Overview of Étale Theta Function

Wojciech Porowski

University of Nottingham

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What we will discuss

In this series of talks we will discuss various constructions from the paper "Étale Theta Function and its Frobenioid-theoretic Manifestations" (sections §1 and §2) as well as from the paper "Inter-Universal Teichmüller Theory II: Hodge-Arakelov-Theoretic Evaluation" by S. Mochizuki.
In the slides, they will be referred to as [EtTh] and [IUT2].

Overview

Here is a very general overview of this series of talks.

- We construct a diagram of covers of a punctured elliptic curve X.
- Then, we define an analytic function on an infinite cover of X (theta function), analyse its properties and give a group theoretic construction of its cohomology class.
- Next, we introduce the notion of mono-theta environment, which serves as a sort of "bridge" between the group theoretic ("étale-like") theta function and its "Frobenius-like" version.
- We discuss the property of multiradiality of theta monoid as well as Galois evaluation, which produces values of the form "q^{j^2}".
- Finally, we globalize this local construction by introducing a (realified) Gaussian Frobenioid.

Notation

Let us fix some notation.
p - a prime number, $p \neq 2$,
K - a p-adic field, i.e., a finite extension of $\mathbb{Q}_p$, such that $\sqrt{-1} \in K$,
$G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ - the absolute Galois group of K,
E - an elliptic curve over K with split multiplicative reduction, i.e., a Tate curve, with $E[2] \subset E(K)$,
O - the origin of the elliptic curve E,
$X = E \setminus \{O\}$ - a hyperbolic curve of type $(1, 1)$.
Denote by

$$\Pi_X = \pi_1^{et}(X) \quad \text{and} \quad \Delta_X = \pi_1^{et}(X_{K^{\mathrm{alg}}})$$

the étale fundamental groups of X.
Similarly, write Π_X^{tp} and Δ_X^{tp} for their tempered fundamental groups.

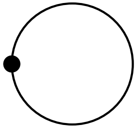
Reminder on the Tate curve

Recall the Tate uniformization:
There exists a unique $q \in K^*, |q| < 1$ such that we have a Galois equivariant isomorphism $E(K^{\text{alg}}) \cong (K^{\text{alg}})^*/q^{\mathbb{Z}} \quad (= \mathbb{G}_m/\mathbb{Z})$
The element q is called the q-parameter of E.
The homomorphism $\mathbb{G}_m \to \mathbb{G}_m/\mathbb{Z}$ is a $\mathbb{Z}$-cover of E (not algebraic).
$\Rightarrow$ not described by the étale fundamental group of E.
On the other hand, it is a *tempered* cover. In fact the map $\mathbb{G}_m \to E$ is a universal topological cover of E.

Geometry of universal covering space, ([EtTh],§1)

Write $Y \to X$ for the tempered $\mathbb{Z}$-cover $\mathbb{G}_m \to \mathbb{G}_m/\mathbb{Z}$.
Another point of view:
Let $\mathfrak{X}$ - a formal scheme obtained by completion of a stable model of X along the closed fibre.
Let $\mathfrak{Y}$ - "the universal topological cover of $\mathfrak{X}$".
The (Raynaud) generic fibre $\mathfrak{Y}_K$ of $\mathfrak{Y}$ is $Y \cong \mathbb{G}_{m,K}$.
The map of formal schemes $\mathfrak{Y} \to \mathfrak{X}$ defines the cover $Y \to X$.
Special fibre of $\mathfrak{Y}$ = infinite chain of projective lines.

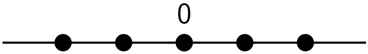
Dual graph of Y, infinite sequence of projective lines $\mathbb{P}^1$.



Dual graph of X, a closed loop.

Double cover of Y

We choose a vertex of the dual graph of Y (i.e., an irreducible component of the special fibre of Y) and label it with "0".



Dual graph of Y with label "0"

Observe that the choice of zero label together with the geometry of the special fibre determines labels in $\mathbb{Z}$ for all vertices, well defined up to ± 1. We also consider a double cover $\ddot{Y} \to Y$ given by the square map

$$\mathbb{G}_m \to \mathbb{G}_m, \quad x \mapsto x^2.$$

Thus we have covers $\ddot{Y} \to Y \to X$ and open subgroups $\Pi^{tp}_{\ddot{Y}} \subset \Pi^{tp}_{Y} \subset \Pi^{tp}_{X}$.

Plan

Recall our plan for the reminder of the talk.

Our goals

- Construct a geometric diagram of covers of X, which plays an important role in the theory,
- Define certain analytic function on $\ddot{Y}$, the *theta* function and discuss some of its properties,
- Construct certain theta function group theoretically from various tempered fundamental groups.

A quick look at the geometry, ([EtTh],§2)

Here is the diagram of covers we are about to construct

$$\begin{tikzcd}
\underline{\underline{\ddot{Y}}} \arrow[r, "\mu_\ell"] \arrow[d, "\mu_2"] & \ddot{Y} \arrow[d, "\mu_2"] & \\
\underline{\underline{Y}} \arrow[r, "\mu_\ell"] \arrow[d, "\ell\mathbb{Z}"] & Y \arrow[d, "\ell\mathbb{Z}"] & \\
\underline{\underline{X}} \arrow[r, "\mu_\ell"] & \underline{X} \arrow[r, "\mathbb{Z}/\ell\mathbb{Z}"] \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] & X \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] \\
 & \underline{C} \arrow[r] & C
\end{tikzcd}$$

(here ℓ is an odd prime number).

Various quotients

First, we have to introduce various quotients of fundamental groups. Define

$$\Delta_X^{\Theta} = \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]].$$

Because Δ_X is a free profinite group on two generators, Δ_X^{Θ} is abstractly isomorphic to the following matrix group

$$\begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \widehat{\mathbb{Z}} \\ 0 & 1 & \widehat{\mathbb{Z}} \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, define the quotient:

$$\Delta_X^{ell} = \Delta_X / [\Delta_X, \Delta_X]$$

(ell $\rightsquigarrow$ covers of X extending to covers of E, i.e. unramified over the cusp).

Various quotients (2)

Further, we define

$$\Delta_{\Theta} = Ker(\Delta_X^{\Theta} \to \Delta_X^{ell}).$$

Thus we have a short exact sequence

$$1 \to \Delta_{\Theta} \to \Delta_X^{\Theta} \to \Delta_X^{ell} \to 1$$

isomorphic to

$$\begin{bmatrix} 1 & 0 & \widehat{\mathbb{Z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \widehat{\mathbb{Z}} \\ 0 & 1 & \widehat{\mathbb{Z}} \\ 0 & 0 & 1 \end{bmatrix} \twoheadrightarrow \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}.$$

Finite covers of X that we will construct are (geometrically) determined by quotients of Δ_X^{Θ}.

More covers (1)

Fix an odd prime number ℓ.
We enhance $\ddot{Y} \to Y \to X$ (i.e., $\mathbb{G}_m \to \mathbb{G}_m \to \mathbb{G}_m/q^{\mathbb{Z}}$) to:

$$\begin{tikzcd} Y \arrow[d] & \\ \underline{X} \arrow[r] & X \end{tikzcd}$$

The subcover $\underline{X} \to X$ is the unique subcover of $Y \to X$ of degree ℓ, corresponding to the subgroup $\ell\mathbb{Z} \subset \mathbb{Z} \cong \mathrm{Gal}(Y/X)$, so we have

$$\mathrm{Gal}(\underline{X}/X) \cong \mathbb{Z}/\ell\mathbb{Z}.$$

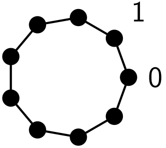
Thus $Y \to \underline{X} \to X$ corresponds to

$$\mathbb{G}_m \to \mathbb{G}_m/q^{\ell\mathbb{Z}} \to \mathbb{G}_m/q^{\mathbb{Z}}.$$

Notice that the cover $\underline{X} \to X$ corresponds to the "canonical multiplicative subspace".

Dual graphs

The cover $\underline{X} \to X$ is also unramifed over the cusp of X
$\Rightarrow \underline{X}$ is a curve of genus one, thus after choosing one cusp of $\underline{X}$ as the origin we get a structure of elliptic curve on the compactification of $\underline{X}$.
The zero cusp on Y determines a cusp of $\underline{X}$, we choose this cusp as the origin and we label it with "0".



Dual graph of $\underline{X}$, here $\ell = 9$

Observe that after the choice of the zero label, the geometry of the special fibre determines labels in the set $\{0, \pm 1, \ldots, \pm \ell^{*}\}$ for all cusps, up to ± 1. (here we define $\ell^{*} = (\ell - 1)/2$.)

Quotient by involution

Write ι (resp. $\underline{\iota}$) for the automorphism of X (resp. $\underline{X}$) given by "multiplication by -1" one the underlying elliptic curve.
We define C and $\underline{C}$ to be stack-theoretic quotients of X and $\underline{X}$ by the involutions ι and $\underline{\iota}$.
Thus we have a (cartesian) diagram:

$$\begin{tikzcd} \underline{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \underline{C} \arrow[r] & C \end{tikzcd}$$

Quotient by involution (2)

We write Π_C and $\Pi_{\underline{C}}$ for étale fundamental groups of C and $\underline{C}$, similarly Δ_C and $\Delta_{\underline{C}}$ for their geometric fundamental groups.
Thus we have $\Pi_{\underline{X}} \subset \Pi_{\underline{C}}$, a normal subgroup of index 2, similarly for the inclusion $\Pi_X \subset \Pi_C$.
Moreover, the covering $\underline{X} \to C$ is also Galois and we denote its Galois group by

$$\mathbb{F}_\ell^{\rtimes\pm} = \mathbb{Z}/\ell\mathbb{Z} \rtimes \{\pm 1\} \cong \mathrm{Gal}(\underline{X} \to C).$$

More covers (2)

Next, we are going to construct a cover $\underline{\underline{X}} \to \underline{X}$ of degree ℓ (which will be totally ramified over cusps of $\underline{X}$).
Write $\Delta_X^\Theta \twoheadrightarrow \overline{\Delta}_X$ for the quotient of Δ_X^Θ by the subgroup of elements which are ℓ-powers. Thus $\overline{\Delta}_X$ is isomorphic to

$$\begin{bmatrix} 1 & \mathbb{Z}/\ell\mathbb{Z} & \mathbb{Z}/\ell\mathbb{Z} \\ 0 & 1 & \mathbb{Z}/\ell\mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the Galois group of the cover $\underline{X} \to X$ factors through the quotient $\overline{\Delta}_X$.

More covers (3)

Write $\overline{\Delta}_{\underline{X}}$ for the kernel of the surjection $\overline{\Delta}_X \twoheadrightarrow \mathrm{Gal}(\underline{X}/X)$. Thus it is a free $\mathbb{Z}/\ell\mathbb{Z}$-module of rank 2.
Consider the action of $\underline{\iota}$ on the group $\overline{\Delta}_{\underline{X}}$. Its eigenvalues are equal to 1 and -1 (note that $1 \neq -1$ in $\mathbb{Z}/\ell\mathbb{Z}$ since ℓ is odd), thus the action is semisimple and we have a splitting

$$\overline{\Delta}_{\underline{X}} \cong \overline{\Delta}_{\underline{X}}^{ell} \oplus \overline{\Delta}_{\Theta}$$

into a product of eigenspaces with eigenvalues -1 and 1, respectively.
Thus we obtain a quotient map $\overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}_{\Theta}$ which determines a cover of $\underline{X}_{K^{\mathrm{alg}}}$ of degree ℓ (totally ramified over cusps). After choosing a decomposition group of a cusp of X, it descends to a cover

$$\underline{\underline{X}} \to \underline{X}$$

More covers (4)

Next, we define covers $\underline{\underline{Y}}$ and $\underline{\underline{\ddot{Y}}}$ of Y such that the squares below are cartesian.

$$\begin{tikzcd}
\underline{\underline{\ddot{Y}}} \arrow[r] \arrow[d] & \ddot{Y} \arrow[d] \\
\underline{\underline{Y}} \arrow[r] \arrow[d] & Y \arrow[d] \\
\underline{\underline{X}} \arrow[r] & \underline{X}
\end{tikzcd}$$

Write $Irr()$ for the set of irreducible components of the special fibre.
$\Rightarrow$ we have $Irr(Y) \cong Irr(\underline{\underline{Y}}) \cong Irr(\underline{\underline{\ddot{Y}}}) \cong Irr(\ddot{Y})$ and $Irr(\underline{\underline{X}}) \cong Irr(\underline{X})$.
$\Rightarrow$ labels for Y induce labels for all covers in the above diagram (up to $\pm$).

More covers (5)

To summarize, here is the full geometric picture

$$
\begin{tikzcd}
\underline{\underline{\ddot{Y}}} \arrow[r, "\mu_\ell"] \arrow[d, "\mu_2"] & \ddot{Y} \arrow[d, "\mu_2"] & \\
\underline{\underline{Y}} \arrow[r, "\mu_\ell"] \arrow[d, "\ell\mathbb{Z}"] & Y \arrow[d, "\ell\mathbb{Z}"] & \\
\underline{\underline{X}} \arrow[r, "\mu_\ell"] & \underline{X} \arrow[r, "\mathbb{Z}/\ell\mathbb{Z}"] \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] & X \arrow[d, "\mathbb{Z}/2\mathbb{Z}"] \\
& \underline{C} \arrow[r] & C.
\end{tikzcd}
$$

Anabelian remark, ([EtTh],§2, 2.6)

We need to mention one more anabelian result. For this, we need to assume additionally that the curve X is not K-arithmetic, equivalently that C is a K-core. We will not discuss the notion further, however we note that this assumption excludes only four j-invariants of E.
Under this assumption we may prove that the (tempered) fundamental groups of all (orbi-)curves from the previous slide can be reconstructed from the topological group $\Pi^{tp}_{\underline{\underline{X}}}$.
In particular, note that it implies that the vertex with label "0" in the dual graph of $\underline{X}$ can also be reconstructed group theoretically from $\Pi^{tp}_{\underline{\underline{X}}}$.

Kummer classes of functions

Next, we would like to consider functions on various (tempered) covers. Let S be a K-variety. From the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

we obtain a boundary map

$$\mathcal{O}(S)^* \to H^1_{et}(S, \mu_n) = H^1(\pi_1(S), \mu_n)$$

Image of $f \in \mathcal{O}(S)^*$ in this cohomology group = Kummer class of f. After taking a limit we have

$$\mathcal{O}(S)^* \to H^1(\pi_1(S), \widehat{\mathbb{Z}}(1)).$$

Theta functions, ([EtTh],§1)

The choice of label zero component determines coordinate functions U and $\ddot{U}$ on $\ddot{Y}$ and Y, satisfying $\ddot{U}^2 = U$.
Define an analytic function on Y:

$$\Theta(U) = \prod_{n\geq 0}(1 - q^n U)\prod_{n>0}(1 - q^n U^{-1})$$

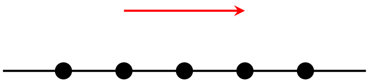
Observe that $\mathrm{Zeroes}(\Theta) = \{q^{\mathbb{Z}}\}$, each zero is of order one.
One easily checks the following two properties:

- (a). $\Theta(U^{-1}) = -U^{-1}\Theta(U)$,
- (b). $\Theta(q^n U) = (-1)^n q^{(n-n^2)/2} U^{-n}\Theta(U)$.

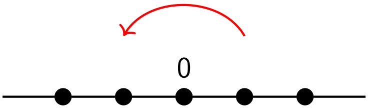
We will modify Θ to obtain a more "symmetric" function.

Geometric remark

Observe that the transformations $U \mapsto q^n U$ and $U \mapsto U^{-1}$ have a clear geometric meaning.



$U \mapsto q^n U$ corresponds to translation by n lines.



$U \mapsto U^{-1}$ corresponds to rotation around the component "0".

Theta functions (2)

Define an analytic function on $\ddot{Y}$:

$$\ddot{\Theta}(\ddot{U}) = \ddot{U}\Theta(\ddot{U}^2) = \ddot{U}\prod_{n\geq 0}(1 - q^n\ddot{U}^2)\prod_{n>0}(1 - q^n\ddot{U}^{-2})$$

Let $\ddot{q}$ be such that $\ddot{q}^2 = q$. From the properties (a) and (b) for Θ we have the following transformation formula for $\ddot{\Theta}$:

- (a) $\ddot{\Theta}(\ddot{U}^{-1}) = -\ddot{\Theta}(\ddot{U})$,
- (b) $\ddot{\Theta}(\ddot{q}^n\ddot{U}) = (-1)^n\ddot{q}^{-n^2}\ddot{U}^{-2n}\ddot{\Theta}(\ddot{U})$,
- (c) $\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$.

Thus $\ddot{\Theta}$ is more "symmetric" (esp. property (a)). We will sketch a characterization of a Kummer class $\ddot{\eta}^{\Theta}$ of $\ddot{\Theta}$ inside $H^1(\Pi^{tp}_{\ddot{Y}}, \widehat{\mathbb{Z}}(1))$

Various quotients (3)

Note that every inertia subgroup $I \hookrightarrow \Delta_X$ of the cusp maps isomorphically (through the surjection $\Delta_X \twoheadrightarrow \Delta_X^{\Theta}$) onto the subgroup $\Delta_{\Theta} \subset \Delta_X^{\Theta}$.
In particular, we have a canonical isomorphism of Galois modules

$$\Delta_{\Theta} \cong \widehat{\mathbb{Z}}(1).$$

Recall that $\Delta_X^{tp} \hookrightarrow \Delta_X$ (and $(\Delta_X^{tp})^{\wedge} \cong \Delta_X$).
We define further quotients:

$$\Delta_X^{tp} \twoheadrightarrow (\Delta_X^{tp})^{\Theta} \twoheadrightarrow (\Delta_X^{tp})^{ell}$$

by taking images of Δ_X^{tp} along the quotients

$$\Delta_X \twoheadrightarrow \Delta_X^{\Theta} \twoheadrightarrow \Delta_X^{ell}.$$

Various quotients (4)

We look at the matrix representation; thus

$$(\Delta_X^{tp})^{\Theta} \twoheadrightarrow (\Delta_X^{tp})^{ell}$$

corresponds to

$$\begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \widehat{\mathbb{Z}} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix} \twoheadrightarrow \widehat{\mathbb{Z}} \times \mathbb{Z}.$$

Note the discrete part corresponding to the topological cover $Y \to X$. Moreover, observe that we have

$$\Delta_{\Theta} = Ker((\Delta_X^{tp})^{\Theta} \twoheadrightarrow (\Delta_X^{tp})^{ell}).$$

Various quotients (5)

Similarly using the inclusion $\Delta_Y^{tp} \hookrightarrow \Delta_X^{tp}$ we define quotients:

$$\Delta_Y^{tp} \twoheadrightarrow (\Delta_Y^{tp})^{\Theta} \twoheadrightarrow (\Delta_Y^{tp})^{ell}$$

by taking images of Δ_Y^{tp} along the (previously constructed) quotients

$$\Delta_X^{tp} \twoheadrightarrow (\Delta_X^{tp})^{\Theta} \twoheadrightarrow (\Delta_X^{tp})^{\Theta}.$$

Thus, the quotient $(\Delta_Y^{tp})^{\Theta} \twoheadrightarrow (\Delta_Y^{tp})^{ell}$ corresponds to

$$\begin{bmatrix} 1 & \widehat{\mathbb{Z}} & \widehat{\mathbb{Z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \twoheadrightarrow \widehat{\mathbb{Z}}.$$

Various quotients (6)

Finally, by using inclusion $\Delta^{tp}_{\ddot{Y}} \hookrightarrow \Delta^{tp}_{Y}$ we define quotients:

$$\Delta^{tp}_{\ddot{Y}} \twoheadrightarrow (\Delta^{tp}_{\ddot{Y}})^{\Theta} \twoheadrightarrow (\Delta^{tp}_{\ddot{Y}})^{ell}$$

by taking images of $\Delta^{tp}_{\ddot{Y}}$ along the (previously defined) quotients

$$\Delta^{tp}_{Y} \twoheadrightarrow (\Delta^{tp}_{Y})^{\Theta} \twoheadrightarrow (\Delta^{tp}_{Y})^{ell}.$$

Thus, the quotient $(\Delta^{tp}_{\ddot{Y}})^{\Theta} \twoheadrightarrow (\Delta^{tp}_{\ddot{Y}})^{ell}$ corresponds to

$$\begin{bmatrix} 1 & 2\widehat{\mathbb{Z}} & \widehat{\mathbb{Z}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow\!\!\!\!\!\rightarrow 2\widehat{\mathbb{Z}}.$$

Characterising theta class

Recall that our goal is to characterise the Kummer class $\ddot{\eta}^{\Theta}$ inside the cohomology module $H^1(\Pi^{tp}_{\ddot{Y}}, \widehat{\mathbb{Z}}(1))$.
Using the canonical isomorphism $\Delta_{\Theta} \cong \widehat{\mathbb{Z}}(1)$ we may consider instead

$$\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_{\Theta}).$$

Define a quotient $\Pi^{tp}_{\ddot{Y}} \twoheadrightarrow (\Pi^{tp}_{\ddot{Y}})^{\Theta}$ which fits in the following s.e.s:

$$1 \to (\Delta^{tp}_{\ddot{Y}})^{\Theta} \to (\Pi^{tp}_{\ddot{Y}})^{\Theta} \to G_K \to 1.$$

Then, the class $\ddot{\eta}^{\Theta}$ belongs to $H^1((\Pi^{tp}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta}) \subset H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_{\Theta})$.
(roughly, it follows from the fact that Θ has zeroes of the same order at each cusp)

Characterising theta class (2)

We have the exact sequence:

$$1 \to H^1(G_K, \Delta_\Theta) \to H^1((\Pi^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta) \to H^1((\Delta^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta).$$

Let us first characterize the image of $\ddot{\eta}^\Theta$ in rightmost group.
From the s.e.s

$$1 \to \Delta_\Theta \to (\Delta^{tp}_{\ddot{Y}})^\Theta \to (\Delta^{tp}_{\ddot{Y}})^{ell} \to 1$$

we obtain

$$1 \to \mathrm{Hom}((\Delta^{tp}_{\ddot{Y}})^{ell}, \Delta_\Theta) \to \mathrm{Hom}((\Delta^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta) \to \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) \to 1.$$

Then the image of $\ddot{\eta}_\Theta$ in $\mathrm{End}(\Delta_\Theta)$ is the identity (this is equivalent to the fact that all zeroes of Θ are of order one).

Characterising theta class (3)

What is the image of $\ddot{\eta}^{\Theta}$ in $\mathrm{Hom}((\Delta^{tp}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta})$?
Let P_0 = projective line with label '0', and let

$$\iota \colon Y \to Y$$

be the "rotation" fixing P_0 (corresponds to $U \mapsto U^{-1}$). The map ι lies over the automorphism $[-1] \colon E \to E$ of the elliptic curve E.
Then, ι induces an automorphism of

$$(\widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}} \cong)\ \mathrm{Hom}((\Delta^{tp}_{Y})^{\Theta}, \Delta_{\Theta}) \hookrightarrow \mathrm{Hom}((\Delta^{tp}_{\ddot{Y}})^{\Theta}, \Delta_{\Theta})\ (\cong \widehat{\mathbb{Z}} \oplus \frac{1}{2}\widehat{\mathbb{Z}})$$

which may be described group theoretically (since the dual graph of the special fibre can be reconstructed).
We claim that the image of $\ddot{\eta}^{\Theta}$ is the unique class invariant with respect to ι and mapping to identity in $\mathrm{Hom}(\Delta_{\Theta}, \Delta_{\Theta})$.

Characterizing theta class (4)

Indeed, using an isomorphism $\mathrm{Hom}((\Delta_Y^{tp})^{\Theta}, \Delta_{\Theta}) \cong \widehat{\mathbb{Z}}^2$, the automorphism ι may be represented by a matrix

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus, $\ddot{\eta}^{\Theta}$ corresponds to the eigenvector $[1, \frac{1}{2}]^T$.
(Note that the appearance of 2 in denominator explains why we need to introduce the double cover $\ddot{Y} \to Y$).
Therefore, we can characterise $\ddot{\eta}^{\Theta}$ up to an element from

$$H^1(G_K, \Delta_{\Theta}) \cong \widehat{K^*}.$$

Reducing $\widehat{K^*}$-indeterminacy

How to reduce this indeterminacy? We normalize $\ddot{\Theta}$ to have a fixed value at a fixed point. If S is a K-scheme, $P \in S(K)$, $D \hookrightarrow \pi_1(S)$ a decomposition group of P then we have a diagram:

$$\begin{array}{ccc} \mathcal{O}(S)^* & \longrightarrow & H^1(\pi_1(S), \widehat{\mathbb{Z}}(1)) \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & H^1(D, \widehat{\mathbb{Z}}(1)) \cong \widehat{K^*}. \end{array}$$

There exists a unique nontrivial 2-torsion point μ_- on E whose reduction lies in the smooth locus (namely the class of -1 in $(K^{\mathrm{alg}})^*/q^{\mathbb{Z}}$).

Reducing $\widehat{K^*}$-indeterminacy (2)

Using a technique called elliptic cuspidalization, together with the reconstruction of the dual graph we can construct a decomposition subgroup of μ_- inside Π_X^{tp}. Then, we lift μ_- to Y and $\ddot{Y}$ to obtain two collections of points:

$$\{-q^n\}_{n\in\mathbb{Z}} \text{ (in } Y) \quad \text{and} \quad \{\pm i\ddot{q}^n\}_{n\in\mathbb{Z}} \text{ (in } \ddot{Y}).$$

(here $i^2 = -1$ and $\ddot{q}^2 = q$, note that $\ddot{q}$ lies in K).
Then we choose points whose reduction lies on the fixed projective line P_0 (namely -1 for Y and $\pm i$ for $\ddot{Y}$), denote them by $\mu_-(Y)$ and $\pm\mu_-(\ddot{Y})$.
Finally, we normalize $\ddot{\Theta}$ be requiring $\ddot{\Theta}(\mu_-(\ddot{Y})) = \pm 1$. Note that since $\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$, this is well defined.
This finishes our reconstruction of $\ddot{\Theta}$, up to a sign.

Final remarks

- In fact, we construct only a $\mathbb{Z} \times \mu_2$-orbit of $\ddot{\Theta}$ since there is no natural choice of irreducible component of special fibre of Y (cf. our choice of P).
- Construction of $\ddot{\Theta}$ in §1 of [EtTh] is slightly different; the function $\ddot{\Theta}$ is obtained as a quotient of two sections of a line bundle associated to cusps.
- Essential properties of $\ddot{\Theta}$: simple zeroes at cusps and symmetry with respect to the transformation $\ddot{U} \mapsto \ddot{U}^{-1}$
- If $\ddot{\Theta}$ is normalized $\ddot{\Theta}(\mu_-(\ddot{Y})) = \pm 1$ then

$$\ddot{\Theta}(\pm i\ddot{q}^n)^{-1} = \pm \ddot{q}^{n^2}$$

  Note that the points $\pm i\ddot{q}^n$ are translations of the evaluation point $\mu_-(\ddot{Y})$ by the group $\mathrm{Aut}(\ddot{Y}/X)$.

ℓth root of theta function, ([EtTh],§2, 2.6.1)

In fact, in the theory we will be using an ℓth root of theta function $\ddot{\Theta}$. Assume that $E[2\ell] \subset E(K)$. Recall that $\ddot{\eta}^{\Theta}$ is a cohomology class in $H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_{\Theta})$. Then, there exists a cohomology class

$$\underline{\underline{\ddot{\eta}}}^{\Theta} \in H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \ell\Delta_{\Theta})$$

which satisfies the property that the image of $\underline{\underline{\ddot{\eta}}}^{\Theta}$ along the map

$$H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \ell\Delta_{\Theta}) \to H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \Delta_{\Theta})$$

induced by $\ell\Delta_{\Theta} \hookrightarrow \Delta_{\Theta}$ is equal to the restriction of $\ddot{\eta}^{\Theta}$ to the open subgroup $\Pi^{tp}_{\underline{\underline{\ddot{Y}}}} \subset \Pi^{tp}_{\ddot{Y}}$ along the map

$$H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_{\Theta}) \to H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \Delta_{\Theta}).$$

ℓth root of theta function (2)

The cohomology class we have obtained, namely

$$\underline{\underline{\ddot{\eta}}}^{\Theta} \in H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}, \ell\Delta_{\Theta})$$

is called the ℓth root of theta function. Note that if the cusp "0" of $\underline{X}$ is fixed, then the class $\underline{\underline{\ddot{\eta}}}^{\Theta}$ is well defined up to an action of $\ell\mathbb{Z} \times \mu_{2\ell}$. Write (temporarily) $\underline{\underline{\theta}}$ for the reciprocal $-\underline{\underline{\ddot{\eta}}}^{\Theta}$ of $\underline{\underline{\ddot{\eta}}}^{\Theta}$. Then, if we define

$$\underline{\underline{q}} = q^{1/2\ell}$$

then the value of $\underline{\underline{\theta}}$ at a "translated cusp" of $\underline{\underline{\ddot{Y}}}$ with label j is equal to

$$\underline{\underline{q}}^{j^2}.$$

Summary and next steps

Let us pause for a moment to discuss where we are at the moment.
We have introduced various covers of the punctured elliptic curve X as well as various theta functions defined on them. We described some of their properties and gave a group theoretic construction of its cohomology class.
As the next step, we will introduce the notion of a mono-theta environment, a collection of data closely related to theta function.
Roughly speaking, the purpose of this new notion is to analyse the rigidity of the Kummer morphism between the "group theoretic" construction of theta function (étale-like) and the "real" theta function (Frobenius-like).

References

Main references:

- S. Mochizuki - "Etale Theta Function and its Frobenioid-theoretic Manifestations"
- S. Mochizuki - "Inter-Universal Teichmüller Theory II: Hodge-Arakelov Theoretic Evaluation"

Regarding the theory of [EtTh], the following additional references are useful:

- D. Mumford, "An analytic construction of degenerated abelian varieties over complete fields"
- J. Silverman, "Advanced Topics in the Arithmetic of Elliptic Curves"

End of the talk

Thank you for your attention!