COMMENTS ON [IUTCHIV], THEOREM 1.10

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In the following, we discuss a *minor error* in the theory of [IUTchIII], [IUTchIV] concerning the precise content of the *"ϵ portion"* of the ABC Conjecture. This error is easily repaired and, moreover, has *no effect on the conclusion constituted by the ABC Conjecture* [i.e., [IUTchIV], Theorem A; [IUTchIV], Corollary 2.3]. That is to say, it only concerns the *somewhat subtle content of the "ϵ term"* that appears in these results.

(1.) In late September 2012, Vesselin Dimitrov and Akshay Venkatesh pointed out to me, in e-mails, the possibility that the inequality of [IUTchIV], Theorem 1.10, contradicts the examples constructed in [Mss]. In fact, I had considered this issue when I wrote [IUTchIV] — cf. the discussion of [IUTchIV], Remark 2.3.2, (ii). At the time I wrote [IUTchIV], I had not studied the proof given in [Mss] in detail. However, the construction given in [Mss] is performed in such a way that there is *no apparent way to bound the contribution at the prime* 2. Since the theory of [IUTchI], [IUTchII], [IUTchIII], depends, in an essential way, on the theory of the *étale theta function* developed in [EtTh], which breaks down in an essential way *at the prime* 2, the bound given in [IUTchIV], Theorem 1.10, does not involve the contribution at the prime 2. In particular,

> at a *purely explicit* level, there is *no contradiction* between the inequality of [IUTchIV], Theorem 1.10, and the examples constructed in [Mss].

This was precisely my understanding when I wrote [IUTchIV].

(2.) On the other hand, it was pointed out to me by Akshay Venkatesh that the argument of [Mss] may be modified in such a way as to obtain examples for which the contribution at the prime 2 may be *bounded*. This led me to reexamine the entire theory of [IUTchI], [IUTchII], [IUTchIII], [IUTchIV] in detail. My conclusions may be summarized as follows:

(a) I continue to believe that the *abstract theory* of [IUTchI], [IUTchII], [IUTchIII] contains no essential errors.

(b) I continue to believe that the *log-volume computations* of [IUTchIV] contain no essential errors.

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(c) On the other hand, I now see that I made a slight error in the *interpretation via log-volume of the abstract theory* of [IUTchI], [IUTchII], [IUTchIII], i.e., in the *"bridge"* between this abstract theory and the log-volume computations discussed in [IUTchIV], §1.

That is to say, at a more technical level, it appears that I made a slight error in the *definition of the constant "C_Θ" in [IUTchIII], Corollary 3.12.*

(3.) The error discussed in (2.) may be explained in more detail as follows. The essence of the abstract theory of [IUTchI], [IUTchII], [IUTchIII] lies in the *computation* — via anabelian geometry, the theory of Frobenioids, etc. — of an *"alien" arithmetic holomorphic structure* in terms of a given *initial arithmetic holomorphic structure* that is related to the "alien" structure via certain *"mono-analytic" data.* This is intended to be an arithmetic analogue of the situation [i.e., in classical complex Teichmüller theory] in which one considers *distinct holomorphic structures* related by a *single underlying real analytic structure* on a topological surface. The current definition of the constant "C_Θ" amounts, in essence, to [an upper bound on] the log-volume of the "alien" structure measured in terms of the **mono-analytic** [i.e., the arithmetic analogue of "underlying real analytic"] data. On the other hand, upon further consideration, I reached the conclusion that the correct definition of this constant "C_Θ" is as [an upper bound on]

> the log-volume of the "alien" structure measured in terms of the given **initial arithmetic holomorphic structure**.

Indeed, this is in essence the content of the crucial argument given in Step (xi) of the proof of [IUTchIII], Corollary 3.12. That is to say, in summary, my current understanding is that

> there is nothing essentially wrong with this argument/proof, but rather that I made an *error* in the *statement of the conclusions* that one should draw from this argument [i.e., in the definition of the constant "C_Θ"].

Although I am quite busy with other work, I hope to post a revised version of [IUTchIII] on my homepage [i.e., with the correct definition of the constant "C_Θ"] in the not so distant future.

(4.) At the level of the computations of [IUTchIV], §1, the effect of the change in the definition of the constant "C_Θ" discussed in (3.) is in fact quite limited. That is to say, in a word, there is in fact no effect on the computations at archimedean primes and at nonarchimedean primes (i.e., of the field "K") that are "moderately ramified", i.e., whose absolute ramification index is $< p - 1$. At nonarchimedean primes (i.e., of the field "K") that are [possibly tamely, but] *not moderately ramified*, one must add a new term arising from the fact that the *radius of convergence of the p-adic log/exp series* is $p^{-1/p-1}$. The main contribution then occurs at the [odd!] **bad primes**, i.e., where there is *tame ramification of index l*, which typically is much larger than p. The total new contribution — say, in the case where the base field is the field of rational numbers $\mathbb{Q}$, the conductor of the *abc*-triple under consideration is denoted N, and we make the assumption (which is possible in the context of Theorem 1.10) that l is $\leq$ a positive constant multiple of $\log(N)$ — is then roughly of the form

$$\omega(N') \cdot \log(\log(N)) - \log(N')$$

— i.e., where "$\omega(-)$" denotes the number of primes that divide the integer in parentheses, and we write N' for the product of p dividing N that are $< l$. Elementary estimates via the prime number theorem then yield asymptotic upper bounds for the new contribution of the form [a positive constant times]

$$\log(N) \cdot (\log(\log(\log(N))))/\log(\log(N))$$

— i.e., which is safely out of range of the lower bound $(\log(N))^{1/2}/\log(\log(N))$ of Masser's examples. Again, although I am quite busy with other work, I hope to post a revised version of [IUTchIV] on my homepage [i.e., with the corrected version of Theorem 1.10 and its proof] in the not so distant future.

(5.) In the context of (4.), it is of interest to note that the contribution involving $\omega(N)$ discussed in (4.) is [not precisely the same as, but nevertheless] *strongly reminiscent* of the many *refinements of the ABC Conjecture* considered by Baker in his 1996 and 2004 papers on the ABC Conjecture.