Multiradial Representations and Log-volume Estimates

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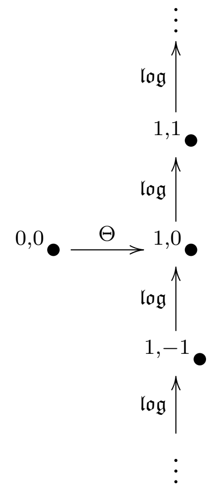
Inter-universal Teichmüller Theory Summit 2021



each of •: a Hodge theater, a miniature model of conventional scheme theory

$\overline{k}$: a fixed algebraic closure of $\mathbb{Q}_p$ for some p

$\Rightarrow \log_{\overline{k}} \colon \mathcal{O}^{\times}_{\overline{k}} \to \overline{k}_+$ induces an isomorphism $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \overset{\text{def}}{=} \mathcal{O}^{\times}_{\overline{k}}/\boldsymbol{\mu}(\overline{k}) \overset{\sim}{\to} \overline{k}_+$.

$$\mathfrak{log}: \qquad {}^{1,0}\overline{k}^{\times} \hookleftarrow {}^{1,0}\mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow {}^{1,0}\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \text{ “} \overset{\log_{\overline{k}}}{\overset{\sim}{\to}} \text{” } {}^{1,1}\overline{k}_+ \hookleftarrow {}^{1,1}\overline{k}^{\times}$$

Arithmetic Line Bundles on $[\mathrm{Spec}(\mathcal{O}_K/\mathrm{Gal}(K/F_{\mathrm{mod}}))]$

Purely Multiplicative Description

An a.l.b. may be defined to be an $F_{\mathrm{mod}}^{\times}$-torsor T equipped with
a suitable collection of trivializations $\big((F_{\mathrm{mod}}^{\times} \to K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times})_* T \xrightarrow{\sim} K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times}\big)_{\underline{v}\in\underline{\mathbb{V}}}$.

$\Rightarrow$ a Frobenioid of a.l.b.'s

$\overset{\text{realified}}{\Rightarrow}$ a realified Frobenioid A.L.B. of a.l.b.'s

$\big($the set of isom. cl.s of ob.s $\cong \big(\bigoplus_{\underline{v}\in\underline{\mathbb{V}}}(K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times})^{\mathbb{R}}\big)/(\text{the prod. form.}) \overset{\text{deg.}}{\xrightarrow{\sim}} \mathbb{R}\big)$

Module/Ideal-theoretic Description

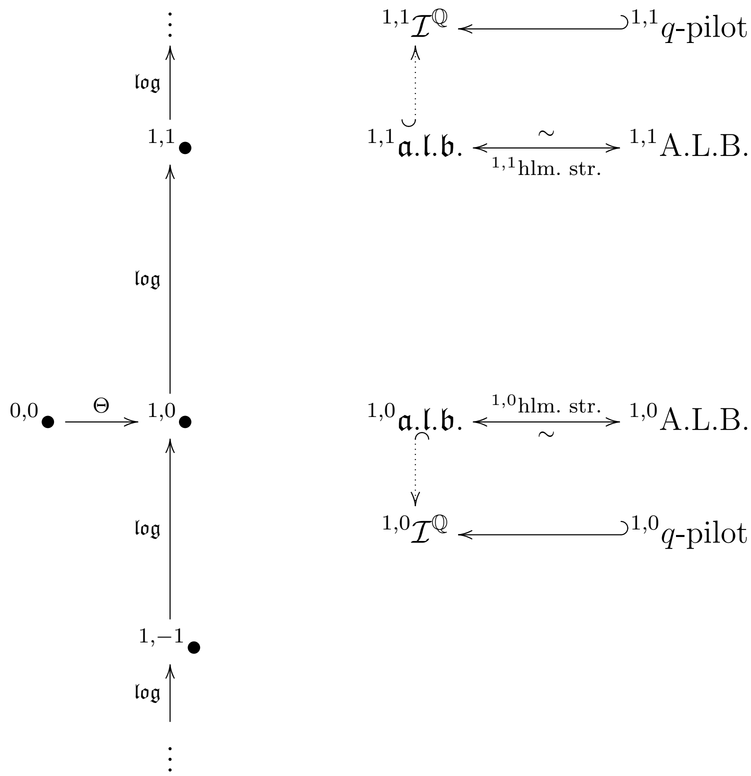
An a.l.b. may be defined to be
a suitable collection $(J_{\underline{v}})_{\underline{v}\in\underline{\mathbb{V}}}$ of fractional ideals $J_{\underline{v}} \subseteq K_{\underline{v}} = \mathcal{I}_{\underline{v}}^{\mathbb{Q}}$.

$\Rightarrow$ a Frobenioid of a.l.b.'s

$\overset{\text{realified}}{\Rightarrow}$ a realified Frobenioid $\mathfrak{a.l.b.}$ of a.l.b.'s

By the arith. hol. str.s involved, we have a natural identification A.L.B. $\overset{\sim}{\leftrightarrow}$ $\mathfrak{a.l.b.}$.

the q-pilot object $\in \mathfrak{a.l.b.} \overset{\text{def}}{=} \big((\text{the } q\text{-parameter at } \underline{v})^{1/2l} \cdot \mathcal{O}_{\underline{v}}\big)_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$
$\Rightarrow$ $-$log-vol(q-pilot) $\approx$ the height of E



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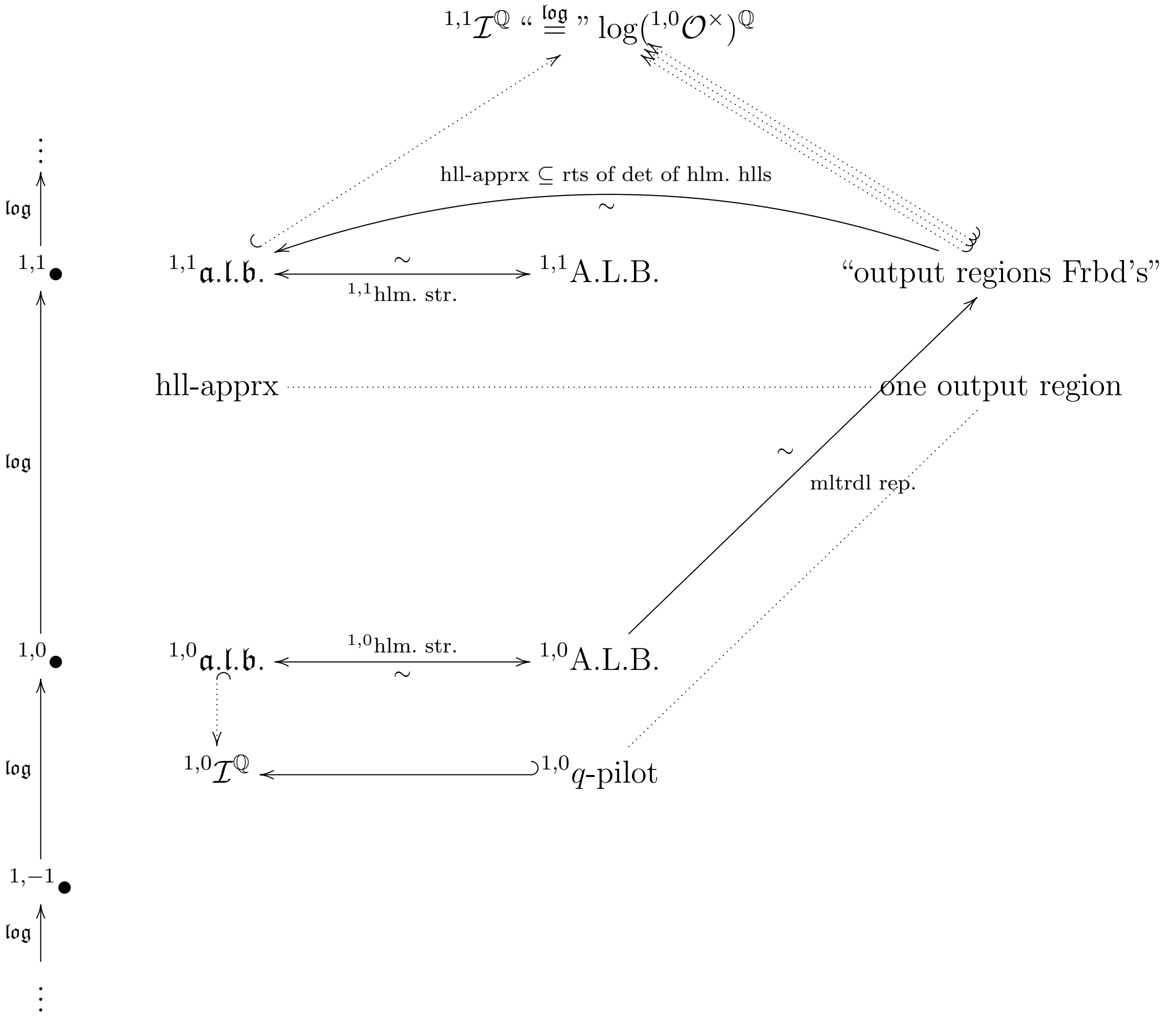
$$\mathfrak{log}: \qquad {}^{1,0}\overline{k}^{\times} \hookleftarrow {}^{1,0}\mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow {}^{1,0}\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \text{ “} \overset{\log_{\overline{k}}}{\overset{\sim}{\to}} \text{” } {}^{1,1}\overline{k}_{+} \hookleftarrow {}^{1,1}\overline{k}^{\times}$$

$\Rightarrow$ $\mathfrak{log}$ induces neither ${}^{1,0}F^{\times}_{\mathrm{mod}} \overset{\sim}{\to} {}^{1,1}F^{\times}_{\mathrm{mod}}$ nor ${}^{1,0}\mathcal{I}_{\underline{v}} \overset{\sim}{\to} {}^{1,1}\mathcal{I}_{\underline{v}}$ at least in a naive sense.

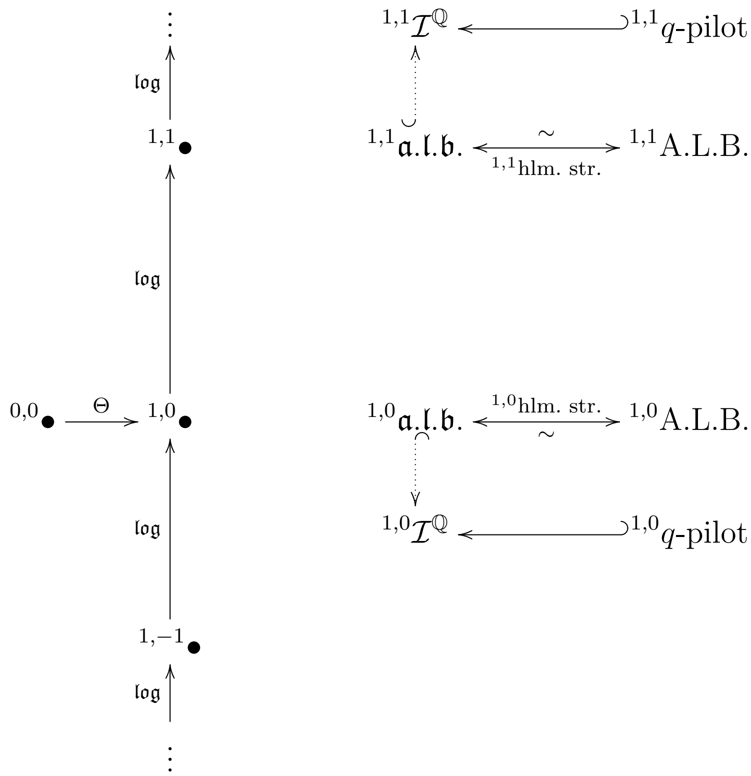


$$\mathfrak{log}: \qquad {}^{1,0}\overline{k}^{\times} \hookleftarrow {}^{1,0}\mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow {}^{1,0}\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \text{ “ } \overset{\log_{\overline{k}}}{\overset{\sim}{\to}} \text{ ” } {}^{1,1}\overline{k}_{+} \hookleftarrow {}^{1,1}\overline{k}^{\times}$$

$\mathcal{H}$ "$(\subseteq \mathcal{I}^{\mathbb{Q}})$":

the holomorphic hull of (i.e., roughly speaking, the $\mathcal{O}$-module generated by) the union of the possible output regions

$\Rightarrow$ "$({}^{1,1}\mathcal{I}^{\mathbb{Q}} \supseteq)$" $\forall$output region $\subseteq {}^{1,1}\mathcal{H}$ "$(\subseteq {}^{1,1}\mathcal{I}^{\mathbb{Q}})$"



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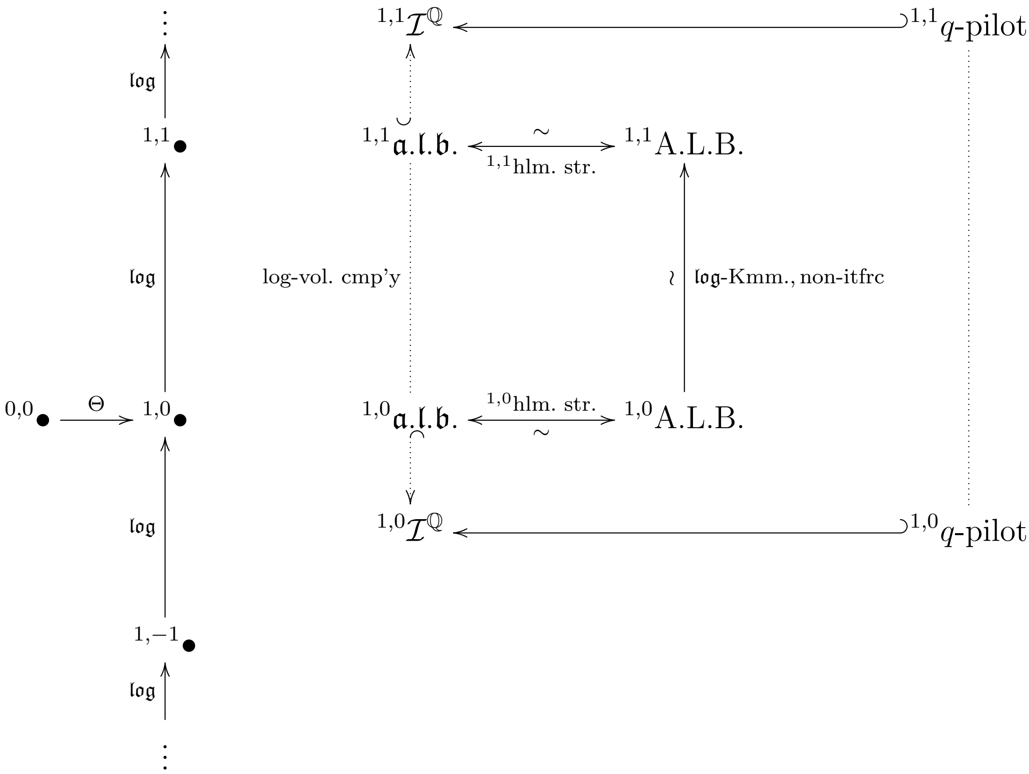

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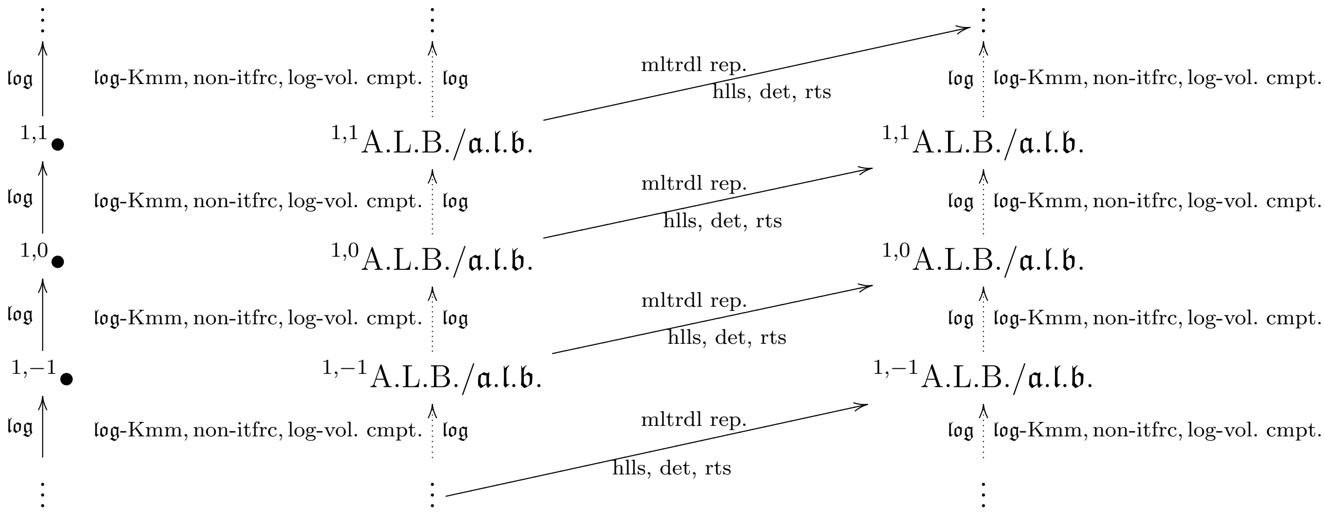
$$\mathfrak{log} : \qquad {}^{1,0}\overline{k}^{\times} \hookleftarrow {}^{1,0}\mathcal{O}^{\times}_{\overline{k}} \twoheadrightarrow {}^{1,0}\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \text{ “ } \overset{\log_{\overline{k}}}{\overset{\sim}{\to}} \text{ ” } {}^{1,1}\overline{k}_{+} \hookleftarrow {}^{1,1}\overline{k}^{\times}$$

$\mathfrak{log}$-Link Compatibility of Log-volumes

$k/\mathbb{Q}_p$: a fin. ext., $S \subseteq \mathcal{O}_k^{\times}$ s.t. $S \overset{\log_k}{\twoheadrightarrow} \log_k(S)$ is bijective $\Rightarrow$ log-vol(S) = log-vol$(\log_k(S))$

Non-interference with Local Integers in Global Kummer Theory

$F^{\times}_{\mathrm{mod}} \cap (\prod_{\underline{v} \in \underline{\mathbb{V}}} \mathcal{O}^{\times}_{K_{\underline{v}}})$ in $\prod_{\underline{v} \in \underline{\mathbb{V}}} K^{\times}_{\underline{v}}$ is $= \boldsymbol{\mu}(F_{\mathrm{mod}})$, whose images by the local log's are $= \{0\}$



a constructed "A.L.B./$\mathfrak{a.l.b.}$" + suitable automorphisms of constructed regions
$\rightsquigarrow$ an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip, to which one may apply
both the original q-intertwining and the intertwining discussed above

the original q-inter. $\curvearrowright$ an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-pr.-st. $\curvearrowleft$ the inter. arising from the multirad. rep.

$$\Rightarrow \quad \text{log-vol}(q\text{-pilot}) \quad \leq \quad \text{log-vol}(\mathcal{H}) \quad (< \infty)$$

(cf. "$({}^{1,1}\mathcal{I}^{\mathbb{Q}} \supseteq)$" $\forall$output region $\subseteq {}^{1,1}\mathcal{H}$ "$(\subseteq {}^{1,1}\mathcal{I}^{\mathbb{Q}})$")

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