Mono-anabelian geometry I: Reconstruction of function fields via Belyi cuspidalization

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The main result (cf. [AbsTpIII], (1.9), (1.11.3))

Theorem

Let k_0 *number field,* Y_0/k_0 *proper smooth curve,* $X_0 \subset Y_0$ *open*
X_0/k_0 *hyperbolic curve, isogenous to genus* 0*,* $k_0 \hookrightarrow k$*, with* k *sub-p-adic*
$\overline{k}$ *algebraic closure of* k*,* $\overline{k}_0 \subset \overline{k}$ *algebraic closure of* k_0
Write $X \stackrel{\mathrm{def}}{=} X_0 \times_{k_0} k$*,* $Y \stackrel{\mathrm{def}}{=} Y_0 \times_{k_0} k$*,* $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$
Then:

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \mathrm{Gal}_k \to 1$$

(regarded as an exact sequence of abstract profinite groups)

$$\rightsquigarrow \textit{the field } \overline{k}_0(Y_0)$$

First, we prove Theorem in the case where Y_0/k_0 of genus ≥ 2

Summary

(i) Belyi cuspidalization and Nakamura,...
$\rightsquigarrow \{\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X)\}$, $\{I_x \subset \pi_1(X \setminus S)\}$, and the set Y_0

(ii) $\rightsquigarrow$ divisors, principal divisors

(iii) synchronization of geometric cyclotomes $\rightsquigarrow I_x \simeq M\ (\simeq \hat{\mathbb{Z}}(1))$

(iv) Kummer theory $\rightsquigarrow \mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S) \hookrightarrow \mathrm{H}^1(\pi_1(Y \setminus S), M)$

(v) $\rightsquigarrow$ the multiplicative group $\bar{k}_0(Y_0)^{\times}$

(vi) Uchida $\rightsquigarrow$ the field $\bar{k}_0(Y_0)$

Belyi cuspidalization (cf. [AbsTpII], (3.6), (3.7), (3.8))

- We may assume without loss of generality that k_0 is algebraic closed in k
- Let $S_0 \subset Y_0^{\mathrm{cl}}$ finite subset ($\mathrm{cl} =$ "the set of closed points")
- Write $Y_{\mathrm{NF}}^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \mathrm{Im}(Y_0^{\mathrm{cl}} \hookrightarrow Y^{\mathrm{cl}})$, $S \stackrel{\mathrm{def}}{=} \mathrm{Im}(S_0 \subset Y_0^{\mathrm{cl}} \stackrel{\sim}{\to} Y_{\mathrm{NF}}^{\mathrm{cl}})$

We want to reconstruct $\{\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X)\}_{S_0 \subset Y_0^{\mathrm{cl}} \text{ finite subset}}$
Since X_0 isogenous to genus 0

$$\begin{array}{ccccc} \exists V & \xrightarrow[\text{étale}]{\exists\ \text{finite Galois}} & Q & \hookrightarrow & \mathbb{P}^1_{\exists k'} \setminus \{0, 1, \infty\} \\ {\scriptstyle \exists\ \text{finite}}\ \big\downarrow\ {\scriptstyle \text{étale}} & & & & \\ X & & & & \end{array}$$

where k'/k finite extension

By the existence of Belyi maps

$$\begin{array}{ccccccc}
 & & & & W_{\exists k''} & \xrightarrow[\text{étale}]{\exists\ \text{finite Galois}} & \exists W \\
 & & & & \Big\downarrow{\scriptstyle \exists\ \text{finite}\ \mid\ \text{étale}} & & \Big\downarrow{\scriptstyle \text{open}} \\
V & \xrightarrow[\text{étale}]{\text{finite Galois}} & Q & \hookrightarrow & \mathbb{P}^1_{k'} \setminus \{0,1,\infty\} & & X \setminus S \\
\Big\downarrow{\scriptstyle \text{finite}\ \mid\ \text{étale}} & & & & & & \Big\downarrow{\scriptstyle \text{open}} \\
X & & & & & & X,
\end{array}$$

where k''/k' finite extension, k''/k Galois

$\rightsquigarrow \{\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X)\}_{S_0 \subset Y_0^{\mathrm{cl}}\ \text{finite subset}}$

Cuspidal inertia groups

For $E \subset Y^{\mathrm{cl}}$ subset, write $\mathrm{Div}(E) \overset{\mathrm{def}}{=} \bigoplus_{x \in E} \mathbb{Z}_x$

- Nakamura,... (cf. [AbsTpI], (4.5))
  $\pi_1(X \setminus S) \rightsquigarrow$ inertia groups $\{I_x \subset \pi_1(X \setminus S)\}_{x \in (Y \setminus (X \setminus S))(\bar{k})}$
- $\pi_1(X \setminus S)$-conj classes of inertia groups $\rightsquigarrow$ the set $Y^{\mathrm{cl}}_{\mathrm{NF}}$
- $\rightsquigarrow \mathrm{Div}(Y^{\mathrm{cl}}_{\mathrm{NF}}) = \bigoplus_{x \in Y^{\mathrm{cl}}_{\mathrm{NF}}} \mathbb{Z}_x$

$\pi_1(\mathrm{Pic}^n_Y)$

- $\rightsquigarrow \pi_1(Y) = \pi_1(X)/\overline{\langle I_x \mid x \in Y(\bar{k})\rangle}$
- $\rightsquigarrow \pi_1(Y_{\bar{k}}) = \mathrm{Ker}(\pi_1(Y) \twoheadrightarrow \mathrm{Gal}_k)$
- $\rightsquigarrow \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) = \pi_1(Y_{\bar{k}})^{\mathrm{ab}}$
- $\rightsquigarrow \pi_1(\mathrm{Pic}^1_Y) = \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) \coprod_{\pi_1(Y_{\bar{k}})} \pi_1(Y)$
- $\pi_1(\mathrm{Pic}^2_Y) = \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) \coprod_{\pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) \times \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}})} (\pi_1(\mathrm{Pic}^1_Y) \times_{\mathrm{Gal}_k} \pi_1(\mathrm{Pic}^1_Y))$, where $\pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) \times \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}}) \to \pi_1(\mathrm{Pic}^1_{Y_{\bar{k}}})$ is multiplication
- $\cdots \rightsquigarrow \pi_1(\mathrm{Pic}^n_Y)$ $(n \in \mathbb{Z})$
- $\rightsquigarrow \pi_1(\mathrm{Pic}^n_{Y_{\bar{k}}}) = \mathrm{Ker}(\pi_1(\mathrm{Pic}^n_Y) \twoheadrightarrow \mathrm{Gal}_k)$ $(n \in \mathbb{Z})$

Decomposition groups and degree map

Let $x \in (Y \setminus (X \setminus S))(\bar{k})$

- $I_x \subset \pi_1(X \setminus S) \rightsquigarrow$ the decomposition group $D_x = N_{\pi_1(X \setminus S)}(I_x)$
- $\rightsquigarrow D_x^{\mathrm{cpt}} \stackrel{\mathrm{def}}{=} D_x / I_x$. Thus,

$$\begin{array}{ccccc} \pi_1(X \setminus S) & \twoheadrightarrow & \pi_1(Y) & \twoheadrightarrow & \mathrm{Gal}_k \\ \cup & & \cup & & \cup \\ D_x & \twoheadrightarrow & D_x^{\mathrm{cpt}} & \xrightarrow{\sim} & \mathrm{Gal}_{\kappa(x)} \end{array}$$

$\rightsquigarrow \deg \colon \mathrm{Div}(Y_{\mathrm{NF}}^{\mathrm{cl}}) \to \mathbb{Z} \colon \sum n_x \cdot x \mapsto \sum n_x \cdot [\mathrm{Gal}_k : \mathrm{Gal}_{\kappa(x)}]$

$\rightsquigarrow Y_{\mathrm{NF}}^{\mathrm{cl}}(k) \stackrel{\mathrm{def}}{=} Y_{\mathrm{NF}}^{\mathrm{cl}} \cap Y(k) = \{x \in Y_{\mathrm{NF}}^{\mathrm{cl}} \mid \deg(x) = 1\}$

Principal divisors (cf. [AbsTpIII], (1.6))

Let $x \in (Y \setminus (X \setminus S))(\bar{k})$, $D \in \mathrm{Div}(Y^{\mathrm{cl}}_{\mathrm{NF}}(k))$

- $s_x \colon D_x \subset \pi_1(X \setminus S) \twoheadrightarrow \pi_1(Y) \twoheadrightarrow \pi_1(\mathrm{Pic}^1_Y)$
- if $\deg(D) = 0$, then $s_D \in \mathrm{H}^1(\mathrm{Gal}_k, \pi_1(\mathrm{Pic}^0_{Y_{\bar{k}}}))$
- D principal $\overset{\mathrm{def}}{\iff} \exists f \in k(Y)^\times \colon \mathrm{div}(f) = D$

$$\iff \begin{cases} \deg(D) = 0 \\ s_D = 0 \text{ in } \mathrm{H}^1(\mathrm{Gal}_k, \pi_1(\mathrm{Pic}^0_{Y_{\bar{k}}})) \end{cases}$$

$\rightsquigarrow \mathrm{PDiv}(Y^{\mathrm{cl}}_{\mathrm{NF}}(k)) \overset{\mathrm{def}}{=} \{D \in \mathrm{Div}(Y^{\mathrm{cl}}_{\mathrm{NF}}(k)) \mid D \text{ principal } \}$

Synchroniz'n of geom. cyclotomes (cf. [AbsTpIII], (1.4))

We know (i.e., can reconstruct) $M \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathrm{H}^2(\pi_1(Y_{\bar{k}}), \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$ $(\simeq \hat{\mathbb{Z}}(1))$
Let $x \in Y^{\mathrm{cl}}_{\mathrm{NF}}(k)$. Write $U \stackrel{\mathrm{def}}{=} Y \setminus \{x\}$
Then we have a natural exact sequence

$$1 \to I_x \to \pi_1^{\mathrm{cc}}(U_{\bar{k}}) \to \pi_1(Y_{\bar{k}}) \to 1$$

where $\pi_1(U_{\bar{k}}) \twoheadrightarrow \pi_1^{\mathrm{cc}}(U_{\bar{k}})$ for the maximal cuspidally central quotient.
Thus,

$$E_2^{i,j} = \mathrm{H}^i(\pi_1(Y_{\bar{k}}), \mathrm{H}^j(I_x, I_x)) \Longrightarrow \mathrm{H}^{i+j}(\pi_1^{\mathrm{cc}}(U_{\bar{k}}), I_x)$$

$$1 \in \hat{\mathbb{Z}} = \mathrm{Hom}(I_x, I_x) = \mathrm{H}^0(\pi_1(Y_{\bar{k}}), \mathrm{H}^1(I_x, I_x))$$

$$\stackrel{d^{0,1}}{\to} \mathrm{H}^2(\pi_1(Y_{\bar{k}}), \mathrm{H}^0(I_x, I_x)) = \mathrm{Hom}(M, I_x) \ni d^{0,1}(1)$$

$\Longrightarrow d^{0,1}(1)\colon M \stackrel{\sim}{\to} I_x$

Kummer theory

Let $S_0 \subset Y_0(k_0)$ finite. Write $S \overset{\mathrm{def}}{=} \mathrm{Im}(S_0 \subset Y_0(k_0) \overset{\sim}{\to} Y^{\mathrm{cl}}_{\mathrm{NF}}(k))$

$\Longrightarrow \quad 1 \to \mu_N \to \mathcal{O}^{\times}_{Y\setminus S} \overset{N}{\to} \mathcal{O}^{\times}_{Y\setminus S} \to 1 \quad$ (on the étale site of $Y \setminus S$)

$$\Longrightarrow \; 1 \longrightarrow \mu_N(k) \longrightarrow \mathcal{O}^{\times}(Y \setminus S) \longrightarrow \mathcal{O}^{\times}(Y \setminus S)$$

$$\longrightarrow \mathrm{H}^1_{\text{ét}}(Y \setminus S, \mu_N) \longrightarrow \mathrm{Pic}_{Y\setminus S} \longrightarrow \mathrm{Pic}_{Y\setminus S}$$

$\Longrightarrow \mathcal{O}^{\times}(Y \setminus S) \to \mathrm{H}^1_{\text{ét}}(Y \setminus S, \mu_N) \simeq \mathrm{H}^1(\pi_1(Y \setminus S), \mu_N)$

$\Longrightarrow \mathcal{O}^{\times}(Y \setminus S) \to \varprojlim_N \mathcal{O}^{\times}(Y \setminus S)/N \hookrightarrow \mathrm{H}^1(\pi_1(Y \setminus S), \hat{\mathbb{Z}}(1))$

How do we recover $\mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S)$?

We want to reconstruct $\mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}(Y \setminus S) \cap \bar{k}_0(Y_0) \subset \bar{k}(Y)$

$$\begin{array}{ccccccc}
1 & \longrightarrow & k^{\times} & \longrightarrow & \mathcal{O}^{\times}(Y \setminus S) & \longrightarrow & \bigoplus_{x \in S} \mathbb{Z} \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
1 & \longrightarrow & \widehat{k^{\times}} & \longrightarrow & \mathcal{O}^{\times}\widehat{(Y \setminus S)} & \longrightarrow & \bigoplus_{x \in S} \hat{\mathbb{Z}}
\end{array}$$

Since k is sub-p-adic ($\Longrightarrow$ "torally Kummer-faithful"), $k^{\times} \to \widehat{k^{\times}}$ is injective

- Since k is sub-p-adic ($\Longrightarrow$ "Kummer-faithful")

$$\mathrm{H}^0(\mathrm{Gal}_k, \mathrm{H}^1(\pi_1(Y_{\bar{k}}), \hat{\mathbb{Z}}(1))) = 0$$

  Then by Hochschild-Serre spectral sequence

$$\mathrm{H}^1(\mathrm{Gal}_k, \hat{\mathbb{Z}}(1)) \hookrightarrow \mathrm{H}^1(\pi_1(Y \setminus S), \hat{\mathbb{Z}}(1)) \longrightarrow \bigoplus_{x \in S} \mathrm{H}^1(I_x, \hat{\mathbb{Z}}(1))$$

- Kummer theory $\Longrightarrow \widehat{k^\times} \simeq \mathrm{H}^1(\mathrm{Gal}_k, \hat{\mathbb{Z}}(1))$

(cf. [AbsTpIII], (1.6))

$$
\begin{array}{ccccc}
k^{\times} & \hookrightarrow & \mathcal{O}^{\times}(Y \setminus S) & \longrightarrow & \bigoplus_{x \in S} \mathbb{Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\widehat{k^{\times}} & \hookrightarrow & \widehat{\mathcal{O}^{\times}(Y \setminus S)} & \longrightarrow & \bigoplus_{x \in S} \hat{\mathbb{Z}} \\
\wr\big\| \text{ Kummer theory} & & \big\downarrow & & \wr\big\| \\
\mathrm{H}^1(\mathrm{Gal}_k, \hat{\mathbb{Z}}(1)) & \hookrightarrow & \mathrm{H}^1(\pi_1(Y \setminus S), \hat{\mathbb{Z}}(1)) & \longrightarrow & \bigoplus_{x \in S} \mathrm{H}^1(I_x, \hat{\mathbb{Z}}(1))
\end{array}
$$

We want to reconstruct $\mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S) \subset \mathcal{O}^{\times}(Y \setminus S)$

Now we proceed to recover $\mathcal{O}_{\mathrm{NF}}^{\times}(Y \setminus S)$

- Since $S \subset Y_{\mathrm{NF}}^{\mathrm{cl}}(k)$, we know

$$\mathrm{PDiv}(S) \subset \mathrm{Div}(S) = \bigoplus_{x \in S} \mathbb{Z} \subset \bigoplus_{x \in S} \mathrm{H}^1(I_x, M)$$

  where $\mathbb{Z} \to \mathrm{H}^1(I_x, M)$: $1 \mapsto d^{0,1}(1)^{-1}$, and $\hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{H}^1(I_x, M)$
- $\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X) \twoheadrightarrow \pi_1(Y) \rightsquigarrow \pi_1(Y \setminus S)$
- $\rightsquigarrow P_{Y \setminus S} \stackrel{\mathrm{def}}{=} \mathrm{H}^1(\pi_1(Y \setminus S), M) \times_{\bigoplus_{x \in S} \mathrm{H}^1(I_x, M)} \mathrm{PDiv}(S)$

  $(\text{“}{=}\text{''}\ \widehat{k^{\times}} \cdot \mathcal{O}_{\mathrm{NF}}^{\times}(Y \setminus S))$

$$
\begin{array}{ccccc}
k_0^\times & \hookrightarrow & \mathcal{O}_{\mathrm{NF}}^\times(Y \setminus S) & \longrightarrow & \mathrm{PDiv}(S) \\
\big\downarrow & & \big\downarrow & & \| \\
\mathrm{H}^1(\mathrm{Gal}_k, M) & \hookrightarrow & P_{Y \setminus S} & \longrightarrow & \mathrm{PDiv}(S) \\
\| & & \big\downarrow & & \big\downarrow \\
\mathrm{H}^1(\mathrm{Gal}_k, M) & \hookrightarrow & \mathrm{H}^1(\pi_1(Y \setminus S), M) & \longrightarrow & \bigoplus_{x \in S} \mathrm{H}^1(I_x, M)
\end{array}
$$

Evaluation

Let $x \in Y_{\mathrm{NF}}^{\mathrm{cl}} \setminus S$

- Kummer theory $\Longrightarrow \mathrm{H}^1(D_x^{\mathrm{cpt}}, M) \simeq \widehat{\kappa(x)^{\times}}$
- Evaluation

$$P_{Y \setminus S} \subset \mathrm{H}^1(\pi_1(Y \setminus S), M) \to \mathrm{H}^1(D_x^{\mathrm{cpt}}, M) \simeq \widehat{\kappa(x)^{\times}}$$

$$\Longrightarrow P_{Y \setminus S} \to \widehat{\kappa(x)^{\times}} : \eta \mapsto \eta(x)$$

Rational functions (cf. [AbsTpIII], (1.8))

- We know $P_{Y\setminus S} \subset \mathrm{H}^1(\pi_1(Y \setminus S), M)$
  $\rightsquigarrow \mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S)$
  $= \{\eta \in P_{Y\setminus S} \mid \exists x \in Y^{\mathrm{cl}}_{\mathrm{NF}} \setminus S,\ \exists n \in \mathbb{Z}_{>0} \colon \eta(x)^n = 1 \in \widehat{\kappa(x)^{\times}}\}$

We want to reconstruct $\mathcal{O}^{\times}((Y_0 \setminus S_0)_{\bar{k}_0})$

- $H \subset \mathrm{Gal}_k$ open subgroup, $H = \mathrm{Gal}_{k'}$, where k'/k finite
  Consider

$$1 \to \pi_1(X_{\bar{k}}) \to \pi_1(X_{k'}) \to H \to 1$$

$\cdots \rightsquigarrow \mathcal{O}^{\times}_{\mathrm{NF}}((Y \setminus S)_{k'})$

$\rightsquigarrow \mathcal{O}^{\times}((Y_0 \setminus S_0)_{\bar{k}_0}) = \varinjlim_{k'/k \text{ finite}} \mathcal{O}^{\times}_{\mathrm{NF}}((Y \setminus S)_{k'})$

$\rightsquigarrow$ the multiplicative group $\bar{k}_0(Y_0)^{\times} = \varinjlim_{S_0} \mathcal{O}^{\times}((Y_0 \setminus S_0)_{\bar{k}_0})$

Already reconstructed

(i) Belyi cuspidalization and Nakamura,...
$\rightsquigarrow \{\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X)\}$, $\{I_x \subset \pi_1(X \setminus S)\}$, and the set $Y^{\mathrm{cl}}_{\mathrm{NF}}$

(ii) $\rightsquigarrow \mathrm{Div}(Y^{\mathrm{cl}}_{\mathrm{NF}})$, $\mathrm{PDiv}(Y^{\mathrm{cl}}_{\mathrm{NF}}(k))$

(iii) synchronization of geometric cyclotomes $\rightsquigarrow I_x \simeq M$ $(\simeq \hat{\mathbb{Z}}(1))$

(iv) Kummer theory

$$\rightsquigarrow \mathcal{O}^{\times}_{\mathrm{NF}}(Y \setminus S) \subset \mathcal{O}^{\times}(Y \setminus S) \hookrightarrow \mathrm{H}^1(\pi_1(Y \setminus S), M)$$

(v) $\rightsquigarrow$ the multiplicative group

$$\bar{k}_0(Y_0)^{\times} = \varinjlim_{S_0} \varinjlim_{k'/k \text{ finite}} \mathcal{O}^{\times}_{\mathrm{NF}}((Y \setminus S)_{k'})$$

We want to reconstruct the field $\bar{k}_0(Y_0)$

Order and divisor maps

Let $x \in (Y_0 \times_{k_0} k_0')(k_0')$, where k_0'/k_0 finite

$$\mathcal{O}^{\times}_{\mathrm{NF}}((Y_0 \times_{k_0} k_0' \setminus \{x\})_{k_0' \times_{k_0} k}) \subset \mathrm{H}^1(\pi_1((Y_0 \times_{k_0} k_0' \setminus \{x\})_{k_0' \times_{k_0} k}), M)$$

$$\underset{\mathrm{res}_x}{\to} \mathrm{H}^1(I_x, M) \overset{\sim}{\leftarrow} \hat{\mathbb{Z}} \supset \mathbb{Z}$$

$\Longrightarrow$ by letting k_0' vary, we reconstruct

$$\mathrm{ord}_x \colon \bar{k}_0(Y_0)^{\times} \to \mathbb{Z}$$

$\rightsquigarrow \bar{k}_0^{\times} = \bigcap_{x \in (Y_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}} \mathrm{Ker}(\mathrm{ord}_x) \subset \bar{k}_0(Y_0)^{\times}$

$\rightsquigarrow \mathrm{div} \colon \bar{k}_0(Y_0)^{\times} \to \mathrm{Div}(Y_0 \times_{k_0} \bar{k}_0) \colon f \mapsto \sum(\mathrm{ord}_x(f) \cdot x)$

Let $D \in \mathrm{Div}(Y_0 \times_{k_0} \bar{k}_0)$

- $\rightsquigarrow \mathrm{Div}^+(Y_0 \times_{k_0} \bar{k}_0) = \{\sum n_x \cdot x \in \mathrm{Div}(Y_0 \times_{k_0} \bar{k}_0) \mid n_x \geq 0\}$
- $\rightsquigarrow \mathrm{H}^0(D) = \{f \in \bar{k}_0(Y_0)^\times \mid \mathrm{div}(f) + D \in \mathrm{Div}^+(Y_0 \times_{k_0} \bar{k}_0)\} \cup \{0\}$
- $\rightsquigarrow \mathrm{h}^0(D)$
  $= \min\{n \mid \exists E \in \mathrm{Div}^+(Y_0 \times_{k_0} \bar{k}_0),\ \deg(E) = n,\ \mathrm{H}^0(D - E) = 0\}$

Proposition (cf. [AbsTpIII], (1.2))

$\exists D \in \mathrm{Div}(Y_0 \times_{k_0} \bar{k}_0)$, $\exists P_1, P_2, P_3 \in (Y_0 \times_{k_0} \bar{k}_0)^{\mathrm{cl}}$ *distinct points such that the following hold:*

(i) $\mathrm{h}^0(D) = 2$

(ii) $P_1, P_2, P_3 \notin \mathrm{Supp}(D)$

(iii) $\mathrm{h}^0(D - P_i - P_j) = 0\ \forall i \neq j \in \{1, 2, 3\}$

Field structure of $\bar{k}_0$ (cf. [AbsTpIII], (1.2))

Write $\bar{k}_0 \stackrel{\mathrm{def}}{=} \bar{k}_0^\times \cup \{0\}$
Let $a, b \in \bar{k}_0^\times$, suppose that $a \neq -b$.
Then we want to reconstruct $a + b \in \bar{k}_0^\times$

- We consider

$$H^0(D) \hookrightarrow \bar{k}_0 \times \bar{k}_0 \times \bar{k}_0 \colon f \mapsto (f(P_1), f(P_2), f(P_3))$$

- $\exists! f \in H^0(D)$, $f(P_1) = 0$, $f(P_2) \neq 0$, $f(P_3) = a$
- $\exists! g \in H^0(D)$, $g(P_1) \neq 0$, $g(P_2) = 0$, $g(P_3) = b$
- $\exists! h \in H^0(D)$ such that $h(P_1) = g(P_1)$, $h(P_2) = f(P_2)$
  $\Longrightarrow h = f + g$

$\rightsquigarrow a + b = h(P_3)$
$\rightsquigarrow$ the field $\bar{k}_0$

Field structure of $\bar{k}_0(Y_0)$ (cf. [AbsTpIII], (1.3))

Write $\bar{k}_0(Y_0) \stackrel{\mathrm{def}}{=} \bar{k}_0(Y_0)^\times \cup \{0\}$

- determine addition by adding values almost everywhere

$$\bar{k}_0(Y_0) = (\varinjlim_{S_0} \mathcal{O}^\times((Y_0 \setminus S_0)_{\bar{k}_0})) \cup \{0\} \hookrightarrow \varinjlim_{S_0} \prod_{P \in (Y_0 \setminus S_0)^{\mathrm{cl}}_{\bar{k}_0}} \bar{k}_0$$

- $\rightsquigarrow$ the field $\bar{k}_0(Y_0)$

This completes the proof of Theorem in the case where Y_0/k_0 of genus ≥ 2

Removal of restriction on the genus of Y_0

- There exists $H \subset \pi_1(X)$ normal open subgroup such that $H = \pi_1(Z_0 \times_{k'_0} k')$, where Z_0 hyperbolic curve, $g_{Z_0^{\mathrm{cpt}}} \geq 2$
- $\rightsquigarrow$ the field $\bar{k}_0(Z_0)$
- $\mathrm{Coker}(\pi_1((Z_0 \times_{k'_0} k')_{\bar{k}}) \hookrightarrow \pi_1(X_{\bar{k}}))$ acts on $\bar{k}_0(Z_0)$ by conjugation
- $\rightsquigarrow$ the field $\bar{k}_0(X_0) = \bar{k}_0(Z_0)^{\mathrm{Coker}(\pi_1((Z_0 \times_{k'_0} k')_{\bar{k}}) \hookrightarrow \pi_1(X_{\bar{k}}))}$

Review

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$\rightsquigarrow \{\pi_1(X \setminus S) \twoheadrightarrow \pi_1(X)\}$, $\{I_x \subset \pi_1(X \setminus S)\}$, and the set $Y_{\mathrm{NF}}^{\mathrm{cl}}$

(ii) $\rightsquigarrow \mathrm{Div}(Y_{\mathrm{NF}}^{\mathrm{cl}})$, $\mathrm{PDiv}(Y_{\mathrm{NF}}^{\mathrm{cl}}(k))$

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$$\rightsquigarrow \mathcal{O}_{\mathrm{NF}}^{\times}(Y \setminus S) \subset \mathcal{O}^{\times}(Y \setminus S) \hookrightarrow \mathrm{H}^1(\pi_1(Y \setminus S), M)$$

(v) $\rightsquigarrow$ the multiplicative group

$$\bar{k}_0(Y_0)^{\times} = \varinjlim_{S_0} \varinjlim_{k'/k \text{ finite}} \mathcal{O}_{\mathrm{NF}}^{\times}((Y \setminus S)_{k'})$$

(vi) Uchida $\rightsquigarrow$ the field $\bar{k}_0(Y_0)$