OVERVIEW OF COMBINATORIAL ANABELIAN GEOMETRY I, II, III

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July 2021

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"Travel and Lectures"

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T0. Semi-graphs of anabelioids
(cf. [GeoAn], [SemiAn], [CbGC])

· *Anabelioids* ([GeoAn]): can consider

finite étale coverings without *specifying basepoints*!

· [SemiAn], §1, §2, §3; [CbGC], §1:

semi-graphs of anabelioids (of PSC-type);
finite étale and *tempered* coverings

· [SemiAn], Thm. 1.2, Cor. 1.6: *Zariski's main theorem for finite semi-graphs*

($\forall$ immersion) = ($\exists$ finite étale covering) $\circ$ ($\exists$ embedding)

(e.g., $\forall$ finitely generated subgroup of a free group of finite rank "almost" arises from a *subbasis*!)

· [SemiAn], Lem. 1.8: *finite groups acting on trees*

fixed point locus is *nonempty*, *simply connected*

· [SemiAn], §2; [CbGC], Prop. 1.2: *verticial/edge-like* s/gps.

satisfy *commensurable terminality* and *independence* properties

· [SemiAn], Thm. 3.7: $\forall$ semi-graph of anabelioids of PSC-type,

$\{$*maximal compact s/gps.*$\}$ = $\{$*verticial s/gps.*$\}$;
$\{$*nontriv.* $\cap$*'s of distinct max. comp. s/gps.*$\}$ = $\{$*edge-like s/gps.*$\}$

· [SemiAn], Cor. 3.11, Thm. 6.6: $\forall$ *hyp. curve*/fin. extn. of $\mathbb{Q}_p$,

geometric tempered fundamental group
$\rightsquigarrow$ *semi-graph* of *special fiber*

. . . in particular, *temp. fund. gp.* $\Longleftrightarrow$
ét. fund. gp. + *special fiber semi-graphs* of fin. et. covs.

T1. <u>**Motivation from differential topology and arithmetic**</u>
(cf. [SemiAn], [CbCsp])

- Semi-graphs of anabelioids and configuration spaces

- The Dehn-Nielsen-Baer Theorem

- The cuspidalization problem (complete solution in discrete case)

- Anabelian encoding of base field ring structure in a tripod

T2. **Combinatorial anabelian results**
(cf. [CbGC], [NodNon], [CbTpIII])

· Combinatorial Groth. Conj.-type results and cuspidalization

· [CbGC], Cor. 2.7(i)(iii): IPSC to IPSC:

gp.-theoretically cuspidal $\Longrightarrow$ graphic

· [NodNon], Thm. A: NN to NN:

gp.-theoretically cuspidal $\wedge$ $^{\exists}$at least one cusp $\Longrightarrow$ graphic

· [CbTpII], Thm. 1.9: IPSC to NN:

gp.-theoretically verticial, nodal

T3. **Combinatorial cuspidalization and "FC = F" results** (cf. [CbCsp], [NodNon], [CbTpI], [CbTpII], [HMT])

· [NodNon], Cor. 6.5: Complete solution in discrete case $\forall\ n \geq 1$:

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \ \stackrel{\sim}{\to}\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

· [NodNon], Thm. B: profinite/pro-l case:

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \ \to\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

injective if $n \geq 1$, *bijective* if $n \geq 3$ (affine) or $n \geq 4$ (proper)

· [CbTpI], Thm. A: profinite/pro-l case:

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\ (\subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n))$$

· [CbTpII], Thm. A(i): profinite/pro-l case:

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \ \to\ \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

injective if $n \geq 1$ (affine) or $n \geq 2$ (proper),
bijective if $n \geq 3$ (affine) or $n \geq 4$ (proper)

· [CbTpII], Thm. A(ii); [HMT], Cor. 2.2: profinite/pro-l case:

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

if $n \geq 2$ (genus 0) or $n \geq 3$ (affine) or $n \geq 4$ (proper)

T4. **Arithmetic applications of combinatorial cuspidalization over number fields and mixed-characteristic local fields** (cf. [NodNon], [CbTpIII])

· [NodNon], Thm. C: X a hyperbolic curve over F an NF or MLF:

$$G_F \to \mathrm{Out}(\Delta) \ \textit{injective}$$

$$(\text{where } 1 \to \Delta \to \pi_1^{\text{ét}}(X) \to G_F \to 1)$$

· [CbTpIII], Thm. B(iii): X a hyperbolic curve over F an NF, $\mathfrak{p}$ a nonarc. prime of F:

$$G_{\mathfrak{p}} = G_F \cap \mathrm{Out}^{\mathrm{M}}(\Delta) \ (\subseteq \ \mathrm{GT} \cap \mathrm{Out}^{\mathrm{M}}(\Delta)$$

$$\subseteq \ \mathrm{GT} \cap \mathrm{Out}^{\mathrm{G}}(\Delta) \ = \ \text{Tsujimura's } p\text{-adic version of GT})$$

$$(\text{where } 1 \to \Delta \to \pi_1^{\text{ét}}(X) \to G_F \to 1)$$

T5. **Synchronization of cyclotomes, profinite Dehn twists, and geometric monodromy anabelian results**
(cf. [CbTpI])

· *Synchronization of cyclotomes*: a sort of profinite orientation
(cf. [CbTpI], §3)

· *Profinite Dehn twists*:
intrinsic definition, structure theorem, coordinates
(cf. [CbTpI], §4, §5)

· *Pro-Σ geo. monodr. "GC"* for $(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}$ when $(g, r) \neq (0, 3)$; $(1, 1)$:

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi)}\left(\text{open subgroup of } \Pi_{(\mathcal{M}_{g,r})_{\overline{\mathbb{Q}}}}\right) = \left(\text{scheme-theoretic aut. group of generic fiber}\right)$$

(cf. [CbTpI], §6)

· Generalization to *Hurwitz stacks*

T6. **Tripod synchronization and the tripod homomorphism** (cf. [CbTpII])

· [CbTpII], Thm. C(i): *Commensurator/centralizer*

$$C_{\Pi_n}(T) = T \times Z_{\Pi_n}(T)$$

of a *tripod* $T \subseteq \Pi_n$; hence we have a natural homomorphism

$$\mathfrak{T}_T : \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \to \mathrm{Out}(T)$$

· *Central tripods*: tripods emanating from the *generic diag. cusp*

· [CbTpII], Thm. C(iv): if $(\Pi_n \twoheadrightarrow)\ \Pi_3 \supseteq T$ is *central*, then we obtain the *tripod homomorphism*

$$\mathfrak{T}_T : \mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{GT}\ (\subseteq \mathrm{Out}(T))$$

for $n \geq 3$ (affine) or $n \geq 4$ (proper)

("$\twoheadrightarrow$ GT" requires theory of *gluing of cuspidalizations*!)

· *Tripod synchronization*: commutative diagram of $\mathfrak{T}_T$'s

T7. **Structure theory and arithmetic subgroups of the Grothendieck-Teichmüller group**
(cf. [MT], [HMM], [HMT])

- [HMM], Thm. A: reconstruction of *generalized fiber subgroups*

- [HMM], Thm. B: profinite/pro-l case:

$$\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times \mathfrak{S}_{n^*}$$

for $n \geq 2$ and (g, r) s.t. $(r, n) \neq (0, 2)$.

- [HMM], Thm. C: in profinite/pro-l (cf. "Σ") $(0, 3)$ case: for $n \geq 2$,

$$\mathrm{Out}(\Pi_n) = \mathrm{GT}^{\Sigma} \times \mathfrak{S}_{n+3}; \quad \mathfrak{S}_{n+3} = Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n));$$
$$\mathrm{GT}^{\Sigma} = Z_{\mathrm{Out}(\Pi_n)}(Z^{\mathrm{loc}}(\mathrm{Out}(\Pi_n)))$$

(also *purely gp.-th. characterization* + *pf.* of $\mathfrak{S}_{n+3} \subseteq \mathrm{Out}(\Pi_n)$)

- [HMT], Thm. A: for BGT $\subseteq$ GT s.t. COF $\wedge$ RGC:

$$C_{\mathrm{GT}}(\mathrm{BGT}) \to G_{\mathbb{Q}_{\mathrm{BGT}}};$$

- [HMT], Thms. B, F: for BGT $\subseteq$ GT s.t. QAA:

$$C_{\mathrm{GT}}(\mathrm{BGT}) \hookrightarrow G_{\mathbb{Q}_{\mathrm{BGT}}}$$

(e.g.: abs. Gal. gp. of *TKND-AVKF-field*)

- [HMT], Thm. C: GT $\supseteq G_{\mathbb{Q}}$ is **maximal AA closed s/gp.**

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