OVERVIEW OF COMBINATORIAL ANABELIAN GEOMETRY I, ~~&~~ II, III

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"Travel and Lectures"

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T0. **Semi-graphs of anabelioids**
(cf. [GeoAn], [SemiAn], [CbGC])

- *Anabelioids* ([GeoAn]): can consider

  finite étale coverings without *specifying basepoints*!

- [SemiAn], §1, §2, §3; [CbGC], §1:

  semi-graphs of anabelioids (of PSC-type);
  finite étale and *tempered* coverings

- [SemiAn], Thm. 1.2, Cor. 1.6: *Zar.'s main thm. for* finite *semi-graphs*

  $\boxed{(\forall \text{ immersion}) = (\exists \text{ finite étale covering}) \circ (\exists \text{ embedding})}$

  (e.g., ∀ fin. gen. s/gp. of a free gp. of fin. rk. almost arises from a *subbasis*!)

- [SemiAn], Lem. 1.8: *finite groups acting on trees*

  fixed point locus is *nonempty*, *simply connected*

- [SemiAn], §2; [CbGC], Prop. 1.2: *verticial/edge-like* s/gps.

  satisfy *commensurable terminality* and *independence* properties

- [SemiAn], Thm. 3.7: ∀ semi-graph of anabelioids of PSC-type,

  {*maximal compact s/gps.*} = {*verticial s/gps.*};
  maxl
  {*intersections of distinct max. comp. s/gps.*} = {*edge-like s/gps.*}
  inside tp'd fund. gp.

- [SemiAn], Cor. 3.11, Thm. 6.6: ∀ *hyp. curve*/fin. extn. of $\mathbb{Q}_p$,

  geometric tempered fundamental group

  $\rightsquigarrow$ *semi-graph of special fiber*

  reconstruct
  . . . in particular, *temp. fund. gp.* $\Longleftrightarrow$
  ét. fund. gp. + *special fiber semi-graphs* of fin. et. covs.

  $\Leftarrow \Pi_v$'s, Π_e's

T1. **Motivation from differential topology and arithmetic** (cf. [SemiAn], [CbCsp])

- Semi-graphs of anabelioids and configuration spaces
- The Dehn-Nielsen-Baer Theorem
- The cuspidalization problem (complete solution in discrete case)
- Anabelian encoding of base field ring structure in a tripod

T2. **Combinatorial anabelian results**
(cf. [CbGC], [NodNon], [CbTpIII])

· Combinatorial Groth. Conj.-type results and cuspidalization

· [CbGC], Cor. 2.7(i)(iii): IPSC to IPSC:

$$\text{gp.-theoretically cuspidal} \implies \text{graphic}$$

requires $\mathrm{Out}^{\mathrm{FC}}$

· [NodNon], Thm. A: NN to NN:

$$\text{gp.-theoretically cuspidal} \wedge {}^{\exists}\text{at least one cusp} \implies \text{graphic}$$

· [CbTpII], Thm. 1.9: IPSC to NN:

$$\text{gp.-theoretically verticial, nodal}$$

allows one to work with $\mathrm{Out}^{\mathrm{FC}}$

T3. <u>**Combinatorial cuspidalization and "FC = F" results**</u>
(cf. [CbCsp], [NodNon], [CbTpI], [CbTpII], [HMT])

- [NodNon], Cor. 6.5: Complete solution in discrete case $\forall\ n \geq 1$:

OutFC

$$\boxed{\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \overset{\sim}{\to} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)}$$

- [NodNon], Thm. B: profinite/pro-l case:

$$\boxed{\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)}$$

injective if $n \geq 1$, *bijective* if $n \geq 3$ (affine) or $n \geq 4$ (proper)

- [CbTpI], Thm. A: <u>profinite/pro-l</u> case:

$$\mathrm{Out}^{\mathrm{F}\cancel{\mathrm{C}}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n) \ (\subseteq \mathrm{Out}^{\mathrm{F}\cancel{\mathrm{C}}}(\Pi_n))$$

- [CbTpII], Thm. A(i): profinite/pro-l case:

OutF

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

injective if $n \geq 1$ (affine) or $n \geq 2$ (proper),
bijective if $n \geq 3$ (affine) or $n \geq 4$ (proper)

- [CbTpII], Thm. A(ii); [HMT], Cor. 2.2: profinite/pro-l case:

$$\boxed{\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)}$$

if $n \geq 2$ (genus 0) or $n \geq 3$ (affine) or $n \geq 4$ (proper)

T4. **Arithmetic applications of combinatorial cuspidalization over number fields and mixed-characteristic local fields**

(cf. [NodNon], [CbTpIII])

- [NodNon], Thm. C: F an NF or MLF: $G_F \to \mathrm{Out}(\Delta)$ *injective*

  (generalization of Belyi, Matsumoto)

- [CbTpIII], Thm. B(iii): F an NF, $\mathfrak{p}$ a nonarc. prime of F:

gen'ztion of André (tripod)

$$\boxed{G_{\mathfrak{p}} = G_F \cap \mathrm{Out}^{\mathrm{M}}(\Delta)} \ (\subseteq \ \mathrm{GT} \cap \mathrm{Out}^{\mathrm{M}}(\Delta)$$

$$\subseteq \ \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Delta) \ = \ \text{Tsujimura's } p\text{-adic version of GT})$$

$\mathrm{Out}(\Delta^{\mathrm{tp}})$

$\cup$ André's version

X : hyp. curve $/F$

$$1 \to \Delta \to \Pi_X \to G_F \to 1$$

geometric π_1 $\pi_1^{\mathrm{et}}(X \times_F \overline{F})$

$\Pi_X = \pi_1^{\mathrm{et}}(X)$

T5. **Synchronization of cyclotomes, profinite Dehn twists, and geometric monodromy anabelian results**
(cf. [CbTpI])

- *Synchronization of cyclotomes*: a sort of profinite orientation

[CbTpI], §3

- *Profinite Dehn twists*:
  intrinsic definition, structure theorem, coordinates

- *Pro-Σ geo. monodr. "GC"* for $(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}$ when $(g,r) \neq (0,3)$; $(1,1)$:

fin

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi)}\left(\text{open subgroup of } \Pi_{(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}}\right) = \left(\text{scheme-theoretic aut. group of generic fiber}\right)$$

$\subseteq \mathrm{Out}(\Pi)$

- Generalization to *Hurwitz stacks*

T6. **Tripod synchronization and the tripod homomorphism** (cf. [CbTpII])

- [CbTpII], Thm. C(i): *Commensurator/centralizer*

$$C_{\Pi_n}(T) = T \times Z_{\Pi_n}(T)$$

of a *tripod* $T \subseteq \Pi_n$; hence we have a natural homomorphism

$\mathrm{Out}^{\mathrm{F}}(\Pi_n)$

$$\mathfrak{T}_T : \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \to \mathrm{Out}(T)$$

Special among tripods

- *Central tripods*: tripods emanating from the *generic diag. cusp*

- [CbTpII], Thm. C(iv): if ($\Pi_n \twoheadrightarrow$) $\Pi_3 \supseteq T$ is *central*, then

tripod homomorphism

$$\mathfrak{T}_T : \mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow GT \ (\subseteq \mathrm{Out}(T))$$

for $n \geq 3$ (affine) or $n \geq 4$ (proper)

("$\twoheadrightarrow GT$" requires theory of *gluing of cuspidalizations*!)

- *Tripod synchronization*: commutative diagram of $\mathfrak{T}_T$'s

(cf. synch. of cycl.)

T7. **Structure theory and arithmetic subgroups of the Grothendieck-Teichmüller group**
(cf. [MT], [HMM], [HMT])

- [HMM], Thm. A: reconstruction of *generalized fiber subgroups*

- [HMM], Thm. B: profinite/pro-l case:

$$\boxed{\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n^*}}$$

for $n \geq 2$ and (g, r) s.t. $(r, n) \neq (0, 2)$.

$\cup$ Z(open subgp.)

- [HMM], Thm. C: in profinite/pro-l (cf. "Σ") $(0, 3)$ case:

pf. requires GC/NF's

$$\boxed{\mathrm{Out}(\Pi_n) = \mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3};} \quad \mathfrak{S}_{n+3} = Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n));$$

$$\mathrm{GT}^{\Sigma} = Z_{\mathrm{Out}(\Pi_n)}(Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)))$$

Out(Π_n)

(also *purely gp.-th. characterization/proof* of s/gp. $\mathfrak{S}_{n+3} \subseteq \cancel{\mathrm{GT}}$)

- [HMT], Thm. A: for BGT $\subseteq$ GT s.t. COF $\wedge$ RGC:

"GT is too large to be $G_{\mathbb{Q}}$?"

e.g. $\quad G_k \subseteq \boxed{C_{\mathrm{GT}}(\mathrm{BGT}) \twoheadrightarrow G_{\mathbb{Q}_{\mathrm{BGT}}};}$

t, l-t
ring str.

- [HMT], Thms. B, F: for BGT $\subseteq$ GT s.t. QAA:

$$C_{\mathrm{GT}}(\mathrm{BGT}) \overset{\sim}{\to} G_{\mathbb{Q}_{\mathrm{BGT}}}$$

(e.g.: abs. Gal. gp. of *TKND-AVKF-field*)

K

- [HMT], Thm. C: GT $\supseteq G_{\mathbb{Q}}$ is **maximal AA closed s/gp.**

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