Hodge-Arakelov evaluation (IUT2)

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Outline

- At bad places $\underline{v}$, evaluate Θ-monoids $\underline{\underline{\theta}}_{\text{env}} \subset {}_{\infty}\underline{\underline{\theta}}_{\text{env}}$ at the evaluation points $g(\mu_-) \in \underline{X}_{\underline{v}}(K_{\underline{v}})$, $g \in \text{Gal}(\underline{X}_{\underline{v}}/C_{\underline{v}})$.
- Synchronization of the conjugacy indeterminacies on the decomposition groups of the different evaluation points.
- Global compatibility of synchronization:
  Need to study profinite conjugates of tempered cuspidal inertia groups.

Conjugacy indeterminacy

- We want to pullback $\underline{\underline{\theta}}_{\text{env}} \subset {}_\infty\underline{\underline{\theta}}_{\text{env}} \subset \lim_J H^1(\Pi_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}|_J, l\Delta_\Theta)$ along the inclusion $G_{\underline{v}} \xrightarrow{\sim} D_{\mu_-,t} \hookrightarrow \Pi_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} \subset \Pi_{\underline{X}_{\underline{v}}}$, where $D_{\mu_-,t}$ is a decomposition group of an evaluation point in $\underline{X}_{\underline{v}}$
- A priori $\Pi_{\underline{X}_{\underline{v}}}$-indeterminacy splits in two:
  Outer $\Pi_{\underline{X}_{\underline{v}}}/\Pi_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} = l\underline{\mathbb{Z}} \times \mu_{2l}$-indeterminacy: get partially rid by chosing specific preimages of the cusps in $Y_{\underline{v}}$.
  Inner indeterminacy.
- Assume we chose a specific $D_{\mu_-,t}$:
  We get submonoids of $\lim_{J\subset G_{\underline{v}}} H^1(J, l\Delta_\Theta)$ up to $G_{\underline{v}}$-conjugacy.
- When doing this for multiple t's, we get a $\prod_t (G_{\underline{v}})_t$ conjugacy indeterminacy.
- We need to reduce the $\prod_t (G_{\underline{v}})_t$ conjugacy indeterminacy to a diagonal conjugacy indeterminacy of $G_{\underline{v}}$.
- $\rightsquigarrow \mathbb{F}_l^{\rtimes\pm} \simeq \mathrm{Gal}(\underline{X}_{\underline{v}}/C_{\underline{v}})$-symmetries.

Profinite conjugates

- Global synchronization of the cusps via the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-bridge:

$$\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}}) \simeq \mathcal{D}_{>,\underline{v}} \to \mathcal{D}^{\circledcirc\pm} \simeq \mathcal{B}(\underline{X}_K).$$

  We want our evaluation decomposition groups and evaluation maps to be parametrized by $\mathrm{LabCusp}(\mathfrak{D}^{\circledcirc}) \simeq \{\text{cusp. inertia subgps of } \Pi_{\underline{\underline{X}}_K}\}/\Pi_{\underline{X}}$.
- Need tempered evaluation for profinite conjugacy classes of cuspidal groups: tempered-profinite conjugacy compatibility issue.
- Global $\mathbb{F}_l^{\rtimes\pm}$-symmetries on $\mathrm{LabCusp}^{\pm}(\mathfrak{D}^{\circledcirc\pm})$ arise from profinite conjugacies in Π_C.

Subgraphs of $\Gamma_{\underline{\underline{X}}_{\underline{v}}}$

$\underline{v} \in \underline{\mathbb{V}}_{\rm bad}$.

- Let $\Gamma^{\blacktriangleright}_{\underline{X}_{\underline{v}}}$ be the subgraph of $\Gamma_{\underline{X}_{\underline{v}}}$ obtained by removing the only edge of $\Gamma_{\underline{X}_{\underline{v}}}$ by the unique involution $\iota_{\underline{X}}$ of $\underline{X}_{\underline{v}}$ extending to $\underline{X}_{\underline{v}}$ (can be recovered group-theoretically from $\Pi_{\underline{v}}$).
- If one choses an involution ι on $Y_{\underline{v}}$, one gets a unique lifting $\Gamma_{\blacktriangleright} \to \Gamma_{Y_{\underline{v}}}$ whose image is invariant by ι.
- Let $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$: t determines a vertex of $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$. Let $\Gamma^{\bullet t}$ be the subgraph of $\Gamma_{\underline{X}}$ consisting of only this vertex.

Decomposition group of subgraph (pointed version)

- Let $\tilde{X}_{\underline{v}} := \text{“}\varprojlim\text{”} X^{\infty}_{\underline{v},i}$ a pro-universal l-tempered cover of $\underline{\underline{X}}_{\underline{v}}$.
- Let $\tilde{\Gamma} := \varprojlim_i \Gamma_{X^{\infty}_{\underline{v},i}}$, $\qquad \mathrm{Aut}(\Pi_{\underline{\underline{X}}_{\underline{v}}}) \circlearrowleft \tilde{\Gamma}$.
- Let $\tilde{\Gamma}_{\blacktriangleright}$ be a closed connected component of the preimage of $\Gamma_{\blacktriangleright}$ (unique up to $\Pi_{\underline{\underline{X}}_{\underline{v}}}$-action).
- Let $\tilde{\Gamma}_{\bullet t}$ be a closed connected component of the preimage of $\Gamma_{\bullet t}$ (unique up to $\Pi_{\underline{\underline{X}}_{\underline{v}}}$-action).
- $\Pi_{\underline{v}\blacktriangleright} := \mathrm{Stab}_{\Pi_{\underline{\underline{X}}_{\underline{v}}}}(\tilde{\Gamma}_{\blacktriangleright})$ (defined up to $\Pi_{\underline{\underline{X}}_{\underline{v}}}$-conjugacy).
- $\Pi_{\underline{v}\ddot{\blacktriangleright}} := \mathrm{Stab}_{\Pi_{\underline{\underline{\ddot{X}}}_{\underline{v}}}}(\tilde{\Gamma}_{\blacktriangleright}) = \Pi_{\underline{v}\blacktriangleright} \cap \Pi_{\underline{\underline{\ddot{X}}}_{\underline{v}}}$.
- $\Pi_{\underline{v}\bullet t} := \mathrm{Stab}_{\Pi_{\underline{\underline{X}}_{\underline{v}}}}(\tilde{\Gamma}_{\bullet t})\ldots$

Profinite-tempered conjugacy compatibility [cor. 2.4]

Notations: $\Pi_{\underline{v}}(\simeq \Pi_{\underline{\underline{X}}_v}) \subset \Pi^{\pm}_{\underline{v}}(\simeq \Pi_{\underline{X}_v}) \subset \Pi^{\mathrm{cor}}_{\underline{v}} := \Pi_{\underline{C}_v}$.

$\mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})^{\sim} = \{$cusp inert. sbgp of $\widehat{\Pi}_{\underline{\underline{X}}_K}\}$

$\mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}}) = \mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})/\widehat{\Pi}^{\pm}_{\underline{v}} = \mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})/\widehat{\Delta}^{\pm}_{\underline{v}}$

Proposition

- *Let* $\gamma, \delta \in \widehat{\Delta}^{\pm}_{\underline{v}}$. *Let* $I \subset \Pi_{\underline{v}}$ *be a cusp. inert. gp s.t.* $I \subset \Delta_{\underline{v},\blacktriangleright}$. *TFAE:*

  $$a)\gamma^{-1}\delta \in \Delta^{\pm}_{\underline{v}\blacktriangleright} \qquad b) I^{\delta} \subset \Pi^{\gamma}_{\underline{v}\blacktriangleright} \qquad c) I^{\delta} \subset (\Pi^{\pm}_{\underline{v}\blacktriangleright})^{\gamma}.$$

- $I \in \mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})^{\sim} \mapsto \Pi_{\underline{v}\blacktriangleright}(I) \in \{\underline{v}\blacktriangleright$*-temp. decomp. sbgp* $\subset \widehat{\Pi}_{\underline{v}}\}$
- $N_{\widehat{\Pi}^{\pm}_{\underline{v}}}(\Pi_{\underline{v}\blacktriangleright}) = \Pi^{\pm}_{\underline{v}\blacktriangleright} \rightsquigarrow$
  $\exists! \Pi_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright})$ *conj of* $\Pi_{\underline{v}}$ *st.* $\Pi_{\underline{v}\blacktriangleright}$ *is a* $\underline{v}\blacktriangleright$*-decomp. sbgp of* $\Pi_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright})$.
- $\Pi_{\underline{v}\blacktriangleright} \mapsto D_{t,\mu_-}(\Pi_{\underline{v}\blacktriangleright}) \subset \Pi_{\underline{v}\blacktriangleright}$ *defined up to* $\Pi^{\pm}_{\underline{v}\blacktriangleright}$*-conjugacy*
  ($\mathrm{Gal}(\Pi^{\pm}_{\underline{v}\blacktriangleright})/\Pi_{\underline{v}\ddot{\blacktriangleright}}) = \mu_{2l}$*-outer indet. if considered as a sbgp of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}$*).*

Compatibility with $\mathbb{F}_l^{\rtimes\pm}$-symmetries [cor. 2.4]

Proposition

- $I_t \in \mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})^{\sim}_t \mapsto \Pi_{\underline{v}\bullet t}(I_t) \in \{\underline{v}\bullet t$*-temp. decomp. sbgp* $\subset \Pi_{\underline{\underline{X}}_K}\}$.
- $\Pi_{\underline{v}\bullet t} \mapsto D_{t,\mu_-}(\Pi_{\underline{v}\bullet t}) \subset \Pi_{\underline{v}\bullet t}$ *defined up to* $\Pi^{\pm}_{\underline{v}\bullet t}$*-conjugacy (*$\mathrm{Gal}(\Pi^{\pm}_{\underline{v}\bullet t})/\Pi_{\underline{v}\ddot{\bullet} t}) = \mu_{2l}$*-outer indet. if considered as a sbgp of* $\Pi_{\underline{v}\ddot{\bullet} t}$*).*

- $\mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})^{\sim} = \{I^l, I$ cusp. inert. sbgp of $\widehat{\Pi}^{\pm}_{\underline{v}}\}$ actually only depends on $\widehat{\Pi}^{\pm}_{\underline{v}}$.
- $\rightsquigarrow \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$ acts by conj. on $\mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}_K})^{\sim}$ and $\mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}_K})$.
- If $\lambda \in \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$, $\Pi_{\underline{v}\bullet t}(I_t)^{\lambda} = \Pi_{\underline{v}\bullet\lambda(t)}(I_t^{\lambda})$.
- $D_{t,\mu_-}(I_t)^{\lambda} = D_{\lambda(t),\mu_-}(I_t^{\lambda})$ up to $\Pi^{\pm}_{\underline{v}\bullet\lambda(t)}$-indeterminacy.

Group-theoretic Theta evaluation

Let $l_t \in \mathrm{LabCusp}(\widehat{\Pi}_{\underline{v}})^{\sim}$, and let $\Pi_{\underline{v}^{\blacktriangleright}} := \Pi_{\underline{v}^{\blacktriangleright}}(l_t); \Pi_{\underline{v}^{\ddot{\blacktriangleright}}} := \Pi_{\underline{v}^{\ddot{\blacktriangleright}}}(l_t)$.
Let $(l\Delta_{\Theta})(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}})$ be the subquotient of $\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}$ determined by the subquotient $(l\Delta_{\Theta})(\Pi_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}}))$ of $\Pi_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}})$.
Let $G_{\underline{v}}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}})$ be the quotient of $\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}$ determined by $\Pi_{\underline{v}} \to G_{\underline{v}}(\Pi_{\underline{v}})$.
By restricting $\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}})) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}}))$ to $\Pi_{\underline{v}^{\ddot{\blacktriangleright}}} \subset \Pi_{\ddot{\underline{\underline{Y}}}}(\Pi_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}}))$, one gets a μ_{2l}-orbit and a μ-orbit:

$$\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \subset \varinjlim_{\widehat{J} \subset \widehat{\Pi}_{\underline{v}}} H^1(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}} \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J}, (l.\Delta)(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}))$$

By further restricting $\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}})$ to
$G_{\underline{v}}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \xrightarrow{\sim} D_{t,\mu_-}(\Pi_{\underline{v}^{\blacktriangleright}}) \subset \Pi_{\underline{v}^{\ddot{\blacktriangleright}}}$, one gets a μ_{2l}-orbit and a μ-orbit:

$$\underline{\underline{\theta}}^{|t|}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \subset {}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \subset \varinjlim_{J_G} H^1(G_{\underline{v}}(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}) \times J_G, (l.\Delta)(\Pi_{\underline{v}^{\ddot{\blacktriangleright}}}))$$

Splitting at zero-labeled evaluation

Isomorph of $\mathcal{O}^{\times}_{K_{\underline{v}}}$ in $\varinjlim_{J_G \subset G_{\underline{v}}(\Pi_{\underline{v}^{\blacktriangleright}})} H^1(J_G, (l.\Delta)(\Pi_{\underline{v}^{\blacktriangleright}}))$.

Pull back along $\Pi_{\underline{v}\infty} \twoheadrightarrow G_{\underline{v}}$: $\overline{\mathcal{O}}^{\times}(\Pi_{\underline{v}^{\blacktriangleright}}) \subset H^1(\Pi_{\underline{v}^{\blacktriangleright}} \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J}, (l.\Delta)(\Pi_{\underline{v}^{\blacktriangleright}}))$

$$\overline{\mathcal{O}}^{\times}(\Pi_{\underline{v}^{\blacktriangleright}}) \cdot \underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\blacktriangleright}}) \subset \overline{\mathcal{O}}^{\times}(\Pi_{\underline{v}^{\blacktriangleright}}) \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\blacktriangleright}}) \subset H^1(\Pi_{\underline{v}^{\blacktriangleright}} \times_{\widehat{\Pi}_{\underline{v}}} \widehat{J}, (l.\Delta)(\Pi_{\underline{v}^{\blacktriangleright}}))$$

If $I_0 \in \mathrm{LabCusp}^{\pm}(\widehat{\Pi}_{\underline{v}})_{\widetilde{0}}$, pullback along $\mathbb{G}_{\underline{v}} \xrightarrow{\sim} D_{0,\mu_-}(I_0) \subset \Pi_{\underline{v}^{\blacktriangleright}}(I_0)$ induces a retraction: $\overline{\mathcal{O}}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\blacktriangleright}}) \to \overline{\mathcal{O}}^{\times}(\Pi_{\underline{v}^{\blacktriangleright}})$ defined up to torsion ($\underline{\underline{\theta}}^{0}(\Pi_{\underline{v}^{\blacktriangleright}}) = \mu_{2l}$).

$\rightsquigarrow$ canonical splitting $\overline{\mathcal{O}}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\blacktriangleright}}) / \overline{\mathcal{O}}^{\mu}(\Pi_{\underline{v}^{\blacktriangleright}}) = \overline{\mathcal{O}}^{\times\mu}(\Pi_{\underline{v}^{\blacktriangleright}}) \times {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}^{\blacktriangleright}})$
(where $\overline{\mathcal{O}}^{\mu} = \mathrm{Torsion}(\overline{\mathcal{O}}^{\times})$ and $\overline{\mathcal{O}}^{\times\mu} = \overline{\mathcal{O}}^{\times} / \overline{\mathcal{O}}^{\mu}$).

MΘ-env. Theta evaluation [cor. 2.8]

Let $\mathbb{M}^{\Theta}_{*}$ be a pro-MΘ-env.
Let $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ be a $\underline{v}\ddot{\blacktriangleright}$-temp. decomp. sbgp of $\widehat{\Pi}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$.
Let
$\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) := \Pi_{\underline{v}\ddot{\blacktriangleright}} \times_{\widehat{\Pi}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})} \widehat{\Pi_{\mathbb{M}^{\Theta}_{*}}} \subset \Pi_{\mathbb{M}^{\Theta}_{*}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) := \Pi_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \times_{\widehat{\Pi}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})} \widehat{\Pi_{\mathbb{M}^{\Theta}_{*}}}$
$\Pi_{\mathbb{M}^{\Theta}_{*}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ can be enriched naturally in a pro-$\mathbb{M}^{\Theta}_{*}$-env $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ st
$\Pi_{\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}\ddot{\blacktriangleright}})} = \Pi_{\mathbb{M}^{\Theta}_{*}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$.
If $(-)(\mathbb{M}^{\Theta}_{*})$ is a subquotient of $\Pi_{\mathbb{M}^{\Theta}_{*}}$, we will denote by $(-)(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ the corresponding subquotient of $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})} \subset \Pi_{\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}\ddot{\blacktriangleright}})}$.
Apply cyclotomic rigidity isom. $l \cdot \Delta_{\Theta}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \to \Pi_{\mu}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ to $\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ and to $\underline{\underline{\theta}}^{|t|}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \subset {}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$, $\rightsquigarrow$

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subset {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subset \varinjlim_{\widehat{J}} H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}))_{|\widehat{J}}, \Pi_{\mu}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}))$$

$$\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subset {}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subset \varinjlim_{J_G} H^1(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|J}, \Pi_{\mu}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}))$$

+ canonical splitting of $\overline{\mathcal{O}}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})/\overline{\mathcal{O}}^{\mu}(\mathbb{M}^{\Theta}_{*})$ by evaluation at 0.

MΘ-env. Theta monoids and constant monoids

Recall:

- Theta monoids:

$$\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) = \overline{\mathcal{O}}^{\times}(\mathbb{M}^{\Theta}_{*}) \cdot \underline{\underline{\theta}}^{\iota}_{\mathsf{env}}(\mathbb{M}^{\Theta}_{*})^{\mathbb{N}}$$

$${}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) = \overline{\mathcal{O}}^{\times}(\mathbb{M}^{\Theta}_{*}) \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathsf{env}}(\mathbb{M}^{\Theta}_{*})^{\mathbb{N}}$$

  \+ splittings (well defined up to μ_{2l} and μ indeterminacy)
- Constant monoid:

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) = \overline{\mathcal{O}}^{\rhd}(\mathbb{M}^{\Theta}_{*})$$

Conjugate synchronization

- For every $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$, D_{t,μ_-} is a copy of $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$ up to inner morphism. The inner indeterminacy on $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$ induces a $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$-indeterminacy on $\Psi_{\mathrm{cns}}(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}))$.
- If we restrict simultaneously $\underline{\underline{\theta}}_{\mathrm{env}}$ to each D_{t,μ_-}, one gets

$$\underline{\underline{\theta}}^{\mathbb{F}^{\circledast}_{l}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\blacktriangleright}) = \prod_{|t|\in\mathbb{F}^{\circledast}_{l}} \underline{\underline{\theta}}^{|t|}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\blacktriangleright}) \subset \prod_{t} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t,$$

  where one has, a priori, on $\prod_t \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t$ a $\prod_t G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_t$-indeterminacy.
- conjugating by $\Delta_C(\mathbb{M}^{\Theta}_{*})$ on $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ permutes cusp. inertia group of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ $\rightsquigarrow$ canonical isom. of $(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_t \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t)_t$ ("$\mathbb{F}^{\rtimes\pm}_{l}$-symmetries").
- Let $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright})_{\langle|\mathbb{F}_l|\rangle} \subset \prod_{t\in\mathbb{F}^{\circledast}_{l}} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright})$ be the subset of elements invariant by the "$\mathbb{F}^{\rtimes\pm}_{l}$-symmetries".

MΘ-env. Gaussian monoid

By functoriality of Ψ_{cns}, the $\mathbb{F}_l^{\rtimes\pm}$-symmetries induce isomorphisms of the copies $\Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)$ in the product: diagonal $\Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)_{\langle|\mathbb{F}_l|\rangle} \subset \prod_t \Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)$.

$\prod_t \Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)$ is well-defined up to $G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^\Theta)_{\langle|\mathbb{F}_l|\rangle}$-indeterminacy.

$\underline{\underline{\theta}}_{\mathrm{env}}^{\mathbb{F}_l^{\circledast}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^\Theta) \subset \prod_{t\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)$ is the set of *value-profiles*.

Definition

$$\Psi_{\mathrm{gau}}(\mathbb{M}_*^\Theta) := \{\Psi_\xi(\mathbb{M}_*^\Theta) := \Psi_{\mathrm{cns}}^\times(\mathbb{M}_*^\Theta)_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{N}}\}_{\xi\in\underline{\underline{\theta}}_{\mathrm{env}}^{\mathbb{F}_l^{\circledast}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^\Theta)}$$

$$_\infty\Psi_{\mathrm{gau}}(\mathbb{M}_*^\Theta) := \{_\infty\Psi_\xi(\mathbb{M}_*^\Theta) := \Psi_{\mathrm{cns}}^\times(\mathbb{M}_*^\Theta)_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{Q}}\}_{\xi\in\underline{\underline{\theta}}_{\mathrm{env}}^{\mathbb{F}_l^{\circledast}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^\Theta)}$$

There are restriction isomorphisms:

$$\Psi_{\mathrm{env}}(\mathbb{M}_*^\Theta) \xrightarrow{\sim} \Psi_{\mathrm{gau}}(\mathbb{M}_*^\Theta) \qquad {}_\infty\Psi_{\mathrm{env}}(\mathbb{M}_*^\Theta) \xrightarrow{\sim} {}_\infty\Psi_{\mathrm{gau}}(\mathbb{M}_*^\Theta).$$

The $\mathbb{F}_l^{\rtimes\pm}$-symmetries also give rise to an isomorphism $\Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)_0 \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\mathbb{M}_*^\Theta)_{\langle\mathbb{F}_l^{\circledast}\rangle}$

Frobenioidal setting

$\underline{\underline{\mathcal{F}}}_{\underline{v}}$:= tempered Frobenioid isomorphic to $\mathcal{F}(\underline{\underline{X}}_{\underline{v}})$.

$\rightsquigarrow \underline{\underline{\mathcal{F}}}_{\underline{v}} \to \mathcal{D}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) :=$ base$(\underline{\underline{\mathcal{F}}}_{\underline{v}})$; $\underline{\underline{\ddot{Y}}}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \in \mathcal{D}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$.

Choose $\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ (def'd up to $\mu_{2l}(\mathbb{T}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ and $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$ indeterminacies)

$\rightsquigarrow$ a monoid $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) := \mathcal{O}^{\times}(\mathbb{T}_{(-)}) \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}|_{(-)}$ on $\mathcal{D}^{\Theta}_{\underline{v}}$ [IUT1, ex. 3.2.(v)].

A^{Θ}_{∞}:= a universal pro-object of $\mathcal{D}_{\underline{v}}$

Definition

- $\Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\mathrm{id}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \mathcal{O}^{\times}_{\mathcal{C}_{\underline{v}}^{\Theta}}(A_{\infty}^{\Theta}) \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}}|_{A_{\infty}^{\Theta}} \subset \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A_{\infty}^{\Theta}})$
- ${}_{\infty}\Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\mathrm{id}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \mathcal{O}^{\times}_{\mathcal{C}_{\underline{v}}^{\Theta}}(A_{\infty}^{\Theta}) \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{Q}_{\geqslant 0}}|_{A_{\infty}^{\Theta}} \subset \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A_{\infty}^{\Theta}})$
- $\Psi_{\mathcal{C}_{\underline{v}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) = \mathcal{O}^{\rhd}_{\mathcal{C}_{\underline{v}}}(A_{\infty}^{\Theta})$ ($\mathcal{C}_{\underline{v}}$ denotes the base-th. hull of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$)

For $\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$, by replacing $\underline{\underline{\Theta}}_{\underline{v}}$ by $\underline{\underline{\Theta}}_{\underline{v}}^{\alpha}$,
$\rightsquigarrow \Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\alpha} \subset {}_{\infty}\Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\alpha} \subset \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A_{\infty}^{\Theta}})$.

Definition (Frobenioid-theoretic Theta monoids)

$\Psi_{\mathcal{F}_{\underline{v}}^{\Theta}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \{\Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\alpha}\}_{\alpha}$ $\quad {}_{\infty}\Psi_{\mathcal{F}_{\underline{v}}^{\Theta}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \{{}_{\infty}\Psi_{\mathcal{F}_{\underline{v}}^{\Theta},\alpha}\}_{\alpha}$

Kummer isomorphism of Theta monoids

Let $\mathbb{M}^{\Theta}_{*} := \mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$.

Frobenioid-theoretical Kummer map [Fr2, def. 2.1]:

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subset \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A^{\Theta}_{\infty}})^{\Pi(\underline{\ddot{Y}})} \cap \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A^{\Theta}_{\infty}})^{N} \to H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}), \mu_N(A^{\Theta}_{\infty})).$$

Projective limit over N & (tautological) isomorphism of cyclotomes $\varprojlim_N \mu_N(A^{\Theta}_{\infty}) \xrightarrow{\sim} \Pi_{\mu}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$, one gets a map:

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}), \Pi_{\mu}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))) \supset \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

Proposition (Prop. 3.3)

For a natural bijection $\{\alpha\} \xrightarrow{\sim} \{\iota\}$, one gets Frob-theoretic Kummer isomorphisms of monoids:

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \xrightarrow{\sim} \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \qquad {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \xrightarrow{\sim} {}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

$$\Psi_{\mathcal{C}_{\underline{v}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

For a value-profile $\xi \in \underline{\underline{\theta}}^{\mathbb{F}_l^{\circledast}}_{\text{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$,

$$\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \subset {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \subset \prod_{|t|} \Psi_{\text{cns}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

By pulling-back along $\prod_{|t|} \Psi_{\mathcal{C}_{\underline{v}}} \overset{\text{Kummer}}{\xrightarrow{\sim}} \prod_{|t|} \Psi_{\text{cns}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$, one gets Frobenioid-th. Gaussian monoids:

$$\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subset {}_{\infty}\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subset \prod_{|t|} \Psi_{\mathcal{C}_{\underline{v}}}$$

$$\begin{array}{ccc}
\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \overset{\text{Kummer}}{\xrightarrow{\sim}} & \Psi^{\iota}_{\text{env}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\downarrow & & \downarrow \\
\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \overset{\text{Kummer}}{\xrightarrow{\sim}} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))
\end{array}$$

$$\left.\begin{array}{lcl} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \circlearrowleft \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) & \rightsquigarrow & \mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) \\ \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \circlearrowleft \Psi_{\mathcal{C}_{\underline{v}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \rightsquigarrow & \mathcal{F}_{\mathcal{C}_{\underline{v}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \end{array}\right\} \begin{array}{l} p_{\underline{v}}\text{-adic Frobenioids} \\ \text{of type } \mathbb{Z} \\ \text{div. monoid} \simeq \mathbb{Q}_{\geqslant 0} \end{array}$$

($\simeq$ components of an $\mathcal{F}$-prime-strip)

$$\mathcal{C}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \overset{\text{taut.}}{\overset{\sim}{\to}} \mathcal{F}_{\mathcal{C}_{\underline{v}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \overset{\text{Kummer}}{\overset{\sim}{\to}} \mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

Action of $\Pi_{\underline{\underline{X}}}$ factors through $G_{\underline{v}}$ $\rightsquigarrow$ mono-an. versions:

$$\mathcal{C}^{\vdash}_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \overset{\text{Kummer}}{\overset{\sim}{\to}} \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

$$\left.\begin{array}{lcl} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \circlearrowleft \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \rightsquigarrow & \mathcal{F}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \\ G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \circlearrowleft \Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \rightsquigarrow & \mathcal{F}_{\hat{\mathcal{F}}_{\underline{v}},\alpha}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \\ G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \circlearrowleft \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) & \rightsquigarrow & \mathcal{F}_{\xi}(\mathbb{M}^{\Theta}_{*}) \\ G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \circlearrowleft \Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \rightsquigarrow & \mathcal{F}_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \end{array}\right\} \begin{array}{l} p_{\underline{v}}\text{-adic Frobenioids} \\ \text{of type } \mathbb{Z} \\ \text{div. monoid} \simeq \mathbb{N} \\ + \textit{splittings} \end{array}$$

($\simeq$ components of an $\mathcal{F}^{\vdash}$-prime strip)

Good non-archimedean places: $\underline{v} \in \underline{\mathbb{V}}_{\mathrm{good}} \cap \underline{\mathbb{V}}_{\mathrm{non}}$

- Group-theoretic:
  - $\Psi_{\mathrm{cns}}(G_{\underline{v}}) \subset \varinjlim_J H^1(J, \mu_{\widehat{\mathbb{Z}}}(G_{\underline{v}}))$; $\quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}}) := \Psi_{\mathrm{cns}}(G_{\underline{v}}(\Pi_{\underline{v}}))$
  - $\Psi^{\mathbb{R}}_{\mathrm{cns}}(G_{\underline{v}}) = (\Psi_{\mathrm{cns}}(G_{\underline{v}})/\Psi^{\times}_{\mathrm{cns}}(G_{\underline{v}}))^{\mathrm{rlf}} \simeq \mathbb{R}_{\geqslant 0}(G_{\underline{v}})$
    $\log^{G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geqslant 0}(G_{\underline{v}})$
  - $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}}) = \Psi^{\times}_{\mathrm{cns}}(G_{\underline{v}}) \times \mathbb{R}_{\geqslant 0}(G_{\underline{v}})$
  - $\mathbb{F}_l^{\rtimes\pm}$-symmetries + diagonal $\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle \mathbb{F}_l^{*} \rangle} \subset \prod_{|t| \in \mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})$;
  - $\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) := \Psi^{\times}_{\mathrm{cns}}(\Pi_{\underline{v}}) \times \mathbb{R}_{\geqslant 0} \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}})$
  - $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) = \Psi^{\times}_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle \mathbb{F}_l^{*} \rangle} \times \{\mathbb{R}_{\geqslant 0} . (j^2 \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}))_j\} \subset \prod_j \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}})$.
- Frob-theoretic and Kummer isomorphisms:
  - unique $G_{\underline{v}}(\Pi_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$-equivariant $\psi_{\underline{\underline{\mathcal{F}}}_{\underline{v}}} \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$.
  - $\widehat{\mathbb{Z}}^{\times}$-orbit of $\Psi^{\times}_{\mathcal{F}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\times}_{\mathrm{cns}}(G_{\underline{v}})$; well-defined $\Psi^{\mathbb{R}}_{\mathcal{F}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\mathbb{R}}_{\mathrm{cns}}(G_{\underline{v}})$
    $\rightsquigarrow \quad \Psi^{\mathrm{ss}}_{\mathcal{F}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}})$
  - $\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} := \Psi_{\mathrm{env}}(\Pi_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})) \qquad \Psi_{\mathcal{F}_{\mathrm{gau}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) := \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$

Archimedean places: $\underline{v} \in \underline{\mathbb{V}}_{\text{arc}}$

$\mathbb{U}_{\underline{v}} := \underline{\mathbb{X}}_{\underrightarrow{\underline{v}}} \to \mathbb{U}^{\pm}_{\underline{v}} := \underline{\mathbb{X}}_{\underline{v}} \to \mathbb{U}^{\text{cor}}_{\underline{v}} := \mathbb{C}_{\underline{v}}$ (aut-hol. spaces)

- $\Psi_{\text{cns}}(\mathbb{U}_{\underline{v}}) := \mathcal{A}^{\triangleright}_{\mathbb{U}_{\underline{v}}}$
- $\log^{\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geqslant 0}(\mathcal{D}^{\vdash}_{\underline{v}})$
- $\Psi^{\text{ss}}_{\text{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) = \Psi^{\times}_{\text{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) \times \mathbb{R}_{\geqslant 0}(\mathcal{D}^{\vdash}_{\underline{v}})$
- $\mathbb{F}_l^{\rtimes\pm}$-symmetries + diagonal $\Psi_{\text{cns}}(\mathbb{U}_{\underline{v}})_{\langle \mathbb{F}_l^{*} \rangle} \subset \prod_{|t| \in \mathbb{F}_l^{*}} \Psi_{\text{cns}}(\mathbb{U}_{\underline{v}})_t$;
- $\Psi_{\text{env}}(\mathbb{U}_{\underline{v}}) := \Psi^{\times}_{\text{cns}}(\mathbb{U}_{\underline{v}}) \times \mathbb{R}_{\geqslant 0} \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}})$
- $\Psi_{\text{gau}}(\mathbb{U}_{\underline{v}}) = \Psi^{\times}_{\text{cns}}(\mathbb{U}_{\underline{v}})_{\langle \mathbb{F}_l^{*} \rangle} \times \{\mathbb{R}_{\geqslant 0}.(j^2 \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}))_j\} \subset \prod_j \Psi^{\text{ss}}_{\text{cns}}(\mathbb{U}_{\underline{v}})$
- ...

$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$ setting [cor. 4.5]

$\mathfrak{D} := \{\mathcal{D}_{\underline{v}}\}$ a $\mathcal{D}$-prime-strip; $\mathfrak{D}^{\vdash} := \{\mathcal{D}^{\vdash}_{\underline{v}}\}$ associated $\mathcal{D}^{\vdash}$-prime-strip

- $\Psi_{\mathrm{cns}}(\mathfrak{D}) : \underline{v} \mapsto \Psi_{\mathrm{cns}}(\mathcal{D}_{\underline{v}}) \qquad (:= \{\Pi_{\underline{v}} \curvearrowright \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})\})$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$);
- $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{D}^{\vdash}) : \underline{v} \mapsto \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) \qquad (:= \{G_{\underline{v}} \curvearrowright \Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}})\}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$);
  + splitting $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{D}^{\vdash})_{\underline{v}} = \Psi^{\times}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) \times \mathbb{R}_{\geqslant 0}(\mathcal{D}^{\vdash}_{\underline{v}})$
  + distinguished element $\log^{\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geqslant 0}(\mathcal{D}^{\vdash}_{\underline{v}})$
- $\mathfrak{D}^{\vdash} \rightsquigarrow \mathcal{D}^{\Vdash}$ st. $\Phi_{\mathcal{D}^{\Vdash}} = \bigoplus_{\underline{v}\in\underline{\mathbb{V}}} \mathbb{R}_{\geqslant 0}(\mathfrak{D}^{\vdash})_{\underline{v}}$;

$$\mathbb{B}_{\mathcal{D}^{\Vdash}} = \left\{ a = \left(a_{\underline{v}} \cdot \log^{\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}})\right)_{\underline{v}} \in \Phi^{\mathrm{gp}}_{\mathcal{D}^{\Vdash}} ; \sum_{\underline{v}} [K_{\underline{v}} : F_{\mathrm{mod}\,v}] a_{\underline{v}} \log(p_{\underline{v}}) = 0 \right\}$$

Let $\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = (\mathfrak{D}_{\succ} \leftarrow \mathfrak{D}_T \rightarrow \mathcal{D}^{\circledcirc\pm})$ be a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater.

- Recall $\zeta_{\succ} : \mathrm{LabCusp}^{\pm}(\mathfrak{D}_{\succ}) \xrightarrow{\sim} T \qquad$ [IUT1, prop. 6.5]
- Global $\mathbb{F}_l^{\rtimes\pm}$-symmetries of $(\Psi_{\mathrm{cns}}(\mathfrak{D}_{\succ})_t)_{t\in\mathrm{LabCusp}^{\pm}(\mathfrak{D}_{\succ})}$ compatible with the $\mathbb{F}_l^{\rtimes\pm}$-symmetry on the Θ^{ell}-bridge $\mathfrak{D}_T \rightarrow \mathcal{D}^{\circledcirc\pm}$
  + diagonal $\Psi_{\mathrm{cns}}(\mathfrak{D}_{\succ})_{\langle\mathbb{F}_l^{*}\rangle} \subset \prod_{|t|} \Psi_{\mathrm{cns}}(\mathfrak{D}_{\succ})$.

$\mathcal{D}$-$\Theta^{\pm\text{ell}}$-th. global rlf'd Θ- & Gau-monoids [cor. 4.5]

- $\Psi_{\text{env}}(\mathfrak{D}_{\succ}) : \underline{v} \mapsto \Psi_{\text{env}}(\mathcal{D}_{\succ,\underline{v}})$ $\quad {}_{\infty}\Psi_{\text{env}}(\mathfrak{D}_{\succ}) : \underline{v} \mapsto {}_{\infty}\Psi_{\text{env}}(\mathcal{D}_{\succ,\underline{v}})$
- $\Psi_{\text{gau}}(\mathfrak{D}_{\succ}) : \underline{v} \mapsto \Psi_{\text{gau}}(\mathcal{D}_{\succ,\underline{v}})$ $\quad {}_{\infty}\Psi_{\text{gau}}(\mathfrak{D}_{\succ}) : \underline{v} \mapsto {}_{\infty}\Psi_{\text{gau}}(\mathcal{D}_{\succ,\underline{v}})$
- $\mathcal{D}^{\Vdash}_{\text{env}}(\mathfrak{D}^{\vdash}_{\succ})$ st. $\Phi_{\mathcal{D}^{\Vdash}_{\text{env}}(\mathfrak{D}^{\vdash}_{\succ})} = \bigoplus_{\underline{v}\in\underline{\mathbb{V}}} \Psi_{\text{env}}(\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$
  ($\Psi_{\text{env}}(\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}} := (\Psi_{\text{env}}(\mathfrak{D}_{\succ})_{\underline{v}}/\Psi^{\times}_{\text{env}}(\mathfrak{D}_{\succ})_{\underline{v}})^{\text{rlf}}$ only depends on $\mathfrak{D}^{\vdash}_{\succ}$)
  $\mathbb{B}_{\mathcal{D}^{\Vdash}_{\text{env}}} =$
  $$\left\{ a = \left(a_{\underline{v}} \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(p_{\underline{v}}) \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}}) \right)_{\underline{v}} ; \textstyle\sum_{\underline{v}} [K_{\underline{v}} : F_{\text{mod}\,v}] a_{\underline{v}} \log(p_{\underline{v}}) = 0 \right\}$$
  (for $\underline{v} \in \underline{\mathbb{V}}_{\text{bad}}$, $\log^{\mathfrak{D}^{\vdash}_{\succ}}(p_{\underline{v}}) \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}}) := \frac{\log(p_{\underline{v}})}{\log(\underline{\underline{q}}_{\underline{v}})} \cdot$ (image of $\underline{\underline{\theta}}_{\text{env}}$))
- $\mathcal{D}^{\Vdash}_{\text{gau}}(\mathfrak{D}^{\vdash}_{\succ}) \subset \prod_j \mathcal{D}^{\Vdash}(\mathfrak{D}^{\vdash}_{\succ})$ st.
  $\Phi_{\mathcal{D}^{\Vdash}_{\text{gau}}(\mathfrak{D}^{\vdash}_{\succ})} = \bigoplus_{\underline{v}\in\underline{\mathbb{V}}} \Psi_{\text{gau}}(\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}} = (j^2)_{j\in\mathbb{F}^{*}_{l}} \cdot \Phi_{\mathcal{D}^{\Vdash}} \subset \prod_j \Phi_{\mathcal{D}^{\Vdash}}$
- Global evaluation isomorphism: $\mathcal{D}^{\Vdash}_{\text{env}}(\mathfrak{D}^{\vdash}_{\succ}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\text{gau}}(\mathfrak{D}^{\vdash}_{\succ})$ by
  $$\left(a_{\underline{v}} \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(p_{\underline{v}}) \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}}) \right)_{\underline{v}} \mapsto \left(\left(j^2 a_{\underline{v}} \cdot \log^{\mathfrak{D}^{\vdash}_{\succ}}(p_{\underline{v}}) \right)_{\underline{v}} \right)_{j\in\mathbb{F}^{*}_{l}}$$

$\mathfrak{F} = \{\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: $\mathcal{F}$-prime-strip ($\rightsquigarrow \mathfrak{F}^{\vdash}, \mathfrak{F}^{\Vdash}$ [IUT1, rmk 5.2.1])

- $\Psi_{\mathrm{cns}}(\mathfrak{F}) : \underline{v} \mapsto \{G_{\underline{v}}(\mathcal{F}_{\underline{v}}) \curvearrowright \Psi_{\mathcal{F}_{\mathcal{C}_{\underline{v}}}}(\mathcal{F}_{\underline{v}})\})$ if $\underline{v} \in \underline{\mathbb{V}}_{\mathrm{non}}$;
- $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{F}^{\vdash}) : \underline{v} \mapsto \Psi^{\mathrm{ss}}_{\mathcal{F}^{\vdash}_{\underline{v}}}$ + splitting + distinguished element
- $\Psi_{\mathrm{cns}}(\mathfrak{F}) \overset{\text{Kummer}}{\overset{\sim}{\rightarrow}} \Psi_{\mathrm{cns}}(\mathfrak{D}(\mathfrak{F}))$
  $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{F}^{\vdash}) \overset{\sim}{\rightarrow} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{D}^{\vdash}(\mathfrak{F}^{\vdash}))$ ($\mathcal{F}^{\vdash}_{\underline{v}}$-th. for $\underline{v} \in \underline{\mathbb{V}}_{\mathrm{non}}$ but not for $\underline{v} \in \underline{\mathbb{V}}_{\mathrm{arc}}$)
- Unique isom. $\mathcal{C}^{\Vdash}(\mathcal{F}) \overset{\sim}{\rightarrow} \mathcal{D}^{\Vdash}(\mathfrak{D}^{\vdash}(\mathfrak{F}^{\vdash}))$ compatible with $\underline{\mathbb{V}}$ and $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{F}^{\vdash}) \overset{\sim}{\rightarrow} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathfrak{D}^{\vdash}(\mathfrak{F}^{\vdash}))$. It is $\mathfrak{F}^{\Vdash}$-theoretic.

$\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}} = (\mathfrak{F}_{\succ} \leftarrow \mathfrak{F}_{T} \rightarrow \mathcal{D}^{\circledcirc\pm})$: $\Theta^{\pm\mathrm{ell}}$-Hodge theater.

$\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-th Θ, Gau. and global rlf'd monoids [cor. 4.6]

$\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$: $\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-Hodge theater.

- - $\underline{v} \mapsto \Psi_{\mathcal{F}_{\mathrm{env}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}} := \Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \quad \rightsquigarrow \quad \mathfrak{F}^{\vdash}_{\mathrm{env}}$
  - $\underline{v} \mapsto \Psi_{\mathcal{F}_{\mathrm{gau}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}} := \Psi_{\mathcal{F}_{\mathrm{gau}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad \rightsquigarrow \quad \mathfrak{F}^{\vdash}_{\mathrm{gau}}$
  - $\Psi_{\mathcal{F}_{\mathrm{env}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}) \xrightarrow{\sim} \Psi_{\mathrm{env}}(\mathfrak{D}_{\succ}) \xrightarrow{\sim} \Psi_{\mathrm{gau}}(\mathfrak{D}_{\succ}) \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$
  - ∞-versions
- - $\mathcal{C}^{\Vdash}_{\mathrm{env}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ st. $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{env}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})} = \bigoplus_{\underline{v} \in \underline{\mathbb{V}}} \Psi_{\mathcal{F}_{\mathrm{env}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})^{\mathbb{R}}_{\underline{v}}$
  - $\mathcal{C}^{\Vdash}_{\mathrm{gau}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ st. $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})} = \bigoplus_{\underline{v} \in \underline{\mathbb{V}}} \Psi_{\mathcal{F}_{\mathrm{gau}}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})^{\mathbb{R}}_{\underline{v}}$
  - $\mathcal{C}^{\Vdash}_{\mathrm{env}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{env}}(\mathfrak{D}^{\vdash}_{\succ}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{gau}}(\mathfrak{D}^{\vdash}_{\succ}) \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{env}}(\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$

$\rightsquigarrow \quad \mathfrak{F}^{\Vdash}_{\mathrm{env}} = (\mathcal{C}^{\Vdash}_{\mathrm{env}}, \mathfrak{F}^{\vdash}_{\mathrm{env}}), \quad \mathfrak{F}^{\Vdash}_{\mathrm{gau}} = (\mathcal{C}^{\Vdash}_{\mathrm{gau}}, \mathfrak{F}^{\vdash}_{\mathrm{gau}})$

\+ evaluation isom. $\quad \mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{gau}}$

$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-th. $\mathbb{F}_l^{\circledast}$-sym. and non-rlf'd glob. str. [cor. 4.7]

$$
\begin{aligned}
\mathcal{D}^{\circledcirc} \quad & \rightsquigarrow \quad \mathcal{D}^{\circledast}; \quad \pi_1(\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc}) \subset \mathbb{M}^{\circledast}(\mathcal{D}^{\circledcirc}) \\
& \rightsquigarrow \quad \pi_1^{\mathrm{rat}}(\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\kappa}(\mathcal{D}^{\circledcirc}) \subset \mathbb{M}^{\circledast}_{\infty\kappa}(\mathcal{D}^{\circledcirc}) \\
& \rightsquigarrow \quad \text{fields } \pi_1(\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc}) \subset \overline{\mathbb{M}}^{\circledast}(\mathcal{D}^{\circledcirc}) \\
& \rightsquigarrow \quad \mathcal{F}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc}), \mathcal{F}^{\circledast}(\mathcal{D}^{\circledcirc}), \mathcal{F}^{\circledcirc}(\mathcal{D}^{\circledcirc}) \\
& \rightsquigarrow \quad \forall j \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc}), \text{ a } \mathcal{F}\text{-prime-strip } \mathcal{F}^{\circledcirc}(\mathcal{D}^{\circledcirc})|_j \\
& \qquad \text{labeled } \mathbb{M}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})_j, \mathbb{M}^{\circledast}_{\infty\kappa}(\mathcal{D}^{\circledcirc})_j, \mathcal{F}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})_j
\end{aligned}
$$

$\mathbb{F}_l^{\circledast}$-poly-action on $\mathcal{D}^{\circledcirc} \rightsquigarrow \mathbb{F}_l^{\circledast}$-symmetrizing isom. of $(\mathcal{F}^{\circledcirc}(\mathcal{D}^{\circledcirc})|_j)_j, (\mathbb{M}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})_j)_j, (\mathbb{M}^{\circledast}_{\infty\kappa}(\mathcal{D}^{\circledcirc})_j)_j, (\mathcal{F}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})_j)_j$.
$\rightsquigarrow$ diagonal objects $(\_)_{\langle \mathbb{F}_l^{\circledast} \rangle}$.

If $\mathcal{D}^{\circledcirc}$ lies in a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater, these $\mathbb{F}_l^{\circledast}$-symm. isomorphisms are compatible with the $\mathbb{F}_l^{\circledast}$-symmetry of the $\mathcal{D}$-NF-bridge.

$$\mathcal{F}^{\circledcirc}(\mathcal{D}^{\circledcirc})|_j \rightsquigarrow \mathfrak{D}_j, \mathfrak{D}_j^{\vdash} \qquad + \qquad \mathcal{D}^{\Vdash}(\mathfrak{D}_j^{\vdash}) \xrightarrow{\sim} (\mathcal{F}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})_j)^{\mathbb{R}}.$$

$\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-th. global Kummer isom. [cor. 4.8]

Let $\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ be a $\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-Hodge theater.
$\pi_1^{\mathrm{rat}}(\mathcal{D}^{\circledast})$-equivariant Kummer isomorphisms $\mathbb{M}^{\circledast}_{\infty\kappa} \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\infty\kappa}(\mathcal{D}^{\circledcirc})$
(defined up to conjugacy; functorial in $\mathcal{F}^{\circledcirc}$)
$\rightsquigarrow \mathbb{M}^{\circledast} \xrightarrow{\sim} \mathbb{M}^{\circledast}(\mathcal{D}^{\circledcirc}); \mathbb{M}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})$;
$\rightsquigarrow \overline{\mathbb{M}}^{\circledast} \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}(\mathcal{D}^{\circledcirc}); \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc})$
$\rightsquigarrow \mathcal{F}^{\circledcirc} \xrightarrow{\sim} \mathcal{F}^{\circledcirc}(\mathcal{D}^{\circledcirc}); \mathcal{F}^{\circledast} \xrightarrow{\sim} \mathcal{F}^{\circledast}(\mathcal{D}^{\circledcirc}); \mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}$
+ labeled (by $j \in J$) Kummer isomorphisms; + Kummer isomorphisms of diagonal submonoids;

Functorially in a *NF*-bridge $\mathfrak{F}_J \to \mathcal{F}^{\circledcirc} \dashrightarrow \mathcal{F}^{\circledast}$:

- Loc. morphisms $(\mathcal{F}^{\circledast}_{\text{mod}})_j \to \mathfrak{F}_j$, $(\mathbb{M}^{\circledast}_{\infty\kappa})_j \to (\mathbb{M}^{\circledast}_{\infty\kappa})_v(\mathfrak{F}_j)$
- $\mathcal{C}^{\Vdash}_j := \mathcal{C}^{\Vdash}(\mathfrak{F}_j) \xrightarrow{\sim} (\mathcal{F}^{\circledast}_{\text{mod}})^{\mathbb{R}}_j$

$\rightsquigarrow \mathcal{C}^{\Vdash}_{\text{gau}}(\mathcal{HT}^{\Theta^{\pm \text{ell}}\text{NF}}) \hookrightarrow \prod_j \mathcal{C}^{\Vdash}_j \xrightarrow{\sim} \prod (\mathcal{F}^{\circledast}_{\text{mod}})^{\mathbb{R}}_j$

Let $\mathfrak{F}^{\vdash} = (\mathcal{F}^{\vdash}_{\underline{v}})_{\underline{v}}$ be an $\mathcal{F}^{\vdash}$-prime-strip; $\mathcal{D}^{\vdash}_{\underline{v}} = \mathrm{base}(\mathcal{F}^{\vdash}_{\underline{v}})$
Focus on $\underline{v} \in \underline{\mathbb{V}}_{\mathrm{bad}}$.
$A_{\underline{v}}$: universal pro-object of $\mathcal{D}^{\vdash}_{\underline{v}}$. $G_{\underline{v}} = \mathrm{Aut}(A_{\underline{v}})$.
$\mathcal{O}^{\perp}(A_{\underline{v}}) \subset \mathcal{O}^{\rhd}(A_{\underline{v}})$ generated by $\mu_{2l}(A_{\underline{v}})$ and the image of the can. splittings of $\mathcal{O}^{\rhd}(A_{\underline{v}})$
$\mathcal{O}^{\blacktriangleright}(A_{\underline{v}}) := \mathcal{O}^{\perp}(A_{\underline{v}})/\mu_{2l}(A_{\underline{v}})$.
$\mathcal{O}^{\blacktriangleright\times\mu}(A_{\underline{v}}) := \mathcal{O}^{\blacktriangleright}(A_{\underline{v}}) \times \mathcal{O}^{\times\mu}(A_{\underline{v}})$.
There exists a unique $\mathbb{Z}^{\times}$-orbit of isom. $\mathcal{O}^{\times}(G_{\underline{v}}) \xrightarrow{\sim} \mathcal{O}^{\times}(A_{\underline{v}})$.
$\rightsquigarrow$ Ism-orbit of isom. $\mathcal{O}^{\times\mu}(G_{\underline{v}}) \xrightarrow{\sim} \mathcal{O}^{\times\mu}(A_{\underline{v}})$ ($\times\mu$-Kummer structure on $\mathcal{F}^{\vdash}_{\underline{v}}$).

Definition (4.9)

- $\mathcal{F}^{\vdash\blacktriangleright\times\mu}_{\underline{v}}$: model frobenioid corresponding to $\mathcal{O}^{\blacktriangleright\times\mu}(A_{\underline{v}})$ endowed with its can. splitting + its $\times\mu$-Kummer str.
  $\mathfrak{F}^{\vdash\blacktriangleright\times\mu} := (\mathcal{F}^{\vdash\blacktriangleright\times\mu}_{\underline{v}})_{\underline{v}\in\underline{\mathbb{V}}}$: $\mathcal{F}^{\vdash\blacktriangleright\times\mu}$-prime-strip. (morphisms of $\mathcal{F}^{\vdash\blacktriangleright\times\mu}$-prime-strip are collections of morphisms of frobenioids compatible with canonical splitting and $\times\mu$-Kummer str.)
- $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-prime-strip: $(\mathcal{C}^{\Vdash}, \mathrm{Prime}(\mathcal{C}) \simeq \underline{\mathbb{V}}, \mathfrak{F}^{\vdash\blacktriangleright\times\mu}, (\rho_{\underline{v}}))$.

$\Theta^{\times\mu}$ and $\Theta^{\times\mu}_{\mathrm{gau}}$-links [cor. 4.10]

Let $\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ be a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater.
For every $t \in \mathrm{LabCusp}^{\pm}(\mathfrak{D}_{\succ})$, $(\Psi_{\mathrm{cns}}(\mathfrak{D}_{\succ})_t) \rightsquigarrow \mathcal{F}$-prime strip
Identify $\Psi_{\mathrm{cns}}(\mathfrak{F}_{\succ})_0$ and $\Psi_{\mathrm{cns}}(\mathfrak{F}_{\succ})_{\langle \mathbb{F}_l^* \rangle}$ by $\mathbb{F}_l^{\rtimes\pm}$-symmetry.
$\rightsquigarrow$ $\mathcal{F}$-prime strip $\rightsquigarrow$ $\mathcal{F}^{\Vdash}$-prime strip $\mathfrak{F}^{\Vdash}_{\triangle} = (\mathcal{C}^{\Vdash}_{\triangle}, \mathfrak{F}^{\vdash}_{\triangle})$
canonically isom. to $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$.

Definition

Let $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ and $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ be two $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theaters.

- The $\Theta^{\times\mu}$-link $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\mu}}{\longrightarrow} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is the full poly-morphism:

$$^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\triangle}$$

- The $\Theta^{\times\mu}_{\mathrm{gau}}$-link $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\mu}}{\longrightarrow} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is the full poly-morphism:

$$^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\triangle}$$

Coricity of $\mathfrak{F}^{\vdash \times \mu}_{\triangle}$

We have natural isomorphisms of $\mathcal{F}^{\vdash \times \mu}$-prime strips (i.e. compatible with $\times\mu$-Kummer structure):

$$ {}^{\dagger}\mathfrak{F}^{\vdash \times \mu}_{\triangle} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash \times \mu}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash \times \mu}_{\mathrm{gau}} $$

By composing with either the $\Theta^{\times \mu}$-link or the $\Theta^{\times \mu}_{\mathrm{gau}}$-link, one gets a (which turn to be full) poly-morphism:

$$ {}^{\dagger}\mathfrak{F}^{\vdash \times \mu}_{\triangle} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\vdash \times \mu}_{\triangle} $$

$\rightsquigarrow$ full poly-morphism ${}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}$ ($\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-link)

$$ \begin{array}{ccc} {}^{\dagger}\mathcal{F}^{\Vdash \times \mu}_{\triangle} & \longrightarrow & {}^{\ddagger}\mathcal{F}^{\Vdash \times \mu} \\ \downarrow & \circlearrowleft_{\mathbb{R}_{>0}} & \downarrow \\ \mathcal{F}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}) & \longrightarrow & \mathcal{F}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}) \end{array} $$