Frobenioids 1

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A **Frobenioid** is a category that is meant to encode the theory of **divisors** and **line bundles** on **"coverings"** i.e. normalizations in various finite separable extensions of the function field of a given normal integral scheme.

Having a sketchy idea of how to formulate IUT, Mochizuki developed the theory of Frobenioids which provided a **unified, intrinsic, category theoretic** language to encode the theory of divisors and line bundles in appropriate categories, general enough to fit in whatever would be developed.

Plan

1. Motivating examples
2. Basic definitions
3. Model Frobenioids

Example (Frobenioid of geometric origin)

V proper normal variety over k
K the function field

- Div_K the set of $\mathbb{Q}$-Cartier divisors on V

For a finite extension L of K put

- Div_L prime divisors of the normalization $V[L]$ that map into Div_K

- effective Cartier divisors of $V[L]$ with support in Div_L

$$\Phi(L) \in \mathfrak{Mon}$$

- (a subgroup of) Cartier divisors

$$\Phi^{\mathrm{gp}}(L) \in \mathfrak{Grp} \subset \mathfrak{Mon}$$

- the group of rational functions on $V[L]$ with zeroes and poles belonging to Div_L

$$\mathbb{B}(L)$$

- principal divisors homomorphism

$$\mathbb{B}(L) \to \Phi^{\mathrm{gp}}(L)$$

Let $\widetilde{K}$ be a Galois extension of K (can be infinite) with the Galois group

$$G \overset{\text{def}}{=} \mathrm{Gal}(\widetilde{K}/K)$$

which has a natural profinite topology.

The connected objects of the category of finite sets with continuous G-action (those which don't split into a disjoint union of non-empty G-sets)

$$\mathcal{D} \overset{\text{def}}{=} \mathcal{B}(G)^0$$

can be identified with finite extensions $K \subset L \subset \widetilde{K}$.

We can consider a category of pairs

$$(L, \mathcal{L})$$

where $K \subset L \subset \widetilde{K}$ is finite and $\mathcal{L}$ is a line bundle on $V[L]$ with morphisms

$$\phi : (L, \mathcal{L}) \to (M, \mathcal{M})$$

consisting of

- $\mathrm{Spec}(L) \to \mathrm{Spec}(M)$ morphism over $\mathrm{Spec}(K)$
- $d \in \mathbb{N}_{\geq 1}$
- $\mathcal{L}^{\otimes d} \to \mathcal{M}|_{V[L]}$ morphism of line bundles whose zero locus is a Cartier divisor supported in Div_L

Example (Frobenioid of arithmetic origin)

L: a number field
$\mathbb{V}(L)$: the set of valuations of L,
L_v: the completion of L at $v \in \mathbb{V}(L)$,

$$\mathcal{O}_v^\times \stackrel{\text{def}}{=} \{|z| = 1\}, \qquad \mathcal{O}_v^\triangleright \stackrel{\text{def}}{=} \{0 < |z| \leq 1\}$$

$$ord(L_v) \stackrel{\text{def}}{=} L_v^\times / \mathcal{O}_v^\times \cong \begin{cases} \mathbb{Z}, & \text{if } v \text{ nonarchimedean} \\ \mathbb{R}, & \text{if } v \text{ archimedean} \end{cases}$$

$$ord(\mathcal{O}_v^\triangleright) \stackrel{\text{def}}{=} \mathcal{O}_v^\triangleright / \mathcal{O}_v^\times \cong \begin{cases} \mathbb{Z}_{\geq 0}, & \text{if } v \text{ non-archimedean} \\ \mathbb{R}_{\geq 0}, & \text{if } v \text{ archimedean} \end{cases}$$

$$ord(L_v) = ord(\mathcal{O}_v^\triangleright)^{\text{gp}}.$$

- effective arithmetic divisors on L

$$\Phi(L) \stackrel{\text{def}}{=} \bigoplus_{v \in \mathbb{V}(L)} ord(\mathcal{O}_v^{\triangleright})$$

- arithmetic divisors on L

$$\Phi(L)^{\text{gp}} = \bigoplus_{v \in \mathbb{V}(L)} ord(L_v)$$

- multiplicative group of L

$$\mathbb{B}(L) \stackrel{\text{def}}{=} L^{\times}$$

- principal divisor homomorphism

$$\mathbb{B}(L) \to \Phi(L)^{\text{gp}}$$

Let F be a number field and let $\widetilde{F}/F$ be a Galois extension with Galois group

$$G \stackrel{\text{def}}{=} Gal(\widetilde{F}/F)$$

G has a natural profinite topology.

The connected objects of the category of finite sets with continuous G-action

$$\mathcal{D} \stackrel{\text{def}}{=} \mathcal{B}(G)^0$$

can be again identified with finite extensions $F \subset L \subset \widetilde{F}$.

We can consider a category of pairs

$$(L, \mathcal{L})$$

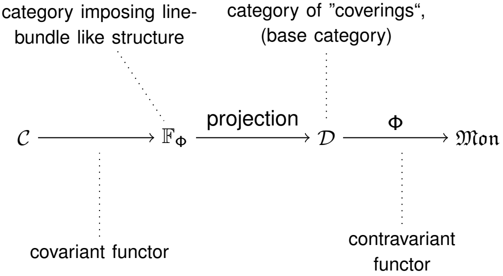
where $F \subset L \subset \widetilde{F}$ is finite and $\mathcal{L}$ is an arithmetic line bundle on $\mathrm{Spec}(\mathcal{O}_L)$ with morphisms

$$\phi : (L, \mathcal{L}) \to (M, \mathcal{M})$$

consisting of

- $\mathrm{Spec}(L) \to \mathrm{Spec}(M)$ morphism over $\mathrm{Spec}(F)$
- $d \in \mathbb{N}_{\geq 1}$
- $\mathcal{L}^{\otimes d} \to \mathcal{M}|_L$ morphism of arithmetic line bundles on L.

A Frobenioid is a category $\mathcal{C}$ which consists of the following data



For a commutative monoid $M \in \mathfrak{Mon}$

- $M^{\pm}$ submonoid of invertible elements of M
- $M^{\mathrm{char}} = M/M^{\pm}$
- M^{gp} groupification of M

Definition

A monoid $M \in \mathfrak{Mon}$ is called

1. *sharp* if $M^{\pm} = 0$
2. *integral* if $\iota : M \to M^{\mathrm{gp}}$ is injective
3. *saturated* if for $a \in M^{\mathrm{gp}}$ if $na \in \iota(M)$ for $n \in \mathbb{N}_{\geq 1}$ then $a \in \iota(M)$
4. *of characteristic type* if fibres of $M \to M^{\mathrm{char}}$ are torsors over $M^{\pm}$
5. *group-like* if M^{char} is trivial

Definition

A monoid is called

- *pre-divisorial* if it is integral, saturated and of characteristic type
- *divisorial* if it is pre-divisorial and sharp

Definition

A morphism

$$M \to N$$

in $\mathfrak{Mon}$ is called characteristically injective if it is injective and the induced morphism

$$M^{\mathrm{char}} \to N^{\mathrm{char}}$$

is also injective.

Definition

- A category is called *connected* if its associated graph

$$\text{vertices} \longleftrightarrow \text{objects}$$

$$\text{edges} \longleftrightarrow \text{morphisms}$$

  is connected.
- A category is called *totally epimorphic* if every morphism in this category is an epimorphism.

Definition

Let $\mathcal{C}$ be a category. An arrow $\beta : B \to A$ is called

- *fiberwise-surjective* if for every arrow $\gamma : C \to A$ there exist arrows $\delta_B : D \to B$ and $\delta_A : D \to A$ such that the following diagram

$$\begin{array}{ccccc} B & \xrightarrow{\beta} & A & \xleftarrow{\gamma} & C \\ & \nwarrow^{\delta_B} & & \nearrow_{\delta_A} & \\ & & D & & \end{array}$$

  commutes.
- FSM-*morphism* if it is a fiberwise-surjective monomorphism.

Definition

Let $\mathcal{D}$ be a category. A *monoid* on $\mathcal{D}$ is a contravariant functor

$$\Phi : \mathcal{D} \to \mathfrak{Mon}$$

such that for every morphism $\alpha : B \to A$ in $\mathcal{D}$

- $\alpha^* : \Phi(A) \to \Phi(B)$ is characteristically injective
- if α is FSM-morphism then α^* is an isomorphism of monoids,

where

$$\alpha^* \Phi(A) \to \Phi(B) := \Phi(\alpha : B \to A).$$

Definition (Elementary Frobenioid)

Let Φ be a monoid on a category $\mathcal{D}$. *Elementary Frobenioid associated to* Φ is a category

$$\mathbb{F}_{\Phi}$$

which objects are just objects of the category $\mathcal{D}$ and morphisms $\phi : A \to B$ are triples

$$\phi = (\phi_{\mathcal{D}}, \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi))$$

where

- $\phi_{\mathcal{D}} : A \to B$ is a morphism of $\mathcal{D}$,
- $\mathrm{Div}(\phi) \in \Phi(A)$ is the *zero-divisor* of ϕ,
- $\deg_{\mathrm{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$ is the *Frobenius degree* of ϕ.

The composite of two morphisms

$$\phi = (\phi_{\mathcal{D}}, Z_{\phi}, n_{\phi}) : A \to B, \quad \psi = (\phi_{\mathcal{D}}, Z_{\psi}, n_{\psi}) : B \to C$$

is given as

$$\psi \circ \phi = \left(\psi_{\mathcal{D}} \circ \phi_{\mathcal{D}}, \psi_{\mathcal{D}}^{*}(Z_{\psi}) + n_{\psi} \cdot Z_{\phi}, n_{\psi} \cdot n_{\phi}\right) : A \to C.$$

Example

Let's consider the elementary Frobenioid $\mathbb{F}_{\Phi_M}$ associated to the functor

$$\begin{array}{rccc} \Phi_M : & \{\bullet\} & \longrightarrow & \mathfrak{Mon} \\ & \Psi & & \Psi \\ & \bullet & \longmapsto & M \end{array}$$

on the one-morphism category $\{\bullet\}$. We have

$$\mathbb{F}_M := \mathbb{F}_{\Phi_M} \cong M \rtimes \mathbb{N}_{\geq 1}.$$

Indeed, the monoid of morphisms consists of triples

$$(\mathrm{id}_{\{\bullet\}}, a, n)$$

where $a \in M$ and $n \in \mathbb{N}_{\geq 1}$.

The composition of $(\mathrm{id}_{\{\bullet\}}, a_1, n_1)$ and $(\mathrm{id}_{\{\bullet\}}, a_2, n_2)$ can be seen as a multiplication

$$(a_1, n_1) \cdot (a_2, n_2) = (a_1 + n_1 \cdot a_2, n_1 \cdot n_2)$$

in the semi-direct product

$$M \rtimes \mathbb{N}_{\geq 1}.$$

Definition (Pre-Frobenioid)

Let

$$\Phi : \mathcal{D} \to \mathfrak{Mon}$$

be a monoid on a connected, totally epimorphic category $\mathcal{D}$.

Let

$$\mathcal{C}$$

be a connected, totally-epimorphic category.

We say that $\mathcal{C}$ is a **pre-Frobenioid** if we have a **covariant** functor

$$\mathcal{C} \to \mathbb{F}_{\Phi}.$$

Model Frobenioids

Let's consider the following data

- $\mathcal{D}$ a connected a totally epimorphic category
- $\Phi : \mathcal{D} \to \mathfrak{Mon}$ a divisorial monoid
- $\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$ a group-like monoid
- $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$ a homomorphism of monoids

Proposition

We have a well defined category $\mathcal{C}$ constructed in the following way

- *the objects of $\mathcal{C}$ are pairs of the form*

$$(A_{\mathcal{D}}, \alpha)$$

where $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$ and $\alpha \in \Phi(A_{\mathcal{D}})^{\mathrm{gp}}$

- *a morphism*

$$\phi : (A_{\mathcal{D}}, \alpha) \to (B_{\mathcal{D}}, \beta)$$

is a collection of data

- $\deg_{\mathrm{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$
- $\mathrm{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$
- $\mathrm{Div}(\phi) \in \Phi(A)$
- $u_{\phi} \in \mathbb{B}(A)$ *such that*

$$\deg_{\mathrm{Fr}} \cdot \alpha + \mathrm{Div}(\phi) = (\Phi^{\mathrm{gp}}(\mathrm{Base}(\phi)))(\beta) + \mathrm{Div}_{\mathbb{B}}(u_{\phi})$$

For given two morphisms $\phi(A_{\mathcal{D}}, \alpha) \to (B_{\mathcal{D}}, \beta)$, $\psi : (B_{\mathcal{D}}, \beta) \to (C_{\mathcal{D}}, \gamma) \in \mathrm{Mor}(\mathcal{C})$ *the composition data*

$$\psi \circ \phi = \big(\deg_{\mathrm{Fr}}(\psi \circ \phi), \mathrm{Base}(\psi \circ \phi), \mathrm{Div}(\psi \circ \phi), u_{\psi \circ \phi}\big)$$

is defined as follows

- $\deg_{\mathrm{Fr}}(\psi \circ \phi) = \deg_{\mathrm{Fr}}(\psi) \cdot \deg_{\mathrm{Fr}}(\phi)$
- $\mathrm{Base}(\psi \circ \phi) = \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi)$
- $\mathrm{Div}(\psi \circ \phi) = \big(\Phi(\mathrm{Base}(\phi))\big)(\mathrm{Div}(\psi)) + \deg_{\mathrm{Fr}}(\psi) \cdot \mathrm{Div}(\phi)$
- $u_{\psi \circ \phi} = \mathbb{B}\big(\mathrm{Base}(\psi)\big)(u_{\phi}) + \deg_{\mathrm{Fr}}(\psi) \cdot u_{\phi}$

There is a natural functor

$$\mathcal{C} \to \mathbb{F}_{\Phi}$$

given by

$$(A_{\mathcal{D}}, \alpha) \mapsto A_{\mathcal{D}}$$

$$\phi = \left(\deg_{\mathrm{Fr}}(\phi), \mathrm{Base}(\phi), \mathrm{Div}(\phi), u_{\phi} \right) \mapsto \left(\mathrm{Base}(\phi), \mathrm{Div}(\phi), \deg_{\mathrm{Fr}}(\phi) \right)$$

so model Frobenioids are in particular pre-Frobenioids.

Example (**Frobenioid of geometric origin**)

V nice variety, K the function field and $\widetilde{K}$ its Galois extension with $G := \mathrm{Gal}(\widetilde{K}/K)$.

- $\mathcal{D} := \mathcal{B}(G)^0$
- divisorial monoid

$$\begin{array}{cccc} \Phi : & \mathcal{D} & \longrightarrow & \mathfrak{Mon} \\ & \Cup & & \Cup \\ & L & \longmapsto & \mathrm{Div}_L \end{array}$$

- group-like monoid

$$\begin{array}{cccc} \mathbb{B} : & \mathcal{D} & \longrightarrow & \mathfrak{Mon} \\ & \Cup & & \Cup \\ & L & \longmapsto & L^{\times} \end{array}$$

- homomorphism of monoids

$$\begin{array}{cccc} \mathrm{Div}_{\mathbb{B}} : & \mathbb{B} & \longrightarrow & \Phi^{\mathrm{gp}} \\ & \Cup & & \Cup \\ & L^{\times} & \longmapsto & \mathrm{PDiv}_L \end{array}$$

We get a model Frobenioid $\mathcal{C}_{\widetilde{K}/K}$

$$\begin{array}{ccccccc} \mathcal{C}_{\widetilde{K}/K} & \longrightarrow & \mathbb{F}_{\Phi} & \longrightarrow & \mathcal{D} & \xrightarrow{\Phi} & \mathfrak{Mon} \\ \Cup & & & & \Cup & & \Cup \\ (L, \mathcal{L}) & \longmapsto & & & L & \longmapsto & \mathrm{Div}_L \end{array}$$

which is exactly the Frobenioid of geometric origin described earlier.

Example (**Frobenioid of arithmetic origin**)

F a number field and $\widetilde{F}$ its Galois extension with $G := \mathrm{Gal}(\widetilde{F}/F)$.

- $\mathcal{D} := \mathcal{B}(G)^0$
- divisorial monoid

$$\begin{array}{cccc} \Phi : & \mathcal{D} & \longrightarrow & \mathfrak{Mon} \\ & \Cup & & \Cup \\ & L & \longmapsto & \bigoplus_{v \in \mathbb{V}(L)} ord(\mathcal{O}_v^{\triangleright}) \end{array}$$

- group-like monoid

$$\begin{array}{cccc} \mathbb{B} : & \mathcal{D} & \longrightarrow & \mathfrak{Mon} \\ & \Cup & & \Cup \\ & L & \longmapsto & L^{\times} \end{array}$$

- homomorphism of monoids

$$\begin{array}{cccc} \mathrm{Div}_{\mathbb{B}} : & \mathbb{B} & \longrightarrow & \Phi^{\mathrm{gp}} \\ & \Cup & & \Cup \\ & L^{\times} & \longmapsto & \mathrm{PDiv}_L \end{array}$$

We get a model Frobenioid $\mathcal{C}_{\widetilde{F}/F}$

$$\begin{array}{ccccccc} \mathcal{C}_{\widetilde{F}/F} & \longrightarrow & \mathbb{F}_{\Phi} & \longrightarrow & \mathcal{D} & \xrightarrow{\Phi} & \mathfrak{Mon} \\ \Cup & & & & \Cup & & \Cup \\ (L, \mathcal{L}) & \longmapsto & & & L & \longmapsto & \mathrm{Div}_L \end{array}$$

which is the Frobenioid of arithmetic origin described earlier.

Plan for tomorrow

1. Torsor-theoretic approach to model Frobenioids.
2. Frobenioids in IUT.
3. The Main Theorem about reconstruction of the functor

$$\mathcal{C} \to \mathbb{F}_{\Phi}.$$

that gives $\mathcal{C}$ structure of a Frobenioid can be reconstructed from $\mathcal{C}$ as a category.