \text{nodal}, \text{edge-like}\}$
  α: group-theoretically $\square$ $\overset{\text{def}}{\Leftrightarrow}$
  $\alpha(\square$ subgp of $\Pi_{\mathcal{G}})$ is $\square$ in $\Pi_{\mathcal{G}'}$, $\alpha^{-1}(\square$ subgp of $\Pi_{\mathcal{G}'})$ is $\square$ in $\Pi_{\mathcal{G}}$

Proposition 1.3 [CbGC, Proposition 1.5, (ii)]

$\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{G}'}$: a continuous outer isomorphism
α: graphic $\Leftrightarrow$
α: gp-theoretically verticial, gp-theoretically cuspidal, gp-theoretically nodal $\Leftrightarrow$
α: gp-theoretically verticial, gp-theoretically edge-like
In this situation, an isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ that induces α is unique.
(follows essentially from Prp 2.2, 2.3, 2.4, 2.5 below)

By Prp 1.3, the natural homomorphism $\mathrm{Aut}(\mathcal{G}) \to \mathrm{Out}(\Pi_{\mathcal{G}})$ is injective.
Let us regard $\mathrm{Aut}(\mathcal{G})$ as a subgroup of $\mathrm{Out}(\Pi_{\mathcal{G}})$.

Observe: $\Pi_{\mathcal{G}}$: topologically finitely generated (cf. Prp 1.2)
$\Rightarrow \bigcap_{N \subseteq \Pi_{\mathcal{G}}\text{: open, characteristic}} N = \{1\}$
$\Rightarrow$ $\mathrm{Out}(\Pi_{\mathcal{G}})$ has a natural structure of profinite group,
i.e., $\mathrm{Out}(\Pi_{\mathcal{G}}) = \varprojlim_{N \subseteq \Pi_{\mathcal{G}}\text{: open, characteristic}} \mathrm{Out}(\Pi_{\mathcal{G}}/N)$,
w.r.t. which $\mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ is a closed subgroup.

§2: Foundations of VCN-subgroups

Σ: a nonempty set of prime numbers
$\mathcal{G}$: a semi-graph of anabelioids of pro-Σ PSC-type
$\widetilde{\mathcal{G}} \to \mathcal{G}$: a universal pro-$\Sigma$ covering

Proposition 2.1 [CbGC, Remark 1.1.3]

(1) $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}}) \Rightarrow \Pi_{\widetilde{e}}$ ($\overset{\sim}{\leftarrow}$ "$\pi_1(X_e)^{\Sigma}$") $\cong \widehat{\mathbb{Z}}^{\Sigma}$
(2) $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \Rightarrow \Pi_{\widetilde{v}}$ ($\overset{\sim}{\leftarrow}$ "$\pi_1(X_v)^{\Sigma}$"): a nonabelian pro-Σ surface group

Proposition 2.2 [CbGC, Proposition 1.2, (ii)]

$\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}}) \Rightarrow \Pi_{\widetilde{z}}$: commensurably terminal in $\Pi_{\mathcal{G}}$,
i.e., $\Pi_{\widetilde{z}} = C_{\Pi_{\mathcal{G}}}(\Pi_{\widetilde{z}}) \overset{\mathrm{def}}{=} \{\, \gamma \in \Pi_{\mathcal{G}} \mid [\Pi_{\widetilde{z}} : \Pi_{\widetilde{z}} \cap \gamma\Pi_{\widetilde{z}}\gamma^{-1} \cap \gamma^{-1}\Pi_{\widetilde{z}}\gamma] < \infty \,\}$

More strongly:

Proposition 2.3 [NodNon, Lemma 1.5]

$\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Edge}(\widetilde{\mathcal{G}})$
$\widetilde{e}_1 = \widetilde{e}_2 \Leftrightarrow \Pi_{\widetilde{e}_1} = \Pi_{\widetilde{e}_2} \Leftrightarrow \Pi_{\widetilde{e}_1} \cap \Pi_{\widetilde{e}_2} \neq \{1\}$

Proposition 2.4 [NodNon, Lemma 1.7]

$\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$, $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$
$\widetilde{e}$ abuts to $\widetilde{v} \Leftrightarrow \Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}} \Leftrightarrow \Pi_{\widetilde{e}} \cap \Pi_{\widetilde{v}} \neq \{1\}$

Proposition 2.5 [NodNon, Lemma 1.9]

$\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$
(1) $\widetilde{v}_1 = \widetilde{v}_2 \Leftrightarrow \Pi_{\widetilde{v}_1} = \Pi_{\widetilde{v}_2}$
(2) $\widetilde{v}_1 \neq \widetilde{v}_2$ but $\exists \widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ s.t. $\widetilde{e}$ abuts both to $\widetilde{v}_1$ and to $\widetilde{v}_2$
$\Leftrightarrow \Pi_{\widetilde{v}_1} \neq \Pi_{\widetilde{v}_2}$, $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} \neq \{1\}$
In this situation, $\Pi_{\widetilde{v}_1} \cap \Pi_{\widetilde{v}_2} = \Pi_{\widetilde{e}}$.

Proposition 2.6 [CbTpII, Propositions 1.4, 1.5]

$H \subseteq \Pi_{\mathcal{G}}$: a closed subgroup, $\square \in \{\mathrm{Vert}, \mathrm{Cusp}, \mathrm{Node}\}$
(a) $\exists \widetilde{z} \in \square(\widetilde{\mathcal{G}})$ s.t. $H \subseteq \Pi_{\widetilde{z}}$ $\Leftrightarrow$ (b) for $\forall \gamma \in H$, $\exists \widetilde{z}_\gamma \in \square(\widetilde{\mathcal{G}})$ s.t. $\gamma \in \Pi_{\widetilde{z}_\gamma}$
$\Leftrightarrow$ (c) for $\forall \Pi_{\mathcal{G}}$-covering $\mathcal{H} \to \mathcal{G}$, $\mathrm{Im}(H \cap \Pi_{\mathcal{H}} \hookrightarrow \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\mathrm{ab}/\square}) = \{0\}$

Proof
(a) $\Rightarrow$ (c): immediate
(b) $\Rightarrow$ (a): omit

(c) $\Rightarrow$ (b) in the case where $\Sigma = \{l\}$:
First, we may assume: $H \cong \mathbb{Z}_l$ (by replacing H by "$\overline{\langle \gamma \rangle}$").

Claim

$\mathcal{H} \to \mathcal{G}$: a Galois $\Pi_{\mathcal{G}}$-covering
$\underline{\mathcal{H}}$: the $\Pi_{\mathcal{G}}$-covering corresponding to the open subgroup $\Pi_{\mathcal{H}} \cdot H \subseteq \Pi_{\mathcal{G}}$
$\Rightarrow \exists z \in \square(\underline{\mathcal{H}})$ s.t. $\mathcal{H} \to \underline{\mathcal{H}}$ is totally ramified at z

Proof of Claim
$H \hookrightarrow \Pi_{\mathcal{H}} \cdot H = \Pi_{\underline{\mathcal{H}}} \twoheadrightarrow (\Pi_{\mathcal{H}} \cdot H)/\Pi_{\mathcal{H}} = \Pi_{\underline{\mathcal{H}}}/\Pi_{\mathcal{H}} = \mathrm{Aut}(\mathcal{H}/\underline{\mathcal{H}})$: surjective
Thus, since $H \cong \mathbb{Z}_l$,

$$\mathrm{Aut}(\mathcal{H}/\underline{\mathcal{H}}) \cong \mathbb{Z}/l^n\mathbb{Z} \text{ for some } n \geq 0 \quad (*_1)$$

Thus, since $\mathrm{Im}(H \hookrightarrow \Pi_{\underline{\mathcal{H}}} \twoheadrightarrow \Pi_{\underline{\mathcal{H}}}^{\mathrm{ab}}) \subseteq \Pi_{\underline{\mathcal{H}}}^{\mathrm{ab}\text{-}\square}$ by (c),

$$\left(\bigoplus_{z \in \square(\underline{\mathcal{H}})} \mathrm{Im}(\Pi_z \text{ in } \Pi_{\underline{\mathcal{H}}}^{\mathrm{ab}}) \twoheadrightarrow \right) \quad \Pi_{\underline{\mathcal{H}}}^{\mathrm{ab}\text{-}\square} \hookrightarrow \Pi_{\underline{\mathcal{H}}}^{\mathrm{ab}} \overset{(*_1)}{\twoheadrightarrow} \mathrm{Aut}(\mathcal{H}/\underline{\mathcal{H}}) \text{ is surjective} \quad (*_2)$$

$(*_1)$, $(*_2)$ $\Rightarrow \exists z \in \square(\underline{\mathcal{H}})$ s.t. $\Pi_z \hookrightarrow \Pi_{\underline{\mathcal{H}}} \twoheadrightarrow \mathrm{Aut}(\mathcal{H}/\underline{\mathcal{H}})$: surjective

By Claim, $\square(\mathcal{H})^H \neq \emptyset$ for $\forall$Galois $\Pi_{\mathcal{G}}$-covering $\mathcal{H} \to \mathcal{G}$
Thus, since $\square(\mathcal{H})^H$ is finite, $\varprojlim_{\widetilde{\mathcal{G}} \to \mathcal{H} \overset{\text{fin. Gal.}}{\to} \mathcal{G}} \square(\mathcal{H})^H \neq \emptyset$.
Then it is immediate: $H \subseteq \Pi_{\widetilde{z}}$ for $\forall \widetilde{z} \in$ this nonempty limit

Corollary 2.7

$\square \in \{\circ, \bullet\}$
$\triangle \in \{\mathrm{Vert}, \mathrm{Cusp}, \mathrm{Node}\}$
$\mathcal{G}_\square$: a semi-graph of anabelioids of pro-Σ PSC-type
$\alpha\colon \Pi_{\mathcal{G}_\circ} \xrightarrow{\sim} \Pi_{\mathcal{G}_\bullet}$: a continuous isomorphism s.t.
for $\forall \Pi_{\mathcal{G}_\circ}$-covering $\mathcal{H}_\circ \to \mathcal{G}_\circ$,
if one writes $\mathcal{H}_\bullet \to \mathcal{G}_\bullet$ for the corresponding $\Pi_{\mathcal{G}_\bullet}$-covering, then

$$\begin{array}{ccccccc}
\Pi_{\mathcal{G}_\circ} & \hookleftarrow & \Pi_{\mathcal{H}_\circ} & \twoheadrightarrow & \Pi^{\mathrm{ab}}_{\mathcal{H}_\circ} & \twoheadrightarrow & \Pi^{\mathrm{ab}/\triangle}_{\mathcal{H}_\circ} \\
\alpha \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \vdots \wr \\
\Pi_{\mathcal{G}_\bullet} & \hookleftarrow & \Pi_{\mathcal{H}_\bullet} & \twoheadrightarrow & \Pi^{\mathrm{ab}}_{\mathcal{H}_\bullet} & \twoheadrightarrow & \Pi^{\mathrm{ab}/\triangle}_{\mathcal{H}_\bullet}
\end{array}$$

$\Rightarrow \alpha$: <u>gp-theretically verticial</u> (resp. <u>cuspidal</u>; <u>nodal</u>) if $\triangle = \mathrm{Vert}$ (resp. Cusp; Node)
(follows from Prp 2.6)

$$\begin{array}{ccc}
\Pi_{\mathcal{G}} + \{\text{verticial subgps}\} & \overset{\text{Prp 2.5}}{\Longleftrightarrow} & \Pi_{\mathcal{G}} + \{\text{verticial subgps}\} + \{\text{nodal subgps}\} \\
\text{Prp 2.6} \Updownarrow & & \Updownarrow \text{Prp 2.6} \\
\Pi_{\mathcal{G}} + \{\Pi_{\mathcal{H}} \twoheadrightarrow \Pi^{\mathrm{ab/Vert}}_{\mathcal{H}}\}_{\mathcal{H}} & & \Pi_{\mathcal{G}} + \{\Pi_{\mathcal{H}} \twoheadrightarrow \Pi^{\mathrm{ab/Node}}_{\mathcal{H}} \twoheadrightarrow \Pi^{\mathrm{ab/Vert}}_{\mathcal{H}}\}_{\mathcal{H}}
\end{array}$$

Proposition 2.8 [CbGC, Theorem 1.6, (i)], also [IUTchI, Remark 1.2.2]

$\Pi_{\mathcal{G}} + \Big(\{\text{open subgps of } \Pi_{\mathcal{G}}\} \ni H \mapsto \#\mathrm{Cusp}(\text{the } \Pi_{\mathcal{G}}\text{-covering corr'g to } H)\Big)$
$\Rightarrow \Pi_{\mathcal{G}} + \{\text{cuspidal subgps}\}$

§3: Cyclotomes Associated to Semi-graphs of Anabelioids of PSC-type

Σ: a nonempty set of prime numbers
$\mathcal{G}$: a semi-graph of anabelioids of pro-Σ PSC-type
$\widetilde{\mathcal{G}} \to \mathcal{G}$: a universal pro-$\Sigma$ covering

Definition

$\mathcal{G}$: <u>strudy</u> $\overset{\mathrm{def}}{\Leftrightarrow}$ $\forall$vertex of $\mathcal{G}$ is "of genus ≥ 2"
(i.e., $\forall \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the fin. free $\widehat{\mathbb{Z}}^{\Sigma}$-module $\mathrm{Im}(\Pi_{\widetilde{v}} \hookrightarrow \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab/Cusp}})$ is of rank ≥ 4)

In the remainder of the present §3, for simplicity (cf. Rmk $*$ below), suppose: $\mathcal{G}$ is <u>sturdy</u>
$\mathcal{G}^{+}$: the semi-graph of anabelioids obtained by <u>removing the open edges</u> form $\mathcal{G}$
$\overset{\mathcal{G}\text{: sturdy}}{\Rightarrow}$ $\mathcal{G}^{+}$: <u>of pro-Σ PSC-type</u>
Moreover, we have a surjective continuous homomorphism $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}^{+}}$
whose kernel is topologically normally generated by the cuspidal subgroups,
which thus induces an isomorphism $\Pi_{\mathcal{G}}^{\mathrm{ab/Cusp}} \overset{\sim}{\to} \Pi_{\mathcal{G}^{+}}^{\mathrm{ab}}$.

Definition

$\Lambda_{\mathcal{G}} \overset{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(H^2(\Pi_{\mathcal{G}^{+}}, \widehat{\mathbb{Z}}^{\Sigma}), \widehat{\mathbb{Z}}^{\Sigma})$: the <u>cyclotome</u> associated to $\mathcal{G}$

Proposition 3.1

$\Lambda_{\mathcal{G}} \cong \widehat{\mathbb{Z}}^{\Sigma}$
(follows from Prp 1.1)

Definition

$\chi_{\mathcal{G}} \colon \mathrm{Aut}(\mathcal{G}) \to \mathrm{Aut}(\Lambda_{\mathcal{G}}) \overset{\text{Prp 3.1}}{=} (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$: the <u>(pro-$\Sigma$) cyclotomic character</u> ass'd to $\mathcal{G}$

Proposition 3.2 [CbGC, Proposition 1.3]

A natural identification $\Pi_{\mathcal{G}^{+}}^{\mathrm{ab}} = \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(H^1(\Pi_{\mathcal{G}^{+}}, \widehat{\mathbb{Z}}^{\Sigma}), \widehat{\mathbb{Z}}^{\Sigma})$ (cf. Prp 1.1) and the pairing $H^1(\Pi_{\mathcal{G}^{+}}, \widehat{\mathbb{Z}}^{\Sigma}) \times H^1(\Pi_{\mathcal{G}^{+}}, \widehat{\mathbb{Z}}^{\Sigma}) \to H^2(\Pi_{\mathcal{G}^{+}}, \widehat{\mathbb{Z}}^{\Sigma})$ determines a commutative diagram

$$\begin{array}{ccccc}
\Pi_{\mathcal{G}^{+}}^{\mathrm{ab}} & \hookleftarrow & \Pi_{\mathcal{G}^{+}}^{\text{ab-Vert}} & \hookleftarrow & \Pi_{\mathcal{G}^{+}}^{\text{ab-Node}} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab}}, \Lambda_{\mathcal{G}}) & \hookleftarrow & \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Node}}, \Lambda_{\mathcal{G}}) & \hookleftarrow & \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Vert}}, \Lambda_{\mathcal{G}}),
\end{array}$$

$$\begin{array}{ccccc}
\Pi_{\mathcal{G}^{+}}^{\mathrm{ab}} & \twoheadrightarrow & \Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Node}} & \twoheadrightarrow & \Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Vert}} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab}}, \Lambda_{\mathcal{G}}) & \twoheadrightarrow & \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Vert}}, \Lambda_{\mathcal{G}}) & \twoheadrightarrow & \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\mathcal{G}^{+}}^{\mathrm{ab/Node}}, \Lambda_{\mathcal{G}}).
\end{array}$$

(follows from Prp 1.1)

Synchronization of Cyclotomes for Cusps

$\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$
$Q_{\widetilde{e}}$: the quotient of $\Pi_{\mathcal{G}}$ by the normal closed subgp topologically normally gen'd by the commutator $[\Pi_{\mathcal{G}}, \Pi_{\widetilde{e}}]$ and the Π_f's w/ $f \in \mathrm{Cusp}(\mathcal{G})$ over which $\widetilde{e}$ does not lie
$J_{\widetilde{e}} \subseteq Q_{\widetilde{e}}$: the image of $\Pi_{\widetilde{e}}$
$\overset{\text{Prp 1.1}}{\Rightarrow}$
- The natural surjective cont. hom. $(\widehat{\mathbb{Z}}^{\Sigma} \cong)\ \Pi_{\widetilde{e}} \twoheadrightarrow J_{\widetilde{e}}$ is an isomorphism.
- $1 \to J_{\widetilde{e}} \to Q_{\widetilde{e}} \to \Pi_{\mathcal{G}^+} \to 1$

Moreover, by Prp 1.1, the image of $\mathrm{id}_{J_{\widetilde{e}}} \in \mathrm{End}_{\widehat{\mathbb{Z}}^{\Sigma}}(J_{\widetilde{e}})$ by the fourth arrow of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Pi_{\mathcal{G}^+}, J_{\widetilde{e}}) & \longrightarrow & H^1(Q_{\widetilde{e}}, J_{\widetilde{e}}) & \longrightarrow & H^1(J_{\widetilde{e}}, J_{\widetilde{e}})^{Q_{\widetilde{e}}} & \longrightarrow & H^2(\Pi_{\mathcal{G}^+}, J_{\widetilde{e}}) \\ & & & & & & {\scriptstyle [\Pi_{\mathcal{G}}, \Pi_{\widetilde{e}}]} \Big\| {\scriptstyle \subseteq \mathrm{Ker}(\Pi_{\mathcal{G}} \twoheadrightarrow Q_{\widetilde{e}})} & & \Big\| \\ & & & & & & \mathrm{End}_{\widehat{\mathbb{Z}}^{\Sigma}}(J_{\widetilde{e}}) & & \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{G}}, J_{\widetilde{e}}) \end{array}$$

is an isomorphism $\Lambda_{\mathcal{G}} \xrightarrow{\sim} J_{\widetilde{e}}$.

$\mathfrak{syn}_{\widetilde{e}}$: $\Pi_{\widetilde{e}} \xrightarrow{\sim} \Lambda_{\mathcal{G}}$:
the composite of the natural isom. $\Pi_{\widetilde{e}} \xrightarrow{\sim} J_{\widetilde{e}}$ and the converse of the resulting isom.

Corollary 3.3

$\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$
$\alpha \in \mathrm{Aut}(\mathcal{G})$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}}))$
$\widetilde{\alpha} \in \mathrm{Aut}(\Pi_{\mathcal{G}})$: a lifting of α
Suppose: $\widetilde{\alpha}(\Pi_{\widetilde{e}}) = \Pi_{\widetilde{e}}$
$\Rightarrow \widetilde{\alpha}|_{\Pi_{\widetilde{e}}} \in \mathrm{Aut}(\Pi_{\widetilde{e}}) \overset{\text{Prp 2.1, (1)}}{=} (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$ is $= \chi_{\mathcal{G}}(\alpha)$
(follows from Synchronization of Cyclotomes for Cusps)

Lemma 3.4

I: a profinite group
$\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$): a continuous homomorphism
Suppose: $\exists l \in \Sigma$ s.t. $\mathrm{Im}(I \overset{\rho}{\to} \mathrm{Aut}(\mathcal{G}) \overset{\chi_{\mathcal{G}}}{\to} (\widehat{\mathbb{Z}}^{\Sigma})^{\times} \to \mathbb{Z}_l^{\times}) \subseteq \mathbb{Z}_l^{\times}$: <u>open</u>
$\Pi_{\mathcal{G}} + (\rho\colon I \to \mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})) \Rightarrow \#\mathrm{Cusp}(\mathcal{G})$

<u>Proof</u> of "$\Pi_{\mathcal{G}} + (\underline{\chi\colon I \overset{\rho}{\to} \mathrm{Aut}(\mathcal{G}) \overset{\chi_{\mathcal{G}}}{\to} (\widehat{\mathbb{Z}}^{\Sigma})^{\times} \to \mathbb{Z}_l^{\times}}) \Rightarrow \#\mathrm{Cusp}(\mathcal{G})$"

First, by Prp 1.1: $\Pi_{\mathcal{G}}$: free pro-$\Sigma \Leftrightarrow \#\mathrm{Cusp}(\mathcal{G}) > 0$
$\Rightarrow$ We may assume: $\#\mathrm{Cusp}(\mathcal{G}) > 0$
V: a finite dimensional $\mathbb{Q}_l$-vector space equipped w/ a continuous action of I
$\Rightarrow$ • $\tau(I, V)$: the sum of the dimensions of the subquot.s V_i/V_{i+1} w/ triv. act. of I w.r.t. a "$\mathbb{Q}_l[I]$-composition series" $\{0\} = V_n \subseteq \ldots \subseteq V_0 = V$
• $\tau(V) = \max_{J \subseteq I\text{:open subgp}} \tau(J, V)$

$\square \in \{\mathcal{G}, \mathcal{G}^+\} \Rightarrow V_{\square} \overset{\mathrm{def}}{=} \Pi_{\square}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}^{\Sigma}} \mathbb{Q}_l(\chi^{-1})$, $W_{\square} \overset{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{\square}^{\mathrm{ab}}, \mathbb{Q}_l)$
$\Rightarrow$ • $\#\mathrm{Cusp}(\mathcal{G}) - 1 \overset{\text{Prp 1.1}}{=} \dim_{\mathbb{Q}_l}(V_{\mathcal{G}}) - \dim_{\mathbb{Q}_l}(V_{\mathcal{G}^+}) \overset{\text{Cor 3.3}}{=} \tau(V_{\mathcal{G}}) - \tau(V_{\mathcal{G}^+})$
• $\tau(V_{\mathcal{G}^+}) \overset{\text{Prp 3.2}}{=} \tau(W_{\mathcal{G}^+})$
• $\tau(W_{\mathcal{G}}) \overset{\text{Prp 1.1, 3.2}}{=} \tau(W_{\mathcal{G}^+})$
$\Rightarrow \#\mathrm{Cusp}(\mathcal{G}) = 1 + \tau(V_{\mathcal{G}}) - \tau(W_{\mathcal{G}})$

Corollary 3.5 [AbTpI, Lemma 4.5]

I: a profinite group
$\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$): a continuous homomorphism
Suppose: $\exists l \in \Sigma$ s.t. $\mathrm{Im}(I \stackrel{\rho}{\to} \mathrm{Aut}(\mathcal{G}) \stackrel{\chi_{\mathcal{G}}}{\to} (\widehat{\mathbb{Z}}^{\Sigma})^{\times} \to \mathbb{Z}_l^{\times}) \subseteq \mathbb{Z}_l^{\times}$: <u>open</u>
$\Pi_{\mathcal{G}} + (\rho\colon I \to \mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})) \Rightarrow \Pi_{\mathcal{G}} + \{$cuspidal subgps$\}$
(follows essentially from Prp 2.8, Cor 3.3, and Lmm 3.4)

Corollary 3.6

$\Box \in \{\circ, \bullet\}$
$\mathcal{G}_{\Box}$: a semi-graph of anabelioids of pro-Σ PSC-type
$I_{\Box}$: a profinite group
$\rho_{\Box}\colon I_{\Box} \to \mathrm{Aut}(\mathcal{G}_{\Box})$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{\Box}})$): a continuous homomorphism
Suppose: $\exists l_{\Box} \in \Sigma_{\Box}$ s.t. $\mathrm{Im}(I_{\Box} \stackrel{\rho}{\to} \mathrm{Aut}(\mathcal{G}_{\Box}) \stackrel{\chi_{\mathcal{G}_{\Box}}}{\to} (\widehat{\mathbb{Z}}^{\Sigma})^{\times} \to \mathbb{Z}_{l_{\Box}}^{\times}) \subseteq \mathbb{Z}_{l_{\Box}}^{\times}$: <u>open</u>
$\alpha\colon \Pi_{\mathcal{G}_{\circ}} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_{\bullet}}$: a continuous isomorphism w/ a commutative diagram

$$\begin{array}{ccc} I_{\circ} & \xrightarrow{\rho_{\circ}} & \mathrm{Out}(\Pi_{\mathcal{G}_{\circ}}) \\ \exists\downarrow & & \downarrow \mathrm{Out}(\alpha) \\ I_{\bullet} & \xrightarrow[\rho_{\bullet}]{} & \mathrm{Out}(\Pi_{\mathcal{G}_{\bullet}}) \end{array}$$

$\Rightarrow \alpha$: <u>group-theretically cuspidal</u>
(follows essentially from Cor 3.5)

Remark $*$

One may define/establish
- the cyclotome,
- the cyclotomic character, and
- synchronization of cyclotomes for cusps

for a general (i.e., not necessarily sturdy) semi-graph of anabelioids of PSC-type (cf. [CbGC, §2], [CbTpI, §3]).

§4: A Combinatorial Anabelian Result for Stable Log Curves over Log Points

Σ: a nonempty set of prime numbers
$\mathcal{G}$: a semi-graph of anabelioids of pro-Σ PSC-type
$\widetilde{\mathcal{G}} \to \mathcal{G}$: a universal pro-$\Sigma$ covering
I: a profinite group
$\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ ($\subseteq \mathrm{Out}(\Pi_\mathcal{G})$): a continuous homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_\mathcal{G} & \longrightarrow & \Pi_I & \longrightarrow & I & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow \rho & & \\ 1 & \xrightarrow{\text{Prp 1.2}} & \Pi_\mathcal{G} & \longrightarrow & \mathrm{Aut}(\Pi_\mathcal{G}) & \longrightarrow & \mathrm{Out}(\Pi_\mathcal{G}) & \longrightarrow & 1 \end{array}$$

Definition

$\widetilde{v} \in \mathrm{Vert}(\mathcal{G}) \Rightarrow I_{\widetilde{v}} \overset{\text{def}}{=} Z_{\Pi_I}(\Pi_{\widetilde{z}}) \subseteq D_{\widetilde{v}} \overset{\text{def}}{=} N_{\Pi_I}(\Pi_{\widetilde{z}}) \subseteq \Pi_I$:
the inertia/decomposition subgroups of Π_I associated to $\widetilde{v} \in \mathrm{Vert}(\mathcal{G})$

Lemma 4.1

$\widetilde{z} \in \mathrm{VCN}(\mathcal{G}) \Rightarrow D_{\widetilde{z}} \cap \Pi_\mathcal{G} = \Pi_{\widetilde{z}}$
(follows from Prp 2.2)

Definition

(1) ρ: <u>of IPSC-type</u> $\overset{\text{def}}{\Leftrightarrow}$
- $\exists k$: an algebraically closed field of characteristic $\notin \Sigma$
- $\exists X^{\log}$: a stable log curve/the standard log point $\mathrm{Spec}(k)^{\log} \overset{\text{def}}{=}$ "$(\mathrm{Spec}(k), \mathbb{N})$"
- $\exists \alpha\colon \mathcal{G}^{\Sigma}_{X^{\log}} \xrightarrow{\sim} \mathcal{G}$ s.t.

$$\begin{array}{ccccccccc} \exists 1 & \longrightarrow & \Pi_{\mathcal{G}^{\Sigma}_{X^{\log}}} & \longrightarrow & \pi_1(X^{\log})^{\Sigma} & \longrightarrow & \pi_1(\mathrm{Spec}(k)^{\log})^{\Sigma} & \longrightarrow & 1 \\ & & {\scriptstyle \Pi_\alpha}\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 1 & \longrightarrow & \Pi_\mathcal{G} & \longrightarrow & \Pi_I & \longrightarrow & I & \longrightarrow & 1 \end{array}$$

(2) ρ: <u>of PIPSC-type</u> $\overset{\text{def}}{\Leftrightarrow} I \cong \widehat{\mathbb{Z}}^{\Sigma}$, $\rho|_{\exists \text{an open subgroup of } I}$ is of IPSC-type

One most important property of a cont. homomorphism of PIPSC-type is as follows:

Lemma 4.2 [CbGC, Proposition 2.6]

Suppose: ρ is <u>of PIPSC-type</u>
$M \subseteq \Pi_\mathcal{G}^{\mathrm{ab}}$: a sub-$\widehat{\mathbb{Z}}^{\Sigma}$-module
$M \subseteq \Pi_\mathcal{G}^{\text{ab-Vert}} \Leftrightarrow \exists$an open subgp $J \subseteq I$ s.t. $J \curvearrowright \Pi_\mathcal{G}^{\mathrm{ab}}$ induces the trivial action on M
(follows essentially from weight-monodromy conj. for Jacobian varieties of curves)

Lemma 4.3 [AbTpII, Proposition 1.3, (iii), (iv)]

Suppose: ρ is <u>of IPSC-type</u>

(1) $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}}) \Rightarrow I_{\widetilde{v}} \hookrightarrow \Pi_I \twoheadrightarrow I$: an isomorphism

(2) $\widetilde{v}, \widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$

$\widetilde{v} = \widetilde{w} \Leftrightarrow I_{\widetilde{v}} = I_{\widetilde{w}}$

(follows from some considerations on the log structures involved)

Lemma 4.4

$\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$

Suppose: ρ is <u>of IPSC-type</u>

(1) $D_{\widetilde{v}} = I_{\widetilde{v}} \times \Pi_{\widetilde{v}}$

(2) $N_{\Pi_I}(I_{\widetilde{v}}) = D_{\widetilde{v}}$ ($\overset{(1);\ \mathrm{Lmm}\ 4.3,\ (1)}{\Rightarrow}$ $N_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}}) = \Pi_{\widetilde{v}}$)

(3) $Z_{\Pi_{\mathcal{G}}}(Z_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}})) = \{1\}$

<u>Proof</u>

(1):

$I_v \cdot \Pi_v \subseteq D_v$ by definition

$\overset{\mathrm{Lmm}\ 4.1;\ 4.3,\ (1)}{\Rightarrow}$ $I_v \cdot \Pi_v = D_v$ (cf. $1 \to \Pi_{\mathcal{G}} \to \Pi_I \to I \to 1$)

Thus, since $Z(\Pi_v) = \{1\}$ (cf. Prp 2.1, (2)), $I_v \times \Pi_v = D_v$.

(2):

$N_{\Pi_I}(I_{\widetilde{v}}) \supseteq D_{\widetilde{v}}$: by (1)

$N_{\Pi_I}(I_{\widetilde{v}}) \subseteq D_{\widetilde{v}}$:

$\gamma \in N_{\Pi_I}(I_{\widetilde{v}})$

$\Rightarrow I_{\widetilde{v}} = \gamma I_{\widetilde{v}} \gamma^{-1} = \gamma Z_{\Pi_I}(\Pi_{\widetilde{v}})\gamma^{-1} = Z_{\Pi_I}(\gamma \Pi_{\widetilde{v}} \gamma^{-1}) = Z_{\Pi_I}(\Pi_{\widetilde{v}^\gamma}) = I_{\widetilde{v}^\gamma}$

$\overset{\mathrm{Lmm}\ 4.3,\ (2)}{\Rightarrow}$ $\widetilde{v} = \widetilde{v}^\gamma \Rightarrow \Pi_{\widetilde{v}} = \Pi_{\widetilde{v}^\gamma} = \gamma \Pi_{\widetilde{v}} \gamma^{-1} \Rightarrow \gamma \in N_{\Pi_I}(\Pi_{\widetilde{v}}) = D_{\widetilde{v}}$

(3):

$\Pi_{\widetilde{v}} \subseteq Z_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}}) \subseteq N_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}}) \overset{(2)}{=} \Pi_{\widetilde{v}}$

$\Rightarrow Z_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}}) = \Pi_{\widetilde{v}}$

$\Rightarrow Z_{\Pi_{\mathcal{G}}}(Z_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}})) = Z_{\Pi_{\mathcal{G}}}(\Pi_{\widetilde{v}}) \overset{\mathrm{Prp}\ 2.2}{=} Z(\Pi_{\widetilde{v}}) \overset{\mathrm{Prp}\ 2.1,\ (2)}{=} \{1\}$

Main Lemma of §4 [CbTpII, Theorem 1.6, (iv)]

Suppose: ρ is <u>of IPSC-type</u>
s: a splitting of $\Pi_I \twoheadrightarrow I$ s.t. $Z_{\Pi_\mathcal{G}}(Z_{\Pi_\mathcal{G}}(\mathrm{Im}(s))) = \{1\}$
$\Rightarrow \exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $\mathrm{Im}(s) = I_{\widetilde{v}}$ ($\overset{\text{Lmm 4.4, (2)}}{\Rightarrow}$ $N_{\Pi_\mathcal{G}}(\mathrm{Im}(s))$: verticial)

<u>Proof</u>

$H \overset{\text{def}}{=} Z_{\Pi_\mathcal{G}}(\mathrm{Im}(s))$

$\overset{\text{Lmm 4.2}}{\Rightarrow}$ for $\forall \Pi_\mathcal{G}$-covering $\mathcal{H} \to \mathcal{G}$, $\mathrm{Im}(H \cap \Pi_\mathcal{H} \hookrightarrow \Pi_\mathcal{H} \twoheadrightarrow \Pi_\mathcal{H}^{\mathrm{ab/Vert}}) = \{0\}$

$\overset{\text{Prp 2.6}}{\Rightarrow}$ $\exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $H \subseteq \Pi_{\widetilde{v}} \Rightarrow$

$$Z_{\Pi_I}(H) \supseteq Z_{\Pi_I}(\Pi_{\widetilde{v}}) = I_{\widetilde{v}} \quad (*_1)$$

$\{1\} = Z_{\Pi_\mathcal{G}}(Z_{\Pi_\mathcal{G}}(\mathrm{Im}(s))) = Z_{\Pi_I}(H) \cap \Pi_\mathcal{G} \Rightarrow Z_{\Pi_I}(H) \hookrightarrow \Pi_I \twoheadrightarrow I$ is injective
Thus, since $\mathrm{Im}(s) \subseteq Z_{\Pi_I}(H)$ by definition,

$$Z_{\Pi_I}(H) = \mathrm{Im}(s) \quad (*_2)$$

$(*_1)$, $(*_2)$ $\Rightarrow I_{\widetilde{v}} \subseteq \mathrm{Im}(s) \overset{\text{Lmm 4.3, (1)}}{\Rightarrow} I_{\widetilde{v}} = \mathrm{Im}(s)$

Main Theorem of §4

Suppose: ρ is <u>of PIPSC-type</u>
$\Pi_\mathcal{G} + (\rho\colon I \to \mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_\mathcal{G})) \Rightarrow \Pi_\mathcal{G} + \{\text{verticial subgps}\}$
($\overset{\text{Prp 2.5}}{\Rightarrow}$ $\Pi_\mathcal{G} + \{\text{verticial subgps}\} + \{\text{nodal subgps}\}$)
(follows essentially from Lmm 4.4, (3), and Main Lmm of §4)

Main Corollary of §4 [CbTpII, Theorem 1.9, (ii)]

$\square \in \{\circ, \bullet\}$
$\mathcal{G}_\square$: a semi-graph of anabelioids of pro-Σ PSC-type
$I_\square$: a profinite group
$\rho_\square\colon I_\square \to \mathrm{Aut}(\mathcal{G}_\square)$ ($\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_\square})$): a continuous homomorphism <u>of PIPSC-type</u>
$\alpha\colon \Pi_{\mathcal{G}_\circ} \overset{\sim}{\to} \Pi_{\mathcal{G}_\bullet}$: a continuous isomorphism w/ a commutative diagram

$$\begin{array}{ccc} I_\circ & \xrightarrow{\rho_\circ} & \mathrm{Out}(\Pi_{\mathcal{G}_\circ}) \\ {\scriptstyle\exists}\downarrow & & \downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ I_\bullet & \xrightarrow[\rho_\bullet]{} & \mathrm{Out}(\Pi_{\mathcal{G}_\bullet}) \end{array}$$

$\Rightarrow \alpha$: <u>group-theretically verticial</u> and <u>group-theretically nodal</u>
(follows from Main Thm of §4)

References

[CbGC] A Combinatorial Version of the Grothendieck Conjecture

[AbTpI] Topics in Absolute Anabelian Geometry I: Generalities

[AbTpII] Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms

[NodNon] On the Combinatorial Anabelian Geometry of Nodally Nondegenerate Outer Representations

[CbTpI] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia Groups and Profinite Dehn Twists

[CbTpII] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization

[IUTchI] Inter-universal Teichmüller Theory I: Construction of Hodge Theaters

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