

**ON THE ESSENTIAL LOGICAL
STRUCTURE OF INTER-UNIVERSAL
TEICHMÜLLER THEORY I, II, III, IV, V**

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“Travel and Lectures”

Parts I, II, III: Origins of IUT ([IUTchIII] \rightsquigarrow [IUTchII] \rightsquigarrow [IUTchI]!)

- §1. Isogs. of ell. curves and global multipl. subspaces/canon. generators
- §2. Gluings via Teichmüller dilations, inter-universality, and logical \wedge/\vee
- §3. Symmetries/nonsymmetries and coricities of the log-theta-lattice
- §4. Frobenius-like vs. étale-like strs. and Kummer-detachment indets.
- §5. Conjugate synchronization and the str. of $(\Theta^{\pm\text{ell}}\text{NF-})$ Hodge theaters
- §6. Multiradial representation and holomorphic hull

Parts IV, V: Technical and logical subtleties of IUT ([EssLgc], §3)

- §7. RCS-redundancy, Frobenius-like/étale-like strs., and Θ -/log-links
- §8. Chains of gluings/logical \wedge relations
- §9. Poly-morphisms, descent to underlying strs., and inter-universality
- §10. Closed loops via multiradial representations and holomorphic hulls

§1. Isogenies of elliptic curves and global multiplicative subspaces/canonical generators

(cf. [Alien], §2.3, §2.4; [ClsIUT], §1; [EssLgc], §3.2)

- A special case of Faltings' isogeny invariance of the height for elliptic curves

Key assumption.

\exists global multiplicative subspace (GMS)

- First key point of proof:
(invalid for isogenies by non-GMS subspaces!!)

$q \mapsto q^l$ (at primes of bad multiplicative reduction)

... cf. positive characteristic Frobenius morphism!

... \rightsquigarrow "Gaussian" values of theta functions in IUT

... \rightsquigarrow need not only GMS, but also

... global canonical generators (GCG) (cf. §5)!

- Second key point of proof:

$$d\log(q) = \frac{dq}{q} \mapsto l \cdot d\log(q)$$

... yields common (cf. $\wedge!$) container (cf. ampleness of $\omega_E!$)
for both elliptic curves!

... \rightsquigarrow log-link, anabelian geometry in IUT

- One way to summarize IUT:

to generalize the above approach to bounding heights

via theta functions + anabelian geometry

to the case of arbitrary elliptic curves

by somehow "simulating" GMS + GCG!

§2. Gluings via Teichmüller dilations, inter-universality, and logical \wedge/\vee

(cf. [Alien], §2.11; [Alien], §3.3, (ii), (vi), (vii); [Alien], §3.11, (iv); [EssLgc], Examples 2.4.5, 2.4.7, 3.1.1; [EssLgc], §3.3, §3.4, §3.8 §3.11; [ClsIUT], §3)

- Naive approach to generalizing Frobenius aspect “ $q^l \approx q$ ” of §1 — i.e., a situation in which, at the level of arithmetic line bundles, one may act as if there exists a “Frobenius automorphism of the number field” $q \mapsto q^l$ that preserves arithmetic degrees, while at the same time multiplying them by l (!):

for $N \geq 2$ an integer, p a prime number, **glue** via “ $*$ ”

(cf. [Alien], §3.11, (iv); [EssLgc], Example 3.1.1; [EssLgc], §3.4):

$$\mathbb{Z} \ni p^N \leftarrow * : \rightarrow p \in \mathbb{Z} \quad \dots \text{ so } (* \mapsto p^N \in \mathbb{Z}) \wedge (* \mapsto p \in \mathbb{Z})$$

... not compatible with ring structures!! (with addition)

Aut(field) $\&$... but compatible with multiplicative structures, actions of Galois groups as abstract groups!!

... **AND** “ \wedge ” depends on distinct labels!!

... ultimately, we want to delete labels (cf. §1!), but doing so naively yields — if one is to avoid giving rise to a contradiction “ $p^N \neq p$ ”! — a meaningless OR “ \vee ” indeterminacy!!

$$(* \mapsto p^N \in \mathbb{Z}) \vee (* \mapsto p \in \mathbb{Z}) \iff * \mapsto ?? \in \{p, p^N\} \subseteq \mathbb{Z}$$

(cf. “contradiction” asserted by “redundant copies school (RCS)”!)

... in IUT, we would like to delete the labels in a somewhat more “constructive” (!) way!

- In IUT, we consider **gluing** via **Θ -link**, for l a prime number (cf. [Alien], §2.11; [Alien], §3.3, (ii), (vii); [EssLgc], §3.4, §3.8):

$(\Theta^{\pm\text{ell}}\text{NF-})$
Hodge theater,
 i.e., another model of
 conventional scheme
 theory surrounding
 given elliptic curve E

non-scheme-

theoretic
 Θ -link

$(\Theta^{\pm\text{ell}}\text{NF-})$
Hodge theater,
 i.e., model of
 conventional scheme
 theory surrounding
 given elliptic curve E

$\downarrow \text{d.m.}$ (loc. unit gps.: $G_{\underline{v}} \curvearrowright \mathcal{O}_{\underline{v}}^{\times\mu} \mapsto G_{\underline{v}} \curvearrowright \mathcal{O}_{\underline{v}}^{\times\mu}$)

$\downarrow \text{d.m.}$ (loc. val. gps.: $(\{q_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*})^{\mathbb{N}} \mapsto (q_{\underline{v}} \stackrel{\text{def}}{=} q_{\underline{v}}^{1/2l})^{\mathbb{N}}$)

glob. val. gps.: corresponding global realfied Frobenioids (s.t. product formula holds!)

compact mult. str.

- ... where $l \geq 5$ a prime number; $l^* \stackrel{\text{def}}{=} \frac{l-1}{2}$;
- $E (= E_F)$ is an elliptic curve over a number field F s.t. ...;
- $E[l] \subseteq \bar{E}$ subgroup scheme of l -torsion points; $K \stackrel{\text{def}}{=} F(E[l])$;
- j_E is the j -invariant of E , so $F_{\text{mod}} \stackrel{\text{def}}{=} \mathbb{Q}(j_E) \subseteq F$;
- $\underline{V} \subseteq \underline{V}(K)$ collection of valuations of K s.t. ...;
- $q_{\underline{v}}$ denotes local q -parameter of E at (nonarch.) $\underline{v} \in \underline{V}$;
- $G_{\underline{v}}$ denotes the (local) absolute Galois group of $K_{\underline{v}}$ regarded **“inter-universally”** as an **abstract top. group**, i.e., **not** as a (“Galois”) group of **field automorphisms** (cf. **incompatibility with ring structure!**);
- $\mathcal{O}_{\underline{v}}^{\times}$: units of the ring of integers $\mathcal{O}_{\underline{v}}$ of an algebraic closure $K_{\underline{v}}$ of the completion $K_{\underline{v}}$ of K at \underline{v} ;
- $\mathcal{O}_{\underline{v}}^{\times\mu} \stackrel{\text{def}}{=} \mathcal{O}_{\underline{v}}^{\times} / \text{tors} + \text{“integral str.” } \{\text{Im}((\mathcal{O}_{\underline{v}}^{\times})^H)\}_{\text{open } H \subseteq G_{\underline{v}}}$

... note

$\square \quad \boxtimes$
two arithmetic/combinatorial dimensions of ring
 = one dilated dimension + another undilated dimension

... cf. cohomological dimension of absolute Galois groups of number fields and mixed characteristic local fields, topological dimension of \mathbb{C}^{\times} !

cf. §5

v: bad reduction.

\mathbb{Q} -v.s.

“lattice”

fin. free \mathbb{Z} -mod

*$\downarrow G_{\underline{v}}$
cdim. = 2*

$= \mathbb{S}^1 \times \mathbb{R} \Rightarrow$

- Concrete example of gluing
(cf. [EssLgc], Example 2.4.7):

the projective line as a gluing of ring schemes along a multiplicative group scheme

... cf. assertions of the RCS!

- Concrete example of gluing
(cf. [EssLgc], Example 3.3.1; [ClsIUT], §3; [Alien], §2.11):

classical complex Teichmüller deformations
of holomorphic structure

... cf. *two combinatorial/arithmetical dimensions of a ring!!*

... cf. assertions of the RCS!

- In IUT, we consider not just Θ -link, but also the log-link, which is defined, roughly speaking, by considering the

p_v -adic logarithm at each v

(cf. [Alien], §3.3, (ii), (vi), Fig. 3.6; [EssLgc], §3.3, (InfH); [EssLgc], §3.11, (Θ ORInd), (logORInd), (Di/NDi)), where we write p_v for the residue characteristic of (nonarch.) v :

apply same principle as above of label deletion via “saturation with all possibilities” on either side of the link”

... but for Θ -link, this yields *meaningless* (Θ ORInd)!!

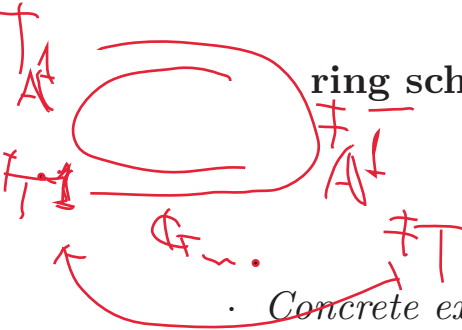
... instead, consider “saturation” (logORInd) for log-link, i.e., but constructing invariants for log-link

... where we recall that

log nondilated unit groups \Leftrightarrow dilated value groups

... i.e., for *invariants*, “nondilated \Leftrightarrow dilated” ... cf. proof of §1!!

TP!



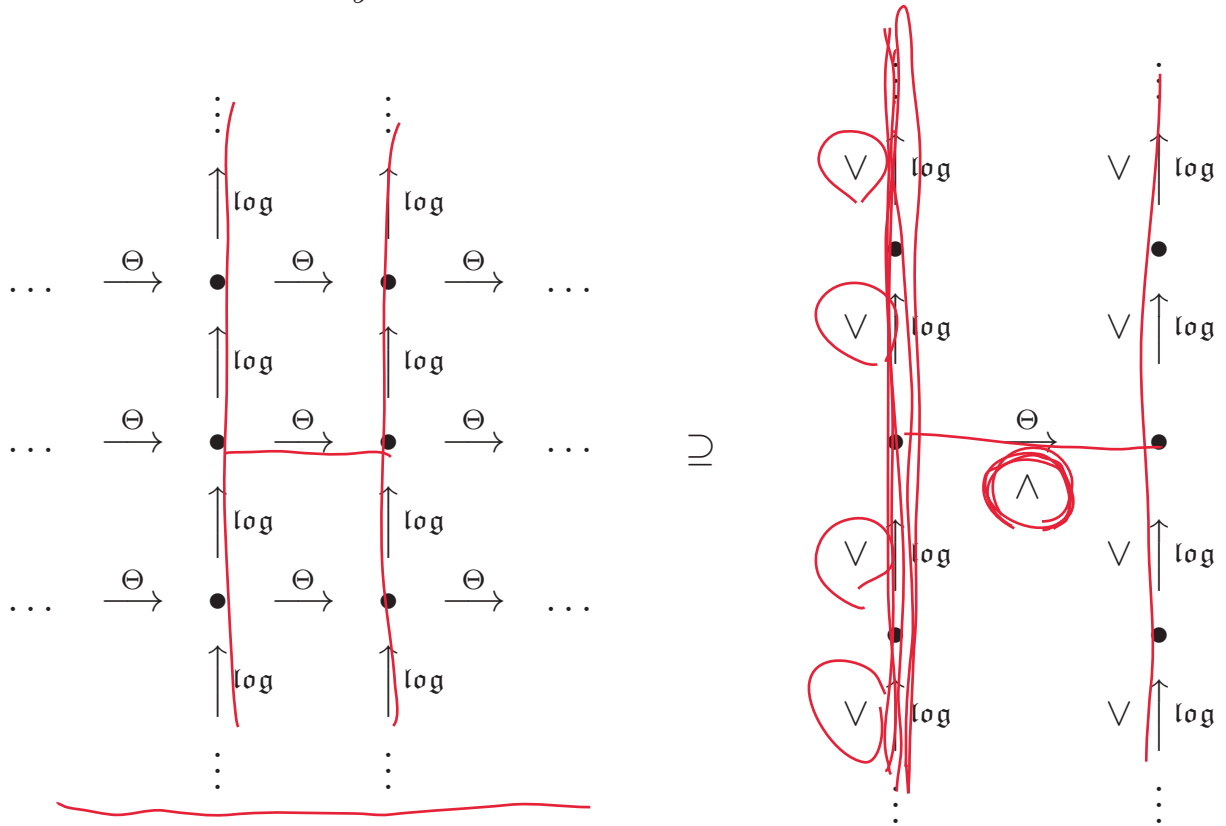
delete labels \Rightarrow

glue/labels
T with
IT

unit
 $\log(1+p) = \dots$
($p \neq 2$)

non-unit

- The entire **log-theta-lattice** and the “**infinite H**” portion that is *actually used*:



$\mathbb{R} = \mathbb{H} \mathbb{T}$

(i.e., not $\begin{matrix} \Theta \\ \rightarrow \\ \downarrow \\ \vee \end{matrix}$!)

§3. Symmetries/nonsymmetries and coricities of the log-theta-lattice

(cf. [Alien], §2.7, §2.8, §2.10, §3.2; [Alien], §3.3, (ii), (vi), (vii); [Alien], §3.6, (i); [EssLgc], §3.2, §3.3; [IUAni2])

- Fundamental Question:
So how do we construct **log-link invariants**?
- Fundamental Observations:
 Θ -link (i.e., “ $q^N \leftarrow q$ ” for some $N \geq 2$) and **log-link** (i.e., “ p -adic logarithm” for some p) clearly satisfy the following:
 - (1) Θ -link, **log-link** are **not compatible** with the **ring structures** in their *domains/codomains*;
 - (2) Θ -link, **log-link** are **not symmetric** with respect to **switching** their *domains/codomains*;
 - (3) **log-link** \circ Θ -link \neq Θ -link \circ **log-link**;
 - (4) **log-link** \circ Θ -link \neq Θ -link
- **Frobenius-like** objects: objects whose definition **depends, a priori**, on the *coordinate* “ $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ ” of the $(\Theta^{\pm\text{ell}}NF\text{-})Hodge$ theater at which they are defined (e.g., *rings, monoids*, etc. that do **not** map **isomorphically** via Θ -link, **log-link**)
- **Étale-like** objects: arise from *arithmetic (étale) fund. groups* regarded as *abstract topological gps.* ... cf. **inter-universality!**
 \implies **mono-anabelian absolute anabelian geometry** may be applied (cf. *ampleness* of ω_E in §1!)
 e.g.: inside each $(\Theta^{\pm\text{ell}}NF\text{-})Hodge$ theater “ \bullet ”, at each \underline{v} , \exists a copy of the *arithmetic/tempered fundamental group*

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$$

of a certain finite étale covering of the *once-punctured elliptic curve* $X_{\underline{v}} \stackrel{\text{def}}{=} E_{\underline{v}} \setminus \{\text{origin}\}$ (where $E_{\underline{v}} \stackrel{\text{def}}{=} E \times_F K_{\underline{v}}$)

- **Étale-like** objects satisfy crucial **coricity**
(i.e., “**common** — cf. $\wedge!$ — to the domain/codomain”)
- each **log-link** induces **indeterminate** (cf. **inter-universality!**)
isomorphisms

$$\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{v}}$$

— cf. the evident *Galois-equivariance* of the (power series defining the) *p-adic logarithm!* — between copies in domain/codomain of the **log-link**

- each **Θ -link** induces **indeterminate** (cf. **inter-universality!**)
isomorphisms

$$G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}$$

— i.e., “(Ind1)” — between copies in domain/codomain of the **Θ -link**

(so **abstract top. gps.** $\Pi_{\underline{v}}, G_{\underline{v}}$ are **coric** for **log-, Θ -links!**) and **symmetry** properties:

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \uparrow \text{log} & & \uparrow \text{log} \\
 \bullet & \xrightarrow{\Theta} & \bullet \\
 \Pi_{\underline{v}} \twoheadrightarrow & G_{\underline{v}} & \leftarrow \Pi_{\underline{v}} \\
 & \circlearrowleft & \\
 & \text{Aut}(G_{\underline{v}}) & \\
 \uparrow \text{log} & & \uparrow \text{log} \\
 \vdots & & \vdots
 \end{array}
 \quad \dots \text{ symmetric w.r.t.}$$

dom./codom.
of Θ -link!

- Thus, in summary,
with regard to the desired **symmetry** and **coricity** properties:

Frobenius-like	FALSE	FALSE
étale-like	TRUE	TRUE

§4. **Frobenius-like vs. étale-like structures and Kummer-detachment indeterminacies**

(cf. [Alien], Examples 2.12.1, 2.12.3, 2.13.1; [Alien], §3.4; [Alien], §3.6, (ii), (iv); [Alien], §3.7, (i), (ii))

- **Kummer theory** yields *isoms.* between corresponding objects:

Frobenius-like objects $\xrightarrow{\sim}$ (mono-anabelian) étale-like objects

... but gives rise to **Kummer-detachment indeterminacies**, i.e., *one must pay some sort of price* for passing from

Frobenius-like objects that do not satisfy *coricity/symmetry* properties to *étale-like objects* that do satisfy *coricity/symmetry* properties

- In IUT, there are *three types* of *Kummer theory*:
 - (a) for **local units** $\mathcal{O}_{\bar{v}}^{\times}$: classical Kummer theory via **local class field theory (LCFT)/Brauer groups** (cf. [Alien], Example 2.12.1);
 - (b) for **local theta values** $\{q_{\bar{v}}^{j^2}\}_{j=1,\dots,l^*}$: Kummer theory via **theta functions** and **Galois evaluation** at ***l*-torsion points** (cf. [Alien], §3.4, (iii), (iv));
 - (c) for **global field of moduli** F_{mod} : Kummer theory via **“ κ -coric” algebraic rational functions** (essentially, non-linear polynomials w.r.t. some “point at infinity”) and **Galois evaluation** at points defined over **number fields** (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))

- In general, “*Kummer theory*” proceeds by:

$$\left(\begin{array}{l} \text{extracting} \\ n\text{-th roots,} \\ \text{for } n \in \mathbb{Z}_{>0}, \text{ of} \\ \text{some element} \\ f \in \text{a multipl.} \\ \text{monoid } M \end{array} \right) \rightsquigarrow \left(\begin{array}{l} \text{Kummer class } \kappa_f \\ \in H^1 \left(\text{some “Gal. group” } \Pi, \mu_n(M) \right) \end{array} \right)$$

... where $\mu_n(M)$ denotes *n*-torsion — i.e., *roots of unity!* — of *M*;
 \rightsquigarrow “ $\widehat{\mathbb{Z}}$ version” by taking \varprojlim_n

- Main Substantive Issue: *eliminating* potential $\widehat{\mathbb{Z}}^\times$ -**indeterminacy** from the conventional **cyclotomic rigidity isomorphism (CRI)**

$$(\widehat{\mathbb{Z}} \cong) \quad \mu_{\widehat{\mathbb{Z}}}(M) \quad \xrightarrow{\sim} \quad \mu_{\widehat{\mathbb{Z}}}(\Pi) \quad (\cong \widehat{\mathbb{Z}})$$

arising from scheme theory (cf. [Alien], §3.4, (i), (ii), (iii), (iv))

... note that this is a *very substantive issue!* indeed,

indeterminate $\widehat{\mathbb{Z}}^\times$ -**multiples/powers** of divs., line blds.,
rational/merom. fns., elts. of number fields/local fields

completely destroy any notion of **positivity/inequalities**
(recall that -1 lies in the closure of the natural numbers in $\widehat{\mathbb{Z}}$!)
for **arithmetic degrees/heights**;

moreover, **inter-universality** — i.e., the property of “**not being anchored to/rigidified by any particular ring/scheme theory**” — means that the $\mathcal{O}_{\widehat{\mathbb{Z}}^\times}^{\times\mu}$ in the Θ -link (cf. §2) is subject to an *unavoidable* $\widehat{\mathbb{Z}}^\times$ -*indeterminacy* “(Ind2)”

$$\widehat{\mathbb{Z}}^\times \quad \curvearrowright \quad \mathcal{O}_{\widehat{\mathbb{Z}}^\times}^{\times\mu}$$

... we shall refer to the **compatibility/incompatibility** — i.e., the **functorial equivariance/nonfunctoriality** — of a given Kummer theory with the “*inter-universality indeterminacies*” (Ind1), (Ind2) as the **multiradiality/uniradiality** of the Kummer theory; thus, the *multiradiality* of the Kummer theory may be understood as a sort of “**splitting/decoupling**” of the Kummer theory from the **unit group** $\mathcal{O}_{\widehat{\mathbb{Z}}^\times}^{\times\mu}$

- Another Substantive Issue for Cyclotomic Rigidity Isomorphisms: **compatibility** with the **profinite/tempered topology**, i.e., the property of admitting *finitely truncated versions*

$$(\mathbb{Z}/n\mathbb{Z} \cong) \quad \mu_n(M) \quad \xrightarrow{\sim} \quad \mu_n(\Pi) \quad (\cong \mathbb{Z}/n\mathbb{Z})$$

... this will be important (cf. [Alien], §3.6, (ii)) since **ring strs.** — which are necessary in order to define the *power series* for the *p-adic logarithm* (cf. **log-link!**) — only exist at “*finite n*”, i.e.,

infinite “*multiplicative Kummer towers* \varprojlim_n ” *destroy additive strs.!*

- In the case of the *three types* (a), (b), (c) of *Kummer theory* that are *actually used* in IUT (cf., especially, [Alien], Fig. 3.10; [Alien], §3.4, (v)):
 - (a) this approach to constructing CRI's is manifestly **compatible** with the **profinite topology**, but is **uniradial** since it depends in an essential way on the *extension of Galois modules* $1 \rightarrow \mathcal{O}_{\underline{v}}^\times \rightarrow K^\times \rightarrow \mathbb{Q} \rightarrow 1$, hence is *fundamentally incompatible* with *indeterminacies* $\widehat{\mathbb{Z}}^\times \curvearrowright \mathcal{O}_{\underline{v}}^\times \twoheadrightarrow \mathcal{O}_{\underline{v}}^{\times\mu}$ (cf. [Alien], §3.4, (i));
 - (b) it follows from the theory of the **étale theta function** — in particular, the symmetries of **theta groups**, together with the **canonical splittings** arising from restriction to 2- (or, alternatively, 6-) torsion points — that this approach to constructing CRI's is both **compatible** with the **profinite/tempered topology** and **multiradial** (cf. [Alien], §3.4, (iii), (iv));
 - (c) it follows from elementary considerations concerning “ **κ -coric**” **algebraic rational functions** that this approach to constructing CRI's is **multiradial**, but **incompatible** with the **profinite topology** (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))
- The *indeterminacies* $\widehat{\mathbb{Z}}^\times \curvearrowright \mathcal{O}_{\underline{v}}^\times \twoheadrightarrow \mathcal{O}_{\underline{v}}^{\times\mu}$ of (a) mean that the **theta values and elts.** $\in F_{\text{mod}}$ obtained by **Galois evaluation**

$$\left(\begin{array}{c} \text{Kummer class of some} \\ \text{sort of function} \end{array} \right) \Big|_{\text{decomposition group of a point}}$$

in (b), (c) are *only meaningful* — i.e., *can only be protected* from the $\widehat{\mathbb{Z}}^\times$ -*indeterminacies* — if they are considered, by applying the “**non-interference**” (up to roots of unity) of the monoids of (a) with those of (b) and (c), in terms of their actions on **log-shells**

$$\{\underline{q}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \stackrel{\text{def}}{=} \frac{1}{2p_{\underline{v}}} \log_{p_{\underline{v}}}(\mathcal{O}_{\underline{v}}^{\times\mu}) \curvearrowright F_{\text{mod}}^\times$$

... whose definition requires one to apply the $p_{\underline{v}}$ -*adic logarithm*, i.e., the **log-link** *vertically shifted* by -1 , relative to the coordin. “ (n, m) ” of the $(\Theta^{\pm\text{ell}}\text{NF-})$ Hodge theater that gave rise to the *theta values* and *elements* $\in F_{\text{mod}}$ under consideration (cf. [Alien], §3.7, (i)).

- Here, we recall that only the **multiplicative monoid** $\mathcal{O}_v^{\times\mu}$ — i.e., *not* the *ring structures*, **log-link**, etc.! — is **accessible**, via the **common data** (cf. “ $\wedge!$ ”) in the gluing of the Θ -link, to the *opposite side* (i.e., domain/codomain) of the Θ -link!

Thus, to overcome the **vertical log-shift** discussed above, it is necessary to construct **invariants** w.r.t. the **log-link** (cf. §2!).

Here, we recall that **étale-like structures** “ \circ ” — such as “ $\Pi_{\underline{v}}$ ” — are indeed **log-link-invariant**, but the diagram — called the **log-Kummer correspondence** — arising from the *vertical column* (written *horizontally*, for convenience) in the *domain* of the Θ -link

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \dots \\
 & & \dots & \searrow & \downarrow & \swarrow & & & \dots \\
 & & & & \circ & & & &
 \end{array}$$

— where the vertical/diagonal arrows in the diagram are **Kummer isomorphisms** — is **not commutative!**

On the other hand, it is **upper semi-commutative** (!), i.e., all composites of **Kummer** and **log-link** morphisms on $\mathcal{O}_{\underline{v}}^{\times}$

$$\mathcal{O}_{\underline{v}}^{\times} \hookrightarrow \mathcal{O}_{\underline{v}} \hookrightarrow \mathcal{I}_{\underline{v}} \leftrightarrow \log_{p_{\underline{v}}}(\mathcal{O}_{\underline{v}}^{\times\mu})$$

have images contained in the **log-shell** $\mathcal{I}_{\underline{v}}$ (cf. [Alien], Example 2.12.3, (iv)). This *very rough* variant of “commutativity” may be thought of as a type of **indeterminacy**, which is called “(Ind3)”. It is (Ind3) that gives rise, ultimately, to the *upper bound* in the **height inequalities** that are obtained in IUT (cf. [Alien], Example 2.12.3, (iv); [Alien], §3.6, (iv); [Alien], §3.7, (i), (ii)).

- Thus, in summary, we have two **Kummer-indeterminacies**, namely,

$$(\text{Ind2}), (\text{Ind3}).$$

§5. Conjugate synchronization and the structure of $(\Theta^{\pm\text{ell}}\mathbf{NF-})$
Hodge theaters

(cf. [Alien], §3.3, (ii), (iv), (v); [Alien], §3.4, (ii), (iii); [Alien], §3.6, (i), (ii), (iii); [AbsTopIII], §1; [EssLgc], §3.3; [EssLgc], Examples 3.3.2, 3.8.2; [ClsIUT], §3, §4; [IUTchI], Fig. I1.2)

· Fundamental Question:

So **how** do we “simulate” **GMS + GCG?**

· In a word, we consider certain *finite étale coverings* over $K = F(E[l])$ of the *hyperbolic orbicurves*

$$X \stackrel{\text{def}}{=} E \setminus \{\text{origin}\}, \quad C \stackrel{\text{def}}{=} X // \{\pm 1\}$$

determined by some *rank one quotient* $E[l]_K \twoheadrightarrow Q$:

$$\underline{X}_K \rightarrow X_K \stackrel{\text{def}}{=} X \times_F K \quad \dots \text{ determined by } E[l]_K \twoheadrightarrow Q$$

$$\underline{C}_K \rightarrow C_K \stackrel{\text{def}}{=} C \times_F K \quad \dots \text{ by taking } C_K \stackrel{\text{def}}{=} X_K // \{\pm 1\}$$

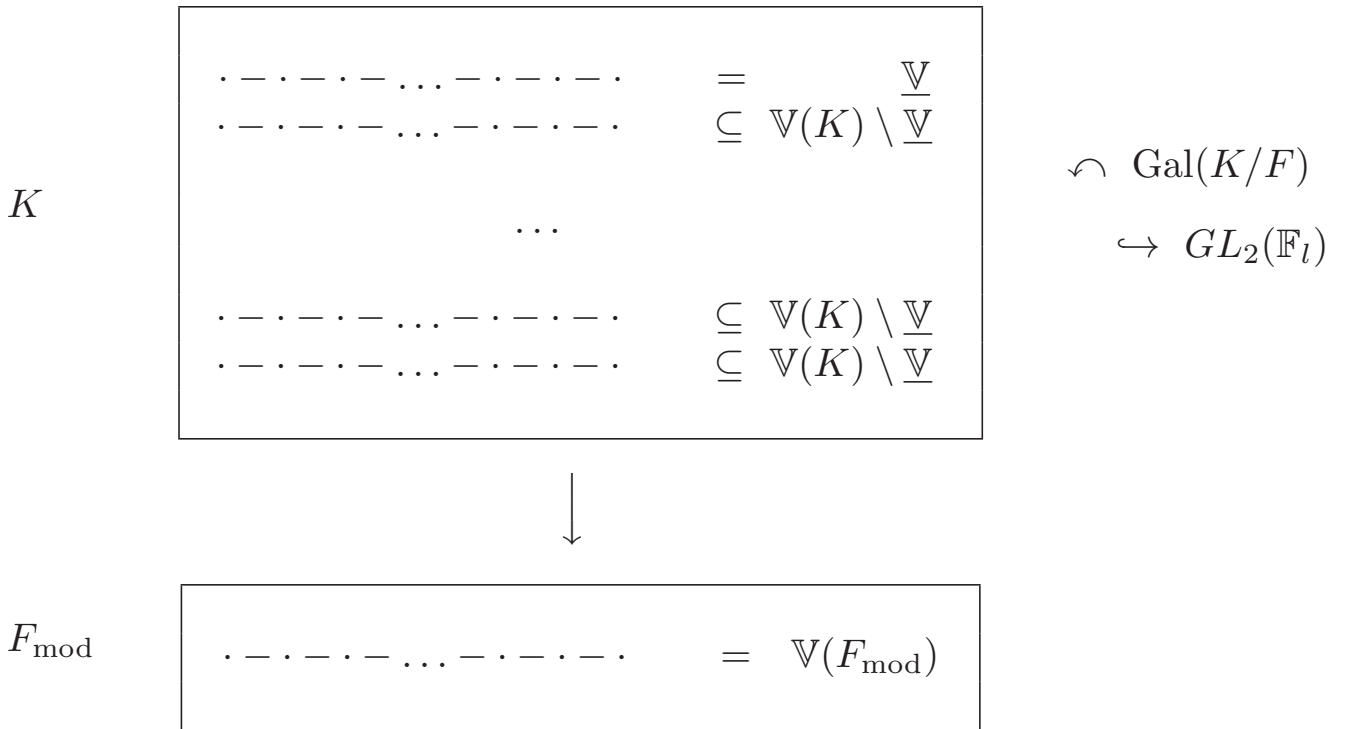
... where “//” denotes the “*stack-theoretic quotient*”

and restrict to “**local analytic sections**” of $\text{Spec}(K) \rightarrow \text{Spec}(F)$ — called “**prime-strips**” (of which there are *various types*, as summarized in [IUTchI], Fig. I1.2), which may be thought of as a sort of *monoid-* or *Galois-theoretic* version of the classical notion of *adèles/idèles* — determined by various $\text{Gal}(K/F)$ -orbits of the *subset/section*

$$\mathbb{V}(K) \supseteq \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\text{mod}}$$

where the *quotient* $E[l]_K \twoheadrightarrow Q$ is indeed the “**multipl. subspace**”, or where some *generator, up to ± 1 , of Q* is indeed the “**canonical generator**”.

Working with such prime-strips means that many conventional objects associated to number fields — such as **absolute global Galois groups** or **prime decomposition trees** — much be *abandoned!* Indeed, this was precisely the *original motivation* (around 2005 - 2006) for the development of the ***p*-adic absolute mono-abelian geometry** of [AbsTopIII], §1 [cf. [Alien], §3.3, (iv)]!



- The hyperbolic orbicurves $\underline{X}_K, \underline{C}_K$ admit **symmetries**

$$\mathbb{F}_l^{\times \pm} \stackrel{\text{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\} \hookrightarrow \text{Aut}_K(\underline{X}_K) \subseteq \text{Aut}(\underline{X}_K)$$

... **additive/geometric!**

$$\text{Aut}(\underline{C}_K) \hookrightarrow \text{Gal}(K/F) \twoheadrightarrow \mathbb{F}_l^* \stackrel{\text{def}}{=} \mathbb{F}_l / \{\pm 1\}$$

... **multiplicative/arithmetic!**

obtained by considering the respective actions on cusps of $\underline{X}_K, \underline{C}_K$ that arise from elements of the *quotient* $E[l]_K \twoheadrightarrow Q$ [cf. [Alien], §3.3, (v); [Alien], §3.6, (i)]. At the level of *arithmetic fundamental groups*, these symmetries may be thought of as **finite groups of outer automorphisms** of

$$\Pi_{\underline{X}_K}, \quad \Pi_{\underline{C}_K}$$

— where we note that since, as is well-known, both the **geometric fundamental group** $\Delta_{\underline{X}_K}$ and the **global absolute Galois group** G_K are *slim* and do *not* admit *finite subgroups of order* > 2 , these finite groups of outer automorphisms *do not lift to finite groups of (non-outer) automorphisms* (cf. [EssLgc], Example 3.8.2)!

Here, we note that since it is of *crucial importance* to **fix** the quotient $E[l]_K \rightarrow Q$, we want to *start from* \underline{C}_K and *descend*, via the *multiplic. \mathbb{F}_l^* -symmetries*, to $C_{F_{\text{mod}}}$ (where $C_{F_{\text{mod}}} \times_{F_{\text{mod}}} F = C$), **not** the other way around, which would obligate us consider *all Galois*, hence, in particular, *all $SL_2(\mathbb{F}_l)$ -conjugates* of Q . Note that this is precisely the **reverse** (!) order to proceed from the point of view of *classical Galois theory* (cf. [Alien], §3.6, (iii); [EssLgc], Example 3.8.2).

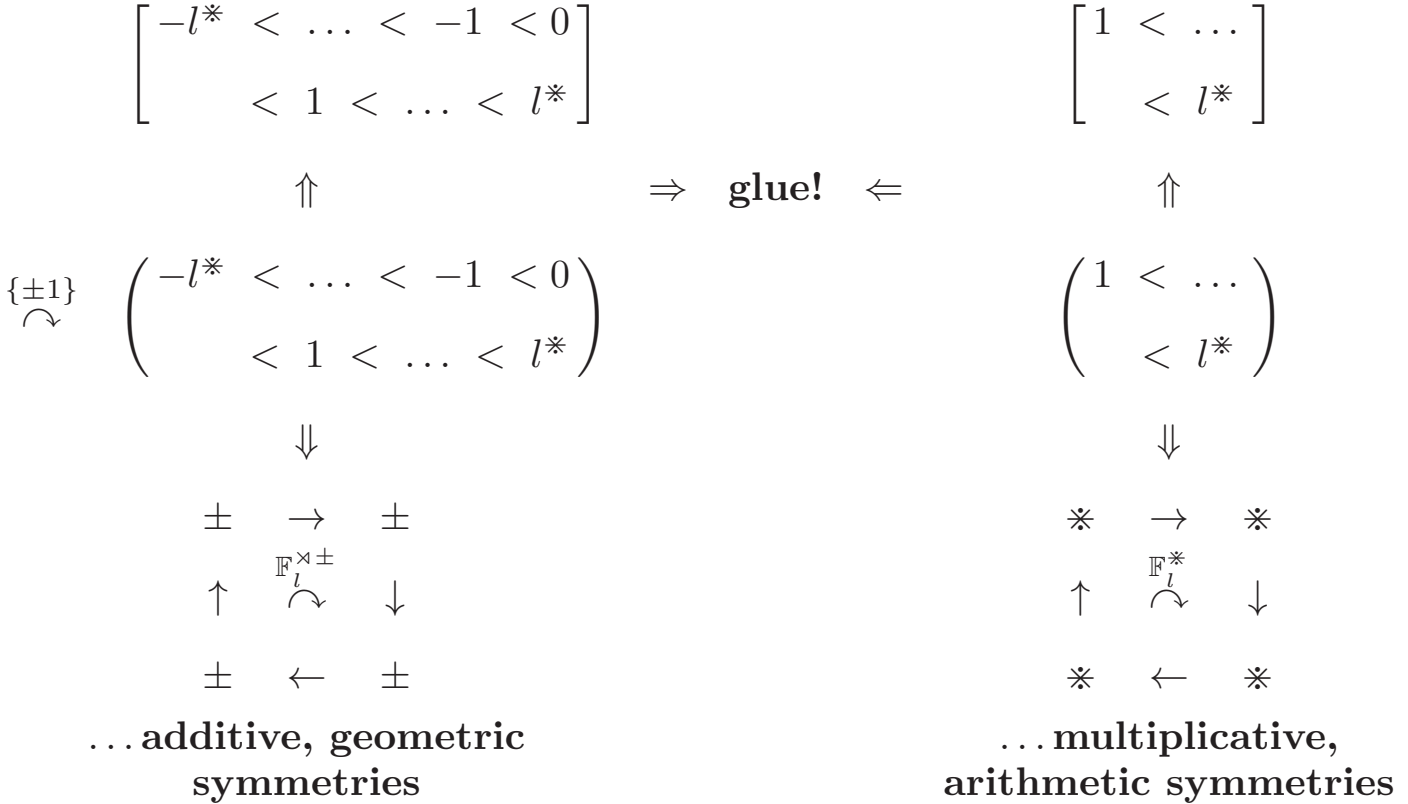
In particular, the “*strictly outer*” nature of the **multiplicative/arith-metic \mathbb{F}_l^* -symmetries** means that various copies of the absolute local Galois groups “ $G_{\underline{v}}$ ” (for, say, nonarch. $\underline{v} \in \underline{\mathbb{V}}$) in the prime-strips that are permuted by these symmetries can only be identified with one another **up to indeterminate inner automorphisms**, i.e., there is *no way to synchronize these conjugate indeterminacies* (cf. [Alien], §3.6, (iii); [EssLgc], Example 3.8.2).

On the other hand, the “ $G_{\underline{v}} \curvearrowright \mathcal{O}_{\underline{v}}^{\times \mu}$ ” that appears in the *gluing data* for the Θ -link (cf. §2) must be **independent** of the “ $j \in \mathbb{F}_l^*$ ” (cf. the “ q^{j^2} ” of §2, where we think of this “ j ” as the smallest integer lifting $\bar{j} \in \mathbb{F}_l^*$). That is to say, we need a “**conjugate synchronized**” $G_{\underline{v}}$ in order to construct the Θ -link, i.e., ultimately, in order to *express the LHS of the Θ -link in terms of the RHS!* This is done by applying the **additive/geometric $\mathbb{F}_l^{\times \pm}$ -symmetries** (cf. [Alien], §3.6, (ii); [EssLgc], Example 3.8.2).

Moreover, these *additive/geometric $\mathbb{F}_l^{\times \pm}$ -symmetries* are **compatible**, relative to the **log-link**, with the *crucial local CRI’s (a), (b)* (but not (c)!) of §4, *precisely* because these local CRI’s (a), (b) are *compatible with the profinite/tempered topology*, which means that they may be computed at a **finite truncated level**, where the **ring structure**, hence also the *power series* for the *p -adic logarithm*, is *well-defined* (cf. [Alien], §3.6, (ii)).

Here, we recall that this *crucial property of compatibility with the profinite/tempered topology* in the case of (b), as opposed to (c), may be understood as a consequence of the fact that the **orders of the zeroes/poles at cusps** of the **theta function** are all equal to 1! Moreover, this phenomenon may in turn be understood as a consequence of the **symmetries of theta groups**, or, alternatively, as a consequence of the **quadratic form/first Chern class “ \square^2 ”** in the exponent of the *classical series representation of the theta function* (cf. [Alien], §3.4, (iii), as well as the discussion below).

By contrast, in the case of (c), the orders of the zeroes/poles at cusps of the **algebraic rational functions** that are used differ from one another by much more arbitrary elements of \mathbb{Q}^\times (cf. [Alien], §3.4, (ii))!



- The properties of **theta functions** in IUT discussed above are *particularly remarkable* when viewed from the point of view of the analogy with the **Jacobi identity** for the **theta function** on the *upper half-plane* (cf. [EssLgc], Example 3.3.2; [ClsIUT], §4). Indeed, on the one hand, the **quadratic form/first Chern class** “ \square^2 ” in the exponent of the *classical series representation of the theta function* (on the imaginary axis of the upper half-plane)

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

gives rise to the **theta group symmetries** that underlie the **rigidity properties** of theta functions that play a *central role* in IUT from the point of view of the ultimate goal in IUT of **expressing the LHS of the Θ -link in terms of the RHS** — i.e., *expressing the “ Θ -pilot” on the LHS of the Θ -link in terms of the “ q -pilot” on the RHS of the Θ -link.*

On the other hand, this **same quadratic form** in the exponent of the classical series representation of the theta function — which in fact appears as “ $t \cdot \square^2$ ”, i.e., with a factor t , where t denotes the standard coordinate on the imaginary axis of the upper half-plane — also underlies the well-known **Fourier transform invariance** of the **Gaussian distribution**, up to a sort of “**rescaling**”

$$t \cdot \square^2 \quad \mapsto \quad t^{-1} \cdot \square^2.$$

It is precisely this rescaling that gives rise to the *Jacobi identity*.

This state of affairs is *remarkable* (cf. [ClsIUT], §3, §4) in that the transformation $t \mapsto t^{-1}$ corresponds to the linear fractional transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which, from the point of view of the analogy between the “**infinite H**” discussed at the end of §2 and the well-known *bijection*

$$\begin{aligned} \mathbb{C}^\times \backslash GL_2^+(\mathbb{R}) / \mathbb{C}^\times &\quad \xrightarrow{\sim} \quad [0, 1) \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} &\quad \mapsto \quad \frac{\lambda-1}{\lambda+1} \end{aligned}$$

(where $\lambda \in \mathbb{R}_{\geq 1}$), may be understood as follows:

$$\begin{aligned} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} &\quad \longleftrightarrow \quad \Theta\text{-link} \quad \dots \text{ cf. “not } \Theta\text{-link-invariants”!} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &\quad \longleftrightarrow \quad \log\text{-link} \quad \dots \text{ cf. “log-link-invariants”!} \end{aligned}$$

(cf. [Alien], §3.3, (ii); [EssLgc], §3.3, (InfH), Example 3.3.2).

• Concluding Question:

So **why** do we need to “**simulate**” **GMS + GCG**?

... in order to secure the **l -torsion points** at which one conducts the **Galois evaluation** of the (reciprocal of the l -th root of the) **étale theta function**, i.e., the *Kummer class* of the *p -adic theta function* (cf. the discussion of the **Θ -link** in §2; §4, (b))

$$\underline{\underline{\Theta}}|_{l\text{-torsion points}} = \{\underline{\underline{q}}^{j^2}\}_{j=1, \dots, l^*}$$

... cf. the *classical series representation of the theta function* on the (imag. axis of the) upper half-plane — i.e., in essence, “ $q = e^{2\pi i(it)}$ ”!

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t} = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}$$

§6. Multiradial representation and holomorphic hull

(cf. [Alien], §3.6, (iv), (v); [Alien], §3.7, (i), (ii); [EssLgc], §3.6, §3.10, §3.11; [ClsIUT], §2; [IUAni1])

· Fundamental Theme:

To *express/describe* the Θ -**pilot** on the LHS of the Θ -**link** in terms of the RHS of the Θ -link, while keeping the Θ -link itself **fixed** (!)

- For instance, the labels “ j ” in “ $\{\underline{q}^{j^2}\}_{j=1,\dots,l^*}$ ” depend on the complicated **bookkeeping system** for these essen'tly **cuspidal labels** (i.e., labels of cuspidal inertia groups in the *geometric fundamental groups* $\Delta_{\underline{v}} \stackrel{\text{def}}{=} \text{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$) furnished (cf. §5) by the structure of the $(\Theta^{\pm\text{ell}}NF\text{-})$ Hodge theater on the LHS, which is **not accessible** from the point of view of the RHS. Thus, it is necessary to express these labels in a way that *is* accessible from the RHS, i.e., by means of **processions of capsules of prime-strips** “/”

$$/ \hookrightarrow // \hookrightarrow /// \hookrightarrow \dots \hookrightarrow / \dots /$$

(i.e., successive inclusions of *unordered* collections of prime-strips of incrementally increasing cardinality) — which still yield **symmetries** between the prime-strips at different labels without “**label-crushing**”, i.e., identifications between distinct labels (cf. [Alien], §3.6, (v)). We then consider the *actions* of (b), (c) (cf. §4) on **tensor-packets** of the *log-shells* arising from the data of (a) (cf. §4) inside each capsule:

$$\{\underline{q}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \curvearrowleft (F_{\text{mod}}^\times)_j$$

— where the “*tensor-packet*” is a tensor product of $j + 1$ copies of $\mathcal{I}_{\underline{v}}$.

- In fact, the various monoids, Galois groups, etc. that appear in the data (a), (b), (c) of §4 — such as $\mathcal{I}_{\underline{v}}$, $\{\underline{q}^{j^2}\}_{j=1,\dots,l^*}$, $(F_{\text{mod}}^\times)_J$ — come in **four types** (cf. [Alien], §3.6, (iv); [Alien], §3.7, (i)):

holomorphic Frobenius-like “ (n, m) ”: monoids etc. on which $\Pi_{\underline{v}} \curvearrowright$ acts, and whose construction involves the **ring structure** associated to the $(\Theta^{\pm\text{ell}}\text{NF-})$ Hodge theater at $(n, m) \in \mathbb{Z} \times \mathbb{Z}$;

holomorphic étale-like “ (n, \circ) ”: similar data to (n, m) , but reconstructed from $\Pi_{\underline{v}}$, hence **independent** of “ m ”;

mono-analytic Frobenius-like “ $(n, m)^{\dagger}$ ”: monoids, etc., on which $G_{\underline{v}} \curvearrowright$ acts; used in the **gluing data** — called an $\mathcal{F}^{\dagger} \blacktriangleright^{\times\mu}$ -**prime-strip** — that appears in the Θ -**link**;

mono-analytic étale-like “ $(n, \circ)^{\dagger}$ ”: similar data to $(n, m)^{\dagger}$, but reconstructed from $G_{\underline{v}}$, hence **independent** of “ m ” (and in fact also of “ n ”).

- Thus, in summary, the **log-Kummer** correspondence yields actions of the monoids of (b), (c) (cf. §4) on tensor-packets of log-shells arising from the data of (a) (cf. §4) up to the indeterminacy (Ind3)

$$\{\underline{q}^{j^2}\}_{j=1, \dots, l^*} \curvearrowright \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \curvearrowright (F_{\text{mod}}^{\times})_J$$

- *first*, at the level of objects of $(0, \circ)$;
- then by “**descent**” (i.e., the observation that reconstructions from *certain input data* may in fact be conducted, up to natural isom., from *less/weaker input data*) up to indeterminacies (Ind1) at the level of objects of $(0, \circ)^{\dagger}$;
- then again by “**descent**” up to indeterminacies (Ind2) at the level of objects of $(0, 0)^{\dagger} \xrightarrow{\sim} (1, 0)^{\dagger}$ (via the Θ -link).

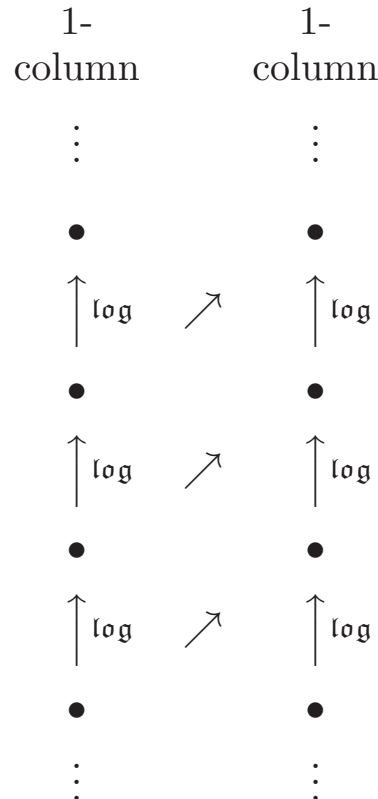
$$(0, 0) \xrightarrow{\text{(Ind3)} \rightsquigarrow} (0, \circ) \xrightarrow{\text{(Ind1)} \rightsquigarrow} (0, \circ)^{\dagger} \xrightarrow{\text{(Ind2)} \rightsquigarrow} (0, 0)^{\dagger} \xrightarrow{\Theta\text{-link} \rightsquigarrow} (1, 0)^{\dagger}$$

(This last step involving (Ind2) plays the role of **fixing** the vertical coordinate, so that (Ind1), (Ind2) are **not mixed** with (Ind3) — cf. the discussion of “ $\mathbb{C}^{\times} \backslash GL_2^+(\mathbb{R}) / \mathbb{C}^{\times}$ ” at the end of §5!)

This is the **multiradial representation of the Θ -pilot** on the LHS of the Θ -link in terms of the RHS (cf. [Alien], §3.7, (i); [EssLgc], §3.10, §3.11). This multiradial representation plays the important role of **exhibiting** the (value group portion of the) Θ -pilot at $(0, 0)$ (i.e., which appears in the Θ -link!) as **one of the possibilities** within a **container** arising from the **RHS** of the Θ -link (cf. the “*infinite H*” at the end of §2; [EssLgc], §3.6, §3.10).

Next, by applying the operation of forming the **holomorphic hull** (i.e., “ \mathcal{O}_v -module generated by””) to the various *output regions* of the multiradial representation, we obtain a module over the local \mathcal{O}_v ’s on the RHS of the Θ -link. Then taking a suitable **root** of “**det**(–)” of this module yields an **arithmetic line bundle** in the *same category* as the category that gives rise to the **q-pilot** on the RHS of the Θ -link — *except for a vertical log-shift* by 1 in the 1-column (cf. the construction of *log-shells* from the “ $\mathcal{O}_v^{\times\mu}$ ”s” that appear in the *gluing data* of the Θ -link!) — cf. [EssLgc], §3.10.

Thus, by **symmetrizing** (i.e., with respect to vertical shifts in the 1-column) the procedure described thus far, we obtain a **closed loop**, i.e.,



a situation in which the **distinct labels** on either side of the Θ -link (cf. the discussion at the beginning of §2!) may be **eliminated**, up to *suitable indeterminacies* (i.e., (Ind1), (Ind2), (Ind3); the holomorphic hull). In particular, by performing an entirely elementary **log-volume** computation, one obtains a **nontrivial height inequality**. This completes the proof of the *main theorems* of IUT (cf. [Alien], §3.7, (ii); [EssLgc], §3.10, §3.11).

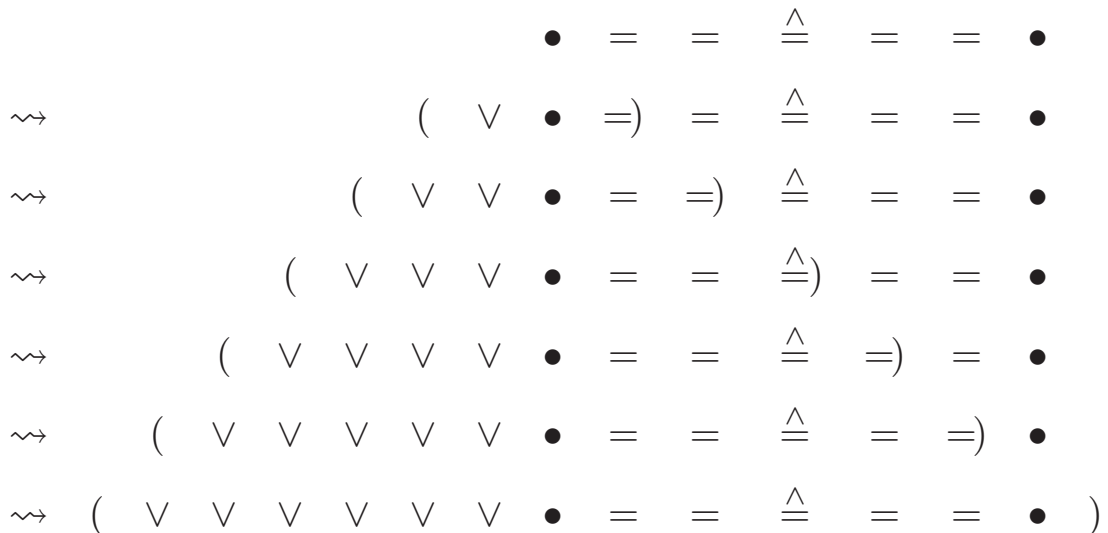
Here, it is important to note that although the term “closed loop” at first might seem to suggest issues of “**diagram commutativity**” or “**log-volume compatibility**” — i.e., issues of

*“How does one conclude a relationship between the **output data** and the **input data** of the **closed loop**?”*

— in fact, such issues **simply do not exist** in this situation! That is to say, the *essential logical structure* of the situation

$$\begin{aligned}
 A \wedge B &= A \wedge (B_1 \vee B_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots \vee B''_1 \vee B''_2 \vee \dots) \\
 &\quad \vdots
 \end{aligned}$$

proceeds by **fixing** the **logical AND** “ \wedge ” relation satisfied by the Θ -link and then adding various **logical OR** “ \vee ” **indeterminacies**, as illustrated in the following diagram (cf. [EssLgc], §3.10):



§7. RCS-redundancy, Frobenius-like/étale-like strs., and Θ -/log-links

(cf. [Alien], §3.3, (ii); [EssLgc], Example 2.4.7; [EssLgc], §3.1, §3.2, §3.3, §3.4, §3.8, §3.11)

- RCS (“redundant copies school”) model of IUT
(i.e., “RCS-IUT” — cf. [EssLgc], §3.1):

This model ignores the various **crucial intertwining**s of two dims. in IUT (such as *addition/multiplication, local unit groups/value groups, Θ -link/log-link*, etc.).

Instead one works relative to a **single rigidified ring structure** by implementing, as described below, various “**RCS-identifications**” of “**RCS-redundant**” copies of objects — i.e., on the grounds that such RCS-identifications may be implemented *without affecting the essential logical structure of the theory*:

- (RC-FrÉt) the **Frobenius-like** and **étale-like** versions of objects in IUT are **identified**;
- (RC-log) the $(\Theta^{\pm\text{ell}}\mathbf{NF}\text{-})$ **Hodge theaters** on opposite sides of the **log-link** in IUT are **identified**;
- (RC- Θ) the $(\Theta^{\pm\text{ell}}\mathbf{NF}\text{-})$ **Hodge theaters** on opposite sides of the **Θ -link** in IUT are **identified**.

Thus, locally, if

$\mathcal{O}_{\bar{k}}$ is the *ring of integers* of an *algebraic closure* \bar{k} of \mathbb{Q}_p ,
 $k \subseteq \bar{k}$ is a *finite subextension* of \mathbb{Q}_p ,
 $q \in \mathcal{O}_k$ is a *nonzero nonunit*,
 $\bar{G} \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$, and

$\Pi (\twoheadrightarrow G)$ is the *étale fundamental group* of some *hyperbolic curve* (say, of strictly Belyi type) over k ,

then we obtain the following situation:

RCS- Θ -link:

$$(k \supseteq) \quad (\underline{q^N})^{\mathbb{N}} \quad \xrightarrow{\sim} \quad \underline{q^{\mathbb{N}}} \quad (\subseteq k)$$

... where the copies of “ \bar{k} ”, “ k ”, and “ $G \curvearrowright \mathcal{O}_{\bar{k}}^{\times}$ ” on opposite sides are **identified** (and in fact $N = 1^2, 2^2, \dots, (l^*)^2$, but we think of N as *some fixed integer* ≥ 2);

RCS-log-link:

$$(\bar{k} \supseteq) \quad \mathcal{O}_{\bar{k}}^{\times} \quad \xrightarrow{\log_R} \quad \bar{k}$$

... where the copies of “ \bar{k} ”, “ k ”, and “ $\Pi \curvearrowright \mathcal{O}_{\bar{k}}^{\times}$ ” on opposite sides are **identified**.

Then the *RCS- Θ -link* identifies

$$(0 \neq) \quad N \cdot \text{ord}(\underline{q}) = \text{ord}(\underline{q^N})$$

with $\text{ord}(\underline{q})$ (where $\text{ord} : k^{\times} \rightarrow \mathbb{Z}$ is the valuation), which yields (since $N \neq 1$) a **“contradiction”**!

- Elementary observation: (cf. [EssLgc], Example 3.1.1)

Let ${}^{\dagger}\mathbb{R}, {}^{\ddagger}\mathbb{R}$ be (*not necessarily distinct!*) copies of \mathbb{R} . Let $0 < x, y \in \mathbb{R}$; write ${}^{\dagger}x, {}^{\ddagger}x, {}^{\dagger}y, {}^{\ddagger}y$ for the corresponding elements of ${}^{\dagger}\mathbb{R}, {}^{\ddagger}\mathbb{R}$. If these two copies ${}^{\dagger}\mathbb{R}, {}^{\ddagger}\mathbb{R}$ of \mathbb{R} are *distinct*, we may glue ${}^{\dagger}\mathbb{R}$ to ${}^{\ddagger}\mathbb{R}$ along

$${}^{\dagger}\mathbb{R} \supseteq \{{}^{\dagger}x\} \xrightarrow{\sim} \{{}^{\ddagger}y\} \subseteq {}^{\ddagger}\mathbb{R}$$

without any *consequences* or *contradictions*. On the other hand, if ${}^{\dagger}\mathbb{R}$ and ${}^{\ddagger}\mathbb{R}$ are the *same copy* of \mathbb{R} , then to assert that ${}^{\dagger}\mathbb{R}$ is glued to ${}^{\ddagger}\mathbb{R}$ along

$${}^{\dagger}\mathbb{R} \supseteq \{{}^{\dagger}x\} \xrightarrow{\sim} \{{}^{\ddagger}y\} \subseteq {}^{\ddagger}\mathbb{R}$$

implies that we have a **contradiction**, unless $x = y$.

- Note that the **RCS-identification** (RC- Θ) discussed above may be regarded as analogous to identifying the two **distinct** copies of the **ring scheme** \mathbb{A}^1 that occur in the conventional gluing of these two distinct copies along the **group scheme** \mathbb{G}_m to obtain \mathbb{P}^1 . That is to say, the RCS-assertion of some sort of **logical equivalence**

$$\text{IUT} \iff \text{RCS-IUT}$$

amounts to an assertion of an equivalence

$$\text{“}\mathbb{P}^1\text{”} \iff \left(\begin{array}{l} \text{“}\mathbb{A}^1 \text{ regarded up to some sort of} \\ \text{identification of the standard coord.} \\ T \text{ with its inverse } T^{-1}\text{”} \end{array} \right)$$

(cf. [EssLgc], Example 2.4.7) — i.e., which is *absurd!*

- **Fundamental Problem with RCS-IUT:**

(cf. [EssLgc], §3.2, §3.4, §3.8, §3.11)

There does **not exist** any **single “neutral” ring structure** with a single element “*” such that

$$(* = \underline{\underline{q}}^N) \quad \wedge \quad (* = \underline{\underline{q}})$$

Of course, there exists a *single “neutral” ring structure* with a single element “*” such that

$$(* = \underline{\underline{q}}^N) \quad \vee \quad (* = \underline{\underline{q}})$$

— but this requires one to contend, in RCS-IUT, with a fundamental (drastic!) **indeterminacy** (Θ ORInd) that renders the entire theory (i.e., RCS-IUT, not IUT!) **meaningless!**

That is to say, the *essential logical structure* of IUT depends, in a very fundamental way, on the crucial **logical AND** “ \wedge ” property of the Θ -link, i.e., that the **abstract** $\mathcal{F}^{\text{I-}\blacktriangleright \times \mu}$ -**prime-strip** in the Θ -link, regarded up to *isomorphism*, is *simultaneously* the Θ -**pilot** on the LHS of the Θ -link **AND** the **q-pilot** on the RHS of the Θ -link.

This is possible precisely because the *realified Frobenioids* and *multiplicative monoids with abstract group actions* that constitute these Θ -/ q -pilot $\mathcal{F}^{\text{tr}} \times \mu$ -prime-strips are **isomorphic** — i.e., unlike the “*field plus distinguished element*” pairs

$$(k, \underline{q}^N) \quad \text{and} \quad (k, \underline{q}),$$

which are *not isomorphic!*

(... cf. the situation with \mathbb{P}^1 : there does **not exist a single ring scheme** \mathbb{A}^1 with a single rational function “ $*$ ” such that

$$(* = T^{-1}) \quad \wedge \quad (* = T).$$

There only exists a *single ring scheme* \mathbb{A}^1 with a single rational function “ $*$ ” such that $(* = T^{-1}) \vee (* = T)$.)

Here, we note that the **RCS-identifications** of

G on opposite sides of the RCS- Θ -link or
 Π on opposite sides of the RCS- \log -link or

— which arise from **Galois-equivariance** properties with respect to the **single “neutral” ring structure** discussed above, i.e., which is subject to the (drastic!) (**Θ ORInd**) **indeterminacies** — yield **false symmetry/coricity** (such as the symmetry of “ $\Pi \rightarrow G \leftarrow \Pi$ ”) properties, i.e., *false* versions of the symm./cor. props. discussed in §3.

Indeed, the various **Galois-rigidifications** — i.e., embeddings of the abstract topological groups involved into the group of automorphisms of **some field** — that *underlie these Galois-equivariance or false symmetry/coricity properties* are **unrelated** to the Galois-rigidifications that underlie the corresponding symmetry/coricity properties of §3.

That is to say, setting up a situation in which these symm./cor. props. of §3 do indeed hold is the whole point of “**inter-universality**”, i.e., working with *abstract groups, abstract monoids, etc.!*

· Finally, we observe that (cf. [Alien], §3.3, (ii); [EssLgc], §3.3)

the **very definition** of the **log-link, Θ -link** (cf. §2;
log : nondilated unit groups \rightleftharpoons **dilated** value groups!)

\implies the **falsity** of (RC-**log**):

Indeed, there is **no natural way** to relate the *two* Θ -links (i.e., the *two horizontal arrows* below) that emanate from the *domain* and *codomain* of the \log -link (i.e., the *left-hand vertical arrow*)

$$\begin{array}{ccc} \bullet & \xrightarrow{\Theta} & \bullet \\ \uparrow \log & & \vdots \\ \bullet & \xrightarrow{\Theta} & \bullet \end{array}$$

— that is to say, there is *no natural candidate* for “??” (i.e., such as, for instance, an *isomorphism* or the \log -link between the two bullets “•” on the *right-hand side* of the diagram) that makes the diagram *commute*. Indeed, it is an easy exercise to show that *neither* of these candidates for “??” yields a commutative diagram.

- Analogy with classical complex Teichmüller theory:
(cf. [EssLgc], Example 3.3.1)

Let $\lambda \in \mathbb{R}_{>1}$. Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory

$$\begin{aligned} \Lambda : \mathbb{C} &\rightarrow \mathbb{C} \\ \mathbb{C} \ni z = x + iy &\mapsto \zeta = \xi + i\eta \stackrel{\text{def}}{=} x + \lambda \cdot iy \in \mathbb{C} \end{aligned}$$

— where $x, y \in \mathbb{R}$. Let $n \geq 2$ be an integer, ω a *primitive n -th root of unity*. Write $(\omega \in) \mu_n \subseteq \mathbb{C}$ for the group of n -th roots of unity. Then *observe* that

$$\text{if } n \geq 3, \text{ then there does not exist } \omega' \in \mu_n \text{ such that } \Lambda(\omega \cdot z) = \omega' \cdot \Lambda(z) \text{ for all } z \in \mathbb{C}.$$

(Indeed, this *observation* follows immediately from the fact that if $n \geq 3$, then $\omega \notin \mathbb{R}$.) That is to say, in words,

Λ is **not compatible** with multiplication by μ_n unless $n = 2$ (in which case $\omega = -1$).

This *incompatibility* with “**indeterminacies**” arising from multiplication by μ_n , for $n \geq 3$, may be understood as one fundamental reason for the *special role* played by **square differentials** (i.e., as opposed to n -th power differentials, for $n \geq 3$) in classical complex Teichmüller theory.

§8. Chains of gluings/logical \wedge relations

(cf. [EssLgc], §3.5, §3.6, §3.11; [ClsIUT], §2)

- Fundamental Question:

Why is the **logical AND** “ \wedge ” relation of the Θ -link so *fundamental* in IUT?

- Consider, for instance, the *classical theory of crystals*

(cf. [ClsIUT], §2; [EssLgc], §3.5, (CrAND), (CrOR), (CrRCS)):

The “*crystals*” that appear in the conventional theory of crystals may be thought of as “ \wedge -**crystals**”. Alternatively, one could consider the (in fact *meaningless!*) theory of “ \vee -**crystals**”. One verifies easily that this theory of “ \vee -*crystals*” is in fact essentially equivalent to the theory obtained by replacing the various **thickenings of diagonals** that appear in the conventional theory of crystals by the “ $(-)$ _{red}” of these thickenings, i.e., by the **diagonals themselves!** Finally, we observe that consideration of “ \vee -*crystals*” corresponds to the **indeterminacy** (Θ ORInd) that appears in RCS-IUT, i.e.:

$$\begin{array}{lcl} \mathbf{IUT} & \longleftrightarrow & \text{“}\wedge\text{-crystals”} \\ \mathbf{RCS-IUT} & \longleftrightarrow & \text{“}\vee\text{-crystals”} \end{array}$$

- Frequently Asked Question:

In IUT, one starts with the fundamental **logical AND** “ \wedge ” relation of the Θ -link, which holds precisely because of the **distinct labels** on the *domain/codomain* of the Θ -link. Then what is the **minimal** amount of **indeterminacy** that one must introduce in order to **delete** the **distinct labels** without invalidating the fundamental *logical AND* “ \wedge ” relation?

In short, the answer (cf. §6!) is that one needs **(Ind1)**, **(Ind2)**, **(Ind3)**, together with the operation of forming the **holomorphic hull**. In some sense, the most fundamental of these indeterminacies is

$$\mathbf{(Ind3)},$$

which in fact in some sense “**subsumes**” the other indeterminacies — at least “**to highest order**”, i.e., in the *height inequalities* that are ultimately obtained (cf. [EssLgc], §3.5, (CnfInd1+2), (CnfInd3); [EssLgc], §3.11, (Ind3>1+2)).

Recall from §4 that (Ind3) is an inevitable consequence of the **non-commutativity** of the **log-Kummer correspondence**

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \dots \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & \dots & & \circ & & \dots & &
 \end{array}$$

(cf. also the discussion of the *falsity* of (RC-log), (RC-FrÉt) in §7!). On the other hand, observe that since automorphisms of the (topological module constituted by the) **log-shell** $\mathcal{I}_{\underline{v}}$ *always preserve* the submodule

$$p^n \cdot \mathcal{I}_{\underline{v}}$$

(where $n \geq 0$ is an integer) — i.e., even if they do *not* necessarily preserve $\mathcal{O}_{\underline{v}} \subseteq \mathcal{I}_{\underline{v}}$ or positive powers of the *maximal ideal* $\mathfrak{m}_{\underline{v}} \subseteq \mathcal{O}_{\underline{v}}$! — it follows immediately that

(Ind1) (or, *a fortiori*, the “ $\Pi_{\underline{v}}$ version” of (Ind1) — cf. the discussion of (Ind1) in §3) and

(Ind2)

(both of which induce automorphisms of $\mathcal{I}_{\underline{v}}$) can **never account for** any sort of “**confusion**” (cf. the definition of the Θ -link) between

$$\underset{=_{\underline{v}}}{“q^{(l^*)^2}”} \text{ and } \underset{=_{\underline{v}}}{“q”}$$

(cf. [EssLgc], §3.5, (CnfInd1+2), (CnfInd3); [EssLgc], Example 3.5.1; [EssLgc], §3.11, (Ind3>1+2))! This is a *common misunderstanding*!

- Now let us return to the *Fundamental Question* posed above.

We begin our discussion by observing (cf. [EssLgc], §3.6) that

(\wedge -Chn) the logical structure of IUT proceeds by *observing a chain of AND relations* “ \wedge ” (*not a chain of intermediate inequalities!* — cf. [EssLgc], §3.6, (Syp3)).

That is to say, one starts with the **logical AND** “ \wedge ” relation of the Θ -link. This *logical AND* “ \wedge ” relation is *preserved* when one passes to the **multiradial representation of the Θ -pilot** as a consequence of the following fact:

(\wedge -Input) the **input data** for this multiradial algorithm consists solely of an **abstract $\mathcal{F}^{\text{!} \blacktriangleright \times \mu}$ -prime-strip**; moreover, this multiradial algorithm is **functorial** with respect to arbitrary isomorphisms between $\mathcal{F}^{\text{!} \blacktriangleright \times \mu}$ -prime-strips.

Indeed, at a more technical level, we make the *fundamental observation* that this multiradial algorithm proceeds by *successive application*, in one form or another, of the following principle of “**extension of indeterminacies**”:

(ExtInd) If A , B , and C are propositions, then it holds (that $B \implies B \vee C$ and hence) that

$$A \wedge B \implies A \wedge (B \vee C).$$

(cf. the final portion of §6!). Applications of (ExtInd) may be further *subclassified* into the following *two types*:

(ExtInd1) (“*set-theoretic*”) operations that consist of simply adding **more possibilites/indeterminacies** (which corresponds to passing from B to $B \vee C$) within some **fixed container**;

(ExtInd2) (“*stack-theoretic*”) operations in which one **identifies** (i.e., “*crushes together*”, by passing from B to $B \vee C$) objects with **distinct labels**, at the cost of passing to a situation in which the object is regarded as being only known **up to isomorphism**

(cf. the discussion of §9 below).

At this point, we recall from §6 that the *ultimate goal* of various applications of (ExtInd) in the algorithms that constitute the **multiradial representation of the Θ -pilot** is to

exhibit the (value group portion of the) **Θ -pilot** at $(0, 0)$ (i.e., which appears in the Θ -link!) as **one of the possibilities** within a **container** arising from the **RHS** of the **Θ -link**

(cf. the situation surrounding *rational functions* on \mathbb{P}^1 , as discussed in [EssLgc], Example 2.4.7, (ii)!).

In particular, any problems in understanding the *essential logical str.* of IUT (i.e., the argument of §6) may be *diagnosed/analyzed* by asking the following **diagnostic question**:

(\wedge -Dgns) **precisely where** in the finite sequence of steps that appear is the **first step** at which the person feels that the **preservation** of the **crucial AND relator** “ \wedge ” is *no longer clear?*

§9. Poly-morphisms, descent to underlying strs., and inter-universality

(cf. [EssLgc], Example 3.1.1; §3.7, §3.8, §3.9, §3.11)

- In IUT, one often considers **poly-morphisms**, i.e., sets of morphisms between objects — such as **full poly-isomorphisms** (the set of all isomorphisms between two objects) — as a tool to keep track explicitly of **all possibilities** that appear. Roughly speaking, consideration of *full poly-isomorphisms* corresponds to the usual notion of “*considering objects up to isomorphism*”. From the point of view of *chains of \wedge 's and \vee 's*

$$\begin{aligned}
 A \wedge B &= A \wedge (B_1 \vee B_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots \vee B''_1 \vee B''_2 \vee \dots) \\
 &\quad \vdots
 \end{aligned}$$

discussed in §6, consideration of poly-morphisms corresponds to adding to the *collection of possibilities*, i.e., to the *collection of \vee 's* that appear (cf. “*set-theoretic*” (*ExtInd1*)!) — cf. [EssLgc], §3.7.

- One fundamental aspect of IUT lies in the use of numerous **functional algorithms** that consist of the construction

$$\textit{input data} \rightsquigarrow \textit{output data}$$

of certain *output data* associated to given *input data*. Often it is natural to regard the “*input data*” as “*original data*” and to regard the “*output data*” as “*underlying data*”:

$$\begin{array}{ccc}
 \textit{input data} & \rightsquigarrow & \textit{output data} \\
 \parallel & & \parallel \\
 \textit{original data} & & \textit{underlying data}
 \end{array}$$

One important example of this sort of situation in IUT involves the notion of “***q-/Θ-intertwinings***” on an $\mathcal{F}^{\text{!} \blacktriangleright \times \mu}$ -*prime-strip* (cf. [EssLgc], §3.9):

original data (“equipped with an intertwining”):

the **q-pilot** $\mathcal{F}^{\text{tr}} \times \mu$ -prime-strip (in the case of the “q-intertwining”) or the **Θ-pilot** $\mathcal{F}^{\text{tr}} \times \mu$ -prime-strip (in the case of the “Θ-intertwining”), equipped with the *auxiliary data* of how this q-/Θ-pilot $\mathcal{F}^{\text{tr}} \times \mu$ -prime-strip is constructed from some $(\Theta^{\pm\text{ell}} NF\text{-})Hodge$ theater;

underlying data:

the *abstract* $\mathcal{F}^{\text{tr}} \times \mu$ -prime-strip associated to the above *original data*, i.e., obtained by *forgetting* the *auxiliary data*.

- In general, in any sort of situation involving *original/underlying data*, it is natural to consider the issue of **descent** to (a functorial algorithm in) the *underlying data* of a **functorial algorithm** in the *original data*: we say that

a *functorial algorithm* Φ in the *original data* **descends** to a *functorial algorithm* Ψ in the *underlying data* if there exists a functorial isomorphism

$$\Phi \xrightarrow{\sim} \Psi|_{\text{original data}}$$

between Φ and the *restriction* of Ψ , i.e., relative to the given construction $\text{original data} \rightsquigarrow \text{underlying data}$.

That is to say, roughly speaking, to say that the functorial algorithm Φ in the original data *descends* to the *underlying data* means, in essence, that although the construction constituted by Φ depends, *a priori*, on the “**finer**” *original data*, in fact, up to *natural isomorphism* (cf. “*stack-theoretic*” (*ExtInd2*)!), the functorial algorithm only depends on “**coarser**” *underlying data*.

- One elementary example of *descent* is the following (cf. [EssLgc], Example 3.9.1):

Let X be a *scheme*, T a *topological space*. Write

- $|X|$ for the *underlying topological space* of X ,
- $\text{Open}(X)$ for the category of *open subschemes* of X and *open immersions* over X ,
- $\text{Open}(T)$ for the category of *open subsets* of T and *open immersions* over T .

Then the *functorial algorithm*

$$X \mapsto \text{Open}(X)$$

— defined, say, on the category of schemes and morphisms of schemes
 — *descends*, relative to the construction $X \rightsquigarrow |X|$, to the *functorial algorithm*

$$T \mapsto \text{Open}(T)$$

— defined, say, on the category of topological spaces and continuous maps of topological spaces. That is to say, there is a *natural functorial isomorphism*

$$\text{Open}(X) \xrightarrow{\sim} \text{Open}(|X|)$$

(i.e., more precisely, following the conventions employed in IUT, a *natural functorial isomorphism class of equivalences of categories*).

- **Inter-universality** in IUT — cf. the *abstract topological groups/monoids* (as opposed to *Galois groups/multiplicative monoids of rings!*) that appear in the Θ -link, as discussed in §2, §3, §4, §7 — arises from the fact that the structures **common** (cf. “ \wedge ”!) to both sides of the Θ -link are **weaker** than ring structures. On the other hand, despite this “*ring str. vs. weaker than ring str.*” difference, at a *purely foundational level*, the resulting indeterminacies (i.e., (Ind1), (Ind2)) are in fact *completely qualitatively similar* to the **inner automorphism indeterminacies** in [SGA1] (cf. [EssLgc], §3.8).

In this context, it is useful to recall the elementary fact that these inner automorphism indeterminacies are *unavoidable* (cf. [EssLgc], Example 3.8.1 (i)!):

Let

k be a *perfect field*;

\bar{k} an *algebraic closure* of k ;

$N \subseteq G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$ a *normal open subgroup* of G_k ;

$\sigma \in G_k$ such that the automorphism $\iota_\sigma : N \xrightarrow{\sim} N$ of N given by *conjugating* by σ is *not inner*.

(One verifies immediately that, for instance, if k is a *number field* or a *mixed-characteristic local field*, then such N, σ do indeed exist.)

Write

$$k_N \subseteq \bar{k} \text{ for the subfield of } N\text{-invariants of } \bar{k},$$

$$G_{k_N} \stackrel{\text{def}}{=} N \subseteq G_k.$$

Then observe that if one assumes that the **functoriality** of the *étale fundamental group* holds *even in the absence of inner automorphism indeterminacies*, then the *commutative diagram of natural morphisms of schemes*

$$\begin{array}{ccc} \text{Spec}(k_N) & \xrightarrow{\sigma} & \text{Spec}(k_N) \\ & \searrow & \swarrow \\ & \text{Spec}(k) & \end{array}$$

induces a *commutative diagram of profinite groups*

$$\begin{array}{ccc} G_{k_N} & \xrightarrow{\iota_\sigma} & G_{k_N} \\ & \searrow & \swarrow \\ & G_k & \end{array}$$

— which (since the natural inclusion $N = G_{k_N} \hookrightarrow G_k$ is *injective!*) implies that ι_σ is the *identity automorphism*, in *contradiction* to our assumption concerning σ !

- As a consequence of the *inter-universality* considerations discussed above (e.g., the need to work with *abstract topological groups!*), one must consider various **reconstruction algorithms** in IUT. Since reconstruction of an object is *never “set-theoretically on the nose”*, but rather always *up to (a necessarily indeterminate!) isomorphism* — whence the use of *full poly-isomorphisms!* — such reconstruction algorithms necessarily lead to **(ExtInd2) indeterminacies**. At first glance, this phenomenon may seem rather strange, but in fact, at a *purely foundational level*, this phenomenon is *completely qualitatively similar* to the indeterminacies that appear in such *classical constructions* as
 - the notion of an **algebraic closure** of a field,
 - **projective/inductive limits**, or
 - **cohomology modules** (i.e., which arise as subquotients of “*some*” *indeterminate resolution*)
- cf. [EssLgc], §3.8, §3.9, §3.11.

- As a result of such **(ExtInd2) indeterminacies**, one does not obtain any *nontrivial consequences/inequalities* (cf. the “Elementary Observation” of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9) at “*stack-theoretic*” *intermediate steps*, i.e., even if one applies the *log-volume*!

In order to obtain *nontrivial consequences/inequalities* (cf. the “Elementary Observation” of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9), it is necessary to obtain a “**set-theoretic**” **closed loop**, i.e., by

- applying the **multiradial representation of the Θ -pilot**, which gives rise to the indeterminacies **(Ind1)**, **(Ind2)**, **(Ind3)**;
 - forming the **holomorphic hull**,
 - symmetrizing with respect to **vertical log-shifts** in the 1-column;
 - and, finally, applying the **log-volume**
- as described in §6.

$$\begin{array}{ccc}
 \Pi_{\underline{v}} \rightarrow & G_{\underline{v}} & \leftarrow \Pi_{\underline{v}} \\
 \curvearrowright & \circlearrowleft & \curvearrowright \\
 & \text{Aut}(G_{\underline{v}}) & \\
 \left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\text{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{domain} \\ \text{of the } \Theta\text{-link} \end{array} \right) & & \left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\text{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{codomain} \\ \text{of the } \Theta\text{-link} \end{array} \right)
 \end{array}$$

§10. Closed loops via multiradial representations and holomorphic hulls

(cf. [EssLgc], Example 2.4.6, (iii); [EssLgc], §3.10, §3.11; [ClsIUT], §2)

- We begin by observing that by *eliminating superfluous overlaps* from the *chain of \wedge 's and \vee 's* that constitutes the *essential logical structure* of IUT (cf. §6) and replacing the various *logical OR* “ \vee 's” by **logical XOR** “ $\dot{\vee}$'s”, we may think of this *essential logical str.* of IUT as consisting of a **chain of \wedge 's and $\dot{\vee}$'s**:

$$\begin{aligned}
 A \wedge B &= A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \dots) \\
 &\implies A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \dots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \dots) \\
 &\implies A \wedge (B_1 \dot{\vee} B_2 \dot{\vee} \dots \dot{\vee} B'_1 \dot{\vee} B'_2 \dot{\vee} \dots \dot{\vee} B''_1 \dot{\vee} B''_2 \dot{\vee} \dots) \\
 &\quad \vdots
 \end{aligned}$$

Recall that from the point of view of the **arithmetic** of the **field** \mathbb{F}_2 ,

$$\begin{array}{ccc}
 \wedge & \longleftrightarrow & \text{multiplication} \\
 \dot{\vee} & \longleftrightarrow & \text{addition}
 \end{array}$$

while from the point of view of the **arithmetic** of the **truncated ring of Witt vectors** $\mathbb{F}_2 \times \mathbb{F}_2$ (i.e., $\mathbb{Z}/4\mathbb{Z}$),

$$\begin{array}{ccc}
 \wedge & \longleftrightarrow & \text{multiplication of Teichmüller reps. of } \mathbb{F}_2 \\
 (\wedge, \dot{\vee}) & \longleftrightarrow & \text{carry-addition on Teichmüller reps. of } \mathbb{F}_2
 \end{array}$$

(cf. [EssLgc], Example 2.4.6, (iii)). That is to say, **carry-addition** — which may thought of as a sort of

“ \wedge stacked on top of an $\dot{\vee}$ ”

— is **remarkably reminiscent** of the *essential logical structure of IUT*, as well as of the fact that IUT itself is a theory concerning the explication of how the two “combinatorial dimensions” of a ring are *mutually intertwined*, i.e., how the *multiplicative structure of a ring is “stacked on top of” the additive structure of a ring!* In the case of the **chain of \wedge 's and $\dot{\vee}$'s** that constitutes the *essential logical structure* of IUT, we observe that:

$$\begin{aligned} \wedge & \longleftrightarrow \left(\begin{array}{l} \mathbf{multiplicative\ \Theta-link}; \\ \text{data } \mathbf{common} \text{ to the} \\ \text{domain/codomain of the} \\ \Theta\text{-link} \end{array} \right) \\ \dot{\vee} & \longleftrightarrow \left(\begin{array}{l} \mathbf{additive\ log-shells} \\ \text{arising from the } \mathbf{log-link}; \\ \text{mutually exclusive distinct} \\ \text{possibilities} \end{array} \right) \end{aligned}$$

Finally, relative to the analogy between IUT and crystals, it is also of interest to observe that:

$$\begin{aligned} \wedge & \longleftrightarrow \left(\begin{array}{l} \text{crystals} \\ = \text{“}\wedge\text{-crystals”} \end{array} \right) \\ \dot{\vee} & \longleftrightarrow \left(\begin{array}{l} \text{mutually exclusive} \\ \text{pull-backs of the} \\ \mathbf{Hodge\ filtration} \end{array} \right) \end{aligned}$$

— where we recall that it is precisely the “*intertwining between these $\wedge / \dot{\vee}$ aspects*” that gives rise to the **Kodaira-Spencer morphism** (cf. [EssLgc], $(\wedge(\dot{\vee})\text{-Chn})$; [ClsIUT], §2).

- We conclude by reviewing once again the discussion of §6, this time taking into account the various subtleties discussed in §7, §8, §9 (cf. also [EssLgc], §3.10, §3.11).

We begin by recalling that the **log-Kummer correspondence**

$$\begin{array}{ccccccc} \dots & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \bullet & \xrightarrow{\log} & \dots \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & \dots & & & & \dots & & \\ & & & & \circ & & & & \end{array}$$

— which **juggles** the **dilated** and **nondilated** underlying arithmetic dimensions of the rings involved (cf. §2)

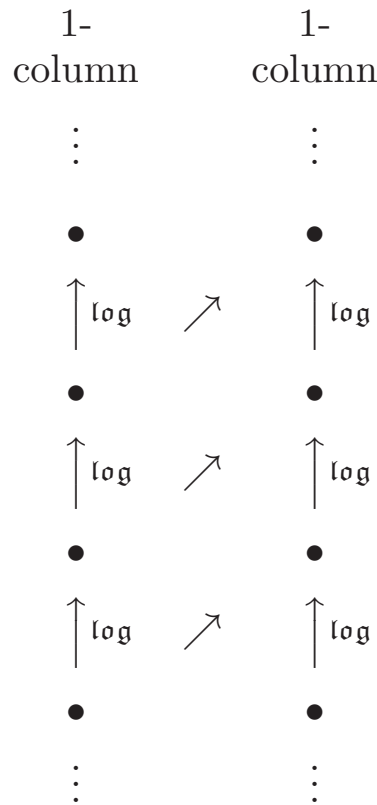
$$\mathbf{log} : \mathbf{nondilated\ unit\ groups} \quad \rightleftharpoons \quad \mathbf{dilated\ value\ groups}$$

yields, by applying various **descent operations**

$$(0, 0) \xrightarrow{\text{(Ind3)}} (0, \circ) \xrightarrow{\text{(Ind1)}} (0, \circ)^{\perp} \xrightarrow{\text{(Ind2)}} (0, 0)^{\perp} \xrightarrow{\Theta\text{-link}} (1, 0)^{\perp}$$

(where we recall that the last step involving (Ind2) plays the role of **fixing** the vertical coordinate, so that (Ind1), (Ind2) are **not mixed** with (Ind3) — cf. the discussion of “ $\mathbb{C}^{\times} \backslash GL_2^+(\mathbb{R}) / \mathbb{C}^{\times}$ ” at the end of §5!), the **multiradial representation of the Θ -pilot**, up to the **indeterminacies (Ind1), (Ind2), (Ind3)**.

Then forming the **holomorphic hull** and symmetrizing with respect to **vertical log-shifts** in the 1-column



yields a **closed loop**, to which we may apply the **log-volume** to obtain **“set-theoretic” consequences/inequalities** (cf. the “Elementary Observation” of §7; [EssLgc], Example 3.1.1; [EssLgc], §3.8, §3.9).

Here, we recall that the repeated introduction of “**stack-theoretic**” (**ExtInd2**) **indeterminacies**

$$\begin{array}{ccc}
 \Pi_{\underline{v}} \twoheadrightarrow & G_{\underline{v}} & \twoheadleftarrow \Pi_{\underline{v}} \\
 \curvearrowright & \circlearrowleft & \curvearrowleft \\
 & \text{Aut}(G_{\underline{v}}) & \\
 \left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\text{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{domain} \\ \text{of the } \Theta\text{-link} \end{array} \right) & & \left(\begin{array}{c} \text{some portion of} \\ \text{the } \textit{Frobenius-like} \\ \text{local data at} \\ \underline{v} \text{ of the} \\ (\Theta^{\pm\text{ell}}\text{NF-}) \\ \text{Hodge theater} \\ \text{in the } \textit{codomain} \\ \text{of the } \Theta\text{-link} \end{array} \right)
 \end{array}$$

— especially in the context of various *reconstruction algorithms* — allows us to achieve the *central goal* of **exhibiting** the (value group portion of the) **Θ -pilot** at $(0,0)$ (i.e., which appears in the Θ -link!) as **one of the possibilities** within a **container** arising from the **RHS** of the Θ -link. Moreover, the *essential logical structure*

$$\begin{aligned}
 A \wedge B &= A \wedge (B_1 \vee B_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots) \\
 &\implies A \wedge (B_1 \vee B_2 \vee \dots \vee B'_1 \vee B'_2 \vee \dots \vee B''_1 \vee B''_2 \vee \dots) \\
 &\quad \vdots
 \end{aligned}$$

underlying the **closed loop** referred to above means that there are **no** issues of “**diagram commutativity**” or “**log-vol. compatibility**” to contend with:

$$\begin{array}{l}
\bullet = = \hat{=} = = \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \bullet =) = \hat{=} = = \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \vee \bullet = =) \hat{=} = = \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \vee \vee \bullet = = \hat{=}) = = \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \vee \vee \vee \bullet = = \hat{=} =) = \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \vee \vee \vee \vee \bullet = = \hat{=} = =) \bullet \\
\rightsquigarrow \quad \quad \quad (\vee \vee \vee \vee \vee \vee \bullet = = \hat{=} = = \bullet)
\end{array}$$

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