# The Grothendieck-Teichmüller group as an open subgroup of the outer automorphism group of the étale fundamental group of a configuration space (joint work with Yuichiro Hoshi and Shinichi Mochizuki)

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## §1 Introduction

$k$: an algebraically closed field of characteristic zero

$X \stackrel{\text{def}}{=} \mathbb{P}^1_k \setminus \{0, 1, \infty\}$

$X_n \stackrel{\text{def}}{=} \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \ (i \neq j)\}$ (the $n$-th conf. sp)

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$\Pi_n$: the étale fundamental group of $X_n$

Note: Since $X_n$ is defined over $\mathbb{Q}$, we have

$$G_{\mathbb{Q}} \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\Pi_n).$$

Drinfeld and Ihara defined a certain explicit subgroup

$$\widehat{\mathrm{GT}} \subseteq \mathrm{Aut}(\widehat{\mathbb{Z}} * \widehat{\mathbb{Z}}),$$

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Open problem: Is $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$ an isomorphism?

Main Theorem (Hoshi-M.-Mochizuki)

*Suppose that* $n \geq 2$*.*

*Then the natural outer actions of* $\widehat{\mathrm{GT}}$ *and* $\mathfrak{S}_{n+3}$ *on* $\Pi_n$ *induce*

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**Corollary (Hoshi-M.-Mochizuki)**

*Suppose that* $n \geq 2$.

(i) $\mathfrak{S}_{n+3} = Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq \mathrm{Out}(\Pi_n)}^{\mathrm{op}} Z_{\mathrm{Out}(\Pi_n)}(H)$.

(ii) $\widehat{\mathrm{GT}} = Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) = Z_{\mathrm{Out}(\Pi_n)}(Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)))$.

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(i) Since $\widehat{\mathrm{GT}} \subseteq \mathrm{Out}(\Pi_n)$ is open, $\mathfrak{S}_{n+3} \subseteq Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n))$ is clear.

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$$\sigma = a \cdot b \qquad (a \in \widehat{\mathrm{GT}},\ b \in \mathfrak{S}_{n+3})$$

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(ii) This follows immediately from the center-freeness of $\mathfrak{S}_{n+3}$.

## §2 Related works

$\overline{\mathcal{M}}_{0,n+3}$: the Deligne-Mumford compactification of $\mathcal{M}_{0,n+3}$

$\Longrightarrow$ Each irreducible component $\delta$ of $\overline{\mathcal{M}}_{0,n+3} \setminus \mathcal{M}_{0,n+3}$ determines the inertia subgroup $\mathbb{I}_\delta \subseteq \Pi_n$ (up to conj.)

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**Theorem (Harbater-Schneps)**

*We have*

$$\widehat{\mathrm{GT}} \xrightarrow{\sim} \mathrm{Out}^\flat(\Pi_n) \cap Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \ \ (\subseteq \mathrm{Out}(\Pi_n)).$$

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**Theorem (Hoshi-M.-Mochizuki)**

*We have*

$$\widehat{\mathrm{GT}} \stackrel{\sim}{\to} \qquad\qquad Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \ \ (\subseteq \ \mathrm{Out}(\Pi_n)).$$

## §3 Results from combinatorial anabelian geometry

**Definition**

Suppose that $n \geq 2$. Note that for any $1 \leq m < n$, the projection morphism $X_n \to X_m$ obtained by "forgetting $n-m$ factors" induces a surjection $\Pi_n \twoheadrightarrow \Pi_m$. We shall refer to

$$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$$

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Next, observe that the projections obt'd by "forgetting the last factors" induce a sequence of surjections

$$\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1.$$

Write $K_m \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$, $\Pi_0 \stackrel{\text{def}}{=} \{1\}$. Then we have

$$\{1\} = K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_1 \subseteq K_0 = \Pi_n.$$

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**Definition**

$\alpha \in \mathrm{Aut}(\Pi_n)$ is F-admissible $\overset{\text{def}}{\Leftrightarrow}$ $\alpha(F) = F$ for $^{\forall}$fiber subgp $F \subseteq \Pi_n$.

$\alpha \in \mathrm{Aut}(\Pi_n)$ is C-admissible $\overset{\text{def}}{\Leftrightarrow}$

(i) $\alpha(K_m) = K_m$ $(0 \leq m \leq n)$;

(ii) $\alpha : K_m/K_{m+1} \overset{\sim}{\rightarrow} K_m/K_{m+1}$ induces a bijection between the set of cuspidal inertia subgps $\subseteq K_m/K_{m+1}$.

$\alpha \in \mathrm{Aut}(\Pi_n)$ is FC-admissible $\overset{\text{def}}{\Leftrightarrow}$ $\alpha$ is F-admissible and C-admissible.

$\mathrm{Out}^{\mathrm{F}}(\Pi_n) \overset{\mathrm{def}}{=} \{$ F-admissible automorphisms of $\Pi_n \}/\mathrm{Inn}(\Pi_n)$

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**Fact**

(i) *The above hom "$\to$" does not depend on the choice of a factor which we forget (cf. [CmbCsp]).*

(ii) *The above hom "$\to$" is injective (cf. [CmbCsp]).*

(iii) *Suppose that $n \geq 2$. Then we have $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ (cf. [CbTpII]).*

## §4 Outline of the proof of Main Theorem

In the present §, we suppose that $n \geq 2$.

Definition

Note that for any $1 \leq m < n$, the projection morphism $X_n \overset{\sim}{\to} \mathcal{M}_{0,n+3} \to \mathcal{M}_{0,m+3} \overset{\sim}{\to} X_m$ obtained by "forgetting $n-m$ marked pts" induces a surjection $\Pi_n \twoheadrightarrow \Pi_m$. We shall refer to

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Note: Since every fiber subgp is a generalized fiber subgp, we have

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**Proposition 1 (Hoshi-M.-Mochizuki)**

*Let* $\alpha \in \mathrm{Aut}(\Pi_n)$; $F \subseteq \Pi_n$ *a generalized fiber subgp of length* $n-m$. *The* $\alpha(F)$ *is a generalized fiber subgp of length* $n-m$. *In particular,* $\alpha$ *induces a permutation on the set*

$$\{\mathrm{Ker}(\Pi_n \stackrel{pr_i}{\twoheadrightarrow} \Pi_{n-1})\}_{i=1,2,\dots,n+3}.$$

*— whose cardinality is* $n+3$ *— of generalized fiber subgps of length* $1$.

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Therefore, we conclude that

$$\widehat{\mathrm{GT}} \overset{\sim}{\to} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \cap Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \quad \text{(cf. [HS]; [CmbCsp])}$$

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&= \mathrm{Out}^{\mathrm{gF}}(\Pi_n),
\end{aligned}$$

In light of Proposition 2, we have

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n+3} \overset{\sim}{\to} \mathrm{Out}(\Pi_n).$$

Observe: Since $\mathfrak{S}_{n+3}$ is center-free, it holds that

$$\mathrm{Out}^{\mathrm{gF}}(\Pi_n) = Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}).$$

Therefore, we conclude that

$$\begin{aligned}
\widehat{\mathrm{GT}} &\overset{\sim}{\to} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \cap Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \quad \text{(cf. [HS]; [CmbCsp])} \\
&= \mathrm{Out}^{\mathrm{F}}(\Pi_n) \cap Z_{\mathrm{Out}(\Pi_n)}(\mathfrak{S}_{n+3}) \quad \text{(cf. Fact, (iii))} \\
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hence that $\widehat{\mathrm{GT}} \times \mathfrak{S}_{n+3} \overset{\sim}{\to} \mathrm{Out}(\Pi_n)$.

Proof of Proposition 1

In the following, for simplicity, we assume that $n = 2$.

For any profinite group $\Box$, write $\Box^{(l)}$ (resp. $\Box^{\mathrm{ab}}$) for the maximal pro-$l$ quotient (resp. abelianization) of $\Box$.

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*Write $\mathrm{Gr}(\Pi_2^{(l)}) \otimes \mathbb{Q}_l$ for the graded Lie algebra over $\mathbb{Q}_l$ assoc. to the lower central series of $\Pi_2^{(l)}$. Then we have*

$$\mathrm{Aut}(\mathrm{Gr}(\Pi_2^{(l)}) \otimes \mathbb{Q}_l) \;\cong\; \mathfrak{S}_5 \times \mathbb{Q}_l^{\times}.$$

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Idea: Using a special property of free prof. gps, reduce to this Thm.

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*Let $G$ be a fin. gen. free profinite group of rk $> 1$. Then every nontrivial fin. gen. closed normal subgroup of $G$ is open in $G$.*

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By replacing $\alpha$ by $\alpha \cdot \beta$ — where $\beta \in \mathrm{Aut}(\Pi_2)$ is a suitable lift. of an $\in \mathfrak{S}_5 \subseteq \mathrm{Out}(\Pi_2)$ — we may assume WLOG that $\widetilde{\alpha} \in \mathbb{Q}_l^{\times}$.

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Let $F = \mathrm{Ker}(p)$ be a generalized fiber subgp, where $p : \Pi_2 \twoheadrightarrow \Pi_1$.

Note: $\Pi_1 \cong \pi_1(\mathbb{P}_k^1 \setminus \{0, 1, \infty\}) \cong \widehat{\mathbb{Z}} * \widehat{\mathbb{Z}}$;

$F \cong \pi_1(\mathbb{P}_k^1 \setminus \{0, 1, \infty, \bullet\}) \cong \widehat{\mathbb{Z}} * \widehat{\mathbb{Z}} * \widehat{\mathbb{Z}}$.

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\Pi_2 & \xrightarrow{\quad\alpha\quad} & \Pi_2 & \xrightarrow{\quad p\quad} & \Pi_1 \\
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(\Pi_2^{(l)})^{\mathrm{ab}} \otimes \mathbb{Q}_l & \xrightarrow{\widetilde{\alpha}} & (\Pi_2^{(l)})^{\mathrm{ab}} \otimes \mathbb{Q}_l & \xrightarrow{\widetilde{p}} & (\Pi_1^{(l)})^{\mathrm{ab}} \otimes \mathbb{Q}_l.
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$\Longrightarrow$ $\mathrm{Im}(\phi_1)$ ($\cong \mathbb{Z}_l \oplus \mathbb{Z}_l$) is finite, a contradiction!

Finally, we consider an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F/\alpha(F) & \longrightarrow & \Pi_2/\alpha(F) & \longrightarrow & \Pi_2/F & \longrightarrow & 1. \\ &&&& {\scriptstyle\alpha}\big\uparrow{\scriptstyle\wr} && {\scriptstyle p}\big\downarrow{\scriptstyle\wr} && \\ &&&& \Pi_2/F && \Pi_1 && \\ &&&& {\scriptstyle p}\big\downarrow{\scriptstyle\wr} &&&& \\ &&&& \Pi_1 &&&& \end{array}$$

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Therefore, again by Lemma, we conclude that $F/\alpha(F) = \{1\}$.