Explicit Estimates in Inter-universal Teichmüller Theory (joint work w/ S. Mochizuki, I. Fesenko, Y. Hoshi, and W. Porowski)

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Notations

$\mathfrak{Primes}$: the set of all prime numbers

F: a number field $\supseteq$ $\mathcal{O}_F$: the ring of integers

Δ_F: the absolute value of the discriminant of F

$\mathbb{V}(F)^{\mathrm{non}}$: the set of nonarchimedean places of F

$\mathbb{V}(F)^{\mathrm{arc}}$: the set of archimedean places of F

$\mathbb{V}(F) \stackrel{\mathrm{def}}{=} \mathbb{V}(F)^{\mathrm{non}} \bigcup \mathbb{V}(F)^{\mathrm{arc}}$

For $v \in \mathbb{V}(F)$, write F_v for the completion of F at v

For $v \in \mathbb{V}(F)^{\mathrm{non}}$, write $\mathfrak{p}_v \subseteq \mathcal{O}_F$ for the prime ideal corr. to v

- Let $v \in \mathbb{V}(F)^{\mathrm{non}}$. Write $\mathrm{ord}_v : F^\times \twoheadrightarrow \mathbb{Z}$ for the order def'd by v. Then for any $x \in F$, we shall write

$$|x|_v \stackrel{\mathrm{def}}{=} \sharp(\mathcal{O}_F/\mathfrak{p}_v)^{-\mathrm{ord}_v(x)}.$$

- Let $v \in \mathbb{V}(F)^{\mathrm{arc}}$. Write $\sigma_v : F \hookrightarrow \mathbb{C}$ for the embed. det'd, up to complex conjugation, by v. Then for any $x \in F$, we shall write

$$|x|_v \stackrel{\mathrm{def}}{=} |\sigma_v(x)|_{\mathbb{C}}^{[F_v:\mathbb{R}]}.$$

Note: (Product formula) For $\alpha \in F^\times$, it holds that

$$\prod_{v \in \mathbb{V}(F)} |\alpha|_v = 1.$$

For an elliptic curve E /a field, write $j(E)$ for the j-invariant of E

Log-volume estimates for Θ-pilot objects (cf. [IUTchIII], Cor 3.12)

Theorem

Write

$$-|\log(\underline{\underline{\Theta}})| \in \mathbb{R} \cup \{\infty\}$$

for the (process.-normalized, mono-an.) log-volume of the "holomorphic hull" of the union of the possible images of a Θ-pilot object, rel. to the relevant Kum. isoms, in the multira'l rep'n of [IUTchIII], Thm 3.11, (i), which we regard as sub. to (Ind1), (Ind2), (Ind3);

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$$-|\log(\underline{\underline{q}})| \in \mathbb{R}$$

for the (process.-normalized, mono-an.) log-volume of the image of a q-pilot object, rel. to the relevant Kum. isoms, in the multirad'l rep'n.

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$$-|\log(\underline{\underline{q}})| \;\in\; \mathbb{R}$$

for the (process.-normalized, mono-an.) log-volume of the image of a $\underline{\underline{q}}$*-pilot object, rel. to the relevant Kum. isoms, in the multirad'l rep'n.*

Then it holds that $-|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$, *and* $-|\log(\underline{\underline{\Theta}})| \geq -|\log(\underline{\underline{q}})|$.

Results in [IUTchIV]

For $\lambda \in \overline{\mathbb{Q}} \setminus \{0,1\}$,

A_λ: the elliptic curve $/\mathbb{Q}(\lambda)$ def'd by "$y^2 = x(x-1)(x-\lambda)$"

$F_\lambda \stackrel{\text{def}}{=} \mathbb{Q}(\lambda, \sqrt{-1}, A_\lambda[3 \cdot 5](\overline{\mathbb{Q}}))$

$\Rightarrow E_\lambda \stackrel{\text{def}}{=} A_\lambda \times_{\mathbb{Q}(\lambda)} F_\lambda$ has at most split multipl. red. at $\forall \in \mathbb{V}(F_\lambda)$

$\mathfrak{q}_\lambda$: the arithmetic divisor det'd by the q-parameters of E_λ/F_λ

$\mathfrak{f}_\lambda$: the "reduced" arithmetic divisor det'd by $\mathfrak{q}_\lambda$

$\mathfrak{d}_\lambda$: the arithmetic divisor det'd by the different of $F_\lambda/\mathbb{Q}$

Theorem (Vojta Conj. — in the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ — for "$\mathcal{K}$")

Let $d \in \mathbb{Z}_{>0}$, $\epsilon \in \mathbb{R}_{>0} \cap \mathbb{R}_{\leq 1}$,

$\mathcal{K} \subseteq \overline{\mathbb{Q}} \setminus \{0, 1\}$: a compactly bounded subset whose "support" $\ni 2, \infty$.

Then $^{\exists}B(d, \epsilon, \mathcal{K}) \in \mathbb{R}_{>0}$ — that depends only on d, ϵ, and $\mathcal{K}$ — s.t. the function on $\mathcal{K}^{\leq d} \stackrel{\text{def}}{=} \{\lambda \in \mathcal{K} \mid [\mathbb{Q}(\lambda) : \mathbb{Q}] \leq d\}$ given by

$$\lambda \mapsto \frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda) - (1 + \epsilon) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda))$$

is bounded by $B(d, \epsilon, \mathcal{K})$.

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Proof: By applying

- the finiteness of $\{\lambda \in \mathcal{K}^{\leq d} \mid \deg(\mathfrak{q}_\lambda) \leq \gamma\}$ $(\gamma \in \mathbb{R}_{>0})$ (cf. Northcott's theorem),

- $\sharp\{j($"arithmetic" elliptic curve over a field of char. zero$)\} = 4$ (cf. Takeuchi's list),
- the prime number theorem,
- the theory of Galois actions on torsion points of elliptic curves (cf. [GenEll]),

we conclude that for all but finitely many $\lambda \in \mathcal{K}^{\leq d}$, there exists a prime number l_λ such that

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we conclude that for all but finitely many $\lambda \in \mathcal{K}^{\leq d}$, there exists a prime number l_λ such that

(i) $^\exists$an initial Θ-data $(\overline{\mathbb{Q}}/F_\lambda,\ E_\lambda,\ l_\lambda,\ \dots)$ s.t. E_λ has good red. at every $\in \mathbb{V}(F_\lambda)^{\mathrm{good}} \cap \mathbb{V}(F_\lambda)^{\mathrm{non}}$ that does not divide $2l_\lambda$

(In the following, we shall write

$\mathfrak{q}_\lambda^{\mathrm{bad}}$: the arithmetic divisor det'd by "restricting $\mathfrak{q}_\lambda$ to $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$".)

(ii) $\frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) \;\leq\; (1 + \frac{20 d_\lambda}{l_\lambda}) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda)) + 20 \cdot \delta_\lambda \cdot l_\lambda$,

where $d_\lambda := [\mathbb{Q}(\lambda) : \mathbb{Q}]$, $\delta_\lambda := 2^{12} \cdot 3^3 \cdot 5 \cdot d_\lambda$ (cf. (i); "Cor 3.12")

(iii) $\mathrm{ord}_{l_\lambda}(q_\Box) \;<\; \deg(\mathfrak{q}_\lambda)^{1/2}$, where $\mathbb{V}(F_\lambda) \ni \Box | l_\lambda$

(iv) $\deg(\mathfrak{q}_\lambda)^{1/2} \;\leq\; l_\lambda \;\leq\; 10 \cdot \delta \cdot \deg(\mathfrak{q}_\lambda)^{1/2} \cdot \log(2 \cdot \delta \cdot \deg(\mathfrak{q}_\lambda))$

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Then it follows from (i), (iii) [cf. also the "compactness" of $\mathcal{K}$] that

$$\lambda \ \mapsto \ \frac{1}{6}\deg(\mathfrak{q}_\lambda) - \frac{1}{6}\deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) - \deg(\mathfrak{q}_\lambda)^{1/2}\log(2\delta\deg(\mathfrak{q}_\lambda))$$

is bounded. On the other hand, it follows from (ii), (iv) that

$$\frac{1}{6}\deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) \;\leq\; (1+\delta\cdot\deg(\mathfrak{q}_\lambda)^{-1/2})(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda)) + 200\delta^2\cdot\deg(\mathfrak{q}_\lambda)^{1/2}\log(2\delta\deg(\mathfrak{q}_\lambda)).$$

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In particular, these two displays imply that $\lambda \mapsto$

$$(1-\tfrac{2}{5}\tfrac{(60\delta)^2\log(2\delta\deg(\mathfrak{q}_\lambda))}{\deg(\mathfrak{q}_\lambda)^{1/2}})\tfrac{1}{6}\deg(\mathfrak{q}_\lambda) - (1+\tfrac{\delta}{\deg(\mathfrak{q}_\lambda)^{1/2}})(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda))$$

is bounded.

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is bounded. By enlarging our "exceptional set", we conclude that

$$\lambda \;\mapsto\; \frac{1}{6}\cdot\deg(\mathfrak{q}_\lambda)-(1+\epsilon)\cdot(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda))$$

is bounded. This completes the proof of Theorem.

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Then, by applying the theory of noncritical Belyi maps, we obtain

($*$): the "version with $\mathcal{K}$ removed" of Theorem (cf. [GenEll]).

Theorem (Corollary of $(*)$ — ABC Conjecture for number fields)

Let $d \in \mathbb{Z}_{>0}$, $\epsilon \in \mathbb{R}_{>0} \cap \mathbb{R}_{\leq 1}$.

Then ${}^{\exists}C(d, \epsilon) \in \mathbb{R}_{>0}$ — that depends only on d and ϵ — s.t. for

- F: a number field — where $d = [F : \mathbb{Q}]$
- (a, b, c) : a triple of elements $\in F^{\times}$ — where $a + b + c = 0$

we have

$$H_F(a, b, c) < C(d, \epsilon) \cdot (\Delta_F \cdot \mathrm{rad}_F(a, b, c))^{1+\epsilon}$$

— where

$$H_F(a, b, c) \overset{\mathrm{def}}{=} \prod_{v \in \mathbb{V}(F)} \max\{|a|_v, |b|_v, |c|_v\},$$

$$\mathrm{rad}_F(a, b, c) \overset{\mathrm{def}}{=} \prod_{\{v \in \mathbb{V}(F)^{\mathrm{non}} \mid \sharp\{|a|_v, |b|_v, |c|_v\} \geq 2\}} \sharp(\mathcal{O}_F/\mathfrak{p}_v).$$

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Note: We do not know the constant "$C(d,\epsilon)$" explicitly.

For instance, it is hard to compute noncritical Belyi maps explicitly.

Computations concerning (ii)

For $\gamma \in \mathbb{R}$, we shall write $\lfloor \gamma \rfloor$ (resp. $\lceil \gamma \rceil$) for the largest integer $\leq \gamma$ (resp. the smallest integer $\geq \gamma$).

$\{k_i\}_{i \in I}$: a finite set of p-adic local fields ($\mathcal{O}_{k_i}$: the ring of integers)

e_i (resp. $\mathfrak{d}_i$): the abs. ram. index (resp. the order of an gen. of δ_{k_i})

$$a_i \overset{\text{def}}{=} \begin{cases} \frac{1}{e_i} \lceil \frac{e_i}{p-2} \rceil & (p > 2) \\ 2 & (p = 2) \end{cases} \qquad b_i \overset{\text{def}}{=} \lfloor \frac{\log(p \cdot e_i/(p-1))}{\log(p)} \rfloor - \frac{1}{e_i}$$

$$a_I \overset{\text{def}}{=} \sum_{i \in I} a_i, \qquad b_I \overset{\text{def}}{=} \sum_{i \in I} b_i, \qquad \mathfrak{d}_I \overset{\text{def}}{=} \sum_{i \in I} \mathfrak{d}_i$$

$\mu^{\log}_{k_I}$: the (nor'd) log-vol. on $k_I \overset{\text{def}}{=} \otimes_{i \in I} k_i$ s.t. $\mu^{\log}_{k_I}(\otimes_{i \in I} \mathcal{O}_{k_i}) = 0$

Lemma

For $\lambda \in \frac{1}{e_i}\mathbb{Z}$, *write* $p^{\lambda}\mathcal{O}_{k_i}$ *for the fractional ideal generated by any element* $x \in k_i$ *s.t.* $\mathrm{ord}(x) = \lambda$. *Let*

$$\phi : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^{\times}) \xrightarrow{\sim} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^{\times})$$

be an automorphism of the finite dimensional $\mathbb{Q}_p$*-vector space that induces an automorphism of the submodule* $\bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^{\times})$.

Lemma

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$$\phi : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^\times) \overset{\sim}{\to} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^\times)$$

be an automorphism of the finite dimensional $\mathbb{Q}_p$*-vector space that induces an automorphism of the submodule* $\bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^\times)$.

(i) *Write* $I^* \overset{\mathrm{def}}{=} \{i \in I \mid e_i > p-2\}$. *For any* $\lambda \in \frac{1}{e_{i_0}}\mathbb{Z}$, $i_0 \in I$,

$$\phi(p^\lambda (\otimes_{i \in I} \mathcal{O}_{k_i})^\sim) \bigcup p^{\lfloor \lambda \rfloor} \bigotimes_{i \in I} \tfrac{1}{2p} \log_p(\mathcal{O}_{k_i}^\times)$$

$$\subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \bigotimes_{i \in I} \log_p(\mathcal{O}_{k_i}^\times) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} (\otimes_{i \in I} \mathcal{O}_{k_i})^\sim.$$

Moreover, we have

$$\mu_{k_I}^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I}(\otimes_{i \in I}\mathcal{O}_{k_i})^{\sim})$$

$$\leq \ (-\lambda + \mathfrak{d}_I + 1)\log(p) + \sum_{i \in I^*}(3 + \log(e_i)).$$

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(ii) Suppose that $p > 2$ and $e_i = 1$ $({}^{\forall} i \in I)$. Then

$$\phi((\otimes_{i \in I} \mathcal{O}_{k_i})^{\sim}) \ \subseteq \ \bigotimes_{i \in I} \tfrac{1}{2p} \log_p(\mathcal{O}_{k_i}^{\times}) \ = \ (\otimes_{i \in I} \mathcal{O}_{k_i})^{\sim}.$$

Moreover, we have

$$\mu_{k_I}^{\log}((\otimes_{i \in I} \mathcal{O}_{k_i})^{\sim}) \ = \ 0.$$

In the following discussion, for simplicity, write

$$(\overline{F}/F,\ E,\ l,\ \dots) \stackrel{\mathrm{def}}{=} (\overline{\mathbb{Q}}/F_\lambda,\ E_\lambda,\ l_\lambda,\ \dots).$$

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($\Rightarrow\ K = F(E[l]) \supseteq F \supseteq F_{\text{mod}}$: the field of moduli of E)

$d_{\text{mod}} \stackrel{\text{def}}{=} [F_{\text{mod}} : \mathbb{Q}] \geq e_{\text{mod}} \stackrel{\text{def}}{=}$ the max. ram. index of $F_{\text{mod}}/\mathbb{Q}$

$d^*_{\text{mod}} \stackrel{\text{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\text{mod}} \geq e^*_{\text{mod}} \stackrel{\text{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\text{mod}}$

$\mathbb{V}_{\mathbb{Q}}^{\text{non}} \stackrel{\text{def}}{=} \mathbb{V}(\mathbb{Q})^{\text{non}} \supseteq \mathbb{V}_{\mathbb{Q}}^{\text{dst}} \stackrel{\text{def}}{=} \{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}} \mid v_{\mathbb{Q}}$ ramifies in $K\}$

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Let us compute an upper bound for the

> *(process.-normalized, mono-an.) log-volume of the "holomorphic hull" of the union of the possible images of a Θ-pilot object, rel. to the relevant Kum. isoms, in the multira'l rep'n of [IUTchIII], Thm 3.11, (i), which we regard as sub. to (Ind1), (Ind2), (Ind3)*

(A) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}$. Fix j $(\in \{1, 2, \ldots, l^*\})$ and the collection

$$\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$$

of [not necessarily distinct] elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$. Write $\underline{v}_i \in \underline{\mathbb{V}}$ $\stackrel{\sim}{\to} \mathbb{V}(F_{\mathrm{mod}})$ for the elem't corr. to v_i.

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of [not necessarily distinct] elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$. Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}(F_{\mathrm{mod}})$ for the elem't corr. to v_i. Then, by applying Lem, (i), we obtain an upper bound on the component of the log-volume in question corresponding to the tensor product of the $\mathbb{Q}$-spans of the log-shells associated to the collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as follows:

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$$(-\lambda + \mathfrak{d}_I + 1)\log(p_{v_{\mathbb{Q}}}) + 4(j+1)\iota_{v_{\mathbb{Q}}} \log(e^*_{\mathrm{mod}} \cdot l)$$

— where $\lambda = \begin{cases} \frac{j^2}{2l}\mathrm{ord}(q_{\underline{v}_j}) & (\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}) \\ 0 & (\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}) \end{cases}$ $\iota_{v_{\mathbb{Q}}} = \begin{cases} 1 & (p_{v_{\mathbb{Q}}} \le e^*_{\mathrm{mod}} l) \\ 0 & (p_{v_{\mathbb{Q}}} > e^*_{\mathrm{mod}} l) \end{cases}$

(B) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}} \setminus \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}$. Fix $j \ (\in \{1, 2, \ldots, l^{*}\})$ and $\{v_i\}_{i \in \mathbb{S}_{j+1}^{\pm}}$ as in (A).

(B) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}} \setminus \mathbb{V}_{\mathbb{Q}}^{\mathrm{dst}}$. Fix $j\ (\in \{1, 2, \dots, l^{*}\})$ and $\{v_i\}_{i \in \mathbb{S}_{j+1}^{\pm}}$ as in (A). Then, by applying Lem, (ii), we obtain an upper bound on the comp. of the log-vol. in question corr. to the tensor prod. of the $\mathbb{Q}$-spans of the log-shells assoc. to $\{v_i\}_{i \in \mathbb{S}_{j+1}^{\pm}}$ as follows:

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$$(j+1) \cdot \log(\pi)$$

After computing a "weighted average upper bound", i.e.,

$$\left(\frac{1}{[F_{\mathrm{mod}}:\mathbb{Q}]}\right)^{j+1} \sum_{v_0,\dots,v_j \in \mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}} \prod_{0 \le i \le j} [(F_{\mathrm{mod}})_{v_i} : \mathbb{Q}_{v_{\mathbb{Q}}}](-)$$

and then a "procession-normalized upper bound", i.e.,

$$\frac{1}{l^*} \sum_{1 \le j \le l^*} (-)$$

for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, by summing over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ these estimates, we obtain an upper bound on $-|\log(\underline{\underline{\Theta}})|$ as follows:

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$$\frac{l+1}{4}\Big\{(1+\frac{12 d_{\mathrm{mod}}}{l})\cdot(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda)) + 10\cdot e^*_{\mathrm{mod}}\cdot l$$

$$-\frac{1}{6}\cdot(1-\frac{12}{l^2})\cdot\log(\mathfrak{q}^{\mathrm{bad}}_\lambda)\Big\} - \frac{1}{2l}\cdot\deg(\mathfrak{q}^{\mathrm{bad}}_\lambda)$$

On the other hand, since $-|\log(\underline{\underline{\Theta}})| \geq -|\log(\underline{\underline{q}})| = -\frac{1}{2l} \cdot \deg(\mathfrak{q}_{\lambda}^{\mathrm{bad}})$, we conclude that

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hence that

$$\frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) \;\leq\; (1 + \frac{20 d_{\mathrm{mod}}}{l}) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda)) + 20 \cdot d^*_{\mathrm{mod}} \cdot l.$$

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Technical Difficulties of Explicit Computations

(i) We cannot use the compactness of "$\mathcal{K}$" at the place 2

$\Rightarrow$ We develop the theory of étale theta functions so that it functions properly at the place 2

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Technical Difficulties of Explicit Computations

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(ii) We cannot use the compactness of "$\mathcal{K}$" at the place ∞

$\Rightarrow$ By restricting our attention to special number fields, we "bound" the archimedean portion of the "height" of the elliptic curve E_λ

Étale Theta Functions

p, l: distinct prime numbers — where $l \geq 5$

k: a p-adic local field $\supseteq$ $\mathcal{O}_k$: the ring of integers

X: an elliptic curve $/k$ which has split multipl. red. $/\mathcal{O}_k$

$q \in \mathcal{O}_k$: the q-parameter of X

$X^{\log} \stackrel{\text{def}}{=} (X, \{o\} \subseteq X)$: the smooth log curve $/k$ assoc. to X

In the following, we assume that

- $\sqrt{-1} \in k$
- $X[2l](\overline{k}) = X[2l](k)$
- $[X^{\log}/\{\pm 1\}]$ is a k-core

Now we have the following sequence of log tempered coverings:

$$\ddot{Y}^{\log} \xrightarrow{\mu_2} Y^{\log} \xrightarrow{l \cdot \underline{\mathbb{Z}}} \underline{X}^{\log} \xrightarrow{\underline{\mathbb{F}}_l} X^{\log}$$

— where

- $Y^{\log} \to \underline{X}^{\log} \to X^{\log}$ is det'd by the [graph-theoretic] universal covering of the dual graph of the special fiber of $X^{\log}$. Write

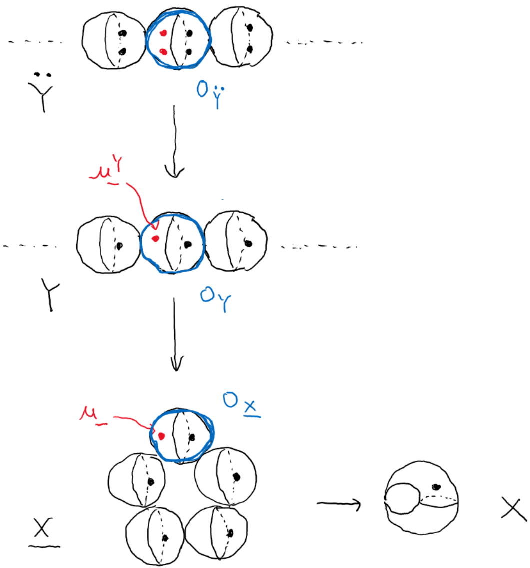
$$\underline{\mathbb{Z}} \stackrel{\text{def}}{=} \text{Gal}(Y^{\log}/X^{\log}) \quad (\cong \mathbb{Z}).$$

- $\underline{X}^{\log} \to X^{\log}$ corresponds to $l \cdot \underline{\mathbb{Z}} \subseteq \underline{\mathbb{Z}}$. Write

$$\underline{\mathbb{F}}_l \stackrel{\text{def}}{=} \text{Gal}(\underline{X}^{\log}/X^{\log}) \quad (\cong \mathbb{F}_l).$$

- $\ddot{Y}^{\log} \to Y^{\log}$ is the double covering det'd by "$u = \ddot{u}^2$".

Special fibers



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$\mathrm{Ver}(-)$: the set of irreducible components of the special fiber of $(-)$

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$\mu_- \in \underline{X}(k)$: the 2-torsion point — not equal to the origin — whose closure intersects $0_{\underline{X}} \in \mathrm{Ver}(\underline{X}^{\log})$

$\mu_-^Y \in Y(k)$: a $^{\exists!}$lift. of μ_- whose closure intersects $0_Y \in \mathrm{Ver}(Y^{\log})$

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Definition

an evaluation point of $\ddot{Y}^{\log}$ labeled by $j \in \underline{\mathbb{Z}}$

$$\overset{\text{def}}{\Leftrightarrow} \text{ a lifting} \in \ddot{Y}(k) \text{ of } \xi_j^Y \in Y(k)$$

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The function

$$\ddot{\Theta}(\ddot{u}) \stackrel{\text{def}}{=} q^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{u}^{2n+1}$$

on $\ddot{Y}^{\log}$ extends uniquely to a meromorphic function $\ddot{\Theta}$ on the stable model of $\ddot{Y}$, and satisfies the following property:

$$\ddot{\Theta}(\xi_j)^{-1} = \pm \ddot{\Theta}(\xi_0)^{-1} \cdot q^{\frac{j^2}{2}}.$$

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Definition

Write

$$\ddot{\Theta}_{\text{st}} \stackrel{\text{def}}{=} \ddot{\Theta}(\xi_0)^{-1} \cdot \ddot{\Theta}$$

and refer to $\ddot{\Theta}_{\text{st}}$ as a theta function of μ_2-standard type.

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Lemma (Well-definedness of the notion of "μ_6-standard type")

$n \in \mathbb{Z}_{>0}$: an even integer

L: an alg. cl. ch. zero fld. $\supseteq$ $\mu_{2n}^{\times}$: the set of pr. $2n$-th roots of unity

Γ_{-} (resp. Γ^{-}): the group of $\sharp = 2$ which acts on $\mu_{2n}^{\times}$ as follows:

$$\zeta \mapsto -\zeta \qquad (\text{resp. } \zeta \mapsto \zeta^{-1})$$

Then the action $\Gamma_{-} \times \Gamma^{-}$ on $\mu_{2n}^{\times}$ is transitive $\Leftrightarrow$ $n \in \{2, 4, 6\}$

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Note: $\ddot{\Theta}(-\ddot{u}) = -\ddot{\Theta}(\ddot{u})$; $\ddot{\Theta}(\ddot{u}^{-1}) = -\ddot{\Theta}(\ddot{u})$; $\ddot{\Theta}(\zeta_{12})$ is unit at $^{\forall}$bad places.

Heights

First, we recall the notion of the Weil height of an algebraic number.

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Definition

Let F be a number field; $\alpha \in F$. Then for $\Box \in \{\mathrm{non}, \mathrm{arc}\}$, we shall write

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Observe: Let $n \in \mathbb{Q}$ be a positive integer. Then we have

$$h_{\mathrm{non}}(n) = 0, \quad h_{\mathrm{arc}}(n) = \log(n).$$

In this work, we introduce a variant of the notion of the Weil height.

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<u>Observe</u>: Let $n \in \mathbb{Q}$ be a positive integer. Then we have

$$h_{\mathrm{non}}(n) = \frac{1}{2}\log(n), \quad h_{\mathrm{arc}}(n) = \frac{1}{2}\log(n).$$

Remark

For $\alpha \in F^{\times}$, it holds that $h(\alpha) = h^{\mathrm{tor}}(\alpha)$.

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F: a mono-complex number field

For $\alpha \in F^{\times}$, it holds that $h^{\mathrm{tor}}_{\mathrm{arc}}(\alpha) \leq h^{\mathrm{tor}}_{\mathrm{non}}(\alpha)$.

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Proof: This follows immediately from the product formula.

Next, we introduce the notion of the “height” of an elliptic curve.

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Definition

$F \subseteq \overline{\mathbb{Q}}$: a number field

E: an elliptic curve $/F$ $\ \overset{\sim}{\to}_{\overline{\mathbb{Q}}}$ "$y^2 = x(x-1)(x-\lambda)$" $\ (\lambda \in \overline{\mathbb{Q}} \setminus \{0,1\})$

<u>Note</u>: $\mathfrak{S}_3 \ {}^{\exists} \curvearrowright \ (\mathbb{P}_{\mathbb{Q}} \setminus \{0,1,\infty\})(\overline{\mathbb{Q}}) \ \overset{\sim}{\to} \ \overline{\mathbb{Q}} \setminus \{0,1\}$

For $\Box \in \{\mathrm{non}, \mathrm{arc}\}$, we shall write

$$h_{\Box}^{\mathfrak{S}\text{-tor}}(E) \ \overset{\mathrm{def}}{=} \sum_{\sigma \in \mathfrak{S}_3} h_{\Box}^{\mathrm{tor}}(\sigma \cdot \lambda),$$

$$h^{\mathfrak{S}\text{-tor}}(E) \ \overset{\mathrm{def}}{=} \ h_{\mathrm{non}}^{\mathfrak{S}\text{-tor}}(E) + h_{\mathrm{arc}}^{\mathfrak{S}\text{-tor}}(E)$$

and refer to $h^{\mathfrak{S}\text{-tor}}(E)$ as the symmetrized toric height of E.

Proposition (Important property of $h_{\square}^{\mathfrak{S}\text{-tor}}$)

Suppose: $\mathbb{Q}(\lambda)$ is mono-complex

Then it holds that $h_{\text{arc}}^{\mathfrak{S}\text{-tor}}(E) \leq h_{\text{non}}^{\mathfrak{S}\text{-tor}}(E)$.

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Now we note that we have an equality "$\deg(\mathfrak{q}_\lambda) = h_{\mathrm{non}}(j(E_\lambda))$".

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Theorem (Comparison between $h_{\square}^{\mathfrak{S}\text{-tor}}(E)$ and $h_{\square}(j(E))$)

We have

$$0 \leq h_{\mathrm{non}}^{\mathfrak{S}\text{-tor}}(E) - h_{\mathrm{non}}(j(E)) \leq 8\log 2,$$
$$-11\log 2 \leq h_{\mathrm{arc}}^{\mathfrak{S}\text{-tor}}(E) - h_{\mathrm{arc}}(j(E)) \leq 2\log 2.$$

Auxiliary numerical results

Theorem (j-invariants of "arithmetic" elliptic curves — due to Sijsling)

j("arithmetic" elliptic curve over a field of char. zero) $\in$

$$\{\tfrac{488095744}{125},\ \tfrac{1556068}{81},\ 1728,\ 0\}.$$

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Theorem (Effective ver. of PNT — due to Axler, Rosser-Schoenfeld)

For $x \in \mathbb{R}_{\geq 2}$, write

$$\pi(x) \stackrel{\text{def}}{=} \sharp\{p \in \mathfrak{Primes} \mid p \leq x\}; \quad \theta(x) \stackrel{\text{def}}{=} \sum_{p \in \mathfrak{Primes};\ p \leq x} \log(p).$$

Then for any real number $x \geq 5 \cdot 10^{20}$ (resp. $\geq 10^{15}$), it holds that

$$\pi(x) \leq 1.022 \cdot \tfrac{x}{\log(x)} \quad \text{(resp. } |\theta(x) - x| \leq 0.00071 \cdot x).$$

Main Results

Theorem (Effective ABC for mono-complex number fields)

Let $d \in \{1, 2\}$, $\epsilon \in \mathbb{R}_{>0} \cap \mathbb{R}_{\leq 1}$. Write

$$h_d(\epsilon) \stackrel{\text{def}}{=} \begin{cases} 3.4 \cdot 10^{30} \cdot \epsilon^{-166/81} & (d = 1) \\ 6 \cdot 10^{31} \cdot \epsilon^{-174/85} & (d = 2). \end{cases}$$

Then for

- F: a mono-complex number field — where $d = [F : \mathbb{Q}]$
- (a, b, c) : a triple of elements $\in F^{\times}$ — where $a + b + c = 0$

we have

$$H_F(a, b, c) \ < \ 2^{5d/2} \cdot \exp(\tfrac{d}{4} \cdot h_d(\epsilon)) \cdot (\Delta_F \cdot \mathrm{rad}_F(a, b, c))^{\frac{3}{2}+\epsilon}.$$

Theorem (Effective version of a conjecture of Szpiro)

Let $\epsilon \in \mathbb{R}_{>0} \cap \mathbb{R}_{\leq 1}$; a, b, c be nonzero coprime integers such that

$$a + b + c \ = \ 0.$$

Then we have

$$|abc| \ \leq \ 2^4 \cdot \exp(1.7 \cdot 10^{30} \cdot \epsilon^{-166/81}) \cdot (\mathrm{rad}(abc))^{3(1+\epsilon)}.$$

Theorem (Effective version of a conjecture of Szpiro)

Let $\epsilon \in \mathbb{R}_{>0} \cap \mathbb{R}_{\leq 1}$; a, b, c be nonzero coprime integers such that

$$a + b + c \ = \ 0.$$

Then we have

$$|abc| \ \leq \ 2^4 \cdot \exp(1.7 \cdot 10^{30} \cdot \epsilon^{-166/81}) \cdot (\mathrm{rad}(abc))^{3(1+\epsilon)}.$$

Corollary (Application to Fermat's Last Theorem)

Let $p > 3.35 \cdot 10^9$ be a prime number. Then there does not exist any triple (x, y, z) of positive integers such that

$$x^p + y^p = z^p$$

holds (cf. [Coppersmith], [Mihăilescu]).