ON THE FORMALIZATION OF IUT: A PRELIMINARY PROGRESS REPORT [JOINT WORK IN PROGRESS WITH Y. HOSHI, G. YAMASHITA, Y. YANG, ...]

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https://www.kurims.kyoto-u.ac.jp/~motizuki/Formalization%20of%20IUT%20(2026-04%20version).pdf

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§1. **Lean formalization ("LeanForm") as a communication tool**

- The significance of LeanForm: verification vs. communication: Classically, the significance of LeanForm is typically regarded as lying in the *verification* of the *logical correctness* of a mathematical theory. In the case of IUT, however, although this verification aspect of LeanForm is *welcome*, in fact the logical structure of the theory is rather *simple*, so this verif. aspect of LeanForm is *not a central focal point* of interest from the point of view of researchers in IUT. Rather, in the case of IUT, the significance of LeanForm lies in producing a *precise record* of the *logical str.* of IUT that is *immune* to *false misinterpretations* and hence can be used, in a *pivotal* way, to *communicate* this *simplicity* in a *maximally efficient/precise* way to other mathematicians.

- The *key* to overcoming formidable *communication barriers* lies in *breaking out* of the following *(closely interrelated!) vicious cycles*:

"The logic of (supposedly difficult portions of) IUT is in fact *very simple* and *elementary*."	$\leftrightarrows$	*Accusations (without any mathematical proof!)*: "Don't *insult* me/my intelligence!" or "It's entirely *inconceivable* that the abc inequality could follow from such simple logic!"
$\Uparrow$		$\Downarrow$
"It's impossible to become sufficiently motivated to study [IUTchI-IV] seriously without *confidence* that the theory is *correct*, e.g., by seeing a *suff'ly transparent synopsis* of the argument!"	$\leftrightarrows$	"In order to understand the logic of IUT in a genuine/rigorous way, it is necessary to *study [IUTchI-IV] in detail* (as many others have already done!)."

LeanForm offers (to my knowledge) the first technology that appears to have the *technical capacity* to allow us to *break free* of such *vicious cycles*, i.e., by providing *confidence* via a *precise and logically verified record* of the logical structure of IUT that can be processed by a machine in a fraction of a second in a way that is *entirely immune* to *nonmathematical accusations* (such as those discussed above) or the *human mental fatigue/stress* that occurs when trying to comprehend an unfamiliar mathematical theory (cf. [RptIUT], §3.2, (LnIm)).

· Unfortunately, the issue of *confidence in correctness* has a very *human, nonmathematical social/political* dimension: That is to say, no one wants to *invest substantial resources* of time and effort in a "*shinking ship*". Ultimately, social/political power may be understood as a reflection of *perceived capacity* to *reliably* deliver/ensure *long-term mathematical accountability* for the mathematical correctness of results/assertions. Thus, to my knowledge, LeanForm appears to be the first technology that has the *technical capacity* do this in a fashion that is *intrinsically independent* of/*unshackled* from *human social/political structures*.

· From a *purely technical* point of view, the *essence of LeanForm* — at least in the case of IUT — lies in undertaking a *fundamental reorganization* of the theory into

suitable purely formal/combinatorial blackboxes

— cf. [RptIUT], §3.2, (MthFm), (BBxFm), (BBxDD); §4 below for technical details of this reorganization in the case of the *final portion* of IUT, which has received the *most public attention*.

· The Lean code to be discussed in §4 below arose precisely from the challenges presented by the task of *communicating* IUT to mathematicians (arithmetic geometers) who are thoroughly familiar with Lean, but who had only a relatively superficial knowledge of IUT (cf. [RptIUT], (LnCom)). This situation was remarkable in that it had the effect of

faithfully recreating in a "laboratory" the vicious cycles

discussed above, but in a generally friendly/productive *nontoxic* environment, while still preserving the *essential dynamics* and *overall sense of utter hopelessness and frustration* that are characteristic of such vicious cycles. The *remarkable conclusion* to the story is that the Lean code discussed in §4 had the power to overcome these vicious cycles and, in particular, to

successfully and efficiently record, in a nonnegatable fashion, the logic of the final portion of IUT.

We note, however, that this Lean code is still in a somewhat *skeletal/bare bones* form, i.e., its success as a *communication tool* depends on the level of understanding of the mathematicians involved (i.e., on the receiving end) of the *basic set-up of IUT* and of the *main issues involved*. In particular, it will still take some more time before sufficiently many details can be added to "*flesh out*" the Lean code to a degree that it is suitable for release to the general public.

§2. First steps toward the LeanForm of IUT

- *The main strategy for the LeanForm of IUT*:

  Stage 1: [IUTchIII] Theorem 3.11 $\Longrightarrow$ Corollary 3.12 (since this has received the *most public attention!*)

  Stage 2: Proof of [IUTchIII] Theorem 3.11 modulo [IUTchI-II]

  Stage 3: [IUTchI-II] modulo earlier results (1995 - 2015) on *anabelian geometry/Frobenioids/theta functions*, etc.

  Stage 4: Earlier results (1995 - 2015): [pGC], [GeoAn], [AbsAnb], [NCBel], [AbSc], [SemiAn], [QuCnf], [CbGC], [Config], [FrdI-II], [EtTh], [GenEll], [NodNon], [AbsTopI-III]

  Stage 5: Numerical aspects ([IUTchIV], [ExpEst])

  (We are currently in the early "*skeletal*" portion of Stage 1.)

  Here, the *current main focus* is on Stages 1 and 2 since Stage 1 has received the *most public attention*, but in fact is not so difficult and ultimately (when understood properly!) just shows that what one is really interested in is Stage 2. Moreover, Stages 1 and 2 most likely require the most *human intervention*, i.e., ultimately (most probably!) in terms of having my research group at RIMS *actually write the skeletal Lean code*, as in the case discussed in detail in §4. By contrast, the mathematical content of Stages 4 and 5, as well as (to a lesser extent) Stage 3, are understood by a *much larger class* of mathematicians and have *never* been a source of confusion/debate, hence should be regarded as *long-term goals* that seem much more amenable to LeanForm by research groups not so closely connected to my research group at RIMS, e.g., (perhaps) via machine-based *autoformalization*.

- The theory of *species/mutations*:

  As was mentioned to me by one Lean expert, it is of fundamental importance to begin any LeanForm project with a formalization of the *basic definitions* underlying the theory to be formalized. In the case of IUT, this amounts to formalizing the notion of a "*functorial algorithm*" (i.e., in more technical language, the notion of a "mutation"), which, in essence, amounts to the theory of *species/mutations*. This notion of a functorial algorithm is of fundamental importance not only in IUT, but also in anabelian geometry (of the sort that is relevant to IUT), as well as, in a certain sense, in pure mathematics as a whole.

  species: a *type of mathematical object* (such as a group, scheme, diagram of schemes, etc.) defined (in a fashion that is *independent* of any particular model ZFC set theory!) by some *set-theoretic formula*

  mutation: a construction of *one species* from another that is defined (in a fashion that is *independent* of any particular ZFC-model!) by some *set-theoretic formula*

· Thus,

a *species* determines a *category of species-objects*, while
a *mutation* determines a *functor* between associated categories of species-objects.

Put another way, one may regard species/mutations as a sort of *natural refinement* of the classical notions of *categories/functors*. Alternatively, and more precisely, if one is willing to go up "one step" in the hierarchy of universes under consideration, then one may regard species/mutations as *internal categories/functors* in the *syntactic category* associated to ZFC regarded as a *first order theory*. In particular, it follows that the LeanForm of species/mutations becomes *essentially immediate* once one has a LeanForm of *ZFC as a first order theory*. In the fall of 2025, I discovered, on the one hand, that such a LeanForm had in fact already recently been achieved by a young Japanese researcher named Shogo Saito, but, on the other hand, (was quite *surprised* to learn!) that such a LeanForm does *not yet exist in MathLib* (the standard library for refereed Lean code). Here, we observe that the technical significance of formalizing *ZFC as a first order theory* lies in the fact that this is necessary in order to be able to work with "*some indeterminate set-theoretic formula* ϕ" (i.e., as opposed to *specific set-theoretic formulas* "$a \in A$", "$B \subseteq C \cap D$", etc., which are "natively" available in Lean).

· Ultimately, as a result of various consultations with Lean experts, I reached the conclusion that in fact the formalization of species/mutations in the abstract using some sort of LeanForm of *ZFC as a first order theory* (though ultimately *theoretically desirable* at some possibly much later stage in the project!) was most likely *not necessary* for the LeanForm of (at least Stages 1∼2 of) IUT. Instead, suitable use of *standard Lean commands* for formalizing types of mathematical objects and relationships between various respective "*component pieces of data*" in the *input/output* of a functor seems to be *sufficient for formalizing the logic of IUT* (at least in the case of Stages 1∼2 — cf. the Lean code discussed below in §4).

· Nevertheless, the *nonexistence* of a LeanForm in MathLib of *ZFC as a first order theory* made a *strong impression* on me, especially considering how fundamental *ZFC as a first order theory* is throughout pure mathematics. Indeed, this situation struck me as a remarkable example of the *power* of Lean as a *communication tool* (cf. §1), namely, by faithfully communicating/reflecting the *nonexistence* — i.e., in the mathematics that has already been formalized in MathLib — of the *algorithmic construction point of view*, which is so *fundamental to the logic of IUT*, as well as *related anabelian geometry*, and which seems to have been one *substantial obstacle* to the study of IUT for many mathematicians who are not familiar with this point of view.

§3. **Brief review of inter-universal Teichmüller theory (IUT)**

- A more detailed exposition of IUT may be found in
  - the *survey texts* [Alien], [EssLgc] (cf. also [IUTchI-IV], [IUAni1], [IUAni2]), well as in
  - the *videos/slides* available at the following URLs: (cf. also my series of *DWANGO LECTURES* on IUT — URLs available to professional mathematicians at request!):

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https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz
      IUT21/WS3/ExpHorizIUT21-InvitationIUT-notes.html
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https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz
      IUT21/WS4/ExpHorizIUT21-IUTSummit-notes.html
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- Let R be an *integral domain* (e.g., $\mathbb{Z} \subseteq \mathbb{Q}$) equipped with the action of a *group* G, $(\mathbb{Z} \ni)\ N \geq 2$. For simplicity, assume that $N = 1 + \dots + 1 \neq 0 \in R$; R has *no nontrivial N-th roots of unity*. Write $R^{\triangleright} \subseteq R$ for the *multiplicative monoid* $R \setminus \{0\}$. Then let us observe that the *N-th power map* on $R^{\triangleright}$ determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$G \curvearrowright R^{\triangleright} \xrightarrow{\sim} (R^{\triangleright})^N\ (\subseteq R^{\triangleright}) \curvearrowleft G$$

that does *not arise* from a *ring homomorphism*, i.e., it is clearly *not compatible* with *addition* (cf. our assumption on N!).

- Let ${}^{\dagger}R$, ${}^{\ddagger}R$ be *two distinct copies* of the integral domain R, equipped with respective actions by *two distinct copies* ${}^{\dagger}G$, ${}^{\ddagger}G$ of the group G. We shall use similar notation for objects with labels "†", "‡" to the previously introduced notation. Then one may use the *isomorphism of multiplicative monoids* arising from the *N-th power map* discussed above to *glue* together

$${}^{\dagger}G \curvearrowright {}^{\dagger}R \supseteq ({}^{\dagger}R^{\triangleright})^N \xleftarrow{\sim} {}^{\ddagger}R^{\triangleright} \subseteq {}^{\ddagger}R \curvearrowleft {}^{\ddagger}G$$

... where the notion of a *gluing* may be understood
  - as a *quotient set* via identifications, or (preferably!)
  - as an *abstract diagram* (cf. graphs of groups/anabelioids!)

the *ring* ${}^{\dagger}R$ to the *ring* ${}^{\ddagger}R$ along the *multiplicative monoids* $({}^{\dagger}R^{\triangleright})^N \xleftarrow{\sim} {}^{\ddagger}R^{\triangleright}$ (with group actions). This gluing is *compatible* with the resp. actions of ${}^{\dagger}G$, ${}^{\ddagger}G$ relative to the isomorphism ${}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$ given by forgetting the labels "†", "‡", but, since the N-th power map is *not compatible* with *addition* (!), this isom. ${}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$ may be regarded either as an isom. of ("*coric*", i.e., *invariant* w.r.t. the N-th power map) *abstract groups* (cf. the notion of "*inter-universality*", as discussed in [EssLgc], §3.2, §3.8!) or as an isomorphism of groups equipped with actions on certain *multiplicative monoids*, but *not* as an isomorphism of ("*Galois*" — cf. the *inner automorphism indeterminacies* of SGA1!) groups equipped with actions on *rings/fields*.

· The problem of *describing (certain portions of the) ring structure* of $^{\dagger}R$ in terms of the *ring structure* of $^{\ddagger}R$ — in a fashion that is *compatible* with the *gluing* and via a *single* algorithm that may be applied to the *common* (cf. *logical AND* $\wedge$*!*) *glued data* to reconstruct *simultaneously* (certain portions of) the ring structures of *both* $^{\dagger}R$ and $^{\ddagger}R$, up to suitable relatively mild *indeterminacies* (cf. the theory of *crystals*!) — seems (at first glance/in general) to be *hopelessly intractable* (cf. the case of $\mathbb{Z}$)!

> . . . where we note that here, considering *portions* is important because we want to *decompose* the above diagram up into *pieces* so that we can consider *symmetry* properties involving these pieces!

One well-known elementary example: when $N = p$, working *modulo p* (cf. *indeterminacies*, analogy with *crystals*!), where there is a *common ring structure* that is *compatible* with the *p-th power map*!

Another important example: Faltings' proof of *invariance* of *height* of elliptic curves under *isogeny*, under the assumption of the existence of a *global multiplicative subspace* (cf. [ClsIUT], §1; [EssLgc], Example 3.2.1)!

. . . This is precisely what is *achieved in IUT* by means of the *multiradial representation for the* Θ*-pilot* via

· *anabelian geometry* (cf. the *abstract groups* $^{\dagger}G$, $^{\ddagger}G$!);
· *the p-adic/complex logarithm, Galois eval. of theta functions*;
· *Kummer theory*, to relate *Frob.-/étale-like* versions of objects.

· *Main point*:

The *multiplicative monoid* and *abstract group* structures (but *not* the ring structures!) are *common* (cf. "*logical AND* $\wedge$*!*") to $\dagger$, $\ddagger$ and can be used as the *input data* for an algorithm to construct the *multiradial representation for the* Θ*-pilot*, i.e., a *common container* for the distinct *ring strs.* (i.e., "*arith. hol. strs.*") $\dagger$, $\ddagger$

$$^{\dagger}R \quad \subseteq \quad \left(\text{multirad. rep. for the } \Theta\text{-pilot}\right) \quad \supseteq \quad ^{\ddagger}R$$

· When $R = \mathbb{Z}$ (or, in fact, more generally, the *ring of integers* "$\mathcal{O}_F$" of a number field F — cf. the multiplicative *norm map* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$), one may consider the "*height/log-volume*"

$$\log(|x|) \in \mathbb{R}$$

for $0 \neq x \in \mathbb{Z}$. Then the *N-th power map* of (i), (ii) corresponds, after passing to *heights*, to *multiplying real numbers by N*; the *multiradial algorithm* corresponds to saying that the height is *unaffected (up to a mild error term!)* by *multiplication by N*, hence that the *height is bounded*!

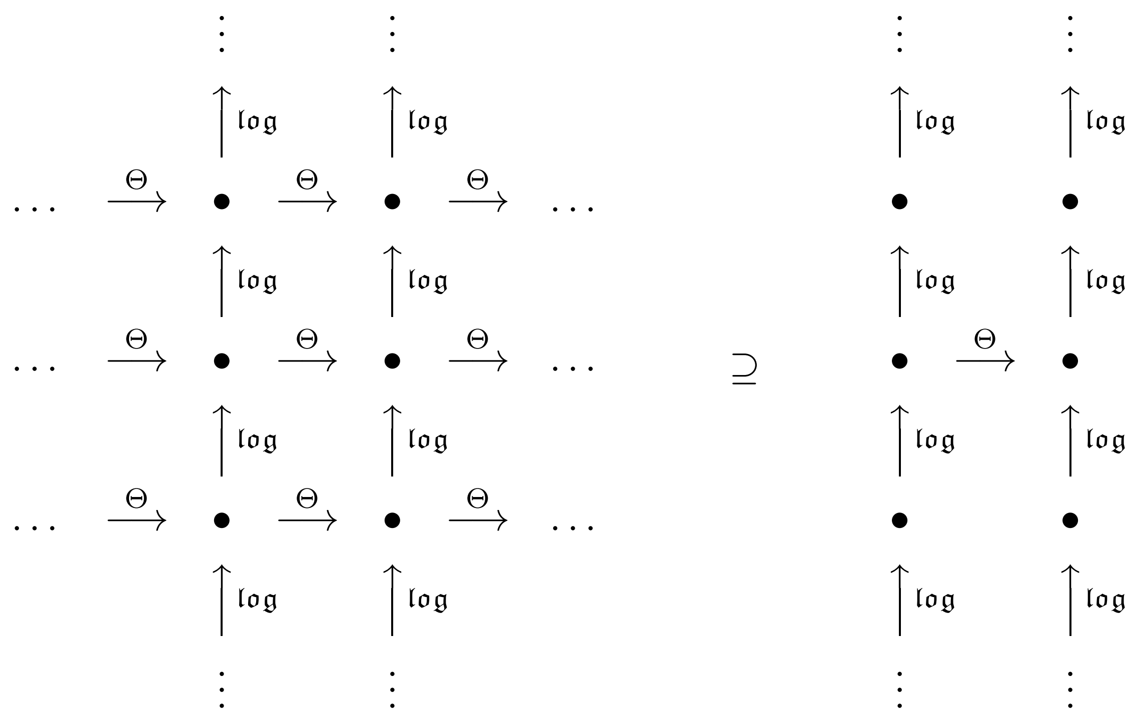
· In the case of IUT, the *multirad. rep. for the Θ-pilot* is obtained by means of a sort of "*analytic continuation*" along a certain "*infinite H*" of the *log-theta-lattice* (cf. the discussion surrounding [EssLgc], §3.3, (InfH))

... where

· the *Θ-link* between distinct ring strs. "•" corresponds to the N-th power map discussed in the present §3, while

· the *log-link* locally at nonarchimedean valuations looks like the p-adic logarithm between distinct ring strs. "•";

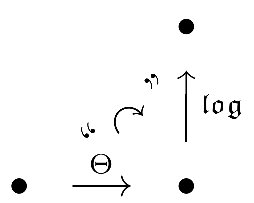
· the *descent operations* revolve around the establishment of certain *coricity/symmetry* properties.



$$(0,0) \overset{\text{(Stp1)}}{\rightsquigarrow} (0,\circ) \overset{\text{(Stp2)}}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{\text{(Stp3)}}{\rightsquigarrow} (0,0)^{\vdash} \overset{\text{(Stp3)}}{\Longleftrightarrow} (1,0)^{\vdash}$$

— which involves a gradual introduction via "*descent*" operations of "*fuzzifications*", corresponding to *indeterminacies* (cf. the discussion of [EssLgc], §3.10).

· At a more technical level, the *multirad. rep. for the Θ-pilot* is obtained by constructing *invariants* with respects to the *log-link*, which has the effect of *juggling addition and multiplication* — i.e., juggling the *dilated* and *non-dilated* portions of the *ring strs.* — and, as a result, effects a sort of "*miraculous rotation*" (the discussion of [EssLgc], §3.11)



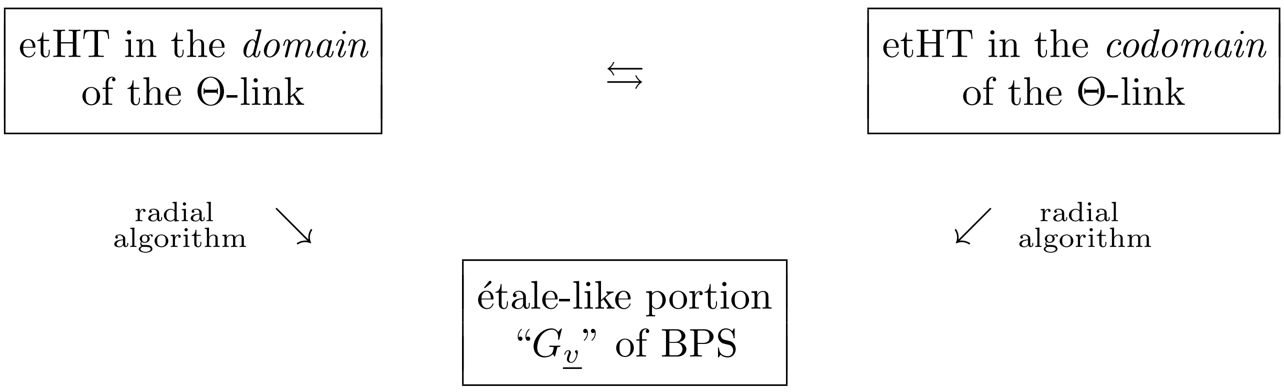
of the

· "*mysterious log-volume-dilating Θ-link gap*" (between the domain/codomain of the Θ-link) onto the

· "*harmless log-volume-preserving log-link gap*" (between the domain/codomain of the log-link) !

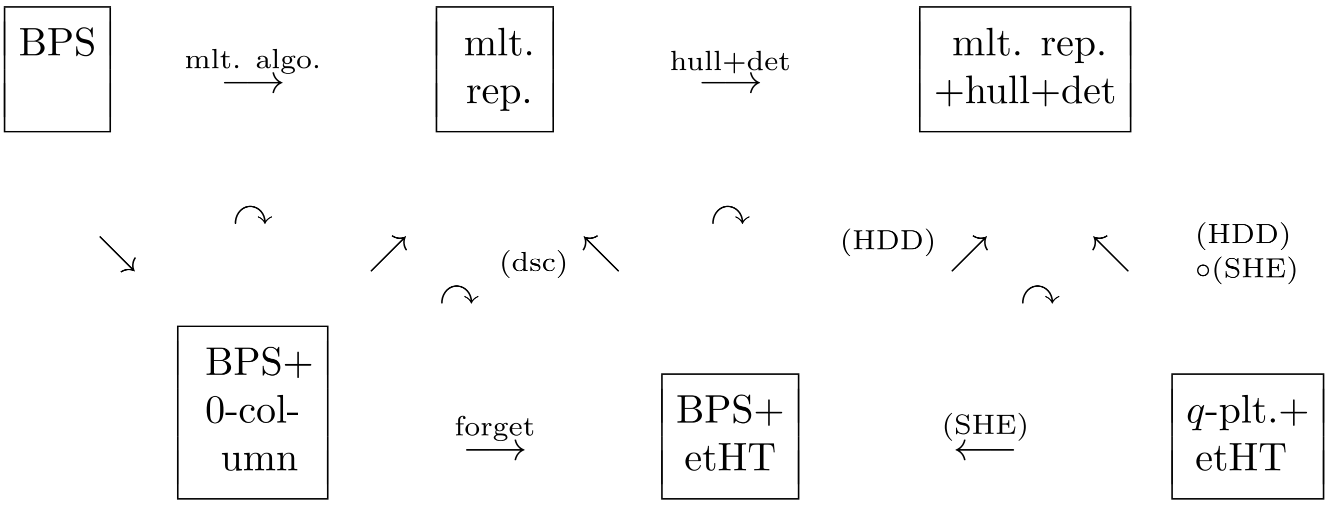
§4. **Skeletal Lean code for "3.11.5 $\Longrightarrow$ 3.12"**

· The significance of the *multiradial representation* lies in the fact that it expresses — by means of the technique of *descent* (cf. the final portion of §3) — the *Θ-pilot* (i.e., in essence, the *image* of the N-th power map) in the *domain* of the Θ-link (i.e., in essence, the N-th power map) in terms of *purely group-theoretic* data — called the *etHT* (i.e., "étale Hodge theater") — that is *vertically coric* (i.e., invariant with respect to the $\mathfrak{log}$-link) and (*unlike* the Θ-link/N-th power map $x \mapsto x^N$!!) is *symmetric*



with respect to the operation of *switching* the *domain/codomain* of the Θ-link, while holding the *gluing data*, called the *BPS* (i.e., "*basic prime-strip*", that is to say, in the original terminology of [IUTchI-IV], "$\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip"), of the Θ-link *fixed*.

· The theory surrounding this operation of *descent* is summarized in the *first two commuting triangles* (i.e., "$\overset{\curvearrowright}{\triangle}$") of the following diagram of *species* (i.e., boxes) and *mutations* (i.e., arrows):



That is to say:

· the *first* $\overset{\curvearrowright}{\triangle}$ expresses the *multiradial algorithm* — whose *input* is a BPS, and whose *output* is the *multiradial representation* — as an algorithm whose *input* may be thought of as a BPS *glued* (via a "full poly-isomorphism", i.e., an arbitrary indeterminate isomorphism) to the underlying BPS of the Θ-pilot (at $(0,0)$) arising from the 0-column of the log-theta-lattice, where we recall that the multiradial representation is constructed as a sort of *quotient of the 0-column*, by regarding it up to *suitable indeterminacies (Ind1,2,3)*;

· the *second* $\overset{\curvearrowright}{\triangle}$ expresses — via the "*descent-arrow (dsc)*" — the *descent* aspect of the multiradial representation, i.e., that if one works up to the indeterminacies (Ind1,2,3), then one may regard the multiradial representation as an object that is constructed from *weaker data* — namely, a BPS whose "étale-like" (i.e., group-theoretic) portion is *glued* (via a "full poly-isomorphism", i.e., an arbitrary indeterminate isomorphism) to the corresponding portion of the (vertically coric!) etHT associated to the 0-column of the log-theta-lattice;

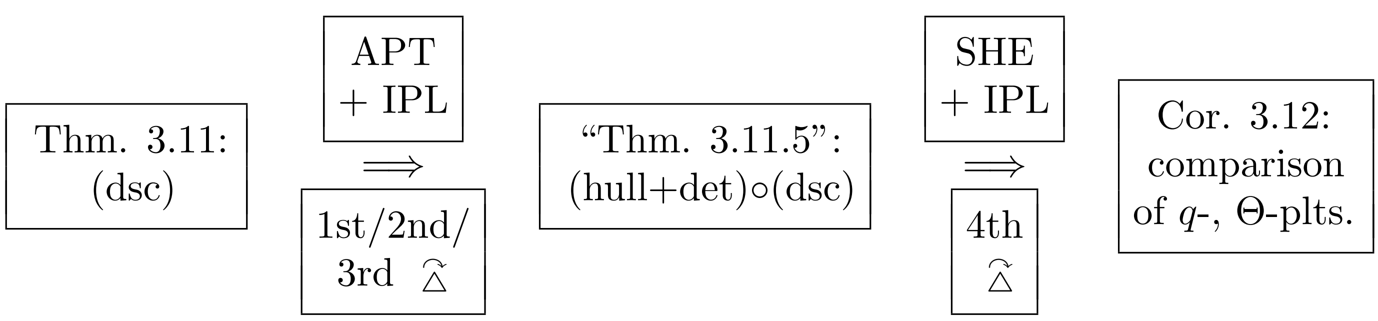
· the *third* $\overset{\curvearrowright}{\triangle}$ simply expresses the definition of the composite arrow (HDD) := (hull+det) ∘ (dsc) obtained by composing the descent-arrow (dsc) (cf. the second $\overset{\curvearrowright}{\triangle}$!) with a certain elementary operation "hull+det" (obtained essentially by taking the determinant of the module over a ring generated by a certain topological module that appears in the multiradial representation);

· the *fourth* $\overset{\curvearrowright}{\triangle}$ expresses the composite (HDD) ∘ (SHE) that is obtained by restricting the arrow (HDD) (cf. the third $\overset{\curvearrowright}{\triangle}$!) to the *special case* where the *input* data "BPS+etHT" of (HDD) is taken — by applying the "*SHE-arrow (SHE)*", where "SHE" stands for *simultaneous holomorphic expressibility* — to be the data given by the *q-pilot* constructed from the etHT (i.e., in essence, the data in the *domain* of the N-th power map).

- The *skeletal Lean code* that was referred to in the discussion of §1 concerns the the *fourth* $\hat{\triangle}$ of the preceding diagram, i.e., the *simultaneous comparison* relative to a *single ring structure* (cf. the discussion of a "*common container*" in §3) of
  - the *Θ-pilot* — which corresponds to (HDD), or, alternatively, to the *image* of the N-th power map — and
  - the *q-pilot* — which corresponds to (SHE), or, alternatively, to to the *domain* of the N-th power map.

  We chose to concentrate on this aspect of the theory *first* since this aspect of the theory — i.e., "Stage 1: [IUTchIII] Theorem 3.11 $\Longrightarrow$ Corollary 3.12" (cf. the discussion at the beginning of §2) — has received the *most public attention*. Indeed, this aspect corresponds to the *essential nontrivial content* of the theory, i.e., that the height of the elliptic curve under consideration is equal to N times the height of the elliptic curve (where N is a large positive number), up to a small discrepancy (arising from (Ind1,2,3) + hull), thus implying a *bound* on the height.

- In fact, the *fourth* $\hat{\triangle}$ corresponds to the final portion of a certain *reorganization/decomposition* of the implication "3.11 $\Longrightarrow$ 3.12" into several portions, which correspond to the *four* $\hat{\triangle}$ discussed above. We shall refer to this final portion as "3.11.5 $\Longrightarrow$ 3.12" (although in fact *"3.11.5 (=3.11+Rmk. 3.9.5)" never appears* in [IUTchI-IV]!). At a more technical level, this reorgan./decomp. is obtained by "*moving*" the "hull+det" in 3.12 to "3.11.5", thus making it possible to concentrate, in "3.11.5 $\Longrightarrow$ 3.12", on the *crucial simultaneous comparison* aspect of 3.12 discussed above. This aspect also involves the *input prime-strip link (IPL)* property, which refers to the relationship between the data contained in the *input* BPS of the multiradial algorithm and the data contained in the *output* of the multiradial algorithm. In fact, IPL also plays an important role in the implication "3.11 $\Longrightarrow$ 3.11.5", i.e., which corresponds to the *first/second/third* $\hat{\triangle}$. The central aspect of the (slightly nontrivial, by comparison to the first $\hat{\triangle}$) *second/third* $\hat{\triangle}$ lies in the *algorithmic parallel transport (APT)* property, which concerns the way in which "*gluings up to indeterminacies*" (such as (Ind1,2,3) + hull) are expressed. Our research group at RIMS is currently making substantial progress with regard to writing *skeletal Lean code* for the *second/third* $\hat{\triangle}$ that formalizes the APT aspect of the theory.



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