Abstract. Let $\Sigma$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. In this paper, we prove that various objects that arise from the geometry of the configuration space of a hyperbolic curve over an algebraically closed field of characteristic zero may be reconstructed group-theoretically from the pro-$\Sigma$ fundamental group of the configuration space. Let $X$ be a hyperbolic curve of type $(g, r)$ over a field $k$ of characteristic zero. Thus, $X$ is obtained by removing from a proper smooth curve of genus $g$ over $k$ a closed subscheme [i.e., the “divisor of cusps”] of $X$ whose structure morphism to Spec($k$) is finite étale of degree $r; 2g - 2 + r > 0$. Write $X_n$ for the $n$-th configuration space associated to $X$, i.e., the complement of the various diagonal divisors in the fiber product over $k$ of $n$ copies of $X$. Then, when $k$ is algebraically closed, we show that the triple $(n, g, r)$ and the generalized fiber subgroups — i.e., the subgroups that arise from the various natural morphisms $X_n \to X_m [m < n]$, which we refer to as generalized projection morphisms — of the pro-$\Sigma$ fundamental group $\Pi_n$ of $X_n$ may be reconstructed group-theoretically from $\Pi_n$ whenever $n \geq 2$. This result generalizes results obtained previously by the first and third authors and A. Tamagawa to the case of arbitrary hyperbolic curves [i.e., without restrictions on $(g, r)$]. As an application, in the case where $(g, r) = (0, 3)$ and $n \geq 2$, we conclude that there exists a direct product decomposition

$$\text{Out}(\Pi_n) = G T^\Sigma \times S_{n+3}$$

— where we write “Out(−)” for the group of outer automorphisms [i.e., without any auxiliary restrictions] of the profinite group in parentheses and GT$^\Sigma$ (respectively, $S_{n+3}$) for the pro-$\Sigma$ Grothendieck-Teichmüller group (respectively, symmetric group on $n+3$ letters). This direct product decomposition may be applied to obtain a simplified purely group-theoretic equivalent definition — i.e., as the centralizer in Out$(\Pi_n)$ of the union of the centers of the open subgroups of Out$(\Pi_n)$ — of GT$^\Sigma$. One of the key notions underlying the theory of the present paper is the notion of a pro-$\Sigma$ log-full subgroup — which may be regarded as a sort of higher-dimensional analogue of the notion of a pro-$\Sigma$ cuspidal inertia subgroup of a surface group — of $\Pi_n$. In the final section of the present paper, we show that, when $X$ and $k$ satisfy certain conditions concerning “weights”, the pro-$l$ log-full subgroups may be reconstructed group-theoretically from the natural outer action of the absolute Galois group of $k$ on the geometric pro-$l$ fundamental group of $X_n$. 

2010 Mathematics Subject Classification. Primary 14H30; Secondary 14H10.

Key words and phrases. anabelian geometry, Grothendieck-Teichmüller group, generalized fiber subgroup, log-full subgroup, hyperbolic curve, configuration space.
INTRODUCTION

Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(k\) an algebraically closed field; \(X\) a hyperbolic curve of type \((g, r)\) over \(k\), i.e., the open subscheme of a proper smooth curve of genus \(g\) over \(k\) obtained by removing \(r\) closed points. In the following discussion, we shall write “\(\Pi(-)\)” for the \([\log]\) étale fundamental group of a connected locally noetherian \([fs\ log]\) scheme [for some choice of basepoint]. If the characteristic of \(k\) is zero, then \(\Pi_X\) is a surface group [cf. the discussion entitled “Topological Groups” in §0]; in particular, if \(r > 0\), then

\[ \Pi_X \text{ is a free profinite group of rank } 2g + r - 1. \]

Thus, at least in the case of \(k\) of characteristic zero,

the isomorphism class of the profinite group \(\Pi_X\) is insufficient to determine \((g, r)\).

[Note that, if the characteristic of \(k\) is positive, then

the isomorphism class of \(\Pi_X\) completely determines \((g, r)\)

— cf. [Tama], Theorem 0.1.] On the other hand, if, instead of just considering the étale fundamental group of the given hyperbolic curve, one considers the étale fundamental groups of the various configuration spaces associated to the hyperbolic curve, then the following Fact is known [cf. [CbTpI], Theorem 1.8; [MT], Corollary 6.3]:

**Fact.** Let \(\Sigma\) be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. For \(\boxsupseteq \in \{\circ, \bullet\}\), let \(X^{\boxsupseteq}\) be a hyperbolic curve of type \((g^{\boxsupseteq}, r^{\boxsupseteq})\) over an algebraically closed field of characteristic zero; \(n^{\boxsupseteq}\) a positive integer; \(X^{\boxsupseteq}_{n^{\boxsupseteq}}\) the \(n^{\boxsupseteq}\)-th configuration space of \(X^{\boxsupseteq}\); \(\Pi^{\boxsupseteq}_{n^{\boxsupseteq}}\) the maximal pro-\(\Sigma\) quotient of \(\Pi_{X^{\boxsupseteq}_{n^{\boxsupseteq}}};\)

\[ \alpha : \Pi^{\circ}_{n^{\circ}} \xrightarrow{\sim} \Pi^{\bullet}_{n^{\bullet}} \]

an isomorphism of profinite groups. Suppose that

\[ \{(g^{\circ}, r^{\circ}), (g^{\bullet}, r^{\bullet})\} \cap \{(0, 3), (1, 1)\} = \emptyset. \]

Then the following hold:
The equality $n^\circ = n^\bullet$ holds. If, moreover, $n \overset{\text{def}}{=} n^\circ = n^\bullet \geq 2$, then $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.

(ii) The isomorphism $\alpha$ induces a bijection between the set of fiber subgroups of $\Pi_n^\circ$ and the set of fiber subgroups of $\Pi_n^\bullet$.

Here, we recall that a fiber subgroup of $\Pi_n^\square$ is defined to be the kernel of the natural [outer] surjection $\Pi_n^\square \twoheadrightarrow \Pi_m^\square$ — where $1 \leq m \leq n$ — induced by a projection $X_n^\square \twoheadrightarrow X_m^\square$ obtained by forgetting some of the factors. In this paper, we first generalize the above Fact to include the case of arbitrary hyperbolic curves [cf. Theorem 2.5 for more details]:

**Theorem A.** (Group-theoreticity of the dimension, genus, number of cusps, and generalized fiber subgroups). Let $\Sigma$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers. For $\square \in \{\circ, \bullet\}$, let $X_n^\square$ be a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of characteristic zero; $n^\square$ a positive integer; $X_n^\square$ the $n^\square$-th configuration space of $X_n^\square$; $\Pi_n^\square$ the maximal pro-$\Sigma$ quotient of $\Pi_{X_n^\square}^\square$; $\alpha : \Pi_n^\circ \overset{\sim}{\rightarrow} \Pi_n^\bullet$ an isomorphism of profinite groups. Then the following hold:

(i) The equality $n^\circ = n^\bullet$ holds. Moreover, if $n \overset{\text{def}}{=} n^\circ = n^\bullet \geq 2$, then $(g^\circ, r^\circ) = (g^\bullet, r^\bullet)$.

(ii) If $n \geq 2$ [cf. (i)], then $\alpha$ induces a bijection between the set of generalized fiber subgroups of $\Pi_n^\circ$ [cf. Definition 2.1, (ii)] and the set of generalized fiber subgroups of $\Pi_n^\bullet$.

Here, note that Theorem A, (ii), fails to hold if one uses [“classical”] fiber subgroups instead of generalized fiber subgroups. Indeed, one verifies immediately [cf. Definition 2.1, (i); Remark 2.1.1] that the following holds:

Let $n \geq 2$ be a positive integer; $X_n$ the $n$-th configuration space of $X$; $\Pi_n \overset{\text{def}}{=} \Pi_{X_n}$. Suppose that $(g, r) \in \{(0, 3), (1, 1)\}$. Then for any [“classical”] fiber subgroup $F \subseteq \Pi_n$, there exists an automorphism $\alpha \in \text{Aut}(\Pi_n)$ — which arises from an $k$-automorphism in $\text{Aut}_k(X_n)$ — such that $\alpha(F) \subseteq \Pi_n$ is not a [“classical”] fiber subgroup of $\Pi_n$.

We also remark that, in Theorem 2.5, below, we give explicit group-theoretic algorithms for reconstructing the triple $(n, g, r)$, as well as the generalized fiber subgroups of $\Pi_n$, from $\Pi_n$.

Next, we apply Theorem A, (ii), to prove the following result [cf. Corollary 2.6 for more details]. This result may be regarded as a generalization of [CbtTpII], Theorem B, (i), to the case of arbitrary hyperbolic curves.

**Corollary B.** (Structure of the group of outer automorphisms of a configuration space group). Let $\Sigma$ be a set of prime numbers which
is either of cardinality one or equal to the set of all prime numbers; \( X \) a hyperbolic curve of type \((g, r)\) over an algebraically closed field of characteristic zero; \( n \geq 2 \) a positive integer; \( X_n \) the \( n \)-th configuration space of \( X \); \( \Pi_n \) the maximal pro-\( \Sigma \) quotient of \( \Pi_{X_n} \); 

\[
 n^* \overset{\text{def}}{=} \begin{cases} 
 n + r & \text{if } (g, r) \in \{(0, 3), (1, 1)\}; \\
 n & \text{if } (g, r) \notin \{(0, 3), (1, 1)\};
\end{cases}
\]

\( \mathcal{S}_{n^*} \) the symmetric group on \( n^* \) letters. Write \( \text{Out}(\Pi_n) \) for the group of outer automorphisms [i.e., without any auxiliary restrictions!] of the profinite group \( \Pi_n \). Let us regard \( \mathcal{S}_{n^*} \) as a subgroup of \( \text{Out}(\Pi_n) \) via the natural inclusion \( \mathcal{S}_{n^*} \hookrightarrow \text{Out}(\Pi_n) \) induced by the natural action of \( \mathcal{S}_{n^*} \) on \( X_n \) [cf. Remark 2.1.1]. Suppose that \((r, n) \neq (0, 2)\). Then we have an equality

\[
\text{Out}(\Pi_n) = \text{Out}_{\text{GF}}(\Pi_n) \times \mathcal{S}_{n^*}
\]

— where we write \( \text{Out}_{\text{GF}}(\Pi_n) \) for the group of outer automorphisms of \( \Pi_n \) that stabilize arbitrary generalized fiber subgroups of \( \Pi_n \).

In this context, we remark that, in Corollaries 2.6, (ii); 2.10, (iv), below, we give explicit group-theoretic algorithms for reconstructing the subgroup \( \mathcal{S}_{n^*} \subseteq \text{Out}(\Pi_n) \) from \( \Pi_n \).

In particular, by restricting Corollary B to the case where \((g, r) = (0, 3)\), we obtain the following result [cf. Corollary 2.8 for more details]:

**Corollary C. (Simplified group-theoretic approach to the pro-\( \Sigma \) Grothendieck-Teichmüller group).** In the notation of Corollary B, suppose that \((g, r) = (0, 3)\). Then \( \text{Out}_{\text{GF}}(\Pi_n) \) may be naturally identified with the **pro-\( \Sigma \) Grothendieck-Teichmüller group** \( \text{GT}^\Sigma \) [cf. Definition 2.7]. In particular, we have an equality

\[
\text{Out}(\Pi_n) = \text{GT}^\Sigma \times \mathcal{S}_{n^*+3}.
\]

Moreover, we have

\[
\mathcal{S}_{n^*+3} = Z_{\text{Out}(\Pi_n)}(\text{GT}^\Sigma) = Z_{\text{Out}(\Pi_n)}^{\text{loc}}(\text{Out}(\Pi_n)),
\]

\[
\text{GT}^\Sigma = Z_{\text{Out}(\Pi_n)}(Z_{\text{Out}(\Pi_n)}^{\text{loc}}(\text{Out}(\Pi_n)))
\]

— where we write \( Z_{\text{Out}(\Pi_n)}^{\text{loc}}(\text{Out}(\Pi_n)) \) for the **local center** of \( \text{Out}(\Pi_n) \) [cf. the discussion entitled “Topological Groups” in §0].

Before proceeding, it is perhaps of interest to consider the significance of Corollary C in the broader context of the considerations that motivated the introduction of the **Grothendieck-Teichmüller group** \( \text{GT} \) [i.e., \( \text{GT}^\Sigma \), in the case where \( \Sigma \) is taken to be the set of all prime numbers]. Let \( \overline{\mathbb{Q}} \) be an algebraic closure of \( \mathbb{Q} \). Write \( \text{G}_{\overline{\mathbb{Q}}} \overset{\text{def}}{=} \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Thus, for a suitable choice of basepoints, one has an embedding

\[
\text{G}_{\overline{\mathbb{Q}}} \hookrightarrow \text{GT} (\subseteq \text{Out}(\Pi_n))
\]

[cf., e.g., [NodNon], Theorem C]. Indeed, historically, one of the main motivating reasons for the introduction of \( \text{GT} \) was the point of view that, relative to the above embedding,
GT should serve as a sort of purely combinatorially/group-theoretically defined approximation of $G_Q$.

That is to say, the conditions used to define GT were thought of as a sort of “first approximation” for [still unknown] conditions for describing $G_Q$ as a subgroup of $\text{Out}(\Pi_n)$. Thus, when regarded from this point of view, Corollary C may be interpreted as asserting that this approach to regarding GT as a “first approximation” to the goal of describing $G_Q$ as a subgroup of $\text{Out}(\Pi_n)$ is fundamentally flawed, in the sense that GT, up to a “trivial finite factor” given by the symmetric group $S_{n+3}$, already essentially coincides with $\text{Out}(\Pi_n)$.

In fact, GT, when regarded as a “first approximation” of $G_Q$ as a subgroup of $\text{Out}(\Pi_n)$, is also fundamentally flawed in a somewhat deeper, more philosophical sense, as follows: The conventional approach to defining GT [cf. the discussion of Definition 2.7 for more details] involves the use of

(S1) the inertia subgroups $\subseteq \Pi_n$ associated to the various $log$ divisors of $X_n^\log$ [cf. the discussion entitled “Curves” in §0] and

(S2) the subgroup $S_{n+3} \subseteq \text{Out}(\Pi_n)$ of Corollaries B, C.

Here, we note that, at least at an a priori level, both (S1) and (S2) are data that arise from [characteristic zero] scheme theory, i.e., scheme theory over $\mathbb{Q}$.

That is to say, this state of affairs — i.e., in which GT is defined within $\text{Out}(\Pi_n)$ by means of conditions that involve data that arise from scheme theory over $\mathbb{Q}$ — is fundamentally philosophically inconsistent with the goal of giving purely combinatorial/group-theoretic conditions — i.e., conditions that are independent of scheme theory over $\mathbb{Q}$ — for describing $G_Q$ [whose construction is immediate, once one allows the construction to depend in a quite essential fashion on scheme theory over $\mathbb{Q}$] as a subgroup of $\text{Out}(\Pi_n)$.

Here, we note that, of course, one can attempt to avoid scheme theory in the construction of the data (S1), (S2) by base-changing via some embedding $\iota : \mathbb{Q} \hookrightarrow \mathbb{C}$ and then constructing the data (S1), (S2) via techniques from classical topology. On the other hand,

this yields a definition of GT that depends essentially on the choice of the embedding $\iota : \mathbb{Q} \hookrightarrow \mathbb{C}$, i.e., the choice of an archimedean valuation of $\mathbb{Q}$.

In order to eliminate this dependence, one must employ some sort of “analytic continuation apparatus” to relate the constructions of the data (S1), (S2) at different archimedean valuations of $\mathbb{Q}$. The conventional approach to resolving this issue is to take the “analytic continuation apparatus” to be the “paths” [i.e., between the étale fundamental groups associated to different basepoints] that arise from the theory of [SGA1]. On the other hand,

such “[SGA1] paths” [i.e., this theory of [SGA1]] are themselves fundamentally dependent on scheme theory over $\mathbb{Q}$,
hence, in particular, insufficient from the point of view of the goal of obtaining a definition of GT that is independent of scheme theory over \( \mathbb{Q} \).

[Indeed, here it is perhaps useful to recall that the subgroup of \( \text{Out}(\Pi_n) \) determined by \( G_\mathbb{Q} \) is precisely the monodromy that arises from such “[SGA1] paths”!] That is to say, the only way to avoid a dependence on scheme theory [over \( \mathbb{Q} \)] is to regard the various profinite groups that are constructed, say, at distinct archimedean valuations of \( \mathbb{Q} \) as mathematical objects that tautologically cannot be related to one another by means of scheme theory [over \( \mathbb{Q} \)], i.e., that can only be related to one another as abstract profinite groups.

This sort of “wall that tautologically obstructs scheme theory” is precisely the sort of situation that arises in the theory of the log-wall/log-link that appears in inter-universal Teichmüller theory [cf. \[AbsTopIII\], §H; \[AbsTopIII\], Remark 3.7.7; the discussion at the beginning of \[Pano\], §4]. Here, we recall that once one takes this point of view of “inter-universal analytic continuation”, one can no longer regard the abstract profinite group “\( \Pi_n \)” as being “equipped with auxiliary data of scheme-theoretic origin” such as (S1), (S2), i.e., one must regard “\( \Pi_n \)” simply as “some abstract profinite group”. Thus, in the context of the present discussion, [the theory surrounding] Corollary C asserts that even if one just regards “\( \Pi_n \)” simply as “some abstract profinite group”, there exist purely combinatorial/group-theoretic algorithms [i.e., that are entirely free of any dependence on data that arises from scheme theory!] for constructing the data (S1), (S2), hence also for constructing GT, from the abstract profinite group \( \Pi_n \).

In our proof of the equality “\( n^\circ = n^* \)” stated in Theorem A, (i), we focus on a certain special kind of point — called a log-full point [cf. Definition 1.1] — of the underlying scheme of the log configuration space of a stable log curve that gives rise to the given hyperbolic curve [cf. the discussion entitled “Curves” in §0]. Roughly speaking, a log-full point is defined to be a closed point of the log configuration space at which the log structure is the “most concentrated”. For instance, if \( X^\log \) is a stable log curve over an algebraically closed field equipped with an fs log structure, then the set of log-full points of \( X^\log \) coincides with the set of cusps and nodes of \( X^\log \). In particular, the notion of a log-full point of the log configuration space of a stable log curve may be considered as a sort of higher-dimensional analogue of the notion of a cusp/node.

In the following discussion, for simplicity, we consider the case of smooth curves. Let \( l \) be a prime number; \( \Sigma \) a set of prime numbers which is equal to either \( \{l\} \) or the set of all prime numbers; \( k \) an algebraically closed field of characteristic \( \notin \Sigma \);

\[
X^\log \rightarrow \text{Spec}(k)
\]
a stable log curve \([i.e., \text{where we regard } \text{Spec}(k)]\ as being equipped with the trivial log structure\] such that the interior \([i.e., \text{the open subscheme of points at which the log structure of } X^\log \text{ is trivial}]\ of X^\log \text{ is affine}. Then let us recall that a cusp of X^\log \text{ determines [up to conjugation]}\)

an inertia subgroup \(\subseteq \Pi^\Sigma_X \log \text{ [noncanonically] isomorphic to } \hat{\mathbb{Z}}^\Sigma\)
— where we write \((-)^\Sigma\) for the maximal pro-\(\Sigma\) quotient of a profinite group \((-)\).

In a similar vein, if \(n\) is a positive integer, then a log-full point of the \(n\)-th log configuration space \(X^\log_n\) of \(X^\log\) determines \([up to conjugation]\ what we shall refer to \([cf. \text{Definition 3.4}]\ as \)

a log-full subgroup \(\subseteq \Pi_n \stackrel{\text{def}}{=} \Pi^\Sigma_{X^\log_n} \text{ [noncanonically] isomorphic to } (\hat{\mathbb{Z}}^\Sigma)^\oplus_n\).

Here, we observe that the dimension "\(n\)" appears as the rank of a log-full subgroup. Thus, a log-full subgroup may be regarded as a sort of group-theoretic manifestation of the dimension. In fact, this point of view plays an important role in the proof of Theorem 1.6.

The notion of a log-full subgroup also plays an important role in our approach to the following Problem:

**Problem.** Can one give a purely group-theoretic algorithm for reconstructing from \(\Pi_n\) the inertia subgroups of \(\Pi_n\) associated to the various log divisors of \(X^\log_n\) \([cf. \text{the discussion entitled } \text{“Curves” in } \S 0]\)?

In fact, since \([as is easily verified] each inertia subgroup \([\cong \hat{\mathbb{Z}}^\Sigma]\) of \(\Pi_n\) associated to a log divisor of \(X^\log_n\) appears as a direct summand of some log-full subgroup \([\cong (\hat{\mathbb{Z}}^\Sigma)^\oplus_n]\) of \(\Pi_n\), it is natural to divide this Problem into steps \((P1), (P2)\), as follows:

\((P1):\) Can one give a purely group-theoretic algorithm for reconstructing from \(\Pi_n\) the log-full subgroups of \(\Pi_n\)?

\((P2):\) Can one give a purely group-theoretic algorithm for reconstructing from \(\Pi_n\), together with the auxiliary data constituted by the set of log-full subgroups of \(\Pi_n\), the direct summands of a given log-full subgroup that arise as inertia subgroups associated to log divisors?

In the present paper, we prove a result that yields a partial affirmative answer to \((P1)\), in the form of a sufficient condition for the reconstrucibility of log-full subgroups \([cf. \text{Theorem 3.8; Corollary 3.9; Proposition 3.11; Corollary 3.12, (ii); Remark 3.12.1; Remark 3.12.2, for more details}]\):

**Theorem D.** \((\text{Group-theoretic preservation of log-full subgroups}).\)

Let \(l\) be a prime number; \(n\) a positive integer. For \(\Box \in \{\circ, \bullet\}, \) let \(k_\Box\) be a field of characteristic \(\neq l\); \(G_\Box\) the absolute Galois group of \(k_\Box\) \([\text{for a suitable choice of algebraic closure of } k_\Box]\); \(\chi_k : G_\Box \to \mathbb{Z}_l^\times\) the \(l\)-adic cyclotomic character associated to \(k_\Box\); \(X^\log_\Box \to \text{Spec}(k_\Box)\) a smooth log curve; \((X^\log_\Box)_n\) the \(n\)-th log configuration space of \(X^\log\); \(\Pi_{(X^\log_\Box)_n/k_\Box}\) the kernel of the natural [outer] surjection \(\Pi_{(X^\log_\Box)_n} \to G_\Box; \Delta_n \stackrel{\text{def}}{=} \Pi_{(X^\log_\Box)_n/k_\Box} ; \Pi^\Box_n \text{ the quotient of} \)}
\[ \Pi(\chi_{\square}^{\log})_n \] by the kernel of the natural surjection \( \Pi(\chi_{\square}^{\log})_n \rightarrow \Delta_n^{\square} \). Thus, the natural conjugation action of \( \Pi(\square)_n \) on \( \Delta_n^{\square} \) determines a natural outer Galois action \( G_{k\square} \rightarrow \text{Out}(\Delta_n^{\square}) \). Set \( n_{\min} = 3 \) if \((g, r) \neq (0, 3)\); \( n_{\min} = 2 \) if \((g, r) = (0, 3)\). Suppose either that \( n \geq n_{\min} \) or that the following conditions are satisfied:

(a) \( k\square \) is strongly \( l \)-cyclo-

tomically full [cf. Definition 3.1, (iii)].

(b) Let \( J \subseteq \Delta_1^{\square} \) be a characteristic open subgroup. Observe that \( \Pi(\square)_1 \) naturally acts on \( J \) by conjugation, hence on \( J^{\text{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) [cf. the discussion entitled “Topological groups” in §0]. Write \( \rho_J : \Pi(\square)_1 \rightarrow \text{Aut}_{\mathbb{Q}_l}(J^{\text{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \) for this action of \( \Pi(\square)_1 \) on \( J^{\text{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \). Then there exists a finite filtration of \( \Pi(\square)_1 \)-stable \( \mathbb{Q}_l \)-subspaces of \( J^{\text{ab/edge}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) such that, for each subquotient of this finite filtration, there exists an element \( g \in \Pi(\square)_1 \) such that the restriction to this subquotient of \( \rho_J(g) \) is \( \chi_{k\square}(g) \)-transverse [cf. Definition 3.1, (ii)], where, by abuse of notation, we write \( \chi_{k\square}(−) \) for the restriction of \( \chi_{k\square}(-) \), as defined above, via the natural [outer] surjection \( \Pi(\square)_1 \rightarrow G_{k\square} \).

Let \( \alpha : \Delta_0^{\square} \sim \rightarrow \Delta_0^{\bullet} \) be an isomorphism of profinite groups that is compatible with the respective natural outer Galois actions \( G^\circ \rightarrow \text{Out}(\Delta_0^{\circ}) \), \( G^{\bullet} \rightarrow \text{Out}(\Delta_0^{\bullet}) \) relative to some isomorphism of profinite groups \( G^\circ \sim \rightarrow G^{\bullet} \). Then for any log-full subgroup \( A \subseteq \Delta_0^{\circ} \) of \( \Delta_0^{\circ} \), \( \alpha(A) \subseteq \Delta_0^{\bullet} \) is a log-full subgroup of \( \Delta_0^{\bullet} \).

Theorem D may be considered as a sort of higher-dimensional analogue of the reconstruction of inertia subgroups of surface groups given in [CmbGC], Corollary 2.7, (i). Finally, we remark that a complete affirmative answer to (P2) may be found in [Higashi].

Acknowledgements

The authors would like to thank N. Takao and A. Tamagawa for their helpful comments concerning the present paper. The first author was supported by the Inamori Foundation and JSPS KAKENHI Grant Number 15K04780. The second author was supported by Grant-in-Aid for JSPS Fellows Grant Number 16J02375.

0. Notations and conventions

Numbers:

The notation \( \mathbb{Q} \) will be used to denote the field of rational numbers. The notation \( \mathbb{Z} \) will be used to denote the set, group, or ring of rational integers. The notation \( \mathbb{N} \) will be used to denote the set or additive monoid of non-negative rational integers. The notation \( \mathfrak{P} \text{rimes} \) will be used to denote the
Let $\Sigma$ be a nonempty subset of $\mathbb{Primes}$. Then we shall write
\[ \hat{\mathbb{Z}}^\Sigma \]
for the pro-$\Sigma$ completion of $\mathbb{Z}$. If $l \in \mathbb{Primes}$, then we shall write $\mathbb{Z}_l \overset{\text{def}}{=} \hat{\mathbb{Z}}\{l\}$. The notation $\mathbb{Q}_l$ will be used to denote the quotient field of the ring $\mathbb{Z}_l$. For a field $k$, we shall denote by $\text{ch}(k)$ the characteristic of $k$.

**Schemes:**

If $S$ is a scheme, and $s$ is a point of the underlying set of the scheme $S$, then, for simplicity, we shall write
\[ s \in S. \]

**Log Schemes:**

We refer to [KK1], [KK2] for basic facts concerning log schemes and log structures. In this paper, log structures are always considered on the étale topoi of schemes [or algebraic stacks]. If $X^\log$ is a log scheme, then we shall write $\mathcal{M}_X$ for the sheaf of monoids that defines the log structure of $X^\log$, $X$ for the underlying scheme of $X^\log$. If $f^\log : X^\log \to Y^\log$ is a morphism of log schemes, then we shall write $f : X \to Y$ for the associated underlying morphism of schemes; we shall refer to the image of $\mathcal{M}_X$ in the cokernel of the morphism induced on groupifications $f^*\mathcal{M}_Y \to \mathcal{M}_X$ by the morphism $f^\log : \mathcal{M}_Y \to \mathcal{M}_X$ determined by $f^\log$ as the relative characteristic of $f^\log$. If $X^\log$ is a log scheme, then we shall refer to the relative characteristic of $X^\log$ over $S$ where $X$ is regarded as a log scheme equipped with the trivial log structure, as the characteristic of $X^\log$. If $P$ is a finitely generated free abelian group, then we shall write $\text{rk}(P)$ for the rank of $P$.

Let $X^\log$ be a log scheme, and $\overline{s}$ a geometric point of $X$. Then we shall denote by $I(\overline{s}, \mathcal{M}_X)$ the ideal of $\mathcal{O}_{X,\overline{s}}$ generated by the image of $\mathcal{M}_{X,\overline{s}} \setminus \mathcal{O}_{X,\overline{s}}^\times$ via the morphism of monoids $\mathcal{M}_{X,\overline{s}} \to \mathcal{O}_{X,\overline{s}}^\times$ induced by the morphism $\mathcal{M}_X \to \mathcal{O}_X$ which defines the log structure of $X^\log$.

If $X^\log$, $Y^\log$ are fs [i.e., fine saturated] log schemes over an fs log scheme $Z^\log$, then we shall denote by $X^\log \times_{Z^\log} Y^\log$ the fiber product of $X^\log$ and $Y^\log$ over $Z^\log$ in the category of fs log schemes; we shall refer to as the interior of $X^\log$ the open subscheme of points at which the log structure of $X^\log$ is trivial.

**Curves:**

Let $n$ be a positive integer; $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$; $X \to S$ a hyperbolic curve of type $(g, r)$ [cf. [MT], §0]; $P_n$ the fiber product of $n$ copies of $X$ over $S$. Then we shall refer to as the $n$-th configuration space of $X \to S$ the $S$-scheme
\[ X_n \to S \]
representing the open subfunctor
\[ T \mapsto \{ (f_1, \ldots, f_n) \in P_n(T) \mid f_i \not= f_j \text{ if } i \not= j \} \]
of the functor represented by $P_n$. 

We shall write $\overline{M}_{g,r}$ for the moduli stack of $r$-pointed stable curves of genus $g$ over $\mathbb{Z}$ [where we assume the marked points to be ordered], $M_{g,r}$ for the open substack of $\overline{M}_{g,r}$ which parametrizes smooth curves, $\overline{M}^\log_{g,r}$ for the log stack obtained by equipping $\overline{M}_{g,r}$ with the log structure determined by the divisor at infinity $\overline{M}_{g,r} \setminus M_{g,r}$, and $C_{g,[r]} \to M_{g,[r]}$ for the stack-theoretic quotient of the morphism $\overline{M}_{g,r+1} \to \overline{M}_{g,r}$ [i.e., determined by forgetting the $(r+1)$-th marked point] by the action of the symmetric group on $r$ letters on the labels of the [first $r$] marked points. Let $k$ be a field. Then we shall write $(M_{g,r})_k \overset{\text{def}}{=} M_{g,r} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k)$.

Let $X^\log \to S^\log$ be a stable log curve of type $(g,r)$ [cf. [CmbGC], §0]. Then we shall refer to as the $n$-th log configuration space of $X^\log \to S^\log$ the $S^\log$-log scheme
\[X^\log_n \to S^\log\]
obtained by pulling back the morphism $\overline{M}^\log_{g,r+n} \to \overline{M}^\log_{g,r}$ determined by forgetting the last $n$ marked points via the classifying morphism $T^\log \to \overline{M}^\log_{g,r}$ of $X^\log \times_S T \to T^\log \overset{\text{def}}{=} S^\log \times_S T$ for a suitable finite étale covering $T$ of $S$ [i.e., over which the divisor of cusps of $X^\log \to S^\log$ becomes split] and then descending from $T$ to $S$. Note that, when the log structure on $S^\log$ is trivial [i.e., “$S^\log = S$”], in which case the interior $U$ of $X^\log$ may be regarded as a hyperbolic curve over $S$, the interior of the $n$-th log configuration space $X^\log_n$ may be identified with the $n$-th configuration space $U_n$ associated to $U$.

When $S^\log$ is the spectrum of a field equipped with the trivial log structure, we shall refer to the divisors of the underlying scheme $X_n$ of $X^\log_n$ that lie in the complement of the interior of $X^\log_n$ as log divisors of $X^\log_n$ [or, depending on the context of the discussion, of $U_n$].

Let $\Sigma$ be a nonempty subset of $\mathbb{Primes}$; $S^\log$ an fs log scheme whose underlying scheme is the spectrum of an algebraically closed field of characteristic $\not\in \Sigma$; $X^\log \to S^\log$ a stable log curve. Then the pointed stable curve determined by the log structure of $X^\log$ defines a semi-graph of anabelioids of pro-$\Sigma$ PSC-type $\mathcal{G}_{X^\log}$ [cf. [CmbGC], Definition 1.1, (i)]. We shall write
\[\mathcal{E}(X^\log)
\]
for the set of closed points of $X$ which correspond to the edges [i.e., “nodes” and “cusps”] of the underlying semi-graph of $\mathcal{G}_{X^\log}$.

**Topological Groups:**

Let $G$ be a topological group. Then we shall use the notation
\[
\text{Aut}(G), \quad \text{Out}(G)
\]
introduced in the discussion entitled “Topological groups” in [CbTpI], §0. Thus, if $G$ is a topologically finitely generated profinite group, then Aut$(G)$ and Out$(G)$ admit a natural profinite topology. Here, we recall from [NS], Theorem 1.1, that, in fact, if $G$ is a topologically finitely generated profinite group, then Aut$(G)$ and Out$(G)$ remain unaffected if one replaces $G$ by the discrete topological group determined by $G$. This result [NS], Theorem 1.1, also implies that, if $G$ is a topologically finitely generated profinite group, then the natural profinite topologies on Aut$(G)$ and Out$(G)$ may be
described as the topologies determined by the subgroups of finite index of $\text{Aut}(G)$ and $\text{Out}(G)$.

Let $\Sigma$ be a nonempty subset of $\mathcal{P}_{\text{Primes}}$, $G$ a profinite group. Then we shall denote by $G^{\Sigma}$ the maximal pro-$\Sigma$ quotient of $G$. For $l \in \mathcal{P}_{\text{Primes}}$, we shall write $G^{(l)} \overset{\text{def}}{=} G^{(l)}$. We shall say that $G$ is almost pro-$l$ if $G$ admits an open pro-$l$ subgroup.

We shall denote by $G^\Sigma$ the maximal pro-$\Sigma$ quotient of $G$. For $l \in \mathcal{P}_{\text{Primes}}$, we shall write $G_{\text{ab}}^l = G^{(l)}$. We shall say that $G$ is almost pro-$l$ if $G$ admits an open pro-$l$ subgroup.

We shall denote by $G_{\text{ab}}$ the abelianization of a profinite group $G$, i.e., the quotient of $G$ by the closure of the commutator subgroup of $G$. If $H$ is a closed subgroup of a profinite group $G$, then we shall write $Z_G(H)$ (respectively, $C_G(H)$) for the centralizer (respectively, commensurator) of $H$ in $G$ [cf. the discussion entitled “Topological groups” in [CbTpI], §0]. We shall say that the closed subgroup $H$ is commensurably terminal in $G$ if $H = C_G(H)$. We shall write $Z_G^{\Sigma}(H) \overset{\text{def}}{=} \lim_{H' \subseteq H} Z_G(H') \subseteq G$ — where $H' \subseteq H$ ranges over the open subgroups of $H$. We shall refer to $Z_G^{\Sigma}(H)$ as the local centralizer of $H$ in $G$; $Z_G^{\Sigma}(G) = Z_G^{\Sigma}(G)$ as the local center of $G$.

Let $\Sigma$ be a nonempty subset of $\mathcal{P}_{\text{Primes}}$. For a connected locally noetherian scheme $X$ (respectively, fs log scheme $X^{\log}$; semi-graph of anabelioids $\mathcal{G}$ of pro-$\Sigma$ PSC-type), we shall denote by $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$; $\Pi_G$) the étale fundamental group of $X$ (respectively, log fundamental group of $X^{\log}$; PSC-fundamental group of $\mathcal{G}$ [cf. [CmbGC], Definition 1.1, (ii)]) [for some choice of basepoint]. We shall denote by $\Pi_G^{\text{ab/edge}}$ the quotient of $\Pi_G^{\text{ab}}$ by the closed subgroup generated by the images in $\Pi_G^{\text{ab}}$ of the edge-like subgroups [cf. [CmbGC], Definition 1.1, (ii)] of $\Pi_G$ [cf. [NodNon], Definition 1.3, (i)].

If $K$ is a field, then we shall write $G_K$ for the absolute Galois group of $K$, i.e., $\pi_1(\text{Spec}(K))$.

Suppose that either $\Sigma = \{l\}$ or $\Sigma = \mathcal{P}_{\text{Primes}}$. We shall say that a profinite group $G$ is a pro-$\Sigma$ surface group (respectively, a pro-$\Sigma$ configuration space group) if $G$ is isomorphic to the maximal pro-$\Sigma$ quotient of the étale fundamental group of a hyperbolic curve (respectively, the configuration space of a hyperbolic curve) over an algebraically closed field of characteristic zero.

1. Group-theoretic reconstruction of the dimension

In this section, we introduce the notion of a log-full point of an fs log scheme [cf. Definition 1.1]. We then discuss some elementary properties of the log-full points of the log configuration space of a stable log curve [cf. Proposition 1.3]. As an application, we give a group-theoretic characterization of the dimension of a log configuration space [cf. Theorem 1.6].
Definition 1.1. Let $X^\log$ be an fs log scheme. Then we shall say that a point $x \in X$ is log-full if, for any geometric point $\overline{x}$ lying over $x$, the equality
\[
\dim(\mathcal{O}_X,\mathcal{I}/(\overline{x},\mathcal{M}_X)) = 0
\]

Proposition 1.2. (Properties of closed points of log configuration spaces). Let $n$ be a positive integer; $\Sigma$ a subset of $\text{Primes}$ which is either of cardinality one or equal to $\text{Primes}$; $k$ an algebraically closed field of characteristic $\neq \Sigma$; $S \overset{\text{def}}{=} \text{Spec}(k)$; $S^\log$ an fs log scheme whose underlying scheme is $S$; $X^\log$ a stable log curve over $S^\log$; $X_n^\log$ the $n$-th log configuration space of $X^\log$; $X_n^\log \overset{\text{def}}{=} S^\log$, $X_n^\log \rightarrow X_n^\log$ the projection morphism obtained by forgetting the factor labeled $n$, when $n \geq 2$; $X_1^\log \rightarrow X_0^\log$ the stable log curve $X^\log \rightarrow S^\log$; $\mathcal{P}_n$ the groupification of the relative characteristic of $X_n^\log \rightarrow S^\log$; $\mathcal{P}_0 \overset{\text{def}}{=} \{0\}$; $x_n \in X_n$ a closed point; $x_{n-1}$ the image of $x_n$ in $X_{n-1}^\log$; $x_n^\log$ (respectively, $x_{n-1}^\log$) the log scheme obtained by restricting the log structure of $X_n^\log$ (respectively, $X_{n-1}^\log$) to the [reduced, artinian] closed subscheme of $X_n$ (respectively, $X_{n-1}$) determined by $x_n$ (respectively, $x_{n-1}$).

Moreover, we shall write
\[
(X_n^\log)_{x_{n-1}} \overset{\text{def}}{=} X_n^\log \times_{X_{n-1}^\log} x_{n-1}^\log \rightarrow x_{n-1}^\log
\]
for the stable log curve obtained by base-changing $X_n^\log \rightarrow X_{n-1}^\log$ via the natural inclusion $x_{n-1}^\log \hookrightarrow X_{n-1}^\log$; $\pi_1(X_n^\log/S^\log)$ (respectively, $\pi_1(x_n^\log/S^\log)$) for the kernel of the natural [outer] surjection $\pi_1(X_n^\log) \twoheadrightarrow \pi_1(S^\log)$ (respectively, $\pi_1(x_n^\log) \twoheadrightarrow \pi_1(S^\log)$); $\Delta_n \overset{\text{def}}{=} \pi_1(X_n^\log/S^\log)$; $\Delta_{x_n} \overset{\text{def}}{=} \pi_1(x_n^\log/S^\log)$; $\Pi_n$ (respectively, $\Pi_{x_n}$) for the quotient of $\pi_1(X_n^\log)$ (respectively, $\pi_1(x_n^\log)$).

Moreover, we shall write
\[
1 \hookrightarrow \Delta_{x_n} \twoheadrightarrow \Pi_{x_n} \rightarrow \pi_1(S^\log) \twoheadrightarrow 1
\]
where the horizontal sequences are exact, and $\iota_{\Delta_{x_n}}, \Pi_{x_n}$ are the [outer] homomorphisms induced by the natural inclusion $x_n^\log \hookrightarrow X_n^\log$. Then the following hold:

(i) Let us regard, by abuse of notation, $x_n$ (respectively, $x_{n-1}$) as a geometric point of $X_n$ (respectively, $X_{n-1}$). Then the following inequalities hold:
\[
\text{rk}(\mathcal{P}_{n-1,x_{n-1}}) \leq \text{rk}(\mathcal{P}_{n,x_n}) \leq \text{rk}(\mathcal{P}_{n-1,x_{n-1}}) + 1.
\]
Here, in the first (respectively, second) inequality, equality holds if and only if $x_n \notin \mathcal{E}((X_n^\log)_{x_{n-1}})$ (respectively, $x_n \in \mathcal{E}((X_n^\log)_{x_{n-1}})$).
(ii) \( \text{rk}(P_{n,x^n}) \leq n \).

(iii) \( \Delta_{x^n} \cong \hat{Z}^\Sigma(1)^{\oplus \text{rk}(P_{n,x^n})} \). [Here, the "(1)" denotes a "Tate twist".]

(iv) The [outer] homomorphism \( \iota_{x^n}^\Delta \) is injective. In particular, \( \iota_{x^n}^\Pi \) is also injective.

Proof. First, we consider assertion (i). To verify the inequalities, it suffices to show that

\[
0 \leq \text{rk}(P_{n/n-1,x^n}) \leq 1
\]

— where we write \( P_{n/n-1} \) for the groupification of the relative characteristic of \( X_n^{\log} \to X_{n-1}^{\log} \). But this follows immediately from [KF], Lemma 1.4. Here, note that \( \text{rk}(P_{n/n-1,x^n}) = 0 \) (respectively, \( \text{rk}(P_{n/n-1,x^n}) = 1 \)) holds if and only if \( x_n \notin \mathcal{E}(X_n^{\log}) \) (respectively, \( x_n \in \mathcal{E}(X_n^{\log}) \)) [cf. [KF], Lemma 1.6; [KF], Proof of Theorem 1.3]. This completes the proof assertion (i).

Assertion (ii) is an immediate consequence of assertion (i). Assertion (iii) follows from [Hsh], Proposition B.5. Finally, we consider assertion (iv). First, we verify the case \( n = 1 \). Suppose that \( x_1 \notin \mathcal{E}(X^{\log}) \). Then since \( \Delta_{x_1} \cong \{1\} \) [cf. assertions (i), (iii)], it follows that \( \iota_{x_1}^\Delta \) is injective. Thus, we may assume that \( x_1 \in \mathcal{E}(X^{\log}) \). Then the injectivity of \( \iota_{x_1}^\Delta \) follows from [the evident pro-\( \Sigma \)-generalization of] [SemiAn], Proposition 2.5, (i) [cf. also [CmbGC], Remark 1.1.3]. This completes the proof of assertion (iv) in the case \( n = 1 \). Thus, it remains to verify assertion (iv) in the case \( n \geq 2 \). To this end, let us first observe that the projection morphism \( X_n^{\log} \to X_{n-1}^{\log} \) induces [outer] surjections

\[
\Delta_n \twoheadrightarrow \Delta_{n-1}; \ \Delta_{x^n} \twoheadrightarrow \Delta_{x_{n-1}}
\]

— where "\( \Delta_{n-1} \)”, "\( \Delta_{x_{n-1}} \)” are defined in the same manner as "\( \Delta_n \)” , “\( \Delta_{x^n} \)”, respectively. Write

\[
\Delta_{n/n-1} \overset{\text{def}}{=} \ker(\Delta_n \twoheadrightarrow \Delta_{n-1}); \ \Delta_{x^n/x_{n-1}} \overset{\text{def}}{=} \ker(\Delta_{x^n} \twoheadrightarrow \Delta_{x_{n-1}}).
\]

Thus, we have a commutative diagram of [outer] homomorphisms of profinite groups as follows:

\[
\begin{array}{cccc}
1 & \xrightarrow{\nu_{x^n/x_{n-1}}} & \Delta_{x^n/x_{n-1}} & \xrightarrow{\iota_{x^n}} \Delta_{x^n} & \xrightarrow{\iota_{x^n}^\Delta} \Delta_{x_{n-1}} & 1 \\
1 & \xrightarrow{\iota_{n/n-1}} & \Delta_{n/n-1} & \xrightarrow{\nu_{x^n}} \Delta_{x^n} & \xrightarrow{\nu_{x^n}^\Delta} \Delta_{x_{n-1}} & 1 \\
\end{array}
\]

— where the horizontal sequences are exact, and \( \nu_{x^n/x_{n-1}} \) (respectively, \( \nu_{n/n-1} \)) is the [outer] homomorphism induced by the natural inclusion \( x_{n-1}^{\log} \hookrightarrow X_n^{\log} \) (respectively, induced by the [outer] homomorphism \( \nu_{x^n}^\Delta \)). [Here, we recall that the image of \( \nu_{n/n-1} \) is commensurably terminal in \( \Delta_{n/n-1} \) [cf. [CmbGC], Proposition 1.2, (ii)]. This implies that \( \nu_{n/n-1} \) is well-defined as an outer homomorphism.] On the other hand, since [cf. [MT], Proposition 2.2, (i); the discussion of "specialization isomorphisms" in the subsection entitled...
“The Etale Fundamental Group of a Log Scheme” in [CmbCusp], §0 one may identify $\iota_{n/n-1}$ with the “$\iota^\Delta_{x_1}$” induced by the natural inclusion

$$x_n^\log \hookrightarrow \varprojlim \{X_n^\log \}_{n-1},$$

it follows [from the case $n = 1$] that $\iota_{n/n-1}$ is injective. Therefore, by induction on $n$, we conclude that $\iota^\Delta_n$ is injective. This completes the proof of assertion (iv).

**Proposition 1.3. (Properties of the log-full points of a log configuration space).** In the notation of Proposition 1.2, the following hold:

(i) $x_n \in X_n$ is log-full if and only if $\text{rk}(P_n, x_n) = n$.

(ii) If $x_n \in X_n$ is log-full, then $x_{n-1}$ is also a log-full point of $X_{n-1}$.

(iii) If $x_n \in X_n$ is log-full, then $\Delta_{x_n} \cong \hat{\mathbb{Z}}^\Sigma(1)^{\mathbb{Z}_n}$.

Proof. First, we verify assertion (i). Fix a clean chart $P \to k$ of $S^\log$ [cf., e.g., [Hsh], Definition B.1, (ii)]. Write $T^\log$ for the log scheme whose underlying scheme is $\text{Spec}(k[[P]])$, and whose log structure is defined by the natural inclusion $P \hookrightarrow k[[P]]$: $S^\log \hookrightarrow T^\log$ for the [strict] closed immersion determined by the maximal ideal of $k[[P]]$. Thus, it follows immediately from the well-known deformation theory of stable log curves that we may assume without loss of generality that there exists a stable log curve

$$Y^\log \hookrightarrow T^\log$$

whose base-change via $S^\log \hookrightarrow T^\log$ is isomorphic to [hence may be identified with] $X^\log \to S^\log$. Write $Y^\log_n$ for the $n$-th log configuration space of $Y^\log$. Since $T^\log$ is log regular [cf. [KK2], Definition 2.1], and $Y^\log_n \to T^\log$ is log smooth, it follows that $Y^\log_n$ is also log regular [cf. [KK2], Theorem 8.2]. Thus, we obtain an equality

$$(*) \quad \dim(\mathcal{O}_{Y^\log_n, x_n}) = \dim(\mathcal{O}_{Y_n, x_n}/I(x_n, M_{Y_n})) + \text{rk}(Q_{n, x_n})$$

where we regard $x_n$ as a geometric point of $Y_n$; $Q_n$ denotes the groupification of the characteristic of $Y^\log_n$ [cf. [KK2], Definition 2.1]. On the other hand, it follows from the various definitions involved that we have equalities

$$\dim(\mathcal{O}_{Y_n, x_n}) = \dim(\mathcal{O}_{T, x_0}) + n, \quad \text{rk}(Q_{n, x_n}) = \dim(\mathcal{O}_{T, x_0}) + \text{rk}(P_{n, x_n}),$$

where we regard $x_0$ as a geometric point of $T$. Combining these equalities with $(*)$, we obtain an equality

$$\dim(\mathcal{O}_{Y_n, x_n}/I(x_n, M_{Y_n})) = n - \text{rk}(P_{n, x_n}).$$

Thus, since $\dim(\mathcal{O}_{Y_n, x_n}/I(x_n, M_{Y_n})) = \dim(\mathcal{O}_{Y_n, x_n}/I(x_n, M_{Y_n}))$, we conclude that $x_n \in X_n$ is log-full if and only if $\text{rk}(P_{n, x_n}) = n$. This completes the proof of assertion (i). Assertion (ii) follows from assertion (i) and the fact that $\text{rk}(P_{n, x_n}) = n$ implies $\text{rk}(P_{n-1, x_{n-1}}) = n - 1$ [cf. Proposition 1.2, (i), (ii)]. Assertion (iii) follows from assertion (i) and Proposition 1.2, (iii). □
Proposition 1.4. (Abelian closed subgroups of a pro-\(l\) surface group). Let \(l\) be a prime number; \(G\) a pro-\(l\) surface group. Then every nontrivial abelian closed subgroup of \(G\) is isomorphic to \(\mathbb{Z}_l\).

Proof. Let \(H \subseteq G\) be a nontrivial abelian closed subgroup of \(G\). Here, note that \(H\) is of infinite index in \(G\). [Indeed, this follows from the fact that every open subgroup of \(G\) is also a pro-\(l\) surface group, hence nonabelian.] Thus, it follows from the well-known fact that the \(l\)-cohomological dimension of a closed infinite index subgroup of a pro-\(l\) surface group is \(\leq 1\) [cf., e.g., the proof of [MT], Theorem 1.5] that \(H\) is a free pro-\(l\) group [cf. [RZ], Theorem 7.7.4]. Since \(H\) (\(\neq \{1\}\)) is abelian, we thus conclude that \(H \cong \mathbb{Z}_l\). \(\square\)

Lemma 1.5. (Upper bound for the rank of free abelian pro-\(l\) closed subgroups of a pro-\(l\) configuration space group). Let \(l\) be a prime number; \(n\) a positive integer; \(s\) a positive integer; \(X\) a hyperbolic curve over an algebraically closed field of characteristic \(\neq l\); \(X_n\) the \(n\)-th configuration space of \(X\); \(\Pi_n \overset{\text{def}}{=} \pi_1(X_n)^{(l)}\); \(H \subseteq \Pi_n\) a closed subgroup such that \(H \cong \mathbb{Z}_l^\oplus s\). Then the inequality

\[ s \leq n \]

holds.

Proof. Let us prove the inequality by induction on \(n\). When \(n = 1\), the inequality follows from Proposition 1.4. Now suppose that \(n \geq 2\), and that the induction hypothesis is in force. Write

\[ p : \Pi_n \twoheadrightarrow \Pi_1 \]

for the natural [outer] surjection induced by the projection morphism \(X_n \rightarrow X\) obtained by forgetting the factors with labels \(\neq n\). Then, by Proposition 1.4, it follows that either

\[ p(H) = \{1\} \quad \text{or} \quad p(H) \cong \mathbb{Z}_l. \]

In particular, \(H' \overset{\text{def}}{=} H \cap \text{Ker}(p) \subseteq \text{Ker}(p)\) satisfies either

\[ H' \cong \mathbb{Z}_l^\oplus s \quad \text{or} \quad H' \cong \mathbb{Z}_l^\oplus s-1. \]

Here, we note that Ker\((p)\) may be identified with the maximal pro-\(l\) quotient of the étale fundamental group of the \((n - 1)\)-st configuration space of some hyperbolic curve over an algebraically closed field of characteristic \(\neq l\) [cf. [MT], Proposition 2.4, (i)]. Thus, by applying the induction hypothesis, we obtain that either

\[ s \leq n - 1 \quad \text{or} \quad s - 1 \leq n - 1, \]

which implies that \(s \leq n\), as desired. \(\square\)
Theorem 1.6. (Group-theoretic characterization of the dimension of a configuration space). Let \( l \) be a prime number; \( n \) a positive integer; \( X \) a hyperbolic curve over an algebraically closed field \( k \) of characteristic \( \neq l \); \( X_n \) the \( n \)-th configuration space of \( X \); \( \Pi_n \overset{\text{def}}{=} \pi_1(X_n)^{(l)} \). Then the following equality holds:

\[
n = \max\{s \in \mathbb{N} \mid \exists \text{ a closed subgroup of } \Pi_n \text{ which is isomorphic to } \mathbb{Z}_l^\oplus s\}.
\]

Proof. In light of Lemma 1.5, to verify the assertion, it suffices to show the following claim:

There exists a closed subgroup \( H \) of \( \Pi_n \) such that \( H \cong \mathbb{Z}_l^\oplus n \).

Thus, we proceed to verify this claim. First, we observe that, by applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCusp], Definition 2.1, as well as [CbTpI], Remark 5.6.1], one may conclude that there exists a stable log curve \( Z^\text{log} \to S^\text{log} \), where \( S \) is the spectrum of an algebraically closed field of characteristic \( \neq l \), and we write \( Z_n^\text{log} \) for the \( n \)-th log configuration space of \( Z^\text{log} \), satisfying the following properties:

\[
\mathcal{E}(Z^\text{log}) \neq \emptyset, \quad \text{Ker}(\pi_1(Z_n^\text{log}) \to \pi_1(S^\text{log}))^{(l)} \cong \Pi_n.
\]

[Here, we note that, although the algebraically closed field “\( k_0 = k \)” of the discussion preceding [CmbCusp], Definition 2.1, is assumed to be of characteristic zero, one verifies immediately that, if one restricts one’s attention to pro-\( l \) fundamental groups, then this discussion generalizes immediately to the case where “\( k_0 = k \)” is of characteristic \( \neq l \).] Thus, by applying Propositions 1.2, (i); 1.3, (i), it follows that there exists a log-full point \( z_n \in Z_n \). In particular, by applying Proposition 1.3, (iii) to \( z_n \), we conclude that there exists a closed subgroup \( H \) of \( \Pi_n \) such that \( H \cong \mathbb{Z}_l^\oplus n \). This completes the proof of the claim, hence also of Theorem 1.6. \( \square \)

2. Group-theoretic reconstruction of the genus, number of cusps, and generalized fiber subgroups

In this section, we prove that, if \( n \) is a positive integer \( \geq 2 \), then the triple \((n, g, r)\) may be recovered group-theoretically from any configuration space group that arises from the \( n \)-th configuration space of a hyperbolic curve of type \((g, r)\) over an algebraically closed field of characteristic zero [cf. Theorem 2.5, (i)]. Moreover, by combining this result with a result of Nakamura and Takao [cf. [NT], Theorem D], we generalize [MT], Corollary 6.3 — by introducing the notion of a generalized fiber subgroup [cf. Definition 2.1, (ii)] — to the case of arbitrary hyperbolic curves [cf. Theorem 2.5, (iii)].

Definition 2.1. Let \((g, r)\) be a pair of nonnegative integers such that \( 2g - 2 + r > 0 \); \( 0 < n' < n \) positive integers; \( \Sigma \) a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; \( k \) an algebraically closed field of characteristic \( \not\in \Sigma \); \( X \) a hyperbolic curve of type \((g, r)\) over \( k \). Write \( n'' \overset{\text{def}}{=} n - n' \); \( X_n \) for the \( n \)-th configuration space of \( X \); \( \Pi_n \overset{\text{def}}{=} \pi_1(X_n)^{\Sigma} \).
We define a *generalized projection morphism of length* $n'$ [or, alternatively, a *generalized projection morphism of co-length* $n''$]

\[ X_n \to X_{n''} \]

to be a morphism that may be described as follows:

(a) Suppose that $(g, r) \in \{(0,3), (1,1)\}$. Then it follows immediately from the definition of $X_n$ [cf. the discussion entitled “Curves” in §0] that the projection morphism $X_{n+1} \to X_n$ to the first $n$ factors of $X_{n+1}$ determines, in a natural way, a(n) [smooth] $(n+r)$-pointed stable curve of genus $g$ over $X_n$. One verifies immediately that our assumption that $(g, r) \in \{(0,3), (1,1)\}$ implies that the [smooth] $r$-pointed stable curve of genus $g$ determined by forgetting any subset of cardinality $n$ of the set of $n+r$ marked points of the $(n+r)$-pointed stable curve of genus $g$ over $X_n$ determined by the projection morphism $X_{n+1} \to X_n$ is naturally isomorphic to the [smooth] $r$-pointed stable curve of genus $g$ determined by $X$. We shall refer to the morphism

\[ X_n \to X_{n''} \]

obtained by forgetting some subset of cardinality $n'$ of the set of $n+r$ marked points of the $(n+r)$-pointed stable curve of genus $g$ over $X_n$ determined by the projection morphism $X_{n+1} \to X_n$ discussed above as a *generalized projection morphism of length* $n'$.

(b) Suppose that $(g, r) \notin \{(0,3), (1,1)\}$. Then we shall refer to the projection morphism $X_n \to X_{n''}$ obtained by forgetting some collection of $n'$ factors as a *generalized projection morphism of length* $n'$.

Note that, in the case where $(g, r) \notin \{(0,3), (1,1)\}$, the notion of a generalized projection morphism coincides with the notion of a *projection morphism* as defined in [MT], Definition 2.1, (ii). By contrast, in the case where $(g, r) \in \{(0,3), (1,1)\}$, one verifies immediately that there exist generalized projection morphisms that are not *projection morphisms* in the sense of [MT], Definition 2.1, (ii).

(ii) We shall refer to

\[ \text{Ker}(\Pi_n \to \Pi_{n''}) \]

— where $\Pi_n \to \Pi_{n''}$ is the natural [outer] surjection induced by a generalized projection morphism of length $n'$ — as a *generalized fiber subgroup of* $\Pi_n$ *of length* $n'$ [or, alternatively, as a *generalized fiber subgroup of* $\Pi_n$ *of co-length* $n''$]. Note that, in the case where $(g, r) \notin \{(0,3), (1,1)\}$, the notion of a generalized fiber subgroup of $\Pi_n$ coincides with the notion of a *fiber subgroup* of $\Pi_n$ as defined in [MT], Definition 2.3, (iii). By contrast, in the case where $(g, r) \in \{(0,3), (1,1)\}$, one verifies immediately that there exist generalized fiber subgroups of $\Pi_n$ that are not *fiber subgroups* of $\Pi_n$ in the sense of [MT], Definition 2.3, (iii).
(iii) We shall refer to a fiber subgroup of $\Pi_n$ of co-length one in the sense of [MT], Definition 2.3, (iii), as a co-surface subgroup of $\Pi_n$. We shall refer to a closed normal subgroup $H$ of $\Pi_n$ such that $\Pi_n/H$ is a [pro-$\Sigma$] surface group which is not isomorphic to a free pro-$\Sigma$ group of rank two as a quasi-co-surface subgroup of $\Pi_n$.

(iv) We shall say that an automorphism $\alpha$ of $\Pi_n$ is $gF$-admissible if, for every generalized fiber subgroup $F$ of $\Pi_n$, it holds that $\alpha(F) = F$. We shall say that an outer automorphism of $\Pi_n$ is $gF$-admissible if it is arises from a $gF$-admissible automorphism. Write $\text{Out}^{gF}(\Pi_n)$ for the group of $gF$-admissible outer automorphisms of $\Pi_n$.

Remark 2.1.1. In the notation of Definition 2.1, let

$$n^* = \begin{cases} n + r & \text{if } (g, r) \in \{(0, 3), (1, 1)\}; \\ n & \text{if } (g, r) \notin \{(0, 3), (1, 1)\}. \end{cases}$$

Then one verifies immediately that the natural action of the symmetric group on $n^*$ letters $S_{n^*}$ on the set of marked points with labels

$$\in \{n + r - n^* + 1, n + r - n^* + 2, \ldots, n + r\}$$

[a set of cardinality $n^*$] of the $(n + r)$-pointed stable curve $X_{n+1} \rightarrow X_n$ [determined by the projection morphism to the first $n$ factors of $X_{n+1}$ — cf. the discussion of Definition 2.1, (i); [MT], Remark 2.1.2] induces a natural action of $S_{n^*}$ on $X_n$ [i.e., the base space of this $(n+r)$-pointed stable curve]. Moreover, this action induces an action of $S_{n^*}$ on the set of generalized fiber subgroups of $\Pi_n$ of co-length $n''$, which is easily verified to be transitive.

The content of the following Proposition is, in essence, well-known.

Proposition 2.2. (Abelianization of a configuration space group). Let $l$ be a prime number; $n \geq 2$ an integer; $k$ an algebraically closed field of characteristic $\neq l$; $X$ a hyperbolic curve of type $(g, r)$ over $k$; $X_n$ the $n$-th configuration space of $X$; $\Pi_n \overset{\text{def}}{=} \pi_1(X_n)^{(l)}$;

$$\varphi : \Pi_n^{ab} \twoheadrightarrow \Pi_1^{ab} \times \cdots \times \Pi_1^{ab}$$

the surjection induced by the natural open immersion $X_n \hookrightarrow X \times_k \cdots \times_k X$; $\Pi_{n/1}$ the kernel of the natural [outer] surjection $\Pi_n \twoheadrightarrow \Pi_1$ induced by the first projection $X_n \rightarrow X$. Then the following hold:
(i) If $g = 0$, then the first projection $X_n \to X$ induces an exact sequence

$$1 \to \Pi_{n/1}^{ab} \to \Pi_n^{ab} \to \Pi_1^{ab} \to 1.$$ 

In particular, it follows that $\Pi_n^{ab}$ is a finitely generated free $\mathbb{Z}_l$-module of rank

$$\text{rk}_{\mathbb{Z}_l}(\Pi_n^{ab}) = n + r - 2 \sum_{j=r-1}^{n-1} j = n(r - 1) + \frac{n(n - 1)}{2} > n(r - 1).$$

(ii) If $g \neq 0$, then $\varphi$ is an isomorphism. In particular, it follows that $\Pi_n^{ab}$ is a finitely generated free $\mathbb{Z}_l$-module of rank

$$\text{rk}_{\mathbb{Z}_l}(\Pi_n^{ab}) = \begin{cases} n(2g + r - 1), & \text{if } r > 0 \\ 2gn, & \text{if } r = 0. \end{cases}$$

(iii) $g \neq 0$ if and only if $\varphi$ is an isomorphism.

(iv) $(g, r) = (1, 1)$ if and only if $\text{rk}_{\mathbb{Z}_l}(\Pi_n^{ab}) = 2n$ [cf. (i), (ii)].

Proof. First, we consider assertion (i). We begin by verifying the exactness of the sequence of the first display in the statement of assertion (i). Write $\Pi_n^{ab}/1$ for the quotient of $\Pi_n$ by the kernel of the natural surjection $\Pi_n/1 \to \Pi_n^{ab}/1$. Thus, we have a tautological exact sequence

$$1 \to \Pi_{n/1}^{ab} \to \Pi_n^{ab} \to \Pi_1^{ab} \to 1.$$ 

Next, recall from the well-known structure theory of surface groups that $\Pi_1$ is a free pro-$l$ group. Thus, the exact sequence of the preceding display admits a splitting $s : \Pi_1 \to \Pi_n^{ab}/1$. Now we claim that

Claim 2.2A: The natural conjugation action of $\Pi_1$ on $\Pi_n^{ab}/1$ is trivial.

Note that the exactness of the sequence of the first display in the statement of assertion (i) follows formally from Claim 2.2A, together with the existence of the splitting $s : \Pi_1 \to \Pi_n^{ab}/1$.

To verify Claim 2.2A, we reason as follows. Let $\overline{K}$ be an algebraic closure of the function field $K$ of $X$. Thus, the geometric fiber $Z \overset{\defeq}{=} X_2 \times_X \text{Spec}(\overline{K})$ of the first projection $X_2 \to X$ is a hyperbolic curve of genus $0$ over $\overline{K}$. For simplicity, we assume that $\overline{K}$ has been chosen so that $\Pi_1$ may be regarded as a quotient of the Galois group $\text{Gal}(\overline{K}/K)$. Now observe that the $(n-1)$-st configuration space $Z_{n-1}$ of $Z$ may be naturally identified with the geometric fiber $X_n \times_X \text{Spec}(\overline{K})$ of the first projection $X_n \to X$, and hence that $\Pi_{n/1}$ may be regarded as the maximal pro-$l$ quotient of the étale fundamental group of $Z_{n-1}$. Since $Z$ is of genus $0$, it follows that $Z_{n-1}$ embeds as an open subvariety of the fiber product over $\overline{K}$ of $n - 1$ copies of the projective line $\mathbb{P}_K^1$ over $\overline{K}$. Since the étale fundamental group of such a fiber product is well-known to be trivial, it follows immediately from the Zariski-Nagata
purity theorem that $\Pi_{n/1}$ is topologically normally generated by the inertia subgroups $\subseteq \Pi_{n/1}$ associated to the various log divisors [cf. the discussion entitled “Curves” in §0] of $Z_{n-1}$. Moreover, one verifies immediately that each such log divisor $D_Z$ arises, by base-changing via the natural morphism $\text{Spec}(K) \to X$, from some log divisor $D_X$ of $X_n$ [where we regard $X_n$ as an object over $X$ by means of the first projection $X_n \to X$] that corresponds to a discrete valuation $v$ of the function field of $X_n$ that induces the trivial valuation of $K$.

Next, let us observe that it follows immediately from the well-known geometry of moduli stacks of pointed stable curves of genus 0 that $K$ embeds as a subfield of the residue field $L$ of the discrete valuation ring $O_v$ determined by $v$ which is algebraically closed in $L$. [Indeed, the log divisor $D_X$ may be identified with a suitable fiber product over $k$ of log configuration spaces associated to smooth log curves of genus 0 over $k$. Moreover, the morphism from $D_X$ to the unique [up to isomorphism] compactification of $X$ induced by the first projection $X_n \to X$ is easily seen to coincide with the [manifestly geometrically connected!] projection morphism that arises from one of the projection morphisms associated to one of the log configuration spaces that appears in such a fiber product over $k$.]

Thus, the image $I_Z \subseteq \Pi_{n/1}^{ab}$ of any inertia subgroup $\subseteq \Pi_{n/1}$ associated to $D_Z$ may be thought of as the image $\subseteq \Pi_{n/1}^{ab}$ of the maximal pro-$l$ quotient $G_v (\cong \mathbb{Z}_l)$ of the absolute Galois group of the quotient field of some strict henselization of $O_v$, whose residue field we denote by $\overline{L}$. Here, we observe that we may assume without loss of generality that $L$, $K$, and $\overline{K}$ are subfields of $\overline{L}$ such that $L \cap K = K$ [so $\text{Gal}(\overline{K}/K)$ may be regarded as a quotient of $\text{Gal}(\overline{L}/L)$]. Since the natural conjugation action of $\text{Gal}(\overline{L}/L)$ on $G_v$ is the action via the cyclotomic character [hence factors through the quotients $\text{Gal}(\overline{L}/L) \to \text{Gal}(\overline{K}/K) \to \Pi_1$], it follows from the fact that $k$ is algebraically closed that this natural conjugation action is trivial, hence that the natural conjugation action of $\Pi_1$ [which we regard as a quotient of $\text{Gal}(\overline{L}/L)$!] on $I_Z \subseteq \Pi_{n/1}^{ab}$ is trivial. Since [cf. the above discussion] $\Pi_{n/1}^{ab}$ is topologically generated by such subgroups $I_Z \subseteq \Pi_{n/1}^{ab}$, we thus conclude that the natural conjugation action of $\Pi_1$ on $\Pi_{n/1}^{ab}$ [cf. Claim 2.2A] is trivial. This completes the proof of Claim 2.2A.

The final portion of assertion (i) now follows formally, by induction on $n$, from the exactness of the sequence of the first display in the statement of assertion (i), together with the well-known structure theory of surface groups. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us prove that $\varphi$ is an isomorphism in the case where $n = 2$. In this case, we shall denote by $\Pi_{2/1}$ (respectively, $M$) the kernel of the natural [outer] surjection $\Pi_2 \to \Pi_1$ induced by the first projection $X_2 \to X$ (respectively, the kernel of $\varphi$). In
particular, we have a commutative diagram as follows:

\[
\begin{array}{c}
1 \\
J \\
\Pi_{2/1}^{ab} \\
\psi \\
1 \\
\end{array}
\quad \alpha \quad \begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
M \\
\Pi_2^{ab} \\
\Pi_1^{ab} \rightarrow 1 \\
\Pi_1^{ab} \times \Pi_1^{ab} \\
\Pi_1^{ab} \rightarrow 1 \\
1 \\
\end{array}
\]

where the middle and lower horizontal sequences are the exact sequences induced by the first projection \(X \times_k X \to X\); \(\psi\) is the surjection, whose kernel we denote by \(J\), induced by \(\varphi\); \(\alpha\) is the surjection induced by the morphism \(\Pi_{2/1}^{ab} \to \Pi_2^{ab}\); \(\beta\) is the injection induced by the morphism \(\Pi_{2/1}^{ab} \to \Pi_2^{ab}\). To verify that \(\varphi\) is an isomorphism, it suffices to show that \(M = \{0\}\). First, let us observe that the structure of \(J\) may be described explicitly by applying the well-known structure theory of surface groups. If \(r = 0\), then [since \(J = \{0\}\)] it follows immediately that \(M = \{0\}\). Thus, it suffices to show that \(M = \{0\}\) under the assumption that \(r > 0\). Suppose that \(M \neq \{0\}\). Then since \(J \cong \mathbb{Z}_l\), it holds that \(M\) is a free \(A\)-module of rank one, where either \(A = \mathbb{Z}_l\) or \(A = \mathbb{Z}_l/l^m\mathbb{Z}_l\) for some \(m > 0\).

Thus, we obtain a commutative diagram as follows:

\[
\begin{array}{c}
1 \\
\Pi_{2/1}^{ab} \\
\Pi_2^{ab} \otimes \mathbb{Z}_l A \\
1 \\
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\Pi_2^{ab} \otimes \mathbb{Z}_l A \\
\Pi_1^{ab} \otimes \mathbb{Z}_l A \\
1 \\
\end{array}
\]

where the vertical arrows are the natural surjections induced by \(1 \in A\); the horizontal sequences are induced by the first projection \(X \times_k X \to X\); the upper horizontal sequence is exact by definition. Next, let us observe that it follows immediately from the definition of \(A\) that \(\alpha \otimes A\) is an isomorphism. This observation, together with the split injectivity of \(\beta : \Pi_1^{ab} \to \Pi_1^{ab} \times \Pi_1^{ab}\), implies that the lower horizontal sequence of the diagram of the preceding display is exact. Thus, it follows immediately that the composite

\[
\rho : \Pi_1 \to \text{Aut}(\Pi_{2/1}^{ab} \otimes \mathbb{Z}_l A)
\]

of the outer representation \(\Pi_1 \to \text{Out}(\Pi_{2/1}^{ab})\) with the natural morphism \(\text{Out}(\Pi_{2/1}^{ab}) \to \text{Aut}(\Pi_{2/1}^{ab} \otimes \mathbb{Z}_l A)\) is trivial. On the other hand, we conclude from the evident “\(\otimes \mathbb{Z}_l A\) version” of [CbTpI], Lemma 1.3, (iv) — which follows immediately [cf. the proof of [CbTpI], Lemma 1.3, (iv)] from the
The final portion of assertion (ii) now follows formally from the fact that from assertions (i), (ii) by considering groups. This completes the proof of assertion (ii). Assertion (iii) follows from the observation that the image of \( \Pi_n \to \Pi_n^{\text{ab}} \) via the natural surjection \( \Pi_n \to \Pi_n^{\text{ab}} \) for the maximal pro-\( l \) quotient of the étale fundamental group of the smooth compactification of \( X \).

In particular, we obtain that
\[
(A \otimes \mathbb{Z}_l) \cong (\Pi^\text{pt}_1)^{ab} \otimes \mathbb{Z}_l A = \{0\}.
\]

Thus, since \( g \neq 0 \), we conclude that \( M = \{0\} \), a contradiction. This completes the proof that \( \varphi \) is an isomorphism in the case where \( n = 2 \).

Next, we consider the case where \( n \geq 3 \). First, we make the following observation, which follows immediately from the Zariski-Nagata purity theorem:

\[
\text{Ker}(\varphi) = \{0\} \text{ if and only if, for every pair of integers } (i, j) \text{ such that } 1 \leq i < j \leq n, \text{ the image, via the natural surjection } \Pi_n \to \Pi_n^{\text{ab}}, \text{ of the inertia subgroup } I[i, j] \subseteq \Pi_n \text{ determined, up to } \Pi_n\text{-conjugacy, by the diagonal divisor } X_n \text{ corresponding to the } i\text{-th and } j\text{-th factors is trivial.}
\]

Denote by \( \Pi[i, j] \) the kernel of the natural outer surjection \( \Pi_n \to \Pi_{n-2} \) determined by the projection \( f_{i,j} : X_n \to X_{n-2} \) obtained by forgetting the factors labeled \( i, j \). In particular, we may identify \( \Pi[i, j] \) with the pro-\( l \) fundamental group of the second configuration space \( Z_2 \) of a hyperbolic curve \( Z \) of type \( (g, r + n - 2) \) over \( k \). Here, we note that, relative to this identification, we may identify the inertia subgroup \( I[i, j] \) determined, up to \( \Pi_n \)-conjugacy, by the diagonal divisor \( X_n \) corresponding to the \( i \)-th and \( j \)-th factors is trivial.

Thus, by applying the fact that \( \varphi \) is an isomorphism in the case where \( n = 2 \), together with the above observation, we conclude that the image of \( I[i, j] \subseteq \Pi[i, j] \) via the natural surjection \( \Pi[i, j] \to (\Pi[i, j])^{ab} \) is trivial, hence that the image of \( I[i, j] \subseteq \Pi_n \) via the natural surjection \( \Pi_n \to \Pi_n^{\text{ab}} \) is trivial.

Therefore, again by the above observation, we obtain that \( \text{Ker}(\varphi) = \{0\} \). This completes the proof that \( \varphi \) is an isomorphism in the case of arbitrary \( n \).

The final portion of assertion (ii) now follows formally from the fact that \( \varphi \) is an isomorphism, together with the well-known structure theory of surface groups. This completes the proof of assertion (ii). Assertion (iii) follows from assertions (i), (ii) by considering ranks.

Finally, we consider assertion (iv). Suppose that \( (g, r) = (1, 1) \). Then since \( \text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab}) = 2 \), and, moreover, \( \varphi \) is an isomorphism [cf. assertion (iii)], it follows that \( \text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab}) = 2n \). Conversely, suppose that \( \text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab}) = 2n \). Then since \( \varphi \) is surjective, it follows that
\[
\text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab} \times \cdots \times \Pi_1^{ab}) \leq 2n.
\]

Thus, since \( \text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab}) \geq 2 \), we have:
\[
\text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab} \times \cdots \times \Pi_1^{ab}) = 2n, \quad \text{rk}_{\mathbb{Z}_l}(\Pi_1^{ab}) = 2.
\]
In particular, the first (respectively, second) equality implies that the surjection \( \phi \) is, in fact, an isomorphism, hence that \( g \neq 0 \) [cf. assertion (iii)] (respectively, that either \((g, r) = (0, 3)\) or \((g, r) = (1, 1)\) holds). Therefore, we conclude that \((g, r) = (1, 1)\). This completes the proof of assertion (iv).

\[ \square \]

Proposition 2.3. (Properties of quasi-co-surface subgroups). Let \( n \geq 2 \) be an integer; \( \Sigma \) a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; \( X \) a hyperbolic curve of type \((g, r)\) over an algebraically closed field of characteristic \( \notin \Sigma \); \( X_n \) the \( n \)-th configuration space of \( X \); \( \Pi_n \equiv \pi_1(X_n)^\Sigma \). Then the following hold:

(i) Every quasi-co-surface subgroup of \( \Pi_n \) contains a co-surface subgroup of \( \Pi_n \).

(ii) It holds that \((g, r) \notin \{(0, 3), (1, 1)\}\) if and only if there exists a quasi-co-surface subgroup of \( \Pi_n \).

(iii) Suppose that \((g, r) \notin \{(0, 3), (1, 1)\}\). Then a subgroup of \( \Pi_n \) is a co-surface subgroup of \( \Pi_n \) if and only if it is a minimal quasi-co-surface subgroup of \( \Pi_n \).

Proof. First, we consider assertion (i). Let \( H \) be a quasi-co-surface subgroup of \( \Pi_n \):

\[ \phi : \Pi_n \to G \equiv \Pi_n/H \]

the natural surjection. Then we consider the following claim:

Claim 2.3A: Any two fiber subgroups \( J_1, J_2 \subseteq \Pi_n \) of length one [cf. [MT], Definition 2.3, (iii)] such that \( \phi(J_1) \neq \{1\} \), \( \phi(J_2) \neq \{1\} \) necessarily coincide.

Indeed, suppose that there exist two distinct fiber subgroups \( J_1, J_2 \subseteq \Pi_n \) of length one such that \( \phi(J_1) \neq \{1\} \), \( \phi(J_2) \neq \{1\} \). Then since \( \phi(J_1) \) and \( \phi(J_2) \) are topologically finitely generated closed normal subgroups of the surface group \( G \), it follows from [MT], Theorem 1.5, that \( \phi(J_1) \) and \( \phi(J_2) \) are open in \( G \). Next, let us observe that it follows immediately from [MT], Proposition 2.4, (v), that there exists a closed normal subgroup \( N \subseteq G \) that is topologically normally generated by a single element such that the images of \( \phi(J_1) \) and \( \phi(J_2) \) in \( G/N \) commute. Thus, we conclude that \( G/N \) has an abelian open subgroup, hence that \( G \) is nearly abelian [cf. [MT], Definition 6.1]. On the other hand, since \( G \) is a surface group, it follows from [MT], Proposition 6.2, that \( G \) is a free pro-\( \Sigma \) group of rank two — in contradiction to the definition of a quasi-co-surface subgroup [cf. Definition 2.1, (iii)]. This completes the proof of Claim 2.3A. It follows formally from Claim 2.3A that there exist \( n - 1 \) [distinct] fiber subgroups of \( \Pi_n \) of length one whose images via \( \phi \) are trivial. Thus, the closed normal subgroup of \( \Pi_n \) topologically generated by these \( n - 1 \) fiber subgroups is a co-surface subgroup of \( \Pi_n \) that is contained in \( H \). This completes the proof of assertion (i).

Next, we consider the following claim:
Claim 2.3B: Suppose that \((g, r) \notin \{(0, 3), (1, 1)\}\). Then every co-surface subgroup of \(\Pi_n\) is a quasi-co-surface subgroup of \(\Pi_n\).

Claim 2.3B follows immediately from the various definitions involved. Next, we consider assertion (ii). The necessity portion of assertion (ii) follows immediately from Claim 2.3B. Thus, it suffices to verify the sufficiency portion of assertion (ii). Suppose that there exists a quasi-co-surface subgroup \(H\) of \(\Pi_n\). Then it follows from assertion (i) that there exists a co-surface subgroup \(N \subseteq \Pi_n\) such that \(N \subseteq H\). Thus, we have a surjection

\[(\Pi_n/N)^{ab} \twoheadrightarrow (\Pi_n/H)^{ab}\]

of free \(\hat{\mathbb{Z}}\Sigma\)-modules. Since \(\text{rk}_{\hat{\mathbb{Z}}\Sigma}(\Pi_n/H)^{ab} > 2\), we thus conclude that

\[\text{rk}_{\hat{\mathbb{Z}}\Sigma}(\Pi_n/N)^{ab} > 2,\]

hence that \((g, r) \notin \{(0, 3), (1, 1)\}\). This completes the proof of the sufficiency portion of assertion (ii).

Finally, we consider assertion (iii). Let \(H\) be a co-surface subgroup of \(\Pi_n\). First, we observe that \(H\) is a quasi-co-surface subgroup of \(\Pi_n\) [cf. Claim 2.3B]. Next, let us verify the minimality of \(H\). Let \(N \subseteq \Pi_n\) be a quasi-co-surface subgroup such that \(N \subseteq H\). Then it follows from assertion (i) that there exists a co-surface subgroup \(K\) of \(\Pi_n\) such that \(K \subseteq N\). Since both \(K\) and \(H\) are co-surface subgroups of \(\Pi_n\), we conclude that \(K = H\) [cf. [MT], Proposition 2.4, (ii)], hence that \(N = H\). This completes the proof of the minimality of \(H\). Conversely, let \(H\) be a minimal quasi-co-surface subgroup of \(\Pi_n\). Then, by assertion (i), there exists a co-surface subgroup \(N \subseteq \Pi_n\) such that \(N \subseteq H\). Since \(N\) is quasi-co-surface subgroup of \(\Pi_n\) [cf. Claim 2.3B], we conclude from the minimality of \(H\) that \(N = H\). In particular, \(H\) is a co-surface subgroup of \(\Pi_n\). This completes the proof of assertion (iii). \(\Box\)

**Definition 2.4.** Let \(n \geq 2\) be an integer.

(i) Write

\[\mathcal{L}_{(0,3,n)}\] (respectively, \(\mathcal{L}_{(1,1,n)}\))

for the graded Lie algebra over \(\mathbb{Q}\) defined by the generators

\[X_{ij}\quad(1 \leq i < j \leq n + 3)\]

(respectively, \(X^{(k)}_1, Y^{(k)}_1\) \(1 \leq k \leq n\))

[all of which are of weight 1] and the relations described in [Nk], Proposition 4.1.1, in the case where the “\(n\)” of loc. cit. is taken to be \(n + 3\) (respectively, described in [NTU], 2.8.2, in the case where the “\(g\)” of loc. cit. is taken to be 1, the “\(n\)” of loc. cit. is taken to be 1, and the “\(r\)” of loc. cit. is taken to be \(n\)).
(ii) For integers $1 \leq a < b < c < d \leq n + 3$ (respectively, $1 \leq e < f \leq n + 1$), we denote by
\[ W_{(0,3,n)}^{(a,b,c,d)} \] (respectively, $W_{(1,1,n)}^{(e,f)}$)
the $\mathbb{Q}_l$-vector subspace of $L_{(0,3,n)}$ generated by the set
\[ \{ X_{ij} \mid 1 \leq i < j \leq n + 3 \text{ and } \{i, j\} \not\subset \{a, b, c, d\} \} \]
(respectively, the $\mathbb{Q}_l$-vector subspace of $L_{(1,1,n)}$ generated by the set
\[ \{ X_1^{(k)}, Y_1^{(k)} \mid 1 \leq k \leq n + 1 \text{ and } k \not\in \{e, f\} \} \]
where $X_1^{(n+1)} \overset{\text{def}}{=} -\sum_{k=1}^{n} X_1^{(k)}$ and $Y_1^{(n+1)} \overset{\text{def}}{=} -\sum_{k=1}^{n} Y_1^{(k)}$.

Write
\[ S_{(0,3,n)} \overset{\text{def}}{=} \{ W_{(0,3,n)}^{(a,b,c,d)} \mid 1 \leq a < b < c < d \leq n + 3 \} \]
(respectively, $S_{(1,1,n)} \overset{\text{def}}{=} \{ W_{(1,1,n)}^{(e,f)} \mid 1 \leq e < f \leq n + 1 \}$).

**Theorem 2.5.** (Group-theoreticity of the dimension, genus, number of cusps, and generalized fiber subgroups). Let $l$ be a prime number; $\Sigma$ a set of prime numbers which is either equal to $\{l\}$ or $\text{Primes}$; $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$; $X$ a hyperbolic curve of type $(g, r)$ over an algebraically closed field $k$ of characteristic $\notin \Sigma$; $n$ a positive integer; $X_n$ the $n$-th configuration space of $X$;
\[ \Pi_n \overset{\text{def}}{=} \pi_1(X_n)^\Sigma. \]

Then:

(i) One may reconstruct $n$ group-theoretically from $\Pi_n$ as the maximum of the set
\[ \{ s \in \mathbb{N} \mid \exists a \text{ closed subgroup of } \Pi_n \text{ which is isomorphic to } \mathbb{Z}_l^{\oplus s} \} \]
[cf. Theorem 1.6].

Moreover, if $n \geq 2$ [cf. (i)], then the following hold:

(ii) One may determine whether or not it holds that
\[ (g, r) \notin \{(0, 3), (1, 1)\} \]
group-theoretically from $\Pi_n$ by considering whether or not there exists a quasi-co-surface subgroup of $\Pi_n$ [cf. Proposition 2.3, (ii)].

Moreover, one may determine whether or not it holds that
\[ (g, r) = (1, 1) \]
group-theoretically from $\Pi_n$ by considering whether or not it holds that
\[ \text{rk}_{\mathbb{Z}_l}(\Pi_n^{(l)})^{\text{ab}} = 2n \]
[cf. (i); Proposition 2.2, (iv)].
(iii) Suppose that \((g, r) \notin \{(0, 3), (1, 1)\}\) [cf. (ii)]. Then the generalized fiber subgroups of \(\Pi_n\) of co-length one may be characterized group-theoretically as the minimal quasi-co-surface subgroups of \(\Pi_n\) [cf. Proposition 2.3, (iii)].

(iv) Suppose that \((g, r) \in \{(0, 3), (1, 1)\}\) [cf. (ii)]. Write

\[ T_{(g, r, n)} \]

for the set of kernels of the natural surjections

\[ (\Pi_n^{(l)})^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \twoheadrightarrow (\Pi_1^{(l)})^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \]

arising from generalized projection morphisms \(X_n \to X\) of co-length one. Then one may reconstruct \(T_{(g, r, n)}\) as the set of images of the \(\mathbb{Q}_l\)-vector subspaces \(W \subseteq L_{(g, r, n)}\) [cf. Definition 2.4, (i)], for \(W \in S_{(g, r, n)}\) [cf. Definition 2.4, (ii)], via the various surjections

\[ L_{(g, r, n)} \twoheadrightarrow \text{Gr}(\Pi_n^{(l)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \twoheadrightarrow (\Pi_n^{(l)})^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \]

where \(\text{Gr}(\Pi_n^{(l)})\) is the graded Lie algebra over \(\mathbb{Z}_l\) associated to the lower central series of \(\Pi_n^{(l)}\); the \(\twoheadrightarrow\) is an isomorphism of graded Lie algebras over \(\mathbb{Q}_l\) [where we recall that such isomorphisms exist — cf. [Nk], Proposition 4.1.1; [NTU], 2.8.2]; the \(\twoheadrightarrow\) is the natural surjection. In particular, one may reconstruct the generalized fiber subgroups of \(\Pi_n\) of co-length one group-theoretically as the topologically finitely generated closed normal subgroups \(H\) of \(\Pi_n\) satisfying the following conditions:

(iv-a) \(\Pi_n / H\) is elastic [cf. [AbsTopI], Definition 1.1, (ii)].

(iv-b) The \(\mathbb{Q}_l\)-vector subspace of \((\Pi_n^{(l)})^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\) generated by the image of \(H\) via the natural morphism \(\Pi_n \to (\Pi_n^{(l)})^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\) coincides with some element of \(T_{(g, r, n)}\).

(v) Let \(m < n\) be a positive integer, \(F\) a generalized fiber subgroup of \(\Pi_n\) of co-length \(m\). Then \(F\) may be naturally identified with

\[ \pi_1(Z_{n-m})^\Sigma \]

where \(Z_{n-m}\) is the \((n-m)\)-th configuration space of a hyperbolic curve \(Z\) of type \((g, r + m)\) over an algebraically closed field of characteristic \(\notin \Sigma\). Moreover, relative to this identification, the generalized fiber subgroups of \(\Pi_n\) of co-length \(m + 1\) which are contained in \(F\) may be identified with the generalized fiber subgroups of \(F \cong \pi_1(Z_{n-m})^\Sigma\) of co-length one [cf. Remark 2.1.1; [MT], Remark 2.1.2; [MT], Proposition 2.2, (i); [MT], Proposition 2.4, (i), (ii)]. In particular, by applying (iii), (iv) inductively, one may reconstruct the generalized fiber subgroups of \(\Pi_n\) [of arbitrary length] group-theoretically from \(\Pi_n\).

(vi) One may reconstruct \((g, r)\) group-theoretically from \(\Pi_n\) as follows:
First, let us observe that, by applying (v), one may reconstruct the quotients
\[
(\Pi_n \to) \quad \Pi_n^{(l)} \to \Pi_2^{(l)} \to \Pi_1^{(l)}
\]
determined, respectively, by some generalized fiber subgroup of co-length 2 and some generalized fiber subgroup of co-length 1 that contains the generalized fiber subgroup of co-length 2. Write \(\Pi_{2/1}^{(l)}\) for the kernel of the surjection \(\Pi_2^{(l)} \to \Pi_1^{(l)}\).

In particular, one may determine whether or not \(r > 0\) by considering whether or not \(\Pi_1^{(l)}\) [cf. (vi-a)] is free.

When \(r = 0\) [cf. (vi-b)], one may reconstruct \((g, r)\) as
\[
g = \frac{1}{2} \cdot \text{rk}_{\mathbb{Z}_l}(\Pi_1^{(l)})_{\text{ab}}, \quad r = 0.
\]

When \(r > 0\) [cf. (vi-b)], note that, by considering the kernel of the natural action \(\Pi_1^{(l)} \to \text{Aut}(\Pi_2^{(l)}/\Pi_1^{(l)})_{\text{ab}}\) induced by the surjection \(\Pi_2^{(l)} \to \Pi_1^{(l)}\) appearing in (vi-a), one may reconstruct
\[
I \overset{\text{def}}{=} \text{Ker}(\Pi_1^{(l)}_{\text{ab}} \to (\Pi_1^{(l)})_{\text{ab}})
\]
— where we write \(\Pi_1\) for the \(\acute{e}tale\) fundamental group of the smooth compactification \(\overline{X}\) of \(X\); the surjection \((\Pi_1^{(l)\text{ab}}) \to (\Pi_1^{(l)})_{\text{ab}}\) is the morphism induced by the natural immersion \(X \hookrightarrow \overline{X}\) [cf. [CbTpl], Lemma 1.3, (iv)]. Thus, one may reconstruct \((g, r)\) as
\[
g = \frac{1}{2} \cdot \{\text{rk}_{\mathbb{Z}_l}(\Pi_1^{(l)})_{\text{ab}} - \text{rk}_{\mathbb{Z}_l}I\}, \quad r = \text{rk}_{\mathbb{Z}_l}I + 1.
\]

Proof. Assertions (i), (ii), (iii), (v), and (vi) follow immediately from the various results cited in the statements of these assertions, together with the various definitions involved. Thus, it remains to verify assertion (iv). First, we consider the reconstruction of generalized fiber subgroups of \(\Pi_n\) of co-length one. It follows from Remark 2.1.1; [MT], Theorem 1.5; [MT], Proposition 2.2, (ii); [MT], Proposition 2.4, (i), that every generalized fiber subgroup of \(\Pi_n\) of co-length one is a topologically finitely generated closed normal subgroup of \(\Pi_n\) satisfying conditions (iv-a), (iv-b). Conversely, let \(H\) be a topologically finitely generated closed normal subgroup of \(\Pi_n\) satisfying conditions (iv-a), (iv-b). Then it follows from condition (iv-b) that there exists a generalized projection morphism \(p_X : X_n \to X\) such that the image of \(H\) via the composite
\[
\Pi_n \overset{p_{\Pi_n}}{\to} \Pi_1 \overset{p_{\Pi_1}}{\to} (\Pi_1^{(l)})_{\text{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l
\]
— where the first arrow is the natural [outer] surjection induced by \(p_X\); the second arrow is the natural morphism — is trivial. Now we claim that \(p_{\Pi_n}(H) = \{1\}\). Indeed, suppose that \(p_{\Pi_n}(H) \neq \{1\}\). Then since \(p_{\Pi_n}(H)\) is a
nontrivial topologically finitely generated closed normal subgroup of Π₁, it follows from [MT], Theorem 1.5, that \( p_{Π}(H) \) is open in Π₁. But this implies that \( p^{ab} \) factors through the finite quotient \( Π₁ \to Π₁/p_{Π}(H) \). Thus, we conclude that \( \text{Im}(p^{ab}) \) is finite — a contradiction. This completes the proof of the claim. In light of the claim, it holds that \( H \subseteq \text{Ker}(p_{Π}) \). Now suppose that \( \text{Ker}(p_{Π})/H \not= \{1\} \). Then since \( \text{Ker}(p_{Π})/H \) is a nontrivial topologically finitely generated closed normal subgroup of \( Π_n/H \), it follows from condition (iv-a) that \( Π_n/\text{Ker}(p_{Π}) \cong Π_1 \) is finite — a contradiction. Thus, we conclude that \( H = \text{Ker}(p_{Π}) \), hence that \( H \) is a generalized fiber subgroup of \( Π_n \) of co-length one, as desired.

Thus, to complete the verification of assertion (iv), it remains to consider the reconstruction of the set \( T_{(g,r,n)} \). We begin by recalling the explicit constructions in the case where

- \((g,r) = (0,3)\): cf. [Nk], (3.1.3); [Nk], Proposition 4.1.1;
- \((g,r) = (1,1)\): cf. [NTU], 2.8.2; [NT], 2.4.1, 2.4.2, 3.2; [NT], Proof of Theorem (3.1) by using (3.3-6), (4).

In particular, let us observe that it follows immediately in the case where

- \((g,r) = (0,3)\), from the explicit description of the “\( A_{ij} \)” in [Nk], (3.1.3) [which correspond to the “\( X_{ij} \)” in [Nk], Proposition 4.1.1],
- \((g,r) = (1,1)\), from the explicit description of the “\( X^{(k)}_i \), "\( X^{(k)}_{g+i} \)” at the beginning of [NT], 3.2 [which correspond to the “\( X^{(k)}_i \), “\( Y^{(k)}_i \)” of [NTU], 2.8.2 in terms of the “\( φ_k \)” of [NT], 2.4.1, 2.4.2, together with the discussion [NT], Proof of Theorem (3.1) by using (3.3-6), (4),

that — relative to the isomorphism of graded Lie algebras over \( Q_l \)

\[
α_{(g,r,n)} : L_{(g,r,n)} \cong \text{Gr}(Π_n^{(l)}) \otimes Z_l Q_l
\]

implicit in these explicit descriptions — each subspace \( W \in S_{(g,r,n)} \) maps into some subspace \( T \in T_{(g,r,n)} \). On the other hand, an elementary computation of the dimensions of \( W \) and \( T \) shows that this implies that \( W \) maps isomorphically onto \( T \). Moreover, one computes immediately that \( S_{(g,r,n)} \) and \( T_{(g,r,n)} \) are of the same cardinality. We thus conclude that \( α_{(g,r,n)} \) induces a bijection \( S_{(g,r,n)} \cong T_{(g,r,n)} \).

Thus, to complete the proof of assertion (iv), it remains to verify the following claim:

**Claim 2.5A:** Let

\[
β : L_{(g,r,n)} \cong \text{Gr}(Π_n^{(l)}) \otimes Z_l Q_l
\]

be an isomorphism of graded Lie algebras over \( Q_l \). Then the set of images of the \( Q_l \)-vector subspaces \( W \subseteq L_{(g,r,n)} \), for \( W \in S_{(g,r,n)} \), via the composite

\[
\overline{β} : L_{(g,r,n)} \cong \text{Gr}(Π_n^{(l)}) \otimes Z_l Q_l \to (Π_n^{(l)})^{ab} \otimes Z_l Q_l
\]

— where the first arrow is \( β \); the second arrow is the natural surjection — coincides with \( T_{(g,r,n)} \).
Let us verify Claim 2.5A. In the following, we write $\mathfrak{S}_\Box$ for the symmetric group on $\Box$ letters.

First, suppose that $(g, r) = (0, 3)$. In this case, recall that we have an isomorphism \[
\text{Aut}(\text{Gr}(\Pi_n^l) \otimes \mathbb{Q}_l) \cong \mathbb{Q}_l^* \times \mathfrak{S}_{n+3}
\] [cf. [NT], Theorem D]. Moreover, the action of $\mathbb{Q}_l^* \times \mathfrak{S}_{n+3}$ on $\text{Gr}(\Pi_n^l) \otimes \mathbb{Z}_l \mathbb{Q}_l$ is described explicitly in the discussion preceding the statement of [Nk], Lemma 4.1.2 [cf. also [Nk], Proof of Lemma 4.1.2]. It follows immediately from this explicit description [cf. also the above discussion of $\alpha(g, r, n)$] that $\beta \circ \alpha_{(g, r, n)}^{-1} \in \text{Aut}(\text{Gr}(\Pi_n^l) \otimes \mathbb{Z}_l \mathbb{Q}_l)$ induces a permutation of the subspaces $W \subseteq L(g, r, n)$, for $W \in S(g, r, n)$, via $\beta$ coincides with $T(g, r, n)$, as desired.

Next, suppose that $(g, r) = (1, 1)$. In this case, recall that we have an isomorphism \[
\text{Aut}(\text{Gr}(\Pi_n^l) \otimes \mathbb{Z}_l \mathbb{Q}_l) \cong \text{GSp}(2, \mathbb{Q}_l) \times \mathfrak{S}_{n+1}
\] — where we write $\text{GSp}(2, \mathbb{Q}_l)$ for the group of symplectic similitudes of $\mathbb{Q}_l \times \mathbb{Q}_l$, equipped with the standard symplectic form [cf. [NT], Theorem D]. Moreover, it follows immediately from the explicit description of the permutation $\tau \in \mathfrak{S}_{r+1}$ in the discussion entitled “(3), (4):” in [NT], Proof of Theorem (3.1) by using (3.3-6), that $\beta \circ \alpha_{(g, r, n)}^{-1} \in \text{Aut}(\text{Gr}(\Pi_n^l) \otimes \mathbb{Z}_l \mathbb{Q}_l)$ induces a permutation of the subspaces $W \subseteq L(g, r, n)$, for $W \in S(g, r, n)$, via $\beta$ coincides with $T(g, r, n)$, as desired.

\[\Box\]

Corollary 2.6. (Structure of the group of outer automorphisms of a configuration space group 1). In the notation of Remark 2.1.1, Theorem 2.5, suppose that $n \geq 2$ and $(r, n) \neq (0, 2)$. Then the following hold:

(i) Every element of $\text{Out}(\Pi_n)$ induces a permutation on the set — whose cardinality is $n^*$ — of generalized fiber subgroups of $\Pi_n$ of length one. Here, we recall from Theorem 2.5, (v), that this set of generalized fiber subgroups may be reconstructed group-theoretically from $\Pi_n$. Thus, by considering such permutations, we obtain the following exact sequence of profinite groups:

\[1 \longrightarrow \text{Out}^F(\Pi_n) \longrightarrow \text{Out}(\Pi_n) \longrightarrow \mathfrak{S}_{n^*} \longrightarrow 1.\]

Moreover, the injection

\[\iota_n : \mathfrak{S}_{n^*} \hookrightarrow \text{Out}(\Pi_n)\]

— induced by the natural action of $\mathfrak{S}_{n^*}$ on $X_n$ [cf. Remark 2.1.1] — determines a section of the above surjection $\text{Out}(\Pi_n) \rightarrow \mathfrak{S}_{n^*}$ and hence a natural isomorphism

\[\text{Out}^F(\Pi_n) \times \mathfrak{S}_{n^*} \sim \text{Out}(\Pi_n).\]
In the following, we shall use ι_n to regard $\mathcal{G}_{n^*}$ as a subgroup of $\text{Out}(\Pi_n)$. In particular, we have inclusions
\[ \mathcal{G}_{n^*} \subseteq Z_{\text{Out}(\Pi_n)}(\text{Out}^F(\Pi_n)) \subseteq Z^{\text{loc}}(\text{Out}(\Pi_n)). \]

(ii) Suppose that $(g, r) \notin \{(0, 4), (1, 1), (1, 2), (2, 0)\}$. Then
\[ \mathcal{G}_{n^*} = Z_{\text{Out}(\Pi_n)}(\text{Out}^F(\Pi_n)) = Z^{\text{loc}}(\text{Out}(\Pi_n)) \]
— i.e., one may reconstruct the subgroup $\mathcal{G}_{n^*} \subseteq \text{Out}(\Pi_n)$ [cf. (i)] group-theoretically from $\Pi_n$ as the local center $Z^{\text{loc}}(\text{Out}(\Pi_n))$ of $\text{Out}(\Pi_n)$.

(iii) If $n^* \geq 3$, then we have an equality
\[ \text{Out}^F(\Pi_n) = Z_{\text{Out}(\Pi_n)}(\mathcal{G}_{n^*}). \]

Proof. First, we consider assertion (i). The fact that $\iota_n$ determines a section of the surjection $\text{Out}(\Pi_n) \to \mathcal{G}_{n^*}$ follows immediately from the definitions. Next, let us recall [cf. [CbTpII], Theorem 2.3, (iv)] that [since it follows immediately from the definitions that $\text{Out}^F(\Pi_n) \subseteq \text{Out}^F(\Pi_n)$]
\[ \iota_n(\mathcal{G}_n) \subseteq Z_{\text{Out}(\Pi_n)}(\text{Out}^F(\Pi_n)) \subseteq Z_{\text{Out}(\Pi_n)}(\text{Out}^F(\Pi_n)). \]

Thus, since $n \geq 2$, by conjugating the composite inclusion of the preceding display by arbitrary elements of $\mathcal{G}_{n^*}$, we conclude that
\[ \iota_n(\mathcal{G}_{n^*}) \subseteq Z_{\text{Out}(\Pi_n)}(\text{Out}^F(\Pi_n)). \]

Now the fact that $\iota_n$ determines a natural isomorphism $\text{Out}^F(\Pi_n) \times \mathcal{G}_{n^*} \to \text{Out}(\Pi_n)$ follows formally.

Next, we consider assertion (ii). In light of the inclusions of the final display of the statement of assertion (i), to verify assertion (ii), it suffices to verify the following claim:

Claim 2.6A: Suppose that $(g, r) \notin \{(0, 4), (1, 1), (1, 2), (2, 0)\}$.
Then we have
\[ Z^{\text{loc}}(\text{Out}(\Pi_n)) \subseteq \mathcal{G}_{n^*}. \]

First, we assume that $(g, r) = (0, 3)$. Let $L$ be a number field;
\[ \rho^\text{tpd}_L : G_L \to \text{Out}(\Pi_n) \]
the natural pro-Σ outer Galois representation associated to the $n$-th configuration space $T_n$ of $T \stackrel{\text{def}}{=} \mathbb{F}^1_L \setminus \{0, 1, \infty\}; \sigma \in Z^{\text{loc}}(\text{Out}(\Pi_n))$. Thus, there exists an open subgroup $H$ of $\text{Out}(\Pi_n)$ such that $\sigma \in Z_{\text{Out}(\Pi_n)}(H)$. In particular, by replacing $L$ by a suitable finite field extension of $L$, we may assume that $\text{Im}(\rho^\text{tpd}_L) \subseteq H$. Then it follows from [Nk], Theorem A; [IN], Theorem C [or, alternatively, from assertion (i); [CbTpII], Theorem A, (i); [LocAn], Theorem A], that
\[ \sigma \in Z_{\text{Out}(\Pi_n)}(\text{Im}(\rho^\text{tpd}_L)) = \mathcal{G}_{n^*} \times Z_{\text{Out}^F(\Pi_n)}(\text{Im}(\rho^\text{tpd}_L)) = \mathcal{G}_{n^*}. \]

This completes the proof of Claim 2.6A in the case where $(g, r) = (0, 3)$. 

Thus, in the remainder of the proof of Claim 2.6A, we may assume that
\((g, r) \notin \{(0, 3), (0, 4), (1, 1), (1, 2), (2, 0)\} \).

Thus, \(n = n^*\), \(\text{Out}^F(\Pi_n) = \text{Out}^F(\Pi_n)\). Let \(\sigma \in Z^{\text{loc}}(\text{Out}(\Pi_n))\). Thus, there exists an open subgroup \(H\) of \(\text{Out}^F(\Pi_n)\) such that \(\sigma \in Z_{\text{Out}(\Pi_n)}(H)\).

Moreover, by assertion (i), we may assume without loss of generality that \(\sigma \in \text{Out}^F(\Pi_n)\). Now we claim that

**Claim 2.6B:** It holds that
\(\sigma \in \text{Out}^{FC}(\Pi_n)\)
— where we write \(\text{Out}^{FC}(\Pi_n)\) for the group of FC-admissible outer automorphisms of \(\Pi_n\) [cf. [CmbCusp], Definition 1.1, (ii)].

Indeed, since, by assertion (i), \(H\) is open in \(\text{Out}(\Pi_n)\), our assumption that \(\sigma \in Z_{\text{Out}(\Pi_n)}(H)\) implies that Claim 2.6B follows from [CmbGC], Proposition 2.4, (v); [CmbGC], Corollary 2.7, (i).

Next, let us write
\[\rho_{g,r,n} : \Pi_{(M_{g,r})_k} \to \text{Out}^{FC}(\Pi_n)\]
for the outer representation induced by the \([1]\)-morphism
\[(\mathcal{M}_{g,r+n})_k \to (\mathcal{M}_{g,r})_k\]
obtained by forgetting the last \(n\) marked points. Then since, by assertion (i), \(H\) is open in \(\text{Out}(\Pi_n)\), there exists an open subgroup \(N\) of \(\Pi_{(M_{g,r})_k}\) such that \(\sigma \in Z_{\text{Out}^{FC}(\Pi_n)}(\rho_{g,r,n}(N))\). Write
\[\phi_n : \text{Out}^{FC}(\Pi_n) \to \text{Out}^{FC}(\Pi_1)\]
for the homomorphism induced by the first projection \(X_n \to X\). Then it follows from [CbTpI], Theorem D, (i), together with our assumption on \((g, r)\), that
\[\phi_n(\sigma) \in Z_{\text{Out}^{FC}(\Pi_1)}(\phi_n(\rho_{g,r,n}(N))) = \{1\} \]
Thus, since \(\phi_n\) is injective [cf. [NodNon], Theorem B], we conclude that \(\sigma = 1\). This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (i) and the well-known fact that, since \(n^* \geq 3\), \(\mathfrak{S}_{n^*}\) is center-free.

**Definition 2.7.** In the notation of Corollary 2.6, suppose that \((g, r) = (0, 3)\). Write
\[\text{Out}^{FCS}(\Pi_n) \subseteq \text{Out}(\Pi_n)\]
for the group of FC-admissible outer automorphisms of \(\Pi_n\) [cf. [CmbCusp], Definition 1.1, (ii)] that commute with the image of \(\iota_n : \mathfrak{S}_{n+3} \to \text{Out}(\Pi_n)\) [cf. [CmbCusp], Definition 1.1, (vi)]. Then [by a slight abuse of the notational conventions established in the discussion entitled “Topological Groups” in §0 concerning the superscript “\(\Sigma\)”] we shall write
\[\text{GT}^\Sigma \overset{\text{def}}{=} \text{Out}^{FCS}(\Pi_2)\]
and refer to $\text{GT}^\Sigma$ as the pro-$\Sigma$ Grothendieck-Teichmüller group [even though it is, in fact, not necessarily a pro-$\Sigma$ group!]. Note that if $\Sigma = \text{Primes}$, then $\text{GT}^\Sigma$ may be identified with the Grothendieck-Teichmüller group as defined in more classical works [cf. [CmbCusp], Definition 1.11, (i); [CmbCusp], Remark 1.11.1; [CmbCusp], Corollary 1.12, (ii); [CmbCusp], Corollary 4.2, (i); [CmbCusp], Remark 4.2.1].

**Corollary 2.8.** (Simplified group-theoretic approach to the pro-$\Sigma$ Grothendieck-Teichmüller group). In the notation of Corollary 2.6, suppose that $(g, r) = (0, 3)$. Then the following hold:

(i) The natural homomorphism $\text{Out}^F(\Pi_n) \to \text{Out}^F(\Pi_2)$ — where we write $\text{Out}^F((-))$ for the group of $F$-admissible outer automorphisms of $(-)$ [cf. [CmbCusp], Definition 1.1, (ii)] — induced by the projection morphism $X_n \to X_2$ to the first two factors of $X_n$ induces a natural isomorphism

$$\text{Out}^gF(\Pi_n) \cong \text{GT}^\Sigma.$$

In particular, we have a natural isomorphism

$$\text{GT}^\Sigma \times S_{n+3} \cong \text{Out}(\Pi_n)$$

— where we recall that

$$S_{n+3} = Z_{\text{Out}(\Pi_n)}(\text{Out}^gF(\Pi_n)) = Z^{\text{loc}}(\text{Out}(\Pi_n))$$

[cf. Remark 2.1.1; Corollary 2.6, (i), (ii)].

(ii) We have a natural isomorphism

$$Z_{\text{Out}(\Pi_n)}(Z^{\text{loc}}(\text{Out}(\Pi_n))) \cong \text{GT}^\Sigma.$$

**Proof.** First, we consider assertion (i). Let us observe that we have

$$\text{Out}^{\text{FCS}}(\Pi_n) = \text{Out}^{\text{FC}}(\Pi_n) \cap Z_{\text{Out}(\Pi_n)}(S_{n+3})$$

$$= \text{Out}^F(\Pi_n) \cap \text{Out}^gF(\Pi_n)$$

$$= \text{Out}^gF(\Pi_n)$$

— where we write $\text{Out}^{\text{FC}}(\Pi_n)$ for the group of FC-admissible outer automorphisms of $\Pi_n$ [cf. [CmbCusp], Definition 1.1, (ii)]; the second equality follows by applying Corollary 2.6, (iii), and [CbTpII], Theorem A, (ii). Thus, assertion (i) follows immediately from [CmbCusp], Corollary 4.2, (ii). Assertion (ii) follows from assertion (i) and Corollary 2.6, (ii), (iii). \qed

**Corollary 2.9.** (Structure of the group of outer automorphisms of a configuration space group $\Pi$). Write $(C_{g,[{r}]})_{\eta} \to \eta$ for the result of base-changing the tautological curve $C_{g,[{r}]} \to M_{g,[{r}]}$ [cf. the discussion entitled “Curves” in §0] via some dominant morphism $\eta \to M_{g,[{r}]}$, where $\eta$ is the spectrum of an algebraically closed field $\Omega$. Then, in the notation of Corollary 2.6, we have inclusions

$$S_{n^*} \subseteq Z^{\text{loc}}(\text{Out}(\Pi_n)) \subseteq S_{n^*} \times G_{g,r}$$
— where we write $G_{g,r} \subseteq \text{Out}(\Pi_n)$ for the subgroup determined by the natural action of the group $\text{Aut}_\overline{\eta}(\mathcal{C}_{g,[r]} \overline{\eta})$ of automorphisms of $(\mathcal{C}_{g,[r]} \overline{\eta})$ over $\overline{\eta}$. Here, we recall that

$$G_{g,r} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) = (0,4); \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) \in \{(1,1), (1,2), (2,0)\}. \end{cases}$$

Proof. The first inclusion follows from Corollary 2.6, (i). Thus, it remains to verify the second inclusion. Let $\sigma \in Z^{\text{loc}}(\text{Out}(\Pi_n))$. Note that, by replacing $\sigma$ by the product of $\sigma$ with an element of $S_n^*$, we may assume, without loss of generality, that $\sigma \in \text{Out}_{\text{FC}}^F(\Pi_n)$ [cf. Corollary 2.6, (i)]. Thus, it follows from [CmbGC], Proposition 2.4, (v); [CmbGC], Corollary 2.7, (i), that $\sigma \in \text{Out}_{\text{FC}}(\Pi_n)$ — where we write $\text{Out}_{\text{FC}}(\Pi_n)$ for the group of $\text{FC-admissible}$ outer automorphisms of $\Pi_n$ [cf. [CmbCusp], Definition 1.1, (ii)]. Next, let us recall from [NodNon], Theorem B, that the natural morphism $\text{Out}_{\text{FC}}(\Pi_n) \to \text{Out}_{\text{FC}}(\Pi_1)$ is injective. Thus, it follows from [LocAn], Theorem A, applied to a suitable sub-$p$-adic subfield of $\Omega$, that $\sigma \in G_{g,r}$. This completes the proof of Corollary 2.9. □

Corollary 2.10. (Structure of the group of outer automorphisms of a configuration space group III). In the notation of Corollary 2.6, the following hold:

(i) Suppose that $(g,r) \in \{(0,4), (1,2)\}$ and $\Sigma = \{l\}$. Then one may reconstruct the subgroup $\mathcal{E}_2 \subseteq \text{Out}(\Pi_2)$ [cf. Corollary 2.6, (i)] group-theoretically from $\Pi_2$ as follows:

(i-a) First, we note that, by applying Theorem 2.5, (v), one may reconstruct the two quotients $p_1, p_2 : \Pi_2 \to \Pi_1$ determined by the two fiber subgroups of co-length one. Here, we note that in fact, Theorem 2.5, (v), only yields a reconstruction of the kernels of these quotients, i.e., not an “identification isomorphism” [cf. the use of the same notation for the codomains of $p_1, p_2$] between the codomains of $p_1, p_2$; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow.

(i-b) In light of the reconstruction of the quotients $p_1, p_2 : \Pi_2 \to \Pi_1$ in (i-a), one may apply [the group-theoretic reconstruction algorithms implicit in the proof of] [CbTpl], Lemma 1.6, to reconstruct the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_1$. 

———

NUMERICAL INVARIANTS AND DISTINGUISHED SUBGROUPS
(i-c) Write $Z$ for the smooth compactification of $X$; $Z^{\log}$ for the log scheme whose underlying scheme is $Z$, and whose log structure is determined by the cusps of $X$; $Z^{\log}_2$ for the second log configuration space of $Z^{\log}$; $Z^{\log}_2 \to Z^{\log}$ for the natural projection corresponding to $p_1$. Recall that, for every cusp $c \in Z(k)$, the log structure on $Z^{\log}_2$ determines on the fiber $(Z_2)_c$ of the morphism $Z^{\log}_2 \to Z^{\log}$ over $c$ a structure of pointed stable curve, which consists of

- precisely one irreducible component of genus 0 with precisely 2 cusps [called the “minor cuspidal component”] $E_c$
- precisely one irreducible component of genus $g$ with precisely $r-1$ cusps [called the “major cuspidal component”] $F_c$ [cf. [CmbCusp], Definition 1.4, (i)].

Write $\Pi^{\log}_2/1 \overset{\text{def}}{=} \text{Ker}(p_1)$. Thus, $\Pi^{\log}_2/1$ may be identified with the PSC-fundamental group of the semi-graph of anabelioids of pro-$\Sigma$ PSC-type determined by the pointed stable curve $(Z_2)_c$ [cf. [CmbGC], Definition 1.1, (i), (ii)].

(i-d) In the notation of (i-c), the vertical subgroups $\Pi_E \subseteq \Pi^{\log}_2/1$ (respectively, $\Pi_F \subseteq \Pi^{\log}_2/1$)

associated to $E_c$ (respectively, $F_c$) [cf. [CmbCusp], Definition 1.4, (ii)] may be characterized as follows: Let $I_c \subseteq \Pi_1$ be a cuspidal inertia subgroup associated to $c$ [cf. (i-b)]. Write $\rho_c : I_c \to \text{Out}(\Pi^{\log}_2/1)$ for the composite of the inclusion $I_c \to \Pi_1$ with the outer representation $\Pi_1 \to \text{Out}(\Pi^{\log}_2/1)$ induced by $p_1$; $\Pi_c^{\log} = \Pi^{\log}_2/1 \rtimes I_c$ [cf. the discussion entitled “Topological groups” in [CbTpI], §0].

Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi^{\log}_2/1 \longrightarrow \Pi^{\log}_E \longrightarrow I_c \longrightarrow 1.$$  

We shall say that a closed subgroup $H \subseteq \Pi^{\log}_2/1$ is a section-centralizer if it may be written in the form $Z_{\Pi^{\log}_E}(s(I))$ for some section $s : I_c \to \Pi^{\log}_E$ of the natural surjection $\Pi^{\log}_E \to I_c$. Then the vertical subgroups $\Pi E \subseteq \Pi^{\log}_2/1$ (respectively, $\Pi F \subseteq \Pi^{\log}_2/1$) may be characterized as the maximal section-centralizers whose rank — as a free pro-$\Sigma$ group — is two (respectively, three) [cf. [CbTpII], Corollary 1.7, (ii)].

(i-e) One may reconstruct the image $J_c$ of $I_c \subseteq \Pi_1$ in $\Pi^{\log}_1$ as the image of $\Pi^{\log}_E \subseteq \Pi^{\log}_2/1$ [cf. (i-c), (i-d)] via the surjection

$$\Pi^{\log}_2/1 \overset{p_2^{\Pi^{\log}_2/1}}{\longrightarrow} \Pi^{\log}_1 \to \Pi^{\log}_1$$

— where the second arrow is the natural surjection. In particular, one may reconstruct the $\hat{Z}^{\Sigma}$-submodule

$$J_c \times J_c \subseteq \Pi^{\log}_1 \times \Pi^{\log}_1.$$
(i-f) One may reconstruct the unique nontrivial element of the subgroup $S_2 \subseteq \text{Out}(\Pi_2)$ as the unique nontrivial element $\alpha \in Z^{\text{loc}}(\text{Out}(\Pi_2))$ — whose image in $\text{Aut}(\Pi_1^{ab} \times \Pi_1^{ab})$ [cf. Theorem 2.5, (v)] we denote by $\overline{\alpha}$ — such that, for each cusp $c \in Z(k)$, i.e., for each cuspidal inertia subgroup $I_c \subseteq \Pi_1$ [cf. (i-b)],

$$\overline{\alpha}(J_c \times J_c) = J_c \times J_c.$$ 

(ii) Suppose that $(g, r) = (1, 1)$ and $\Sigma = \{l\}$. Then one may reconstruct the subgroup $S_2 \subseteq S_3 \subseteq \text{Out}(\Pi_2)$ [cf. Corollary 2.6, (i)] — i.e., which is well-defined up to conjugation in $S_3$ — group-theoretically from $\Pi_2$ as follows:

(ii-a) First, we note that, by applying Theorem 2.5, (v), one may reconstruct the three quotients

$$p_1, p_2, p_3 : \Pi_2 \twoheadrightarrow \Pi_1$$

determined by the three generalized fiber subgroups of co-length one. Here, we note that in fact, Theorem 2.5, (v), only yields a reconstruction of the kernels of these quotients, i.e., not an “identification isomorphism” [cf. the use of the same notation for the codomains of $p_1, p_2, p_3$, respectively]; on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow. Write

$$\overline{p}_1, \overline{p}_2, \overline{p}_3 : \Pi_2^{ab} \twoheadrightarrow \Pi_1^{ab}$$

for the natural surjections induced by $p_1, p_2, p_3$, respectively.

(ii-b) Observe that $\overline{p}_3$ factors as the composite of the isomorphism

$$(\overline{p}_1, \overline{p}_2) : \Pi_2^{ab} \cong \Pi_1^{ab} \times \Pi_1^{ab}$$

[cf. Proposition 2.2, (ii)] with some surjection

$$p_- : \Pi_1^{ab} \times \Pi_1^{ab} \twoheadrightarrow \Pi_1^{ab}.$$ 

whose kernel coincides with the diagonal

$$\delta \subseteq \Pi_1^{ab} \times \Pi_1^{ab}.$$ 

In particular, one may reconstruct $\delta$ as $\text{Ker}(p_-)$ and hence the natural identification between the copy of $\Pi_1^{ab}$ that appears as the codomain of $\overline{p}_1$ and the copy of $\Pi_1^{ab}$ that appears as the codomain of $\overline{p}_2$ as the composite isomorphism

$$\Pi_1^{ab} \cong \delta \cong \Pi_1^{ab}$$

— where the first (respectively, second) arrow denotes the arrow induced by $\overline{p}_1$ (respectively, $\overline{p}_2$).
(ii-c) One may reconstruct the unique nontrivial element of the sub-

\[ S_2 \subseteq \text{Out}(\Pi_2) \]

which is well-defined up to conjugation in \( S_3 \), i.e., which corresponds to permutation of the projections \( p_1, p_2, p_3 \) as the unique element \( \alpha \in Z^\text{loc}(\text{Out}(\Pi_2)) \) — whose image in \( \text{Aut}(\Pi_1^\text{ab} \times \Pi_1^\text{ab}) \) we denote by \( \overline{\alpha} \) — such that, for any \( (x, y) \in \Pi_1^\text{ab} \times \Pi_1^\text{ab} \),

\[ \overline{\alpha}(x, y) = (y, x). \]

(iii) Suppose that \((g, r) = (2, 0)\) [which implies that \( n \geq 3 \)] and \( \Sigma = \{l\} \).

Note that one may reconstruct the quotients

\[ \Pi_n \twoheadrightarrow \Pi_3 \twoheadrightarrow \Pi_2 \]

determined by fiber subgroups of \( \Pi_n \) of co-length three and two, respectively [cf. Theorem 2.5, (v)]. In the remainder of the present item (iii), we fix such quotients and write \( F_1 \subseteq \Pi_3 \) for the kernel of the surjection \( \Pi_3 \twoheadrightarrow \Pi_2 \). Then one may reconstruct the subgroup \( S_2 \subseteq \text{Out}(\Pi_2) \) [cf. Corollary 2.6, (i)] group-theoretically from \( \Pi_3 \) as follows:

(iii-a) Let \( F_2 \neq F_1 \) be a fiber subgroup of \( \Pi_3 \) of co-length two; \( F_{12} \) the unique fiber subgroup of \( \Pi_3 \) of co-length one that contains \( F_1 \) and \( F_2 \) [cf. Theorem 2.5, (v)]. Here, let us recall that \( \Pi_3 \) may be identified with the maximal pro-\( l \) quotient of the étale fundamental group of the third configuration space \( X_3 \) of \( X \). Moreover, relative to this identification, \( F_{12}/F_1, F_{12}/F_2 \) may be naturally identified with the maximal pro-\( l \) quotient of the étale fundamental group of a hyperbolic curve \( X^* \) of type \((2, 1)\) over \( k \) [i.e., a geometric fiber of a projection morphism \( X_2 \to X \)]; \( F_{12} \) may be naturally identified with the maximal pro-\( l \) quotient of the étale fundamental group of the second configuration space \( X^*_2 \) of \( X^* \). In particular, relative to these identifications, one may reconstruct the quotients

\[ q_1, q_2 : \Pi_2 \overset{\text{def}}{=} \pi_1(X_2^*)(l) \twoheadrightarrow \Pi_1^* \overset{\text{def}}{=} \pi_1(X^*)(l) \]

corresponding to the quotients \( F_{12} \to F_{12}/F_2, F_{12} \to F_{12}/F_1 \), respectively [i.e., corresponding to the two projection morphisms \( X^*_2 \to X^* \)]. Here, we note that in fact, Theorem 2.5, (v), only yields a reconstruction of the kernels of these quotients, i.e., not an “identification isomorphism” [cf. the use of the same notation for the codomains of \( q_1, q_2 \)] between the codomains of \( q_1, q_2 \); on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow.
(iii-b) In light of the reconstruction of the quotients \( q_1, q_2 : \Pi_2^* \to \Pi_1^* \) in (iii-a), one may apply [the group-theoretic reconstruction algorithms implicit in the proof of] [CbTpI], Lemma 1.6, to reconstruct the conjugacy class of cuspidal inertia subgroups of \( \Pi_1^* \) associated to the unique cusp of \( X^* \).

(iii-c) One may reconstruct the conjugacy class of decomposition groups of \( \Pi_2 \) associated to the diagonal divisor \( \subseteq X \times_k X \) as the set of normalizers in \( \Pi_2 \) of the cuspidal inertia subgroups of \( \Pi_1^* = F_{12}/F_1 = \Pi_3/F_1 = \Pi_2 \) associated to the unique cusp of \( X^* \) [cf. (iii-b)].

(iii-d) Write \( F_{13} \) for the unique fiber subgroup of \( \Pi_3 \) of co-length one that contains \( F_1 \) but is \( \neq F_{12} \) [cf. Theorem 2.5, (v)]; \( p_2, p_3 : \Pi_2 = \Pi_3/F_1 \to \Pi_1 \) for the quotients determined by the quotients \( \Pi_3/F_1 \to \Pi_3/F_{13}, \Pi_3/F_1 \to \Pi_3/F_{12} \), respectively;
\[
\bar{p}_2, \bar{p}_3 : \Pi_2^{ab} \to \Pi_1^{ab}
\]
for the natural surjections induced by \( p_2, p_3 \), respectively. Here, we note that in fact, Theorem 2.5, (v), only yields a reconstruction of the kernels of these quotients, i.e., not an “identification isomorphism” [cf. the use of the same notation for the codomains of \( p_2, p_3 \)] between the codomains of \( p_2, p_3 \); on the other hand, one verifies easily that this abuse of notation [which was made for the sake of keeping the notation simple] does not have any substantive effect on the argument to follow. Then one may reconstruct the diagonal
\[
\delta \subseteq \Pi_1^{ab} \times \Pi_1^{ab}
\]
as the image of any decomposition group associated to the diagonal divisor of \( X \times_k X \) [cf. (iii-c)] via the surjection
\[
\Pi_2 \to \Pi_2^{ab} \xrightarrow{(p_2, p_3)} \Pi_1^{ab} \times \Pi_1^{ab}
\]
— where the first arrow is the natural surjection. In particular, one may reconstruct the natural identification between the copy of \( \Pi_1^{ab} \) that appears as the codomain of \( \bar{p}_2 \) and the copy of \( \Pi_1^{ab} \) that appears as the codomain of \( \bar{p}_3 \) as the composite isomorphism
\[
\Pi_1^{ab} \xrightarrow{\zeta} \delta \xrightarrow{\varepsilon} \Pi_1^{ab}
\]
— where the first (respectively, second) arrow denotes the arrow induced by \( \bar{p}_2 \) (respectively, \( \bar{p}_3 \)).

(iii-e) One may reconstruct the unique nontrivial element of the subgroup \( S_2 \subseteq \text{Out}(\Pi_2) \) as the unique element \( \alpha \in Z^{\text{loc}}(\text{Out}(\Pi_2)) \) — whose image in \( \text{Aut}(\Pi_1^{ab} \times \Pi_1^{ab}) \) [cf. Theorem 2.5, (v)] we denote by \( \overline{\alpha} \) — such that, for any \( (x, y) \in \Pi_1^{ab} \times \Pi_1^{ab} \),
\[
\overline{\alpha}(x, y) = (y, x).
\]
(iv) Suppose that \((g, r) \in \{(0, 4), (1, 1), (1, 2), (2, 0)\}\). Then one may reconstruct the subgroup \(\mathfrak{S}_n^r \subseteq \text{Out}(\Pi_n)\) as the subgroup generated by the elements \(\alpha \in \mathbb{Z}^{\text{loc}}(\text{Out}(\Pi_n))\) for which there exists a generalized fiber subgroup \(F\) of \(\Pi_n^{(l)}\) of co-length two \([\text{cf. Theorem 2.5, (v)}]\) satisfying the following conditions:

- The outer automorphism of \(\Pi_n^{(l)}\) induced by \(\alpha\) preserves \(F\) and induces the identity permutation on the set of generalized fiber subgroups of \(\Pi_n^{(l)}\) contained in \(F\) \([\text{cf. Theorem 2.5, (v)}]\).
- The outer automorphism of \(\Pi_2^{(l)}\) induced by \(\alpha\) via the surjection \(\Pi_n^{(l)} \twoheadrightarrow \Pi_2^{(l)}\) determined by \(F\) is contained in \(\mathfrak{S}_2 \subseteq \text{Out}(\Pi_2^{(l)})\) \([\text{cf. the descriptions of (i-f), (ii-c), (iii-e)}]\).

Proof. Assertion (i) (respectively, (ii); (iii)) follows immediately — with the exception of (i-f) (respectively, (ii-c), (iii-e)) — from the various results cited in the statement of assertion (i) (respectively, (ii); (iii)), together with the various definitions involved. On the other hand, assertion (iv), as well as (i-f), (ii-c), (iii-e), follow immediately, by considering the various transpositions in \(\mathfrak{S}_n^r\), from Corollary 2.9. \(\square\)

3. Group-theoretic reconstruction of log-full subgroups

Let \(l\) be a prime number; \(n\) a positive integer; \(\Sigma\) a set of prime numbers which is either equal to \(\{l\}\) or \(\text{Primes}\); \((g, r)\) a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(k\) a field of characteristic \(\notin \Sigma\); \(\overline{k}\) an algebraic closure of \(k\); \(S \overset{\text{def}}{=} \text{Spec}(k)\); \(S^{\text{log}}\) an fs log scheme whose underlying scheme is \(S\); \(X^{\text{log}}\) a stable log curve of type \((g, r)\) over \(S^{\text{log}}\); \(X_n^{\text{log}}\) the \(n\)-th log configuration space of \(X^{\text{log}}\); \(\Delta_n\) the maximal pro-\(\Sigma\) quotient of the kernel of the natural [outer] surjection \(\pi_1(X_n^{\text{log}}) \twoheadrightarrow \pi_1(S^{\text{log}})\). In this section, we introduce the notion of a pro-\(\Sigma\) log-full subgroup of \(\Delta_n\) \([\text{cf. Definition 3.4}]\) and show that, when \(\Sigma = \{l\}\), and, moreover, \(X^{\text{log}}\) and \(k\) satisfy certain conditions concerning “weights”, the pro-\(\Sigma\) log-full subgroups may be reconstructed group-theoretically from the natural outer action of \(\pi_1(S^{\text{log}})\) on \(\Delta_n\) \([\text{cf. Theorem 3.8}]\). We also introduce \([\text{cf. Definition 3.10}]\) and study \([\text{cf. Corollary 3.12}]\) the notion of a log-full subgroup of smooth type and prove that, in the case where \((g, r) = (1, 1)\), log-full subgroups of smooth type \([\text{cf. Definition 3.10}]\) may be distinguished from log-full subgroups of singular type \([\text{cf. Definition 3.10}]\) by means of a very simple group-theoretic criterion \([\text{cf. Proposition 3.14}]\), which makes essential use of the assumption \((g, r) = (1, 1)\).

Definition 3.1. In the notation introduced above:

(i) Write \(\mathcal{G}\) for the semi-graph of anabelioids of pro-\(\Sigma\) PSC-type \([\text{cf. [CmbGC], Definition 1.1, (i)}]\) determined by the stable log curve \(X^{\text{log}} \to S^{\text{log}}\). [Thus, \(\Delta_1\) may be identified with the PSC-fundamental group \([\text{cf. [CmbGC], Definition 1.1, (ii)}]\) \(\Pi_\mathcal{G}\) of \(\mathcal{G}\).] Write \(\text{Out}^{\text{FC}}(\cdot)\).
for the group of \textit{FC-admissible} outer automorphisms of \((-\rangle\) [cf. [CmbCusp], Definition 1.1, (ii)],
\[
\chi_1 : \text{Out}^{\text{FC}}(\Delta_1) \to \mathbb{Z}_l^\times
\]
for the \textit{pro-}\(l\) \textit{cyclo-}tomic character associated to \textit{“G−Node(G)“} [cf. [CbTpI], Definitions 2.8 and 3.8], and
\[
\chi_n : \text{Out}^{\text{FC}}(\Delta_n) \to \mathbb{Z}_l^\times
\]
for the composite with \(\chi_1\) of the natural homomorphism \(\text{Out}^{\text{FC}}(\Delta_n) \to \text{Out}^{\text{FC}}(\Delta_1)\) [cf. [CmbCusp], Proposition 1.2, (iii); [CbTpI], Corollary 1.9, (i)] induced by any projection morphism \(X_{\log}^n \to X_{\log}^1\) of co-length one.

(ii) Let \(V\) be a finite dimensional \(\mathbb{Q}_l\)-vector space, \(\alpha\) a \(\mathbb{Q}_l\)-linear automorphism of \(V\), \(\beta \in \mathbb{Q}_l^\times\). Then we shall say that \(\alpha\) is \(\beta\)-\textit{transverse} if, for every positive integer \(N\) and every eigenvalue \(\lambda \in \mathbb{Q}_l^\times\) of \(\alpha^N\), it holds that \(\lambda \neq \beta^N\). [Thus, if \(\alpha\) is \(\beta\)-\textit{transverse}, and \(M\) is a positive integer, then \(\alpha^M\) is \(\beta^M\)-\textit{transverse}.]

(iii) We shall say that a field \(K\) is \(l\)-cyclotomically \textit{full} if the image of the \(l\)-adic cyclotomic character \(\chi_K : G_K \to \mathbb{Z}_l^\times\) associated to \(K\) is infinite. We shall say that a field \(K\) is \textit{strongly} \(l\)-cyclotomically \textit{full} if there exists a normal noetherian domain \(A\) such that, if we write \(K_{\text{prim}} \subseteq K\) for the algebraic closure in \(K\) of the prime field of \(K\), then \(K_{\text{prim}}\) is a subfield of the field of fractions \(L\) of \(A\), and, moreover, the subset
\[
\{ p \in \text{Spec}(A) \mid \text{ch}(k(p)) > 0, \text{ and } k(p) \text{ is } l\text{-cyclotomically full}\}
\]
— where we write \(k(p)\) for the residue field of \(p\) — of \(\text{Spec}(A)\) is Zariski dense in \(\text{Spec}(A)\).

\textbf{Remark 3.1.1.} Observe that it follows formally from Definition 3.1, (iii), that any subfield of a strongly \(l\)-cyclotomically full field is itself strongly \(l\)-cyclotomically full.

\textbf{Remark 3.1.2.} In the notation of Definition 3.1, (iii), let \(K'\) be a finitely generated field extension of \(K\). Then since \(K_{\text{prim}} \subseteq K\) is a regular extension [cf. [Bbk], Chapter \(V\), \S 17, Proposition 9], we conclude that the natural outer homomorphism \(G_K \to G_{K_{\text{prim}}}\) is surjective, and that \(K_{\text{prim}} \subseteq K'_{\text{prim}}\) is a finite extension.

\textbf{Proposition 3.2. (Computation of characters). Set}
\[
m = \begin{cases} 
2n(g + r - 1), & \text{if } g > 0 \text{ and } r > 0, \\
2ng, & \text{if } g > 0 \text{ and } r = 0, \\
2n(r - 1) + n(n - 1), & \text{if } g = 0.
\end{cases}
\]
Write
\[
\delta_n : \text{Out}^{\text{FC}}(\Delta_n) \to \mathbb{Z}_l^\times
\]
for the square of the character obtained by forming the composite of the natural morphism \( \text{Out}^\text{FC}(\Delta_n) \to \text{Aut}(\Delta_n^{\text{ab}}) \) with the determinant homomorphism \( \text{Aut}(\Delta_n^{\text{ab}}) \to \mathbb{Z}_l^\times \). Then \( \delta_n \) coincides with the \( m \)-th power of \( \chi_n \).

Proof. This follows immediately from Proposition 2.2, (i), (ii); [CmbGC], Proposition 1.3. \( \square \)

Proposition 3.3. (Basic properties of [strongly] \( l \)-cyclotomically full fields). Let \( K \) be a field, \( K' \) a finitely generated field extension of \( K \). Then the following hold:

(i) If \( K \) is \( l \)-cyclotomically full, then \( \text{ch}(K) \neq l \).

(ii) If \( K \) is strongly \( l \)-cyclotomically full, then \( K \) is \( l \)-cyclotomically full.

(iii) Suppose that \( \text{ch}(K) > 0 \). Then \( K \) is \( l \)-cyclotomically full if and only if \( K \) is strongly \( l \)-cyclotomically full.

(iv) \( K \) is \( l \)-cyclotomically full if and only if \( K' \) is.

(v) \( K \) is strongly \( l \)-cyclotomically full if and only if \( K' \) is.

(vi) Suppose that \( K \) is strongly \( l \)-cyclotomically full. Write \( T \) \( \overset{\text{def}}{=} \mathbb{P}^1_K \setminus \{0, 1, \infty\} \). Let \( Y \) be a hyperbolic curve over \( K \) that admits a finite étale Galois covering \( Y \to T \) of degree \( d \), where \( d \) is invertible in \( K \). Write \( Z \) for the \( [K]\)-smooth compactification of \( Y \). Then there exists an element \( g \in G_K \) such that \( \rho(g) \) is \( \chi_K(g) \)-transverse.

(vii) Suppose that \( K \) is a mixed characteristic, nonarchimedean local field, i.e., a finite extension of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. Then \( K \) is \( l \)-cyclotomically full, but not strongly \( l \)-cyclotomically full.

(viii) Suppose that \( K \) is either the field \( \mathbb{Q} \) of rational numbers or a finite field of characteristic \( \neq l \). Then every finitely generated field extension of \( K \) is strongly \( l \)-cyclotomically full.

Proof. Assertion (i) follows immediately from the well-known fact that if \( \text{ch}(K) = l \), then the cyclotomic character \( \chi_K : G_K \to \mathbb{Z}_l^\times \) is trivial. Next, we consider assertion (ii). Suppose that \( K \) is strongly \( l \)-cyclotomically full. We apply the notation of Definition 3.1, (iii). Then let us observe that to verify that \( K \) is \( l \)-cyclotomically full, it suffices to show that \( L \) is \( l \)-cyclotomically full [cf. Remark 3.1.2]. Also, we observe that we may assume without loss of generality [cf. assertion (i)] that \( l \) is invertible in \( A \). Since \( A \) is normal, this assumption implies that the \( l \)-adic cyclotomic character \( \chi_L : G_L \to \mathbb{Z}_l^\times \)
associated to \(L\) may be thought of as the restriction via the natural [outer]
surjection \(G_L \to \pi_1(\text{Spec}(A))\) induced by the injection \(A \hookrightarrow L\) of the \(l\)-adic
cyclotomic character \(\chi_A : \pi_1(\text{Spec}(A)) \to \mathbb{Z}_l^\times\) associated to \(A\). On the other
hand, since [cf. Definition 3.1, (iii)] there exists a prime ideal \(p \in \text{Spec}(A)\)
such that \(k(p)\) is \(l\)-cyclotomically full, we thus conclude that the image of
\(\chi_A\), hence also of \(\chi_L\), is infinite. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii) and Definition 3.1, (iii).

Since the natural outer homomorphism \(G_{K'} \to G_K\) is [well-known to be]
\(open\), assertion (iv) follows immediately from Definition 3.1, (iii).

Next, we consider assertion (v). We apply the notation of Definition 3.1, (iii), together with assertion (iv). This
completes the proof of assertion (v), one may
descend the diagram \(Z \supseteq Y \to T\) of smooth curves over \(K\) to a diagram \(Z_{\text{prm}} \supseteq Y_{\text{prm}} \to T_{\text{prm}}\) of smooth curves over \(K_{\text{prm}}\).

In particular, by Remark 3.1.2, we may assume without loss of generality that \(K = K_{\text{prm}}\). Moreover, one may base-change the diagram \(Z \supseteq Y \to T\) of smooth curves over \(K = K_{\text{prm}}\) to a diagram \(Z_L \supseteq Y_L \to T_L\) of smooth curves over \(L\). Next, we observe that, by possibly replacing \(A\) by \(A[1/f]\) for
a suitable nonzero \(f \in A\), we may assume without loss of generality that this diagram \(Z_L \supseteq Y_L \to T_L\) of smooth curves over \(L\) arises from a diagram \(Z_A \supseteq Y_A \to T_A\) of smooth curves over \(A\), and that \(d\) is invertible in \(A\). Thus, by restricting to a point \(p \in \text{Spec}(A)\) as in the display of Definition
3.1, (iii), we obtain a diagram \(Z_p \supseteq Y_p \to T_p\) of smooth curves over \(k(p)\).

In particular, by replacing \(K\) by a finite extension of \(k(p)_{\text{prm}}\), we conclude that we may assume without loss of generality [cf. [SGA1], EXPOSÉ X, Corollaire 3.9] that \(K\) is an \(l\)-cyclotomically full algebraic extension of a
finite field \(K_0\). Moreover, it follows immediately from well-known properties of absolute Galois groups of finite fields that we may assume without loss of
generality, after possibly replacing \(K\) and \(K_0\) by finite extensions [cf. assertion (iv)] of \(K\) and \(K_0\), respectively, that the natural [outer] injection
\(G_K \to G_{K_0}\) induces an isomorphism on maximal pro-\(l\) quotients, and that \(\rho\) factors through the maximal pro-\(l\) quotient of \(G_K\). But then assertion (vi)
follows immediately from the “Riemann hypothesis” for abelian varieties over finite fields [cf., e.g., [Mumf], p. 206]. This completes the proof of assertion (vi).

Next, we consider assertion (vii). When \(l \neq p\), the fact that \(K\) is \(l\)-cyclotomically full follows immediately from the fact that the residue field
of $K$ is finite [together with well-known facts concerning the structure of finite fields]. When $l = p$, the fact that $K$ is $l$-cyclotomically full follows immediately from well-known facts concerning ramification in cyclotomic extensions of $\mathbb{Q}_p$ [cf. also assertion (iv)]. Next, assume that $K$ is strongly $l$-cyclotomically full. Observe that, by assertion (v), from the point of view of deriving a contradiction to this assumption, we may always replace $K$ by a finite extension of $K$. Let $E$ be a Tate curve over $K$ [i.e., a smooth, proper curve of genus 1 over $K$ that has stable, multiplicative reduction over the ring of integers of $K$] that is defined over a number field. Thus, by applying the theory of Belyi maps [cf. [Belyi]], we conclude that there exists a dense open subscheme $U \subseteq E$ that admits a finite étale morphism $U \to T$, where $T$ is as in assertion (vi). Note, moreover, that, by possibly replacing $K$ by a finite extension of $K$, we may assume without loss of generality that there exists a finite étale covering $Y \to U$ of hyperbolic curves over $K$ such that the composite finite étale morphism $Y \to U \to T$ is Galois. On the other hand, the well-known appearance of the cyclotomic character in the Galois action on the Tate module of a Tate curve [cf. [Serre], IV, §A.1.2] implies that this finite étale Galois covering $Y \to T$ yields a contradiction to the conclusion of assertion (vi). This completes the proof of assertion (vii). Finally, since finite fields of characteristic $\neq l$ are clearly $l$-cyclotomically full, assertion (viii) follows immediately from assertions (iii), (v) [cf. also Definition 3.1, (iii); Remark 3.1.2]. □

Definition 3.4. Let $x_n \in X_n \times_k \overline{k}$ be a log-full closed point [cf. Definition 1.1]. Then we shall refer to any $\Delta_n$-conjugate of the image of the natural inclusion $\Delta_{x_n} \hookrightarrow \Delta_n$ of Proposition 1.2, (iv), as a $[\text{pro-}\Sigma]$ log-full subgroup of $\Delta_n$ associated to $x_n$. We shall refer to an open subgroup of a log-full subgroup of $\Delta_n$ associated to $x_n$ as a $[\text{pro-}\Sigma]$ quasi-log-full subgroup of $\Delta_n$ associated to $x_n$.

Proposition 3.5. (Projections of log-full subgroups). Let $n''$ be a positive integer such that $n'' < n$, $A$ a log-full subgroup of $\Delta_n$. Write $p : \Delta_n \to \Delta_{n''}$ for the natural [outer] surjection induced by the projection morphism $X_n^{\text{log}} \to X_{n''}^{\text{log}}$ obtained by forgetting $n - n''$ of the $n$ factors of $X_n^{\text{log}}$. Then $p(A)$ is a log-full subgroup of $\Delta_{n''}$.

Proof. Proposition 3.5 follows immediately from Proposition 1.3, (ii) [cf. also the commutative diagram in the proof of Proposition 1.2]. □

Remark 3.5.1. Observe that, in Proposition 3.5, the natural [outer] surjection $p : \Delta_n \to \Delta_{n''}$ may in fact be taken to be the natural [outer] surjection induced by a “generalized projection morphism”. That is to say, although the notion of a “generalized projection morphism” is only defined in Definition 2.1, (i), in the case where $X^{\text{log}}$ is smooth, one verifies immediately that a similar definition may be made in the case of arbitrary $X^{\text{log}}$ [i.e., as in the present §3]. Alternatively, in the notation of Remark 2.1.1, one may observe
that, by a similar argument to the argument applied in Remark 2.1.1, the symmetric group on \( n^* \) letters \( S_n^* \) acts naturally on \( X_n^{\text{log}} \), hence induces a natural outer action of \( S_n^* \) on \( \Delta_n \); thus, in Proposition 3.5, one may take the natural \([\text{outer}]\) surjection \( p : \Delta_n \to \Delta_n^{n^*} \) to be the composite of a surjection \( p \) as in the statement of Proposition 3.5 with the outer automorphism of \( \Delta_n \) determined by an element of \( S_n^* \).

**Lemma 3.6.** (Commensurable terminality). Every log-full subgroup of \( \Delta_n \) is commensurably terminal in \( \Delta_n \).

**Proof.** Let us first observe that, to verify Lemma 3.6, we may assume without loss of generality that \( k \) is algebraically closed. Let \( A \) be a log-full subgroup of \( \Delta_n \). Note that the inclusion \( A \subseteq C_{\Delta_n}(A) \) is clear. We prove the inclusion \( C_{\Delta_n}(A) \subseteq A \) by induction on \( n \). The case where \( n = 1 \) follows from [CmbGC], Proposition 1.2, (ii). Next, suppose that \( n > 1 \), and that the induction hypothesis is in force. Write \( p : \Delta_n \to \Delta_{n-1} \) for the natural \([\text{outer}]\) surjection induced by the projection morphism \( X_n^{\text{log}} \to X_{n-1}^{\text{log}} \) obtained by forgetting the last factor. Then since \( p(A) \) is a log-full subgroup of \( \Delta_{n-1} \) [cf. Proposition 3.5], by applying the induction hypothesis, we conclude that \( C_{\Delta_{n-1}}(p(A)) \subseteq p(A) \). Thus, we obtain that

\[
p(A) \subseteq p(C_{\Delta_n}(A)) \subseteq C_{\Delta_{n-1}}(p(A)) \subseteq p(A),
\]

hence that \( p(A) = p(C_{\Delta_n}(A)) \). Therefore, to verify the inclusion \( C_{\Delta_n}(A) \subseteq A \), it suffices to show that

\[
\Delta_{n/n-1} \cap A = \Delta_{n/n-1} \cap C_{\Delta_n}(A)
\]

where we write \( \Delta_{n/n-1} \) as the kernel of the projection \( p : \Delta_n \to \Delta_{n-1} \). Let \( x_n \in X_n \) be a log-full point of \( X_n \) such that \( A = \Delta_{x_n} \) [cf. Definition 3.4]. Write \( x_{n-1} \in X_{n-1} \) for the image of \( x_n \) via \( X_n^{\text{log}} \to X_{n-1}^{\text{log}} \). For \( \Delta_n \) determined by \( x_{n-1} \), and \( (X_n^{\text{log}})_{x_{n-1}} \) is the sub-scheme of \( X_n \) determined by \( x_{n-1} \), and \( (X_n^{\text{log}})_{x_{n-1}} \) is the sub-scheme of \( X_n \) determined by \( x_{n-1} \). Then we observe that — by applying [Hsh], Theorem 2, to the completion of the natural projection morphism \( X_n^{\text{log}} \to X_{n-1}^{\text{log}} \) along the natural projection morphism \( (X_n^{\text{log}})_{x_{n-1}} \to X_{n-1}^{\text{log}} \) — we may identify \( \Delta_{n/n-1} \) with the maximal pro-\( \Sigma \) quotient of the kernel of the \([\text{outer}]\) surjection \( \pi_1((X_n^{\text{log}})_{x_{n-1}}) \to \pi_1(X_{n-1}^{\text{log}}) \). Next, let us observe that \( \Delta_{n/n-1} \cap A \) is a log-full subgroup of \( \Delta_{n/n-1} \) that arises from the log-full point of the underlying scheme of \( (X_n^{\text{log}})_{x_{n-1}} \) determined by \( x_n \) [cf. the commutative diagram in the proof of Proposition 1.2]. Thus, it follows from the case where \( n = 1 \) that \( C_{\Delta_n}(\Delta_{n/n-1} \cap A) \subseteq \Delta_{n/n-1} \cap A \), hence that

\[
\Delta_{n/n-1} \cap A \subseteq \Delta_{n/n-1} \cap C_{\Delta_n}(A) \\
\subseteq C_{\Delta_{n-1}}(\Delta_{n/n-1} \cap A) \subseteq \Delta_{n/n-1} \cap A.
\]

This completes the proof of the equality \( \Delta_{n/n-1} \cap A = \Delta_{n/n-1} \cap C_{\Delta_n}(A) \) and hence of Lemma 3.6. \( \square \)
Lemma 3.7. (Open subgroups of closed subgroups of profinite groups via commensurable terminality). For $i \in \{1, 2, 3\}$, let $G_i$ be a profinite group; $H_i \subseteq G_i$ an abelian closed subgroup of $G_i$; $I_2 \subseteq G_2$ an abelian closed subgroup of $G_2$. Suppose that $H_1$ is commensurably terminal in $G_1$, and that there exists a commutative diagram of profinite groups

$$
\begin{array}{ccccccc}
1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1 
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are the natural injections. Suppose, moreover, that $I_1 \overset{\text{def}}{=} I_2 \cap G_1$ (respectively, $I_3 \overset{\text{def}}{=} p(I_2)$) is an open subgroup of $H_1$ (respectively, of $H_3$). Then it holds that $Z_{p^{-1}(H_3)}^\text{loc}(H_1) = H_2$. In particular, $I_2$ is an open subgroup of $H_2$.

Proof. Write $K_2 \overset{\text{def}}{=} p^{-1}(H_3)$. In particular, it holds that $H_1 \subseteq H_2 \subseteq K_2$. In light of our assumption that $I_1$ (respectively, $I_3$) is an open subgroup of $H_1$ (respectively, of $H_3$), to complete the verification of Lemma 3.7, it suffices to show that

$$I_2 \subseteq Z_{K_2}^\text{loc}(H_1) = H_2.$$ 

First, let us observe that the inclusion $H_2 \subseteq Z_{K_2}^\text{loc}(H_1)$ follows from our assumption that $H_2$ is abelian. On the other hand, since $H_1$ is abelian and commensurably terminal in $G_1$, the following inclusions hold:

$$H_1 \subseteq G_1 \cap Z_{K_2}^\text{loc}(H_1) = Z_{G_1}^\text{loc}(H_1) \subseteq C_{G_1}(H_1) = H_1.$$ 

Thus, we obtain that $H_1 = G_1 \cap Z_{K_2}^\text{loc}(H_1)$. In particular, we obtain a commutative diagram of profinite groups

$$
\begin{array}{ccccccc}
1 & \longrightarrow & G_1 & \longrightarrow & K_2 & \longrightarrow & H_3 & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_1 & \longrightarrow & Z_{K_2}^\text{loc}(H_1) & \longrightarrow & p(Z_{K_2}^\text{loc}(H_1)) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 1 
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are the natural injections. It follows immediately from this commutative diagram that the inclusion $H_3 \hookrightarrow p(Z_{K_2}^\text{loc}(H_1))$ is, in fact, bijective, hence that $Z_{K_2}^\text{loc}(H_1) = H_2$. Finally, since $I_1$ is open in $H_1$, and $I_2$ is abelian, we conclude that

$$I_2 \subseteq Z_{K_2}^\text{loc}(H_1) = H_2.$$ 

This completes the proof of Lemma 3.7. □
Theorem 3.8. (Group-theoretic characterization of log-full subgroups). Let \( l \) be a prime number; \( n \) a positive integer; \( k \) a field of characteristic \( \neq l \); \( \chi_k : G_k \to \mathbb{Z}_l^\times \) the \( l \)-adic cyclotomic character associated to \( k \); \( S \) def = Spec(\( k \)); \( S^{\log} \) an fs log scheme whose underlying scheme is \( S \); \( X^{\log} \) a stable log curve over \( S^{\log} \); \( \mathcal{G} \) the semi-graph of anabelioids of pro-\( l \) PSC-type determined by \( X^{\log} \to S^{\log} \) [cf. [CmbGC], Definition 1.1, (i)]; \( X_n^{\log} \) the \( n \)-th log configuration space of \( X^{\log} \); \( G \) def = \( \pi_1(S^{\log}) \); \( \pi_1(X_n^{\log}/S^{\log}) \) the kernel of the natural \[\text{[outer]}\] surjection \( \pi_1(X_n^{\log}) \to G \); \( \Delta_n = \pi_1(X_n^{\log}/S^{\log})(l) \); \( \Pi_n \) the quotient of \( \pi_1(X_n^{\log}) \) by the kernel of the natural surjection \( \pi_1(X_n^{\log}/S^{\log}) \to \Delta_n \). Moreover, suppose that the following two conditions are satisfied:

(a) \( k \) is strongly \( l \)-cyclotomically full.

(b) Let \( \Pi_G \subseteq \Pi_\mathcal{G} \cong \Delta_1 \) be a characteristic open subgroup. Observe that \( \Pi_1 \) naturally acts on \( \Pi_G \) by conjugation, hence on \( \Pi_G^{ab/edge} \otimes \mathbb{Z}_l \mathbb{Q} \) [cf. the discussion entitled “Topological groups” in §70]. Write \( \rho_G : \Pi_1 \to \text{Aut}_{\mathbb{Q}_l}(\Pi_G^{ab/edge} \otimes \mathbb{Z}_l \mathbb{Q}) \) for this action of \( \Pi_1 \) on \( \Pi_G^{ab/edge} \otimes \mathbb{Z}_l \mathbb{Q} \). Then there exists a finite filtration of \( \Pi_1 \)-stable \( \mathbb{Q}_l \)-subspaces of \( \Pi_G^{ab/edge} \otimes \mathbb{Z}_l \mathbb{Q} \) such that, for each subquotient of this finite filtration, there exists an element \( g \in \Pi_1 \) such that the restriction to this subquotient of \( \rho_G(g) \) is \( \chi_k(g) \)-transverse [cf. Definition 3.1, (ii)], where, by abuse of notation, we write \( \chi_k(\cdot) \) for the restriction of \( \chi_k(\cdot) \), as defined above, via the natural \[\text{[outer]}\] surjection \( \Pi_1 \to G \to G_k \).

Let \( A \) be a closed subgroup of \( \Delta_n \). Then \( A \) is quasi-log-full (respectively, log-full) if and only if \( A \) satisfies the following two conditions:

(i) The pro-\( l \) group \( A \) is abelian [hence necessarily free of rank \( \leq n \) — cf. Proposition 1.4; Theorem 1.6; [MT], Proposition 2.2, (i)] and free of rank \( n \) (respectively, is a maximal free abelian closed subgroup of \( \Delta_n \) of rank \( n \)). [Here, we recall that, by Theorem 1.6, \( n \) is determined by the isomorphism class of the pro-\( l \) group \( \Delta_n \).]

(ii) There exists a closed subgroup \( H \) of \( \Pi_n \) such that

(ii-1) the image of \( H \) via the surjection \( \Pi_n \to G \) is an open subgroup of \( G \);

(ii-2) the conjugation action of \( H \) on \( \Delta_n \) preserves \( A \subseteq \Delta_n \), and the resulting action of \( H \) on \( A \) is given by a character \( \chi_H : H \to \mathbb{Z}_l^\times \);

(ii-3) the \( m \)-th power of \( \chi_H \) [cf. (ii-2)], where we write \( m \) for the positive integer “\( m \)” appearing in the statement of Proposition 3.2, coincides with the restriction \( \delta_n|_H \) of \( \delta_n \) [cf. Proposition 3.2] to \( H \) [via the homomorphism \( H \to \text{Out} FC(\Delta_n) \) determined by the conjugation action of \( H \) on \( \Delta_n \)].
Here, we note that, in the situation of (ii-3), the definition of \( \delta_n|_H \) is manifestly group-theoretic, and that a group-theoretic characterization of the integer \( m \) may be obtained by applying Theorem 2.5, (i), (vi), together with [when \( n = 1 \)] assumption (a); Proposition 3.3, (ii); [CmbGC], Corollary 2.7, (i).

Proof. First, we note that it follows from Lemma 3.6 that the equivalence

\[ A \text{ is log-full} \iff A \text{ satisfies the resp'd portion of conditions (i) and (ii)} \]

follows immediately from the equivalence

\[ A \text{ is quasi-log-full} \iff A \text{ satisfies the non-resp'd portion of conditions (i) and (ii)}. \]

Thus, in the remainder of the proof of Theorem 3.8, we prove the non-resp'd equivalence. Suppose that \( A \) satisfies conditions (i), (ii). Then we prove that \( A \) is quasi-log-full by induction on \( n \). First, we consider the case where \( n = 1 \). In this case, since \( A \) is nontrivial procyclic [cf. condition (i)], it follows from assumption (b); condition (ii); Proposition 3.2; [NodNon], Lemma 1.6, that \( A \) is contained in a log-full subgroup \( \Delta_1 \), hence that \( A \) is quasi-log-full.

Next, we suppose that \( n \geq 2 \), and that the induction hypothesis is in force. Let \( \overline{k} \) be an algebraic closure of \( k \), \( S \overset{\text{def}}{=} \text{Spec}(\overline{k}) \). In this case, we observe that we have an exact sequence of profinite groups

\[ 1 \longrightarrow \Delta_n/1 \longrightarrow \Delta_n \overset{p}{\longrightarrow} \Delta_1 \longrightarrow 1 \]

where we write \( p \) for the [outer] surjection induced by the first projection \( X_n^{\log} \to X^{\log} \) and \( \Delta_n/1 \overset{\text{def}}{=} \ker(p) \). In particular, it follows from Lemma 1.5 and condition (i) [cf. also Proposition 1.4; [MT], Proposition 2.4, (i)] that \( p(A) \) is a free, abelian closed subgroup of \( \Delta_1 \) of rank 1. Moreover, since \( Z_{l}^{\times} \) admits a torsion-free open subgroup, one verifies immediately, by applying Proposition 3.2, that, after possibly replacing \( H \) [cf. condition (ii)] by a suitable open subgroup of \( H \), it holds that the image of \( H \) via the natural surjection \( \Pi_n \to \Pi_1 \) induced by the first projection \( X_n^{\log} \to X^{\log} \) satisfies “condition (ii) for the closed subgroup \( p(A) \subseteq \Delta_1 \)”. In particular, it follows from the case where \( n = 1 \) [which has already been verified] that \( p(A) \) is quasi-log-full, hence, that there exists a log-full point \( \overline{x} \in X \times_{S} \overline{S} \) such that \( p(A) \) is an open subgroup of a log-full subgroup \( \Delta_{\overline{x}} \subseteq \Delta_1 \) associated to \( \overline{x} \).

Write \( x \in X \) for the image of \( \overline{x} \) via the natural morphism \( X \times_{S} \overline{S} \to X \), \( S^{\log}_x \) for the log scheme obtained by restricting the log structure of \( X^{\log} \) to the closed subscheme of \( X \) determined by \( x \). Then, by considering the base-change of the first projection \( X^{\log}_2 \to X^{\log} \) via the natural inclusion \( X^{\log}_2 \hookrightarrow X^{\log} \), we obtain a stable log curve

\[ X^{\log}_x \overset{\text{def}}{=} X^{\log}_2 \times_{X^{\log}} S^{\log}_x \to S^{\log}_x. \]

Next, observe that the stable log curve \( X^{\log}_x \) may be obtained by “attaching” the tripod \( \mathbb{P}^1_{S_x} \setminus \{0, 1, \infty\} \), in a suitable fashion, to the stable log curve \( X^{\log} \times_{S_x^{\log}} S^{\log}_x \). Thus, by applying Proposition 3.3, (ii), (v), (vi), to
suitable connected finite étale coverings of the tripod, we conclude from assumptions (a), (b) [cf. also [NodNon], Lemma 1.4; [CmbGC], Remark 1.1.3] that “assumptions (a), (b) for the stable log curve $X_x^{\log}$” hold.

Next, let us observe that the fs log scheme over $S_x^{\log}$

$$X_n^{\log} \times_{X^{\log}} S_x^{\log} \rightarrow S_x^{\log}$$

— where the fiber product denotes the base-change of the first projection $X_n^{\log} \rightarrow X^{\log}$ via the natural inclusion $S_x^{\log} \rightarrow X^{\log}$ — may be naturally identified with the $(n - 1)$-st log configuration space $(X_x)^{\log}_{n-1}$ of $X_x^{\log}$. On the other hand, since the kernel of the natural [outer] surjection $\pi_1((X_{n-1})^{\log}) \rightarrow \pi_1(S_x^{\log})$ may be identified with the kernel of the natural [outer] surjection $\pi_1(X^{\log}_{n-1}) \rightarrow \pi_1(X^{\log})$ [cf. [Hsh], Theorem 2, which we think of as being applied to the completion of the natural projection morphism $X^{\log}_n \rightarrow X^{\log}$ along the natural projection morphism $(X_x)^{\log}_{n-1} \rightarrow S_x^{\log}$, we obtain a commutative diagram

$$
\begin{array}{cccc}
1 & \rightarrow & \Delta_{n/1} & \rightarrow & \Pi_n & \rightarrow & \Pi_1 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \Delta_{n/1} & \rightarrow & \Pi_{n-1} & \rightarrow & G^x & \rightarrow & 1
\end{array}
$$

— where we write $G^x \overset{\text{def}}{=} \pi_1(S_x^{\log})$ and $\Pi_{n-1}^x$ for the quotient of $\pi_1((X_{n-1})^{\log})$ by the kernel of the natural surjection $\text{Ker} (\pi_1(X^{\log}_{n-1}) \rightarrow \Delta_{n/1}$; the horizontal sequences are exact; the middle (respectively, right-hand) vertical arrow is the [outer] homomorphism induced by $(X_{n-1})^{\log} \rightarrow X_n^{\log}$ (respectively, $S_x^{\log} \rightarrow X^{\log}$).

Thus, $A \cap \Delta_{n/1}$ is a free abelian closed subgroup of $\Delta_{n/1}$ of rank $n - 1$; moreover, one verifies immediately, by applying Proposition 3.2, that [since $\mathbb{Z}_n^x$ admits a torsion-free open subgroup] there exists an open subgroup of the closed subgroup of $\Pi_{n-1}^x$ obtained by pulling back $H \subseteq \Pi_n$ via the arrow $\Pi_{n-1}^x \rightarrow \Pi_n$ of the above commutative diagram with respect to which “condition (ii) for the closed subgroup $A \cap \Delta_{n/1} \subseteq \Delta_n^{1/I}$” is satisfied. In particular, we conclude from the induction hypothesis that $A \cap \Delta_{n/1}$ is quasi-log-full, hence that there exists a log-full point $\overline{y} \in (X_x)^{\log}_{n-1} \times_{S_x} \overline{S}$, such that $A \cap \Delta_{n/1}$ is an open subgroup of a log-full subgroup $\Delta_{\overline{y}} \subseteq \Delta_{n/1}$ associated to $\overline{y}$.

Next, let us write $\overline{x} \in X_n \times_S \overline{S}$ for the image of $\overline{y} \in (X_x)^{\log}_{n-1} \times_{S_x} \overline{S}$ via the morphism

$$(X_x)^{\log}_{n-1} \times_{S_x} \overline{S} \rightarrow (X_n \times S_x) \times_{S_x} \overline{S} = X_n \times S \overline{S}$$

obtained by considering the base-change of the natural morphism $(X_x)^{\log}_{n-1} \rightarrow X_n \times S_x$ via the natural morphism $\overline{S} \rightarrow S_x$. Here, note that since $\overline{x} \in X \times_S \overline{S}$ and $\overline{y} \in (X_x)^{\log}_{n-1} \times_{S_x} \overline{S}$ are log-full, it follows from Proposition 1.3, (i), that $\overline{x} \in X_n \times S \overline{S}$ is log-full. Moreover, one verifies easily that there exists a log-full subgroup $\Delta_{\overline{x}} \subseteq \Delta_n$ — among its various $\Delta_n$-conjugates —
Corollary 3.9. (Group-theoretic preservation of log-full subgroups).

Let $l$ be a prime number; $n$ a positive integer. For $\square \in \{\circ, \bullet\}$, let $S^\log_\square$ be an fs log scheme whose underlying scheme is the spectrum of a field $k_\square$ of characteristic $\neq l$; $X^\log_\square$ a stable log curve over $S^\log_\square$; $(X^\log_\square)_n$ the $n$-th log configuration space of $X^\log_\square$; $\pi_1((X^\log_\square)_n/S^\log_\square)$ the kernel of the natural [outer] surjection $\pi_1((X^\log_\square)_n/S^\log_\square) \to \operatorname{Out}(S^\log_\square)$. Thus, the natural conjugation action of $\Pi^n \square$ on $\Delta_n^\square$ determines a natural outer Galois action $G^\square \to \operatorname{Out}(\Delta_n^\square)$. Suppose that, for each $\square \in \{\circ, \bullet\}$, the above collection of data [i.e., with “\$’s” omitted] satisfies assumptions (a), (b) in the statement of Theorem 3.8. Let

$$\alpha : \Delta_n^\circ \cong \Delta_n^\bullet$$

be an isomorphism of profinite groups that is compatible with the respective natural outer Galois actions $G^\circ \to \operatorname{Out}(\Delta_n^\circ), G^\bullet \to \operatorname{Out}(\Delta_n^\bullet)$ relative to some isomorphism of profinite groups $G^\circ \cong G^\bullet$. Then for any quasi-log-full (respectively, log-full) subgroup $A \subseteq \Delta_n^\circ$ of $\Delta_n^\circ$, $\alpha(A) \subseteq \Delta_n^\bullet$ is a quasi-log-full (respectively, log-full) subgroup of $\Delta_n^\bullet$. 

associated to $\tau$ that fits into a commutative diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & \Delta_{n/1} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_n^\tau
\end{array}
$$

where the horizontal sequences are exact [cf. the exactness discussed above of the sequence $1 \to \Delta_{n/1} \to \Pi_n \to \Pi_1 \to 1$; the commutative diagram in the proof of Proposition 1.2], and the vertical arrows are the natural injections. Therefore, since $p(A)$ (respectively, $A \cap \Delta_{n/1}$) is an open subgroup of $\Delta_{n/1}^\tau$ (respectively, $\Delta_n^\tau$), it follows from Lemmas 3.6, 3.7 [where we take the “commutative diagram” to be the commutative diagram of the above display and “$I_\tau$” to be $A$] that $A$ is an open subgroup of $\Delta_n^\tau$, hence that $A$ is quasi-log-full, as desired.

Conversely, suppose that $A$ is quasi-log-full, i.e., that $A$ is an open subgroup of a log-full subgroup $\Delta_n \subseteq \Delta_n^\tau$ associated to a log-full point $\tau \in X_n \times_S \overline{S}$. Then since $\Delta_n^\tau \cong \mathbb{Z}_l(1)^{\oplus n}$ [cf. Proposition 1.3, (iii)], we conclude that $A$ satisfies condition (i). Write $S^\log_2$ for the log scheme obtained by restricting the log structure of $X^\log_n$ to the image of $\tau$ via the natural projection $X_n \times_S \overline{S} \to X_n$, $H$ for the image of the composite

$$\pi_1(S^\log_2) \to \pi_1(X^\log_n) \to \Pi_n$$

— where the first arrow is the [outer] homomorphism induced by the natural morphism $S^\log_2 \to X^\log_n$; the second arrow is the natural surjection. Then one verifies easily [cf. Proposition 3.2] that $H$ satisfies condition (ii). This completes the proof of Theorem 3.8. 

$\square$ 

$$1 \longrightarrow \Delta_{n/1} \longrightarrow \Delta_n \xrightarrow{p} \Delta_1 \longrightarrow 1$$
Proof. Corollary 3.9 follows immediately from Theorem 3.8. □

Finally, in the remainder of the present §3, we introduce [cf. Definition 3.10, below] and study [cf. Corollary 3.12, below] the notion of a log-full subgroup of smooth type and show that, at least in the case of once-punctured elliptic curves, the condition that a log-full subgroup be of smooth type admits a simple group-theoretic characterization [cf. Proposition 3.14, Corollary 3.15, below].

Definition 3.10. Let \( A \subseteq \Delta_n \) be a log-full subgroup [cf. the discussion at the beginning of the present §3; Definition 3.4] of \( \Delta_n \). For \( i \in \{1, \ldots, n^*\} \) [cf. Remark 2.1.1], write \( p_i : \Delta_n \to \Delta_1 \) for the natural [outer] surjection induced by the generalized projection morphism [cf. Definition 2.1, (i); Remark 3.5.1] \( X_n^{\log} \to X^{\log} \) determined by forgetting the marked points with labels \( \in \{1, \ldots, n^*\} \setminus \{i\} \). We shall say that \( A \) is of smooth type if, for every \( i \in \{1, \ldots, n^*\} \), the log-full subgroup [cf. Proposition 3.5; Remark 3.5.1] \( p_i(A) \subseteq \Delta_1 \) is a cuspidal edge-like subgroup [cf. Definition 3.1, (i); [CmbGC], Definition 1.1, (ii)] of \( \Delta_1 \). We shall refer to a log-full subgroup of \( \Delta_n \) which is not of smooth type as a log-full subgroup of singular type.

Remark 3.10.1. Observe that it follows immediately from Definition 3.10 that if there exists a log-full subgroup of \( \Delta_n \) which is of smooth type, then \( r > 0 \).

Proposition 3.11. (Alternative characterization of log-full subgroups of smooth type). Fix a clean chart \( P \to k \) of \( S^{\log} \) [cf., e.g., [Hsh], Definition B.1, (ii)]. Write \( T^{\log} \) for the log scheme whose underlying scheme is \( \text{Spec}(k[[P]]) \), and whose log structure is defined by the natural inclusion \( P \to \text{Spec}(k[[P]]) \); \( S^{\log} \hookrightarrow T^{\log} \) for the [strict] closed immersion determined by the maximal ideal of \( k[[P]] \). Let \( Y^{\log} \to T^{\log} \) be a generically smooth stable log curve whose base-change via \( S^{\log} \hookrightarrow T^{\log} \) is isomorphic to [hence may be identified with] \( X^{\log} \to S^{\log} \); \( Y_\eta \) the hyperbolic curve determined by the interior of some geometric generic fiber of \( Y^{\log} \to T^{\log} \); \( (Y_\eta)_n \) the \( n \)-th configuration space of \( Y_\eta \); \( A \) a log-full subgroup of \( \Delta_n \). Here, recall that we have a natural specialization isomorphism

\[
\pi_1((Y_\eta)_n)^{(l)} \xrightarrow{\sim} \Delta_n^{(l)}
\]

[cf. the discussion preceding [CmbCusp], Definition 2.1, as well as [CbTpI], Remark 5.6.1]. Then the following conditions are equivalent:

(i) \( A \) is of smooth type.

(ii) For each \( l \in \Sigma \), the image of \( A \) in \( \Delta_n^{(l)} \) coincides with the image, via the above specialization isomorphism, of some log-full subgroup of \( \pi_1((Y_\eta)_n)^{(l)} \).
Proof. Proposition 3.11 follows immediately from Proposition 3.5; [CmbGC], Proposition 1.2, (i), together with the fact that a stable log curve is smooth in a suitable neighborhood of a cusp. □

Remark 3.11.1. Note that it follows immediately from a similar argument to the argument given in the proof of Proposition 3.11 that, in fact, in Definition 3.10, an equivalent definition is obtained if one replaces “$n^*$” by “$n$”.

Corollary 3.12. (Preservation of log-full subgroups of smooth type by FC-admissible outer automorphisms). Let $A \subseteq \Delta_n$ be a log-full subgroup of smooth type, $\sigma$ an automorphism of $\Delta_n$. Set $n_{\text{min}} = 3$ if $(g, r) \neq (0, 3)$; $n_{\text{min}} = 2$ if $(g, r) = (0, 3)$. Then:

(i) If $\sigma$ is FC-admissible [cf. [CmbCusp], Definition 1.1, (ii)], then $\sigma(A) \subseteq \Delta_n$ is a log-full subgroup of smooth type.

(ii) If $n \geq n_{\text{min}}$, then $\sigma(A) \subseteq \Delta_n$ is a log-full subgroup of smooth type [i.e., even if $\sigma$ is not assumed to be FC-admissible!].

Proof. We begin by observing that, by applying the alternative characterization of Proposition 3.11, we may assume without loss of generality that $X^\log$ is a smooth log curve. Then assertion (i) follows immediately from the well-known structure of fibers of $X^\log_n \to X^\log_{n-1}$ over log-full points [cf. [CbTpI], Lemma 5.4, (i), (ii)] — which allows one to interpret log-full subgroups of $\Delta_{n-1}$ as groups “Dehn(−)” of profinite Dehn-multi-twists [cf. [CbTpI], Proposition 5.6, (ii)] — by applying induction on $n$, together with [CbTpI], Theorem 5.14, (iii) [concerning the normalizer of “Dehn(−)”]; [CmbGC], Proposition 2.4, (v) [concerning the graphic fullness of such normalizers]; [CmbGC], Corollary 2.7, (ii) [concerning the graphicity implied by this graphic fullness]. Finally, assertion (ii) follows formally from assertion (i), together with Remark 3.10.1; Corollary 2.6, (i); [CbTpII], Theorem A, (ii). □

Remark 3.12.1. At first glance, it may appear to be unclear whether or not Corollary 3.12, (ii), is applicable to situations such as the situation in the statement of Theorem D of the Introduction, i.e., situations involving two distinct collections of data, labeled by $\circ$ and $\bullet$. In fact, however, Corollary 3.12, (ii), may be applied to such situations, by considering the following argument: The existence of the isomorphism $\alpha$ of Theorem D implies [cf. Theorem 2.5, (vi)] that the type “$(g, r)$” associated to the data labeled by $\square \in \{\circ, \bullet\}$ is independent of the choice of $\square$. In particular, it follows from the [well-known!] connectedness of the moduli stack of hyperbolic curves of type $(g, r)$ that there exists at least one isomorphism of profinite groups

$$\alpha_{\text{mod}} : \Delta_0^\circ \xrightarrow{\sim} \Delta_n^\bullet$$

[i.e., arising from the classical theory of the étale fundamental group!] that maps log-full subgroups to log-full subgroups. On the other hand, the given
isomorphism $\alpha$ differs from $\alpha_{\text{mod}}$ by composition with an automorphism $\alpha_0$ of $\Delta^0_n$. Thus, the fact that $\alpha_0$, hence also $\alpha$, maps log-full subgroups to log-full subgroups follows from a direct application of Corollary 3.12, (ii).

Remark 3.12.2. In some sense, a more “conceptually satisfying” approach [i.e., by comparison to the approach discussed in Remark 3.12.1] to seeing that Corollary 3.12, (ii), is applicable to situations such as the situation in the statement of Theorem D of the Introduction may be obtained simply by observing that the various reconstruction algorithms of [CbTpI], [CbTpII], and [CmbGC] that are applied in the proof of Corollary 3.12, (ii), to show the preservation of log-full subgroups by automorphisms are, in fact, entirely functorial and group-theoretic in nature [i.e., require as input data only the abstract profinite group $\Delta_n$]. We leave the routine details to the interested reader.

Lemma 3.13. (Group-theoretic characterization of edge-like subgroups of smooth type for once-punctured elliptic curves). Suppose that $(g,r) = (1,1)$, and that $X$ is singular. Let $A$ be an edge-like subgroup [cf. Definition 3.1, (i); [CmbGC], Definition 1.1, (ii)] — i.e., a log-full subgroup — of $\Delta_1 \cong \Pi_G$. Then the following conditions are equivalent:

(i) $A$ is a cuspidal edge-like subgroup [cf. Definition 3.1, (i); [CmbGC], Definition 1.1, (ii)] of $\Pi_G$.

(ii) The image of $A$ via the natural surjection $\Pi_G \twoheadrightarrow \Pi_G^{\text{ab}}$ is trivial.

Proof. Lemma 3.13 follows immediately from the following well-known fact: If $A$ is a cuspidal (respectively, nodal) edge-like subgroup of $\Pi_G$, then the natural surjection $\Pi_G \twoheadrightarrow \Pi_G^{\text{ab}}$ maps $A$ to $\{0\}$ (respectively, $A$ isomorphically onto its [nonzero] image).

In the following, we give a brief sketch of a proof of this well-known fact, for the convenience of the reader. First, observe that we may assume without loss of generality that $\Sigma = \{l\}$. Write $p_G : \Pi_G \rightarrow \Pi_G^{\text{ab}}$ for the natural surjection. One verifies easily that $p_G$ maps cuspidal edge-like subgroups of $\Pi_G$ to $\{0\}$ [cf. the specialization isomorphism reviewed in the statement of Proposition 3.11; the well-known structure theory of surface groups]. Thus, to complete the verification of the fact of the above display, it remains to consider the case where $A$ is nodal. In this case, let us verify that $A \cong p_G(A)$. Suppose that $p_G(A)$ is finite, hence trivial [cf. the specialization isomorphism reviewed in the statement of Proposition 3.11; Proposition 2.2, (ii)]. Write $G$ for the underlying semi-graph of $G$. Let $\Pi_v \subseteq \Pi_G$ be a vertical subgroup [cf. [CmbGC], Definition 1.1, (ii)] associated to the unique vertex of $G$. Then it follows immediately from the well-known structure theory of surface groups [i.e., in this case, the maximal pro-$l$ quotient of the étale fundamental group of the affine line!] that $p_G(\Pi_v) = \{0\}$. Thus, we conclude that the rank two free $\mathbb{Z}_l$-module $\Pi_G^{\text{ab}}$ [cf. the specialization isomorphism reviewed in the statement of Proposition 3.11; Proposition 2.2, (ii)] is isomorphic to the
abelianization of the pro-$l$ completion of the topological fundamental group of $G$, i.e., $\mathbb{Z}$ [cf. [CmbGC], Remark 1.1.4], a contradiction.

Proposition 3.14. (Group-theoretic characterization of log-full subgroups of smooth type). Suppose that $(g, r) = (1, 1)$, and that $X$ is singular. Let $A \subseteq \Delta_n$ be a log-full subgroup of $\Delta_n$. Then the following conditions are equivalent:

(i) $A$ is of smooth type [cf. Definition 3.10].

(ii) The image of $A$ via the natural surjection $\Delta_n \twoheadrightarrow \Delta_n^{\text{ab}}$ is trivial.


Corollary 3.15. (Group-theoretic preservation of log-full subgroups of smooth type). Let $l$ be a prime number; $n$ a positive integer; $\Sigma$ a set of prime numbers which is either equal to $\{l\}$ or $\mathfrak{Primes}$. For $\Box \in \{\circ, \bullet\}$, let $S_{\Box}$ be an fs log scheme whose underlying scheme is the spectrum of an algebraically closed field of characteristic $\not\in \Sigma$; $X_{\Box}^{\log}$ a stable log curve of type $(1, 1)$ over $S_{\Box}$ such that $X_{\Box}$ is singular; $(X_{\Box}^{\log})_n$ the $n$-th log configuration space of $X_{\Box}^{\log}$; $\Delta_n^{\Box}$ the maximal pro-$\Sigma$ quotient of the kernel of the natural [outer] surjection $\pi_1((X_{\Box}^{\log})_n) \twoheadrightarrow \pi_1(S_{\Box}^{\log})$; $\alpha : \Delta_n^{\circ} \isom \Delta_n^{\bullet}$ an isomorphism of profinite groups satisfying the following condition:

a closed subgroup $A \subseteq \Delta_n^{\circ}$ of $\Delta_n^{\circ}$ is log-full if and only if $\alpha(A) \subseteq \Delta_n^{\bullet}$ is log-full [cf. Definition 3.4].

Then a log-full subgroup $A \subseteq \Delta_n^{\circ}$ of $\Delta_n^{\circ}$ is of smooth type (respectively, singular type) if and only if $\alpha(A) \subseteq \Delta_n^{\bullet}$ is of smooth type (respectively, singular type) [cf. Definition 3.10].


References


(Yuichiro Hoshi) **RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN**

*E-mail address: yuichiro@kurims.kyoto-u.ac.jp*

(Arata Minamide) **RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN**

*E-mail address: minamide@kurims.kyoto-u.ac.jp*

(Shinichi Mochizuki) **RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN**

*E-mail address: motizuki@kurims.kyoto-u.ac.jp*