Overview of some of aspects IUT

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Overview of the talk

The purpose of this talk is to give an informal overview over some aspects of IUT theory. In particular, we will discuss étale-like and Frobenius-like structures, cyclotomic rigidity isomorphisms and multiradiality.
Throughout the talk we will use some notation from the first week of IUT workshop.

Structure of the theory

How to briefly summarise the structure of IUT?
Roughly speaking, the main result of the theory is a construction of certain group-theoretic *algorithm* and description of its properties. In particular, this algorithm is compatible with *Kummer-theories* (up to certain indeterminacies), which link *Frobenius-like* structures with its *étale-like* counterparts and with the Θ-link.
We will try to discuss some of the expressions appearing above and explain how they fit into the IUT theory.

Mono-anabelian geometry

Let us first discuss the notion of a group-theoretic algorithm, which is used extensively in IUT. Recall that in anabelian geometry one can often consider two types of questions:

- (bi-anabelian) Given two schemes X and Y and a (iso-)morphism $\Pi_X \to \Pi_Y$ between their fundamental groups, is it induced by a (iso-)morphism of schemes $X \to Y$?
- (mono-anabelian) Given a fundamental group Π_X of a scheme X, can we reconstruct the scheme X (or some weaker data) from the topological group Π_X?

Most of the time a bi-anabelian result is obtained as a corollary of the corresponding mono-anabelian result. The main example where it is not the case is Neukirch-Uchida theorem ($G_K \cong G_L \Rightarrow K \cong L$ for number fields)

Algorithms in IUT

We will not attempt to give a precise definition of a "group theoretical algorithm". However, we note here the following important point: in IUT, in various statements concerning the existence of a particular group theoretic algorithm, the construction of this algorithm itself constitutes an important of the statement. Informally, one could say that the way we construct an object is as important as the (isomorphism class) of the considered object.
Let us also recall that groups that we use in the theory to construct various objects are either (étale or tempered) fundamental groups of (orbi-)curves or absolute Galois groups of fields (local or global).

Containers and cyclotomes

A common theme in IUT theory is constructing various objects of interest from either fundamental groups or Frobenioids by embedding them inside some kind of container. This container is often (some subset of) a first cohomology group

$$H^1(G, C)$$

where G is some topological group equipped with a "cyclotomic character"

$$\phi\colon G \to \widehat{\mathbb{Z}}^\times$$

and C is a "cyclotome" (i.e., $C \cong \widehat{\mathbb{Z}}$ with the action of G given by ϕ).

Étale-like constant monoid

Let K be a local field and write G_K for the absolute Galois group $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ of K. In the following discussion we will concentrate on the simple example of the Galois monoid

$$G_K \curvearrowright \mathcal{O}^{\triangleright},$$

where we write $\mathcal{O}^{\triangleright} = \mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$. Recall that we have seen a group theoretic construction of a monoid $\mathcal{O}^{\triangleright}(G_K)$, equipped with the action of G_K, which is isomorphic to $\mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$. This algorithm proceeds as follows: first we construct a canonical surjection $G_K^{ab} \twoheadrightarrow \widehat{\mathbb{Z}}$ and take the preimage of $\mathbb{N} \subset \widehat{\mathbb{Z}}$; this submonoid corresponds to $\mathcal{O}^{\triangleright}_K$. Then, we repeat this construction for every open subgroup of G_K and take a colimit with respect to the transfer map.

Étale-like constant monoid (2)

The construction of the monoid $\mathcal{O}^{\triangleright}(G_K)$ can be alternatively presented as follows: write $\mu_n(G_K)$ for the subgroup of n-torsion elements and $\mu_{\widehat{\mathbb{Z}}}(G_K)$ for the inverse limit of $\mu_n(G_K)$. Then, we have a natural isomorphism

$$G_K^{ab} \cong H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)),$$

thus after taking a colimit the monoid $\mathcal{O}^{\triangleright}(G_K)$ may be regarded as a submonoid of the cohomology group

$${}_{\infty}H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)).$$

Recall our notation ${}_{\infty}H^1(G, A) = colim_{H \subset G, open} H^1(H, A)$. Thus we have constructed an étale-like object $\mathcal{O}^{\triangleright}(G_K)$.

Étale-like structures

Let us pause for a moment to discuss one particular type of an object appearing in IUT, namely the notion of étale-like objects. Étale-like objects include various fundamental groups and Galois groups, e.g.,

$$(\Pi_X, \Delta_X, G_K, \dots),$$

as well as objects constructed from them group-theoretically, e.g.,

$$(\mathbb{M}_*(\Pi_X), \ell\Delta_\Theta, \mathcal{O}^{\triangleright}(G_K), \dots).$$

Moreover, the actual construction of an object is a part of the data defining this object. For example, the G_K-module $\mathcal{O}^{\triangleright}(G_K)$ is not just a copy of the monoid $\mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$ (i.e., one cannot make an identification "$\mathcal{O}^{\triangleright}(G_K) = \mathcal{O}^{\triangleright}_{K^{\mathrm{alg}}}$").

Frobenius-like constant monoid

Let us go back to the case of the étale-like monoid $G_K \curvearrowright \mathcal{O}^{\triangleright}(G_K)$. The next step is to augment this group theoretic situation by introducing additional structure (e.g., monoid) equipped with an action of the group G_K and relate this structure to the corresponding étale-like structure. Suppose that we start with a pair consisting of a group G_K acting on a monoid M, which is isomorphic to

$$G_K \curvearrowright \mathcal{O}^{\triangleright}.$$

Write $M[n]$ for the subgroup of n-torsion of M and define the inverse limit

$$\mu_{\widehat{\mathbb{Z}}}(M) = lim_n M[n],$$

thus $\mu_{\widehat{\mathbb{Z}}}(M)$ is isomorphic to $\widehat{\mathbb{Z}}(1)$ (thus $\mu_{\widehat{\mathbb{Z}}}(M)$ is a "cyclotome").

Frobenius-like constant monoid (2)

Then, we have seen the construction of a cyclotomic rigidity isomorphism

$$\mu_{\widehat{\mathbb{Z}}}(M) \cong \mu_{\widehat{\mathbb{Z}}}(G_K),$$

which is the unique isomorphism that makes the following diagram commutative

$$\begin{array}{ccc} M & \hookrightarrow & {}_{\infty}H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(M)) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathcal{O}^{\triangleright}(G_K) & \hookrightarrow & {}_{\infty}H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)). \end{array}$$

Here both vertical arrows are isomorphisms. The above relation is an example of a comparison between étale-like and Frobenius-like structures, through their Kummer theories and an isomorphism of cyclotomes.

Frobenius-like structures

Recall that the data

$$G_K \curvearrowright \mathcal{O}^{\triangleright}$$

is equivalent to the data of the p-adic Frobenioid $\mathcal{F}$ over the category $\mathcal{D}$ of connected covers of $Spec(K)$. Thus we have a forgetful functor to the base category

$$\mathcal{F} \to \mathcal{D}.$$

Moreover, recall that in the definition of a Frobenioid we specified various functors Ψ ("divisors") and $\mathbb{B}$ ("functions") from $\mathcal{F}$ to categories of monoids and groups as well as a natural transformation *div* ("divisor") between them.

Finally, by a Frobenius-like object we mean an object constructed from the "monoid" part of a Frobenioid, i.e., the images of the functors Ψ and $\mathbb{B}$. For example, regarding the pair $G_K \curvearrowright \mathcal{O}^{\triangleright}$ as a p-adic Frobenioid we see that the monoid $\mathcal{O}^{\triangleright}$ is Frobenius-like.

Étale-like units

Let us now make similar discussion concerning the monoid $G_K \curvearrowright \mathcal{O}^\times$. Namely, there is a group-theoretic construction

$$\mathcal{O}^\times(G_K) \hookrightarrow H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K))$$

of a group $\mathcal{O}^\times$; clearly we have a containment $\mathcal{O}^\times(G_K) \subset \mathcal{O}^\triangleright(G)$ of étale-like objects.
Suppose now that $M^\times$ is a group equipped with an action of G_K and isomorphic to $\mathcal{O}^\times$. Note that one can still construct the cyclotome $\mu_{\widehat{\mathbb{Z}}}(M^\times)$ (in fact we have "$\mu_{\widehat{\mathbb{Z}}}(M^\times) = \mu_{\widehat{\mathbb{Z}}}(M)$"). Thus, we may also consider the Kummer map

$$M^\times \hookrightarrow {}_\infty H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(M^\times)).$$

Frobenius-like units

Then we have the comparison of the étale-like units $\mathcal{O}^{\times}(G_K)$ with the corresponding Frobenius-like structure

$$\begin{array}{ccc} M^{\times} & \hookrightarrow & {}_{\infty}H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(M^{\times})) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{O}^{\times}(G_K) & \hookrightarrow & {}_{\infty}H^1(G_K, \mu_{\widehat{\mathbb{Z}}}(G_K)). \end{array}$$

induced by a cyclotomic rigidity isomorphism. However, note that even though the limit $\mu_{\widehat{\mathbb{Z}}}(M^{\times})$ can be constructed from the group $M^{\times}$, the construction of the isomorphism $\mu_{\widehat{\mathbb{Z}}}(M) \cong \mu_{\widehat{\mathbb{Z}}}(G_K)$ required the full data of the monoid M. In particular, starting from the data of the group $M^{\times}$ one can only construct the isomorphism $\mu_{\widehat{\mathbb{Z}}}(M^{\times}) \cong \mu_{\widehat{\mathbb{Z}}}(G_K)$ up to an action of $\widehat{\mathbb{Z}}^{\times}$ (i.e., $\widehat{\mathbb{Z}}^{\times} \curvearrowright \widehat{\mathbb{Z}}$).

Uniradiality of constant monoids

The previous example of the group G_K acting on monoids $\mathcal{O}^{\triangleright}$ and $\mathcal{O}^{\times}$ together with the relationship between their Kummer theories can be presented in the following way. Write $G_K \curvearrowright {}^{\dagger}\mathcal{O}^{\triangleright}$ and $G_K \curvearrowright {}^{\dagger\dagger}\mathcal{O}^{\triangleright}$ for two isomorphs of the monoid $G_K \curvearrowright \mathcal{O}^{\triangleright}$ and suppose that they are glued along corresponding unit subgroups

$$ {}^{\dagger}\mathcal{O}^{\times} \cong {}^{\dagger\dagger}\mathcal{O}^{\times}. $$

Then, the construction of $\mathcal{O}^{\triangleright}(G_K)$ is not compatible simultaneously with Kummer theories of ${}^{\dagger}\mathcal{O}^{\times}$ and ${}^{\dagger\dagger}\mathcal{O}^{\times}$ (since automorphisms of $\mathcal{O}^{\times}$ do not lift to automorphisms of $\mathcal{O}^{\triangleright}$). Thus, the construction of the monoid $\mathcal{O}^{\triangleright}(G_K)$ is *uniradial* with respect to the above link (isomorphism of unit groups).

Multiradiality of theta function

On the other hand, let us recall the multiradiality of theta function and its construction from a mono-theta environment. Consider a tempered Frobenioid $\underline{\underline{\mathcal{F}}}$ over the base category $\mathcal{D} = \mathcal{B}(\underline{\underline{X}})^0$. Recall that one can construct a mono-theta environment $\mathbb{M}_*$ either from the tempered Frobenioid or from the base category, denoted by $\mathbb{M}_*(\underline{\underline{\mathcal{F}}})$ and $\mathbb{M}_*(\Pi^{tp}_{\underline{\underline{X}}})$, respectively. Furthermore, we have seen a construction (from a mono-theta environment $\mathbb{M}_*$) of the theta monoid

$$\mathcal{O}^{\times} \cdot \underline{\underline{\theta}} \subset {}_{\infty}H^1(\Pi^{tp}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*), \Pi_{\mu}(\mathbb{M}_*))$$

(i.e., the mono-theta theoretic version of the monoid $\mathcal{O}^{\times} \cdot \Theta^{1/\ell}$).

Multiradiality of theta function (2)

Suppose now that one considers two isomorphs ${}^{\dagger}\underline{\underline{\mathcal{F}}}$ and ${}^{\dagger\dagger}\underline{\underline{\mathcal{F}}}$ of the Frobenioid $\underline{\underline{\mathcal{F}}}$ lying over the base category $\mathcal{D}$, which we again glue along the corresponding subgroup of units

$$ {}^{\dagger}\mathcal{O}^{\times} \cong {}^{\dagger\dagger}\mathcal{O}^{\times}. $$

Then, the construction of the theta monoid is *multiradial*, i.e., this construction is compatible simultaneously with the Kummer theories and with the link. This follows from the fact that the required cyclotomic rigidity isomorphism can be reconstructed from a mono-theta environment and that the only Frobenius-like portion of a mono-theta environment consists of roots of unity. (cf. the usage of the value group portion in the case of a constant monoid)

Remark about the Hodge Theatre

We have seen that some examples of the following pattern: one has a Frobeniod $\mathcal{F}$ over the base category $\mathcal{D}$, thus also a functor $\mathcal{F} \to \mathcal{D}$. Then we construct various objects from $\mathcal{F}$ and $\mathcal{D}$ and connection between them (Kummer theory and cyclotomic rigidity).

In fact, the Hodge Theatre is also has this type of structure, where $\mathcal{D}$ is now a collection of base categories, similarly $\mathcal{F}$ is a collection of Frobenioids. Elements of this collections are linked by various natural functors (e.g., localization) and the whole structure possesses certain symmetries.

Thus, (very roughly!) one may think of the Hodge Theatre as a globalization of local pieces of the form $G_{K_{\underline{v}}} \curvearrowright \mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ and $\underline{\underline{\mathcal{F}}}_{v}$.

Why (Ind1) and (Ind2)?

We have seen that the Θ-link involves a poly-isomorphism of a unit part (modulo torsion) on both sides of the link, i.e., an identification of the form

$$ {}^{\dagger}\mathcal{O}^{\times\mu} \cong {}^{\dagger\dagger}\mathcal{O}^{\times\mu}. $$

One can ask, what is the reason that we have to consider the unit part up to an arbitrary automorphism? To answer this question, recall that an important part of the main algorithm consists of constructing the theta monoid

$$ "\mathcal{O}^{\times} \cdot \Theta^{\mathbb{N}}". $$

However, the unit part of the theta monoid is not equipped with any additional structure. Therefore, if one wants to treat this étale-like object as a Frobenius-like object, it is only possible by treating the unit part as isomorphic to $\mathcal{O}^{\times}$ through some indeterminate isomorphism.

Why (Ind1) and (Ind2)? (2)

Therefore, informally, the fact that we use the theta monoid algorithm in the theory already forces the introduction of the indeterminacy within the isomorphism class of $\mathcal{O}^{\times\mu}$ (i.e., (Ind2)).
Moreover, if we want to keep the usual interpretation of Galois group G_K as a group of automorphisms (over K) of a certain field K^{alg}, the we have to treat the Galois groups G_K on both sides of the theta link as only abstractly isomorphic. Indeed, it follows from the fact that there is no field isomorphism ${}^{\dagger}K^{\mathrm{alg}} \cong {}^{\dagger\dagger}K^{\mathrm{alg}}$ lying over the (poly)-isomorphism of groups ${}^{\dagger}\mathcal{O}^{\times} \cong {}^{\dagger\dagger}\mathcal{O}^{\times}$.
Therefore, one can say (informally) that (Ind2) implies (Ind1).

Why (Ind3)?

For completeness, let us also briefly mention how the indeterminacy (Ind3) arises. Recall that in the construction of log-link we consider a diagram of the form

$$\mathcal{O}^{\times\mu} \twoheadleftarrow \mathcal{O}^{\times} \hookrightarrow \mathcal{O}^{\triangleright} \hookrightarrow (K^{\mathrm{alg}})^{\times} \cong \mathcal{O}^{\times\mu},$$

where the isomorphism is given by the p-adic logarithm. Moreover, recall that by evaluating the theta function at various special points we obtained values

$$\underline{\underline{q}}^{j^2} \subset \mathcal{O}^{\triangleright}.$$

However, we have seen that the construction of the monoid $\mathcal{O}^{\triangleright}$ cannot be transported to the other side of the theta link (uniradiality of $\mathcal{O}^{\triangleright}$). Then, the idea is to use the logarithm to embed $\mathcal{O}^{\triangleright}$ inside $\mathcal{O}^{\times\mu}$, as in the above diagram.

Final remarks

Let us finish with a few general remarks.

- From our discussion it may seem that the main (global) result is obtained as a sum of local results, i.e., as a kind of implication:

$$\text{"}\textit{locally } \underline{\underline{q}}^{j^2} \approx \underline{\underline{q}}\text{"} \;\Rightarrow\; \text{"}\textit{globally } \underline{\underline{q}}^{j^2} \approx \underline{\underline{q}}\text{"}.$$

  However, this is not the case. To obtain the main result one needs to replace the collection of regions inside (tensor packets of) log-shells by an element of a global (realified) Frobenioid; this is achieved by the portion of the multiradial algorithm consisting of a multiplicative group of a global field F_{mod}.

Final remarks (2)

- In fact, (Ind1) and (Ind2) have little effect on log-volume at local places. To see it, note that they fix the filtration

$$\ldots, \subset p^{-1}\mathcal{I} \subset \mathcal{I} \subset p\mathcal{I} \subset, \ldots$$

  (here $\mathcal{I}$ is a log shell).
- The existence of non-geometric automorphisms of G_K is not used in the argument.

End of the talk

Thank you for your attention!