First Steps

Most of my research from the summer of 1992 when I earned by Ph.D. up until the summer of 2000 falls into one of the following three categories:

(a) **p-adic Teichmüller Theory** : (1993～1996)

This theory may be regarded, on the one hand, as a p-adic analogue of Koebe's uniformization by the upper half-plane of hyperbolic Riemann surfaces over the complex numbers and Bers' uniformization of the corresponding moduli space, and, on the other hand, as an analogue for hyperbolic curves of Serre-Tate theory for ordinary abelian varieties. For a more detailed discussion of this topic, the reader is referred to the following papers:

A Theory of Ordinary p-adic Curves

An Introduction to p-adic Teichmüller Theory

(b) **p-adic Anabelian Geometry** : (1995～1996)

One typical theorem of this theory asserts, in a relative setting over a "sub-p-adic field" (i.e., a subfield of a finitely generated extension of a p-adic local field), that there is a natural bijection between the nonconstant maps from an arbitrary variety to a hyperbolic curve and the open outer homomorphisms between the respective arithmetic fundamental groups. For a more detailed discussion of this topic, the reader is referred to the following paper:

The Local Pro-p Anabelian Geometry of Curves

(c) **The Hodge-Arakelov Theory of Elliptic Curves** : (1998～2000)

The goal of this theory is to realize, in an Arakelov-theoretic setting, an analogue for elliptic curves over number fields of Hodge theory as it is known over the complex and p-adic numbers. A typical theorem asserts that there is a bijection between a certain space of functions on the universal extension of an elliptic curve over a number field and a space of functions on the torsion points of the elliptic curve that is compatible, up to a certain discrepancy, with metrics at all of the primes of the number field. The theory may also be thought of as a sort of "scheme-theoretic discretization" of the classical **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}.$$

For a more detailed discussion of this topic, the reader is referred to the following papers:

A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, II

Efforts Toward a New Framework

Hodge-Arakelov theory exhibits many attractive features, such as the construction of an **arithmetic Kodaira-Spencer map**, which suggest some sort of relationship to the ABC Conjecture. Hodge-Arakelov theory alone, however, is not

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sufficient to yield a proof of the ABC Conjecture on account of certain fundamental obstacles. My intuition led me to believe that to overcome these obstacles would require the use of some sort of

> **framework that lies fundamentally beyond the scheme-theoretic framework of conventional arithmetic geometry.**

Thus, between the summer of 2000 and the summer of 2006, I began to investigate just what would be necessary to construct such a framework and to develop the mathematical infrastructure that would serve as the foundations for such a framework. Such research activities were supported by the following fundamental philosophy:

> The essence of arithmetic geometry lies not in the various specific schemes that occur in a specific arithmetic-geometric setting, but rather in the **abstract combinatorial patterns**, along with the **combinatorial algorithms** that describe these patterns, that govern the dynamics of such specific schemes.

I decided to refer to the geometry based on this philosophy as **"inter-universal (IU) geometry"**. Three fundamental examples of this phenomenon that I had in mind are the following:

- the **monoids** that appear in the geometry of log schemes
- the arithmetic fundamental groups, i.e., **Galois categories**, that appear in anabelian geometry
- the abstract structure of **graphs**, such as the dual graph of a degenerate stable curve.

The "monoids", "Galois categories", and "graphs" that appear in these three examples may be regarded as special cases of the notion of a **category**. (For instance, in the case of a graph, one obtains a category by considering the paths on the graph.) Thus, one (but not the only) important aspect of inter-universal geometry may be thought of as being manifested in the

"geometry of categories".

In particular, in the case of anabelian geometry, this "geometry of categories" corresponds to

absolute anabelian geometry

(i.e., anabelian geometry conducted in a setting in which one does not regard the absolute Galois group of the base field as being given).

Typical examples of my research during these six years (i.e., the summer of 2000 to the summer of 2006) concerning the main themes of

the **"geometry of categories"** and **absolute anabelian geometry**

are the following:

· **The geometry of anabelioids** (2001)

I investigate the properties of the category of finite étale coverings that arises by treating slim profinite groups (i.e., profinite groups for which the center of every open subgroup is trivial) as geometric objects. In particular, when the profinite groups in question arise as arithmetic fundamental groups of hyperbolic curves over p-adic fields, this category exhibits various interesting properties, such as a sort of **absolute and canonical "boundedness"** that is reminiscent of the geometry of the upper half-plane.

· **The absolute anabelian geometry of canonical curves** (2001)

I verify a theorem of absolute anabelian type, which is the first such theorem to be proven over p-adic fields, for the canonical curves that appear in p-adic Teichmüller theory.

· **Categorical representation of locally noetherian log schemes** (2002)

I obtain a fundamental result, which looks as though it might have been discovered in the 1960's, to the effect that a scheme or log scheme may be reconstructed naturally from the category of (log) schemes of finite type over the given (log) scheme.

· **Semi-graphs of anabelioids** (2004)

I prove that various scheme-theoretic "patterns" are faithfully reflected in the geometry of "semi-graphs of anabelioids" — a minor generalization of classical "graphs of groups" — and obtain various related results of an anabelian flavor.

· **A combinatorial version of the Grothendieck conjecture** (2004)

I treat the "semi-graphs of anabelioids" that are associated to degenerate stable curves in an abstract combinatorial framework in which scheme theory does not appear explicitly and prove various "reconstruction theorems" of an anabelian flavor.

· **Conformal and quasiconformal categorical representation of hyperbolic Riemann surfaces** (2004)

I give category-theoretic descriptions of the geometry of hyperbolic Riemann surfaces via two approaches. The first approach originates from the uniformization by the upper half-plane, while the second arises from considering "rectangles" (which correspond to the conformal structure) and "parallelograms" (which correspond to the quasiconformal structure) on the Riemann surface.

· **Absolute anabelian cuspidalizations of proper hyperbolic curves** (2005)

I develop a theory of reconstruction of the arithmetic fundamental groups of open subschemes from the arithmetic fundamental group of a given proper hyperbolic curve. By applying this theory to absolute anabelian geometry over finite and p-adic fields, I settle various unknown conjectures.

• The geometry of Frobenioids I, II (2005)

I study how **étale-like** categorical structures such as Galois categories and **Frobenius-like** category-theoretic structures such as monoids (of the sort that appear in the theory of log schemes) operate upon one another, as well as how these two types of structure may be distinguished from one another.

Teichmüller Theory for Number Fields

By the latter half of 2006, my ideas concerning the theory that I was seeking to develop had congealed into a somewhat more definite form, and, as a result, my efforts to render these ideas in written form gained momentum. The "more definite form" of these ideas was, in a word, the following:

> to develop a **"pattern-wise" analogous theory** for **number fields equipped with a one-pointed elliptic curve** to the p-adic Teichmüller theory that I had developed for **positive characteristic hyperbolic curves equipped with a nilpotent, ordinary indigenous bundle**.

Here, the "one-pointed elliptic curve" (over a number field) is regarded as including the **Hodge-Arakelov theory** which may be conducted over this one-pointed elliptic curve. I decided to refer to this theory as **"inter-universal Teichmüller theory"**, i.e., "IUTeich" for short. During the development of IUTeich, I was frequently impressed by the extent and detail of the structural, "pattern-wise" similarity of IUTeich with p-adic Teichmüller theory (i.e., "pTeich") — a similarity that exists despite the fact that IUTeich is a theory that is formulated in a way that lies essentially outside the framework of scheme theory (i.e., in an "inter-universal framework").

My papers devoted to "preparing for IUTeich" during the period from 2006 to the spring of 2008 are the following:

• The étale theta function and its Frobenioid-theoretic manifestations (2006)

Let us refer to as the **étale theta function** the Kummer class associated to the theta function which is defined on a certain covering of a degenerating elliptic curve, i.e., a Tate curve, over a p-adic local field. This étale theta function, together with a certain Kummer-theoretic object associated to the theta trivialization, satisfies various interesting absolute anabelian and rigidity properties. Certain of these properties only become meaningful when examined in relation to the theory of Frobenioids. Moreover, this étale theta function is expected to give rise, in IUTeich, to an object that corresponds to the **canonical Frobenius lifting** of pTeich. It is by "differentiating this object which is analogous to a Frobenius lifting" that I expect that the inequality of the ABC Conjecture may be verified. This sort of argument involving the derivation of an inequality in this way may be regarded as a sort of inter-universal version of the following classical argument:

If one assumes that a smooth, proper genus g curve over the ring of Witt vectors associated to a perfect field of positive characteristic is equipped with a Frobenius lifting, then by differentiating this Frobenius lifting and computing degrees of sheaves of differentials, one obtains the inequality

$$g \leq 1.$$

- Topics in absolute anabelian geometry I: generalities (2008)

The main theme of this series (I, II, III) is the idea that one should pursue absolute anabelian geometry not from the point of view of attempting to obtain "fully faithfulness results in the style of the Grothendieck Conjecture", but rather from the point of view of developing **"group-theoretic algorithms, or software"**. In this first paper of the series, we develop various preparatory results that will be of use in the further development of the theory. One typical theorem obtained is a "Hom version" in the style of the Grothendieck Conjecture, which is the first of its kind in (semi-)absolute p-adic geometry, and which is obtained by applying a certain unpublished result that was related to me by A. Tamagawa. We remark in passing that this last result has no direct relation to IUTeich.

- Topics in absolute anabelian geometry II: decomposition groups (2008)

In addition to various "preparations for IUTeich", we develop the absolute anabelian geometry of configuration spaces — which has no direct logical relation to IUTeich — as well as a theory of reconstruction of the additive structure of the base field from the decomposition groups of points in an absolute p-adic anabelian setting. This last p-adic theory makes use of an argument of the sort discussed above involving the **"derivation of an inequality by differentiating a Frobenius lifting"**, hence exhibits certain aspects that are philosophically related to IUTeich.

- Topics in absolute anabelian geometry III: global reconstruction algorithms (2008)

We develop **"mono-anabelian geometry"** — which is to be regarded as being in sharp contrast with "bi-anabelian geometry", which has as its goal the proof of "fully faithfulness results in the style of the Grothendieck Conjecture" — in a global setting over a number field. This is precisely

the sort of anabelian geometry that we expect to use in IUTeich.

For a more detailed discussion of the content of this theory and its relation to the idea of IUTeich, we refer to the Introduction of this paper.

At this point, let us recall the following interesting fact. Grothendieck's principal original motivation for proposing his "anabelian philosophy" in the famous "letter to Faltings" apparently lay precisely in the possibility of applications to diophantine geometry. That is to say, at first glance, the fact that anabelian geometry plays a central role in IUTeich (which I expect to have applications to the ABC Conjecture) appears to confirm Grothendieck's intuition. On the other hand, a closer examination reveals that things are not so simple. For instance, in the approach to such applications that Grothendieck apparently had in mind, the observation that the "Section Conjecture" over number fields makes it possible to treat limits of sequences of rational points over number fields plays a central role. By contrast, in IUTeich, it is not the Section Conjecture over number fields, but rather

> the existence of **mono-anabelian algorithms** (which constitute a sort of absolute anabelian geometry) that hold over **both number fields and p-adic fields** in a compatible fashion that plays a central role.

These "mono-anabelian algorithms" correspond to the Frobenius invariants of the $\mathcal{MF}^{\nabla}$-objects that appear in pTeich — that is to say, they may be regarded as inter-universal analogues of the

Teichmüller representatives of Witt rings and **canonical curves of pTeich**

that appear in the p-adic theory. Put another way, these "mono-anabelian algorithms" define a sort of **canonical lifting, or splitting**. Moreover, this state of affairs in which (mono-anabelian) objects of Galois-theoretic origin correspond to crystals (i.e., the underlying crystals of $\mathcal{MF}^{\nabla}$-objects) in the p-adic theory is reminiscent of the "arithmetic Kodaira-Spencer map" of Hodge-Arakelov theory (which arises by considering Galois actions).

I plan to begin writing up the "main body" of IUTeich in April 2008. This task consists, roughly speaking, of gluing together the following three theories:

- The geometry of Frobenioids I, II
- The étale theta function and its Frobenioid-theoretic manifestations
- Topics in absolute anabelian geometry III

Incidentally, if one regards the scheme-theoretic Hodge-Arakelov theory that I studied until the summer of 2000 as being a sort "scheme-theoretic discretization" of the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi},$$

then IUTeich may be regarded as a sort of

"global Galois-theoretic or inter-universal version of this Gaussian integral".

Moreover, the coordinate transformation between **Cartesian coordinates** and **polar coordinates** that appears in the classical computation of the Gaussian integral may be regarded as corresponding (in the inter-universal version) to the **"Frobenius-like structures"** and **"étale-like structures"** that were studied in "The geometry of Frobenioids I, II". At the present time, I plan to write up the "main body" of the theory in the following two papers:

- **Inter-universal Teichmüller theory I: Hodge-Arakelov-theoretic aspects**
(scheduled for completion (?) in 2009)

I plan to construct the **inter-universal versions** of the **canonical liftings** "modulo p^n" of curves and the Frobenius morphism that appear in p-adic Teichmüller theory.

· **Inter-universal Teichmüller theory II: limits and bounds** (scheduled for completion (?) in 2010)

I plan to construct the **"inter-universal limits"** that correspond to p-adic limits that arise when one allows the n of the "mod p^n liftings" discussed above to vary and then to compute the objects that correspond to the **derivative of the Frobenius lifting** in pTeich.