Frobenioids 2

Weronika Czerniawska

The Univeristy of Nottingham

19.07.2016

Recall

The notion of a Frobenioid involves several functors

$$\mathcal{C} \to \mathbb{F}_\Phi \xrightarrow{\textit{projection}} \mathcal{D} \xrightarrow{\Phi} \mathfrak{Mon}$$

A model Frobenioid can be constructed from the following data:

- base category
$$\mathcal{D}$$
- divisorial monoid
$$\Phi : \mathcal{D} \to \mathfrak{Mon}$$
- group like monoid (**rational function** monoid)
$$\mathbb{B} : \mathcal{D} \to \mathit{Grp} \subset \mathfrak{Mon}$$
- homomorphism of monoids
$$\mathit{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$$

Therefore model Frobenioids may be thought of as a collection of the following data

- the **base category**

$$\mathcal{D}$$

  of ”coverings”, a topological group i.e. [possibly tempered] arithmetic fundamental group
- the divisorial monoid

$$\Phi : \mathcal{D} \to \mathfrak{Mon}$$

  giving a **divisor monoid** to each open supgroup of $\mathcal{D}$ (Weil divisors on the covering)
- the group like monoid

$$\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$$

  giving the **rational function monoid** to each open subgroup of $\mathcal{D}$ (multiplicative group of rational functions on the covering)
- the homomorphism of monoids

$$\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi$$

  giving **principal divisors** homomorphism to each open subgroup of $\mathcal{D}$

In particular Frobenioids can be thought of as category-theoretic abstractions of various aspects of the **multiplicative** portion of the ring structure of the function field a normal integral scheme.

The most important result

Under some technical conditions (satisfied in IUT context) the functor

$$\mathcal{C} \to \mathbb{F}_\Phi$$

*determining the structure of $\mathcal{C}$ as a **Frobenioid** may be reconstructed from the structure of $\mathcal{C}$ as a **category**.*

Plan

1. Torsor-theoretic approach to model Frobenioids.
2. Frobenioids in IUT.
3. The Frobenius-like and étale-like dichotomy example and its relationship with the Main Theorem.
4. The Main Theorem.
5. Frobenioids and Arithmetic Toposes.

Proposition (Tautological Torsor-theoretic Approach to Model Frobenioids)

Let's $\mathcal{C}^{\mathrm{tor}}$ be a category defined as follows

- *objects are triples of the form*

$$(A_{\mathcal{D}}, T_A, \tau_A)$$

where

- $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$
- T_A *is a* $\mathbb{B}(A_{\mathcal{D}})$*-torsor*
- τ_A *is a trivialization of the of the* $\Phi(A_{\mathcal{D}})^{\mathrm{gp}}$*-torsor obtained from* T_A *by executing the 'change of the structure group' operation determined by the homomorphism*

$$\mathrm{Div}_{\mathbb{B}}(A_{\mathcal{D}}) : \mathbb{B}(A_{\mathcal{D}}) \to \Phi(A_{\mathcal{D}})^{\mathrm{gp}}.$$

Therefore, for any $d \in \mathbb{Z}$, we obtain an object

$$(A_{\mathcal{D}}, T_A^{\otimes d}, \tau_A^{\otimes d})$$

by executing the 'change of structure group' operation determined by the homomorphism

$$\mathbb{B}(A_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$$

determined by multiplication by d.

- *morphisms*

$$\phi : (A_{\mathcal{D}}, T_A, \tau_A) \to (B_{\mathcal{D}}, T_B, \tau_B)$$

consists of the following

1. $d \in \mathbb{N}_{\geq 1}$
2. $$\mathrm{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$$

   which determines by executing the 'change of structure group' operations determined by homomorphisms

   $$\mathbb{B}(A_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$$
   $$\Phi(B_{\mathcal{D}})^{\mathrm{gp}} \to \Phi(B_{\mathcal{D}})^{\mathrm{gp}}$$

   an object

   $$\phi^* B = (A_{\mathcal{D}}, \phi^* T_B, \phi^* \tau_B)$$
3. *an isomorphism of* $\mathbb{B}(A_{\mathcal{D}})$*-torsors*

   $$T_A^{\otimes d} \xrightarrow{\sim} \phi^* T_B$$

   that maps $\tau_A^{\otimes d}$ *to an element in the* $\Phi(A_D)$*-orbit of* $\phi^* \tau_B$*.*

There is a natural equivalence of categories

$$\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\mathrm{tor}}$$

defined as follows

- **objects**

$$(A_{\mathcal{D}}, \alpha) \mapsto (A_{\mathcal{D}}, T_A, \tau_A)$$

where T_A is the trivial $\mathbb{B}(A_{\mathcal{D}})$-module obtained by shifting the tautological trivialization of T_A by the element $-\alpha \in \Phi(A_{\mathcal{D}})^{\mathrm{gp}}$

- **morphisms**

$$\begin{array}{ccc} (A_{\mathcal{D}}, \alpha) & & (A_{\mathcal{D}}, T_A, \tau_A) \\ \downarrow & \longmapsto & \downarrow \\ (B_{\mathcal{D}}, \beta) & & (B_{\mathcal{D}}, T_B, \tau_B) \end{array}$$

The morphism transformation

$$(n, A_{\mathcal{D}} \xrightarrow{\phi} B_{\mathcal{D}}, Z, u) \quad \begin{array}{c}(A_{\mathcal{D}}, \alpha)\\ \downarrow \\ (B_{\mathcal{D}}, \beta)\end{array} \longmapsto \begin{array}{c}(A_{\mathcal{D}}, T_A, \tau_A)\\ \downarrow \\ (B_{\mathcal{D}}, T_B, \tau_B)\end{array} \quad (n, A_{\mathcal{D}} \xrightarrow{\phi} B_{\mathcal{D}}, T_A^{\otimes d} \xrightarrow{\sim} \phi^* T_B)$$

where

$$T_A^{\otimes d} \xrightarrow{\sim} \phi^* T_B$$

is an isomorphism of trivial $\mathbb{B}(A_{\mathcal{D}})$-torsors determined by multiplication by the element u.

Reminder: $\phi^* : \Phi(B_{\mathcal{D}}) \to \Phi(A_{\mathcal{D}}) = \Phi(\phi)$.

Example (p-adic Frobenioid)

L: a finite extension of $\mathbb{Q}_p$,

$$\mathcal{O}_L^{\times} \stackrel{\text{def}}{=} \{|z| = 1\} \subset \mathcal{O}_L^{\triangleright} \stackrel{\text{def}}{=} \{0 < |z| \leq 1\} \subset \mathcal{O}_L$$

$$ord(L) \stackrel{\text{def}}{=} L^{\times}/\mathcal{O}_L^{\times} \cong \mathbb{Z}$$

$$ord(\mathcal{O}_L^{\triangleright}) \stackrel{\text{def}}{=} \mathcal{O}_L^{\triangleright}/\mathcal{O}_L^{\times} \cong \mathbb{Z}_{\geq 0}$$

$$ord(L) = ord(\mathcal{O}_L^{\triangleright})^{\text{gp}}.$$

Let $\mathcal{D}$ be the full subcategory of connected objects of the Galois category of finite étale coverings of $Spec(\mathbb{Q}_p)$.
We have monoids:

- $\Phi_0 : \mathcal{D} \ni Spec(L) \mapsto ord(\mathcal{O}_L^{\triangleright})^{\text{rlf}} \in \mathfrak{Mon}, \;\; ord(\mathcal{O}_L^{\triangleright})^{\text{rlf}} \cong \mathbb{R}_{\geq 0}$
- $\mathbb{B}_0 : \mathcal{D} \ni Spec(L) \mapsto L^{\times} \in Grp \subset \mathfrak{Mon}$

and natural homomorphism of monoids:

- $Div_{\mathbb{B}_0} : \mathbb{B}_0 \to \Phi^{\text{gp}}$

We obtain a model Frobenioid

$$\mathcal{C}_0$$

$$Ob(\mathcal{C}_0) = \left\{ \begin{array}{l} \text{metrized line bundles } V \\ \text{on } Spec(L),\ F \in \mathcal{D} \end{array} \right\}$$

$$Mor(\mathcal{C}_0) = \left\{ V \to W = \left(Spec(L) \to Spec(M), d \in \mathbb{N}_{\geq 1}, V^{\otimes d} \xrightarrow{\sim} W|_L \right) \right\}$$

where a metrized line bundle is a $L^{\times}$-torsor V with a choice of an element

$$\mu_V \in ord(V)^{\mathrm{rlf}} \stackrel{\mathrm{def}}{=} (V/\mathcal{O}_L^{\times})^{\mathrm{rlf}}.$$

and $V^{\otimes d} \xrightarrow{\sim} W|_L$ is an isomorphism of $L^{\times}$-torsors such that the element $\mu_{V^{\otimes d}} \in ord(V^{\otimes d})^{\mathrm{rlf}}$ maps to an element of the $ord(\mathcal{O}_L^{\triangleright})^{\mathrm{rlf}}$-orbit of $\mu_{W|_L} \in ord(W|_L)^{\mathrm{rlf}}$.

- It is possible to present the theory of Frobenioids used in IUT in a very efficient way.
- As we will see, almost all of Frobenioids appearing in IUT are model Frobenioids.
- Those which are not can be easily dealt with without notion of Frobenioid.

Types of Frobenioids appearing in IUT

- **archimedean Frobenioids** i.e. Frobenioids at archimedean places **(not model!)**.

Archimedean Frobenioids that show up in IUT can be viewed as

$$\mathcal{O}_{\mathbb{C}}^{\triangleright} = \left\{ \begin{array}{c} \text{the multiplicative topological monoid of non-zero} \\ \text{complex numbers of the norm smaller or equal than 1} \end{array} \right\}$$

Therefore using a Frobenioid notion is not unavoidable in this case.

All the Frobenioids listed from this point on will be **model**.

- **nonarchimedean Frobenioids** i.e Frobenioids at nonarchimedean places (except tempered Frobenioids!), are essentially equivalent to

  (v1) an abstract ind-topological monoid equipped with continuous action by an abstract topological group

  $$G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$$

  (v2) an abstract ind-topological monoid equipped with continuous action by an abstract topological group

  $$\Pi_X \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$$

  (Π_X can possibly stand for a tempered fundamental group)

  (v3) data obtained from (v1) or (v2) by replacing $\mathcal{O}^{\triangleright}_{\overline{k}}$ by some subquotient of $\mathcal{O}^{\triangleright}_{\overline{k}}$.

- **tempered Frobenioids** The only portion used in IUT are

  (t1) the **theta monoids** $\theta^{-1/\mathrm{ell}}$ generated by local units and non-negative powers of roots of the **theta functions** that are constructed from **tempered Frobenioids**

  (t2) the **mono-theta environments** constructed from **tempered Frobenioids** which are related to the monoids of **(t1)** (they share the same submonoids of roots of unity)

- the **global Frobenioids** associated to number fields (or realifications of global Frobenioids). They admit a simple elementary description as categories of arithmetic line bundles on number fields

Remark

Global and tempered Frobenioids cannot be reduced to an ind-topological monoid equipped with a continuous action by a topological group.

Their **Picard groups** admit **non-torsion** elements;

- in the **global** case Frobenioids contain objects corresponding to arithmetic line bundles whose arithmetic degree $\neq 0$
- in the **tempered** case Frobenioids contain objects corresponding to line bundles for which arbitrary positive tensor powers are non-trivial

Frobenius-like and étele-like dichotomy

Remark

One way to think about the reconstruction of the functor $\mathcal{C} \to \mathbb{F}_\Phi$ is that the structure of the base category $\mathcal{D}$ is fundamentally combinatorially different from the structure of 'Frobenius portion' $\mathbb{F}_\Phi$ of a Frobenioid.

Proposition

Let G be a finite group. Any homomorphism of monoids $\mathbb{F}_{\mathbb{Z}_{\geq 0}} \to G$ *factors through the natural surjection* $\mathbb{F}_{\mathbb{Z}_{\geq 0}} \twoheadrightarrow \mathbb{N}_{\geq 1}$.

Proposition

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Proof.

Indeed, we have a natural inclusion

$$\mathbb{Z}_{\geq 0} \hookrightarrow \mathbb{F}_{\mathbb{Z}_{\geq 0}} \cong \mathbb{Z}_{\geq 0} \rtimes \mathbb{N}_{\geq 1}.$$

The image of this inclusion can be identified with elements of the form $(d, 1)$, $d \in \mathbb{Z}_{\geq 0}$. For all $d \in \mathbb{N}_{\geq 1}$ we have

$$(1, 1)^d = (d, 1)$$

and

$$(0, d)(1, 1) = (d, d) = (d, 1)(0, d).$$

Therefore the image γ of $(1, 1)$ in G satisfies the equation

$$\delta_d \cdot \gamma \cdot \delta_d^{-1} = \gamma^d$$

where δ_d is the image of $(0, d)$ in G. Taking d to be the order of γ we get that $\gamma = 1_G$. So the image of $\mathbb{Z}_{\geq 0}$ lies in the kernel of our homomorphism.

□

The Main Theorem

Definition

An **isomorphism of functors** is a natural transformation, such that all the component morphisms are isomorphisms.

An **automorphism of a functor** is an isomorphism of functors from the functor into itself.

A functor is called **rigid** if its group of automorphisms is trivial.

Definition

We say that a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Phi_1} & \mathcal{C}_2 \\ \Phi_2 \downarrow & & \downarrow \Phi_4 \\ \mathcal{C}_3 & \xrightarrow{\Phi_3} & \mathcal{C}_4 \end{array}$$

is 1-commutative, if we have an isomorphism of functors

$$\Phi_4 \circ \Phi_1 \cong \Phi_3 \circ \Phi_2.$$

The Main Theorem (Rigidity)

Let $\Phi_i : \mathcal{D}_i \to \mathfrak{Mon}$ *be a monoid, let* $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *be a Frobenioid,* $i = 1, 2$ *and let*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

be an equivalence of categories.

- There exists a category equivalence

  $$\Psi^{\mathrm{Base}} : \mathcal{D}_1 \to \mathcal{D}_2$$

  that fits into a 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ {\scriptstyle \text{natural projection}} \downarrow & & \downarrow {\scriptstyle \text{natural projection}} \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\mathrm{Base}}} & \mathcal{D}_2 \end{array}$$

- *There exists an isomorphism of functors*

$$\Psi^{\Phi} : \Phi_1 \xrightarrow{\sim} \Phi_2$$

lying over the equivalence of categories $\Psi^{Base} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$*;*

i.e. for all $A, B \in Ob(\mathcal{D}_1)$ *and every morphism* $\phi : A \to B$ *there is a commutative diagram*

$$\begin{array}{ccc} \Phi_1(A) & \xleftarrow{\Phi_1(\phi)} & \Phi_1(B) \\ \Big\downarrow \Psi^{\Phi}_A & & \Big\downarrow \Psi^{\Phi}_B \\ \Phi_2(\Psi^{Base}(A)) & \xleftarrow{\Phi_2(\Psi^{Base}(\phi))} & \Phi_2(\Psi^{Base}(B)) \end{array}$$

where the vertical arrows are isomorphisms.

In particular, Ψ^{Base} *and* Ψ^{Φ} *induce an equivalence of categories*

$$\Psi^{\mathbb{F}} : \mathbb{F}_{\Phi_1} \xrightarrow{\sim} \mathbb{F}_{\Phi_2}.$$

- *There exists a 1-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathbb{F}_{\Phi_1} & \xrightarrow{\Psi^{\mathbb{F}}} & \mathbb{F}_{\Phi_2} \end{array}$$

where the vertical arrows are the functors that define the Frobenioid structure on $\mathcal{C}_1, \mathcal{C}_2$, and the horizontal arrows are equivalences of categories.

Moreover, each of the composite functors of this diagram is rigid.

Frobenioids and Arithmetic Toposes

In IUT nonarchimedean Frobenioids and number fields (global) Frobenioids of $\mathbb{Z}$-type are of the following type

- **objects**

$$a_K \in A_K$$

  where A_K is the set of G_K-fixed elements for an open subgroup $G_K \subset G$ of a topological group G acting on an abelian group A

- **morphisms**

$$(a_K, a_L, n)$$

  for a ring homomorphism $K \to L$ and $n \in \mathbb{Z}_{\geq 1}$ such that

$$a_L^n \in a_K \mathcal{O}^{\triangleright}(L)$$

  where $\mathcal{O}^{\triangleright}(L)$ stands for the non-zero integral elements of A_L (the archimedean integral elements is the unit ball)

Recently, Connes and Consani computed the space of points of the topos

$$[0, \infty) \rtimes \mathbb{N}_{\geq 1}$$

which can be viewed as a category

- **objects**

$$[0, x)$$

- **morphisms**

$$n \in \mathbb{N}_{\geq 1}$$

  such that

$$n[0, x_1) \subset [0, x_2)$$

is canonically isomorphic to the quotient of

$$\mathbb{A}_{\mathbb{Q}} / \mathbb{Q}^{\times}$$

by the action of

$$\widehat{\mathbb{Z}}^{\times}$$

Summary

- Model Frobenioids have a very intuitive structure
- Almost all Frobenioids that appear in IUT are model
- The theory of Frobenioids used in IUT can be summarized in a compact way
- The functor $\mathcal{C} \to \mathbb{F}_\Phi$ can be reconstructed from the category $\mathcal{C}$
- Structures similar to Frobenioids show up in other mathematical theories