Hodge theaters and label classes of cusps

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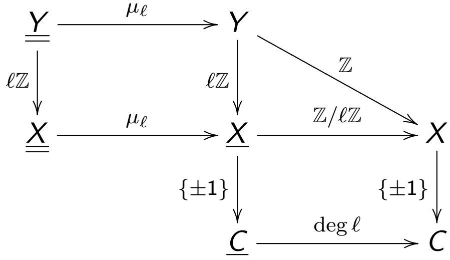
Initial Θ-data $(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \underline{\mathbb{V}}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\varepsilon})$

Recall the geometry:



- $\mathrm{Aut}_K(\underline{C}) \simeq \mathrm{Aut}_K(\underline{C}/C) \simeq \{1\}$.
- $\mathrm{Aut}_K(\underline{\underline{X}}) \simeq \mu_\ell \times \{\pm 1\}$.
- $\mathrm{Aut}_K(\underline{X}) \simeq \mathbb{Z}/\ell\mathbb{Z} \rtimes \{\pm 1\}$.

Note: $\Pi_{\underline{C}_K} \rightsquigarrow \Pi_{\underline{X}_K}$. We apply AAG to $\Pi_{\underline{X}_K}$ (called the "Θ-**approach**") so that

$$\mathrm{Hom}(H^2(\Delta_X, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}) = M_X \xrightarrow{\sim} \Delta_\Theta = [\Delta_X, \Delta_X]/[[\Delta_X, \Delta_X], \Delta_X]$$

$$\rightsquigarrow M_{\underline{X}} \xrightarrow{\sim} \ell \cdot \Delta_\Theta. \quad (\textbf{cyclotomic rigidity})$$

- $\mathbb{V}_{\mathrm{mod}} \xrightarrow{\sim} \underline{\mathbb{V}} \subset \mathbb{V}(K)$, a chosen section of $\mathbb{V}(K) \to \mathbb{V}_{\mathrm{mod}}$.
- $\underline{\epsilon}$ is a cusp of $\underline{C}_K$ arising from an element of Q. For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, it is chosen that $\underline{\epsilon_v}$ corresponds to the canonical generator "± 1" of the quotient $\Pi_X^{\mathrm{tp}} \twoheadrightarrow \mathbb{Z}$.

C_K is a K-core $\rightsquigarrow$ a unique model over F_{mod}:

$$C_{F_{\mathrm{mod}}}.$$

- $L := F_{\mathrm{mod}}$ or $(F_{\mathrm{mod}})_v$ with $v \in \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}$. (similarly for archimedean places)
- L_C: =function field of C_L ($\rightsquigarrow$ algebraic closures $\overline{L}_C, \overline{L}$).
- Note $|C_L| \simeq \mathbb{A}^1_L$.
- For the proper smooth curve determined by some finite extension of L_C, a closed point is called critical if it maps to (a closed point of $|C_L|^{\mathrm{cpt}}$ coming from) the 2-torsion points of E_F. Among them, those not mapping to the cusp of C_L is called strictly critical.

A rational function $f \in L_C$ is κ-coric if

(i) If $f \notin L$, then it has exactly 1 pole and ≥ 2 (distinct) zeros.

(ii) The divisor of f is defined over a finite extension of F_{mod} and avoids the critical points.

(iii) f restricts to roots of unity at any strictly critical point of $|C_L|^{\mathrm{cpt}}$.

Every element of L can be realized as a value of a κ-coric function on C_L at some non-critical L-valued point.

- $f \in \overline{L}_C$ is ${}_\infty\kappa$-coric if f^n is κ-coric for some $n \in \mathbb{Z}_{>0}$.
- $f \in \overline{L}_C$ is ${}_\infty\kappa\times$-coric if $c \cdot f$ is ${}_\infty\kappa$-coric for some $c \in \overline{F}^\times_{\mathrm{mod}}$ (resp. $\mathcal{O}^\times_{\overline{F}_{\mathrm{mod},v}}$).

- Tempered Frobenoid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, $\rightsquigarrow \underline{\underline{\mathcal{F}}}^{\mathrm{birat}}_{\underline{v}}$.
- Base category

$$\mathcal{D}_{\underline{v}} := \mathcal{B}^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}})^{\circ} \supset \mathcal{B}(K_{\underline{v}})^{\circ} =: \mathcal{D}^{\vdash}_{\underline{v}}.$$

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$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \longrightarrow \mathcal{D}_{\underline{v}} \xrightarrow{\text{left adjoint to the inclusion}} \mathcal{D}^{\vdash}_{\underline{v}}.$$

- The reciprocal of ℓ-th root of the Frobenioid-theoretic theta function

$$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}) = \ker\left(\mathrm{Aut}(\mathbb{T}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}) \to \mathrm{Aut}(\underline{\underline{\ddot{Y}}}_{\underline{v}})\right) \simeq K^{\times}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$$

  is determined category-theoretically by $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ up to $\mu_{2\ell}(\mathbb{T}^{\mathrm{birat}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}) \times \ell\mathbb{Z}$-indeterminacy. $(\ell\mathbb{Z} \subset \mathrm{Aut}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}))$
- $\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow p_{\underline{v}}$-adic Frobenioid (base-field-theoretic hull)

$$\mathcal{C}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}_{\underline{v}} \text{ over } \mathcal{D}_{\underline{v}}.$$

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$$\underline{\underline{\Theta}}_{\underline{v}}(\sqrt{-q_{\underline{v}}}) = q_{\underline{v}}^{\frac{1}{2\ell}} =: \underline{\underline{q}}_{\underline{v}} \in \mathcal{O}^{\rhd}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})(\simeq \mathcal{O}^{\rhd}_{K_{\underline{v}}}).$$

- Constant section of the divisor monoid: $\mathbb{N} \cdot \log_{\Phi}(\underline{\underline{q}}_{\underline{v}}) \subset \Phi_{\mathcal{C}_{\underline{v}}}$.
- $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} := \mathbb{N} \cdot \log_{\Phi}(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}}$, $\rightsquigarrow p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$.
- $\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}} \rightsquigarrow \mu_{2\ell}(-)$-orbit of characteristic splittings $\tau^{\vdash}_{\underline{v}}$ on $\mathcal{C}^{\vdash}_{\underline{v}}$.

  ($\tau^{\vdash}_{\underline{v}}$ is a subfunctor of $\mathcal{O}^{\rhd}(-) : (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{lin}} \to \mathfrak{Mon}$.)

- $\mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni A \mapsto A^{\Theta} := A \times \underline{\underline{\ddot{Y}}}_{\underline{v}} \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$

$$\rightsquigarrow \text{full subcategory } \mathcal{D}^{\Theta}_{\underline{v}} \subset \mathcal{D}_{\underline{v}}|_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}.$$

- $\mathcal{O}^{\times}(\mathbb{T}^{\mathrm{birat}}_{A^{\Theta}}) \supset$

$$\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}} : A^{\Theta} \mapsto \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}}) \cdot \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}.$$

- $$\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}, \quad \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

  compatible with

$$\underline{\underline{q}}_{\underline{v}}|_{\mathbb{T}_A} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_A}, \quad \mathcal{O}^{\times}(\mathbb{T}_A) \xrightarrow{\sim} \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}}).$$

- $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}} \rightsquigarrow p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}}$ over $\mathcal{D}^{\Theta}_{\underline{v}}$ (a subcategory of $\underline{\underline{\mathcal{F}}}^{\mathrm{birat}}_{\underline{v}}$).
- $\underline{\underline{\Theta}}_{\underline{v}} \rightsquigarrow \mu_{2\ell}(-)$-orbit of characteristic splittings $\tau^{\Theta}_{\underline{v}}$ on $\mathcal{C}^{\Theta}_{\underline{v}}$.
- Split Frobenioids: $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) =: \mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$.

Note: $\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow \mathcal{C}_{\underline{v}} \rightsquigarrow \mathcal{F}^{\vdash}_{\underline{v}}$ since $\mathcal{D}_{\underline{v}} \rightsquigarrow$ theta values, but $\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow$ (up to the $\ell\mathbb{Z}$-indeterminacy in $\underline{\underline{\Theta}}_{\underline{v}}$) $\mathcal{F}^{\Theta}_{\underline{v}}$.

Frobenioid at a good nonarchimedean place $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$

- $\mathcal{B}(\underset{\to}{X})^\circ = \mathcal{D}_{\underline{v}} \supset \mathcal{D}^{\vdash}_{\underline{v}} := \mathcal{B}(K_{\underline{v}})^\circ$.
- Monoid on $\mathcal{D}_{\underline{v}}$:

$$\Phi_{\mathcal{C}_{\underline{v}}} : \mathrm{Ob}(\mathcal{D}_{\underline{v}}) \ni \mathrm{Spec}\, L \mapsto \mathrm{ord}(\mathcal{O}^{\rhd}_L)^{\mathrm{pf}}$$

$$\rightsquigarrow \mathcal{C}_{\underline{v}} := \underline{\underline{\mathcal{F}}}_{\underline{v}}.$$

- Monoid on $\mathcal{D}^{\vdash}_{\underline{v}}$:

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} : \mathrm{Ob}(\mathcal{D}_{\underline{v}}) \ni \mathrm{Spec}\, L \mapsto \mathrm{ord}(\mathbb{Z}^{\rhd}_{p_{\underline{v}}})^{\mathrm{pf}}$$

$$\rightsquigarrow \mathcal{C}^{\vdash}_{\underline{v}}.$$

$$p_{\underline{v}} \rightsquigarrow \text{characteristic splitting } \tau^{\vdash}_{\underline{v}}.$$

- Formal symbol $\log p_{\underline{v}} \cdot \log \underline{\underline{\Theta}}$

$$\rightsquigarrow \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}} := \mathcal{O}^{\times}_{\mathcal{C}_{\underline{v}}} \times \mathbb{N} \cdot \log p_{\underline{v}} \cdot \log \underline{\underline{\Theta}}$$

$$\simeq \mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}} := \mathcal{O}^{\times}_{\mathcal{C}_{\underline{v}}} \times \mathbb{N} \cdot \log p_{\underline{v}}.$$

- $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}} \rightsquigarrow \mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}}$.
- Split Frobenioids: $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) =: \mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$.

Note: $\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow \mathcal{F}^{\vdash}_{\underline{v}}, \mathcal{F}^{\Theta}_{\underline{v}}$.

- $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$=realification of the Frobenioid associated to $(F_{\mathrm{mod}}, $ trivial Galois extension$)$.
- $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \simeq \mathbb{V}_{\mathrm{mod}} \simeq \underline{\mathbb{V}}$.
- $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \supset \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod,v}}} \simeq \mathrm{ord}(\mathcal{O}^{\rhd}_{F_{\mathrm{mod},v}})^{\mathrm{pf}} \otimes \mathbb{R}_{\geq 0}$.
- $p_v \rightsquigarrow \log^{\vdash}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod,v}}}$.
- $\forall \underline{v} \in \underline{\mathbb{V}}$, have $\log_{\Phi}(p_{\underline{v}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ given by $p_{\underline{v}}$.
- The restriction functor $\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{mod}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$ induces isomorphism of top. monoids

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod,v}}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}, \quad \log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}} : F_{\mathrm{mod},v}]}.$$

- Formal symbol $\log \underline{\underline{\Theta}} \rightsquigarrow \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}} := \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \cdot \log \underline{\underline{\Theta}}$,

$$\rightsquigarrow \mathcal{C}^{\Vdash}_{\mathrm{tht}}, \quad \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}.$$

- $\log^{\vdash}_{\mathrm{mod}}(p_v) \rightsquigarrow \log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log \underline{\underline{\Theta}} \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht},v}} \subset \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}}$.
- The restriction functor $\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{tht}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$ induces isomorphism of top. monoids

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht},v}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}},$$

$$\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log \underline{\underline{\Theta}} \mapsto \frac{\log_{\Phi}(p_{\underline{v}}) \cdot \log \underline{\underline{\Theta}}}{[K_{\underline{v}} : F_{\mathrm{mod},v}]}, \quad \text{at } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}},$$

$$\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log \underline{\underline{\Theta}} \mapsto \frac{\log_{\Phi}(p_{\underline{v}}) \cdot \log \underline{\underline{\Theta}}_{\underline{v}}}{[K_{\underline{v}} : F_{\mathrm{mod},v}] \log_{\Phi}(\underline{\underline{q}}_{\underline{v}})}, \quad \text{at } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}.$$

Data:

- ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \simeq \underline{\underline{\mathcal{F}}}_{\underline{v}}$ a category, for any $\underline{v} \in \underline{\mathbb{V}}$. (${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}, {}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}, {}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$)
- ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}} \simeq \mathcal{C}^{\Vdash}_{\mathrm{mod}}$ a category ($\rightsquigarrow$category-theoretically constructible Frobebioid strucuture)
- $\mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}$ a bijection of sets.
- $\forall \underline{v} \in \underline{\mathbb{V}}$, an isomorphism of top. monoids

$$ {}^{\dagger}\rho_{\underline{v}} : \Phi_{{}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod,v}}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} $$

- Require:

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} := \left({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}, \{{}^{\dagger}\rho_{\underline{v}}\}\right) $$
$$ \simeq \mathfrak{F}^{\Vdash}_{\mathrm{mod}} := \left(\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \simeq \underline{\mathbb{V}}, \{\mathcal{F}^{\vdash}_{\underline{v}}\}, \{\rho_{\underline{v}}\}\right). $$

$$ {}^{\dagger}\mathcal{HT}^{\Theta} := \left(\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}\right). $$

- ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \rightsquigarrow$

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad (\simeq \mathfrak{F}^{\Vdash}_{\mathrm{tht}} := (\mathcal{C}^{\Vdash}_{\mathrm{tht}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \simeq \underline{\mathbb{V}}, \{\mathcal{F}^{\Theta}_{\underline{v}}\}, \{\rho^{\Theta}_{\underline{v}}\})). $$

- The <u>Θ-link</u>

$$ (\cdots \xrightarrow{\Theta}) \quad {}^{\dagger}\mathcal{HT}^{\Theta} \xrightarrow{\Theta} {}^{\ddagger}\mathcal{HT}^{\Theta} \quad (\xrightarrow{\Theta} \cdots) $$

is defined to be the full poly-isomorphism

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}. $$

${}^{n}\underline{\underline{\Theta}}_{\underline{v}} \xrightarrow{\Theta} {}^{n+1}\underline{\underline{q}}_{\underline{v}}$ is NOT a conventional evaluation map!

- ${}^\dagger\mathcal{D}_{\underline{v}} \simeq \mathcal{D}_{\underline{v}}$ a category ($\underline{v} \in \underline{\mathbb{V}}^{\rm non}$) or an Aut-holomorphic orbispace ($\underline{v} \in \underline{\mathbb{V}}^{\rm arc}$).
- ${}^\dagger\mathcal{D}^{\vdash}_{\underline{v}} \simeq \mathcal{D}^{\vdash}_{\underline{v}}$ a category ($\underline{v} \in \underline{\mathbb{V}}^{\rm non}$) or an object of $\mathbb{TM}^{\vdash}$ ($\underline{v} \in \underline{\mathbb{V}}^{\rm arc}$).
- $\mathcal{D}$-prime-strip

$$ {}^\dagger\mathfrak{D} := \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}. $$

- $\mathcal{D}^{\vdash}$-prime-strip

$$ {}^\dagger\mathfrak{D}^{\vdash} := \{{}^\dagger\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}. $$

  $({}^\dagger\mathfrak{D} \rightsquigarrow {}^\dagger\mathfrak{D}^{\vdash})$
- Morphisms between prime-strips := collection of morphisms between the constituent objects of prime-strips, indexed by $\underline{\mathbb{V}}$.
- $$ {}^\dagger\mathcal{D}_{\underline{v}} \rightsquigarrow {}^\dagger\underline{\mathcal{D}}_{\underline{v}} \quad (\leftrightarrow \underline{C}_{v}) $$

  .
- Recall that $\{$cusps of ${}^\dagger\mathcal{D}_{\underline{v}}\}, \{$cusps of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}\}$ are group-theoretic via $\pi_1({}^\dagger\mathcal{D}_{\underline{v}}), \pi_1({}^\dagger\underline{\mathcal{D}}_{\underline{v}})$.

A label class of cusps of ${}^\dagger\mathcal{D}_{\underline{v}}$:= the set of cusps of ${}^\dagger\mathcal{D}_{\underline{v}}$ over a nonzero cusp of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ (arising from a nonzero element of Q).

$(\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \leftrightarrow \mathrm{Aut}_{K_{\underline{v}}}(\underline{\underline{X}}_{\underline{v}}/\underline{C}_{\underline{v}})$-orbits of nonzero cusps of $\underline{\underline{X}}_{\underline{v}}$, for $\underline{v}$ bad.)

- $$ \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \simeq \mathbb{F}_\ell^{*} \quad \text{as an } \mathbb{F}_\ell^{*}\text{-torsor.} \quad (\mathbb{F}_\ell^{\times} \curvearrowright Q) $$
- ${}^\dagger\mathcal{D}_{\underline{v}} \rightsquigarrow$ a canonical element ${}^\dagger\underline{\eta}_{\underline{v}} \in \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$ determined by $\underline{\epsilon}_{\underline{v}}$.

- $\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K)^{\circ}$.
- ${}^{\dagger}\mathcal{D}^{\circledcirc} \simeq \mathcal{D}^{\circledcirc}$ a category,

$$\rightsquigarrow \overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad (\simeq \mathbb{V}(\overline{F})),$$

$$\mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})/\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad (\simeq \mathbb{V}(K)).$$

- Recall that $\{$cusps of ${}^{\dagger}\mathcal{D}^{\circledcirc}\}$ is group-theoretic via $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$.

A $\underline{\text{label class of cusps of } {}^{\dagger}\mathcal{D}^{\circledcirc}}$ = a nonzero cusp of ${}^{\dagger}\mathcal{D}^{\circledcirc}$.

- $$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \simeq \mathbb{F}_{\ell}^{*} \quad \text{as an } \mathbb{F}_{\ell}^{*}\text{-torsor.}$$
- For ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ a $\mathcal{D}$-prime-strip, a poly-morphism ${}^{\dagger}\mathfrak{D} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ is a collection $\{{}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\underline{v}\in\underline{\mathbb{V}}}$ of poly-morphisms ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$.
- $\forall \underline{v}, \underline{w} \in \underline{\mathbb{V}}$, $\exists!$ isomorphism of $\mathbb{F}_{\ell}^{*}$-torsors

$$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{w}}), \quad \text{s.t. } {}^{\dagger}\underline{\eta}_{\underline{v}} \mapsto {}^{\dagger}\underline{\eta}_{\underline{w}}.$$

$$\text{identify them, } \rightsquigarrow \mathrm{LabCusp}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \mathbb{F}_{\ell}^{*}.$$

Model base-NF-bridges

- $\mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_\ell) \simeq \mathrm{GL}_2(\mathbb{F}_\ell)$ with chosen basis adapted to $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_\ell \simeq E_{\overline{F}}[\ell] \twoheadrightarrow Q$. (Note $\Pi_{\underline{C}_K} \rightsquigarrow \Delta_X^{\mathrm{ab}}$.)
- $\mathrm{Aut}(\underline{C}_K)(\simeq \mathrm{Out}(\Pi_{\underline{C}_K})) \to \mathrm{GL}_2(\mathbb{F}_\ell)/\{\pm 1\}$. $(\mathrm{Inn}(\Pi_{\underline{C}_K})$ acts by $\cdot \pm 1$.)
- The model $C_{F_{\mathrm{mod}}} \rightsquigarrow$

$$\begin{array}{rcl} \mathrm{Gal}(K/F_{\mathrm{mod}}) & \longrightarrow & \mathrm{GL}_2(\mathbb{F}_\ell)/\{\pm 1\} \\ (\mathrm{Aut}(\mathcal{D}^{\circledcirc}) \simeq)\, \mathrm{Aut}(\underline{C}_K) & \xrightarrow{\sim} & \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}/\{\pm 1\} \bigcap \mathrm{Im}(\mathrm{Gal}(K/F_{\mathrm{mod}})) \\ (\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \simeq)\, \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) & \xrightarrow{\sim} & \left\{\begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}\right\}/\{\pm 1\} \bigcap \mathrm{Im}(\mathrm{Gal}(K/F_{\mathrm{mod}})) \end{array}$$

- $$\mathrm{Aut}(\mathcal{D}^{\circledcirc})/\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \simeq \mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathbb{F}_\ell^{\circledast}.$$

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$: (Look at the covers $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$ and $\underset{\rightarrow}{X}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$.)

$$\begin{array}{ccc} \mathcal{D}_{\underline{v}} & \xrightarrow{\phi^{\mathrm{NF}}_{\bullet,\underline{v}}} & \mathcal{D}^{\circledcirc} \\ \uparrow {\scriptstyle \alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}})} & & \downarrow {\scriptstyle \beta \in \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})} \\ \mathcal{D}_{\underline{v}} & \xrightarrow{\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha} & \mathcal{D}^{\circledcirc} \end{array}$$

$$\phi^{\mathrm{NF}}_{\underline{v}} = \{\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha\},$$

$$\phi^{\mathrm{NF}}_{j} : \mathfrak{D}_j = \{\mathcal{D}_{\underline{v}_j = (\underline{v}, j)}\} \xrightarrow{\{\phi^{\mathrm{NF}}_{\underline{v}}\}} \mathcal{D}^{\circledcirc} \xrightarrow{\cdot j} \mathcal{D}^{\circledcirc}, \quad \forall j \in \mathbb{F}_\ell^{\circledast},$$

$$\phi^{\mathrm{NF}}_{\circledast} = \{\phi^{\mathrm{NF}}_{j}\}_{j \in \mathbb{F}_\ell^{\circledast}} : \mathfrak{D}_{\circledast} = \{\mathfrak{D}_j\}_{j \in \mathbb{F}_\ell^{\circledast}} \to \mathcal{D}^{\circledcirc}, \quad \mathbb{F}_\ell^{\circledast}\text{-equivariant}.$$

(analogue for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

Model base-Θ-bridges

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$:

- $|\mathbb{F}_\ell| = \mathbb{F}_\ell/\{\pm 1\} = \mathbb{F}_\ell^{\circledast} \cup \{0\} \leftrightarrow \{\text{cusps of } \underline{C}_{\underline{v}}\}$.
- $\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$ =image of -1 under Tate uniformization.
- Evaluation points of $\underline{X}_{\underline{v}}$=$\mu_-$-translations of the cusps with labels in $|\mathbb{F}_\ell|$.
- $$\underline{\underline{\Theta}}_{\underline{v}}(\text{a point over an evaluation point with label } j \in |\mathbb{F}_\ell|)$$
  $$\in \mu_{2\ell}\text{-orbit of } \left\{ \underline{\underline{q}}_{\underline{v}}^{\underline{j}^2} \right\}_{\underline{j} \equiv j, \quad \underline{j} \in \mathbb{Z}}.$$
- By definition of $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$, the points of $\underline{\underline{X}}_{\underline{v}}$ over the evaluation points of $\underline{X}_{\underline{v}}$ are all defined over $K_{\underline{v}}$. We call them the evaluation points of $\underline{\underline{X}}_{\underline{v}}$.

  $$\rightsquigarrow \underline{\text{evaluation sections}} \quad G_{\underline{v}} \to \Pi_{\underline{v}} := \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}, \quad (\text{group-theoretic via } \Pi_{\underline{v}}).$$

- $\mathfrak{D}_> = \{\mathcal{D}_{>,\underline{v}}\}$ a copy of the $\mathcal{D}$-prime-strip $\mathfrak{D} = \{\mathcal{D}_{\underline{v}}\}$. For any $j \in \mathbb{F}_\ell^{\circledast}$,

$$\begin{array}{ccccccc} \phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} & \xrightarrow{\text{arbitrary iso.}} & \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{v}})^{\circ} & \longrightarrow & \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{v}})^{\circ} & \xrightarrow{\text{arbitrary iso.}} & \mathcal{D}_{>,\underline{v}} \\ & & \Big\downarrow \text{natural sur.} & \nearrow \text{ev. sec. with label } j & & & \\ & & \mathcal{B}(G_{\underline{v}})^{\circ} & & & & \end{array}$$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$:

- $\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}$ is defined to be the full poly-isomorphism.

Finally,

$$\phi^{\Theta}_j = \{\phi^{\Theta}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}} : \mathfrak{D}_j = \{\mathcal{D}_{\underline{v}_j = (\underline{v}, j)}\} \to \mathfrak{D}_> = \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}},$$

$$\phi^{\Theta}_{\circledast} = \{\phi^{\Theta}_j\}_{j \in \mathbb{F}_\ell^{\circledast}} : \mathfrak{D}_{\circledast} = \{\mathfrak{D}_j\}_{j \in \mathbb{F}_\ell^{\circledast}} \to \mathfrak{D}_>.$$

Transport of label classes of cusps via model base-bridges

- For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, look at $\Pi_{\underline{\underline{X}}_{\underline{v}}} \hookrightarrow \Pi_{\underline{X}_{\underline{v}}} \hookrightarrow \Pi_{\underline{C}_K}$ and $\Pi_{\underline{\overrightarrow{X}}_{\underline{v}}} \hookrightarrow \Pi_{\underline{X}_{\underline{v}}} \hookrightarrow \Pi_{\underline{C}_K}$.
- The arrow $\phi^{\mathrm{NF}}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}^{\circledcirc}$

$$\rightsquigarrow \pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc}),$$

$\rightsquigarrow$ isomorphism of $\mathbb{F}_\ell^{\circledast}$-torsors on cusp labels

$$\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j}).$$

(Consider the cuspidal inertias $I \subset \pi_1(\mathcal{D}^{\circledcirc})$ whose unique index ℓ subgroup $\subset \mathrm{Im}(\Pi_{\underline{\underline{X}}_{\underline{v}}})$ (resp. $\mathrm{Im}(\Pi_{\underline{\overrightarrow{X}}_{\underline{v}}})$).

- Similarly for archimedean places.

In summary, for any $\underline{v} \in \underline{\mathbb{V}}$ and $j \in \mathbb{F}_\ell^{\circledast}$, have isomorphisms of $\mathbb{F}_\ell^{\circledast}$-torsors:

$$\begin{array}{ccc} \mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) & \xrightarrow{\phi^{\mathrm{NF}}_{\underline{v}_j}} & \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j}) \\ & & \downarrow {=} \\ & & \mathrm{LabCusp}(\mathfrak{D}_j) \\ & \searrow^{\phi^{\mathrm{LC}}_j} & \downarrow \phi^{\Theta}_j \\ & & \mathrm{LabCusp}(\mathfrak{D}_{>}) \end{array}$$

$$\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\phi^{\mathrm{LC}}_j} \mathrm{LabCusp}(\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_\ell^{\circledast}, \quad [\underline{\epsilon}] \mapsto j.$$

Base-NF-bridges

- A <u>$\mathcal{D}$-NF-bridge</u> is a poly-morphism ${}^{\dagger}\phi_{※}^{\mathrm{NF}} : {}^{\dagger}\mathfrak{D}_{J} = \{{}^{\dagger}\mathcal{D}_{j}\}_{j \in J} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ from a capsule of $\mathcal{D}$-prime-strips to a category equivalent to $\mathcal{D}^{\circledcirc}$, which fits into the following commutative diagram:

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{D}_{J} & \xrightarrow{{}^{\dagger}\phi_{※}^{\mathrm{NF}}} & {}^{\dagger}\mathcal{D}^{\circledcirc} \\ \uparrow \exists\simeq & & \uparrow \exists\simeq \\ \mathfrak{D}_{※} & \xrightarrow{\phi_{※}^{\mathrm{NF}}} & \mathcal{D}^{\circledcirc} \end{array}$$

- A <u>morphism of $\mathcal{D}$-NF-bridges</u> is a pair of poly-morphisms fitting into the following diagram:

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{D}_{J} & \xrightarrow{{}^{\dagger}\phi_{※}^{\mathrm{NF}}} & {}^{\dagger}\mathcal{D}^{\circledcirc} \\ \text{capsule-full poly-iso.} \downarrow & & \downarrow \mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\circledcirc})\text{-orbit of iso.} \\ {}^{\ddagger}\mathfrak{D}_{J'} & \xrightarrow{{}^{\ddagger}\phi_{※}^{\mathrm{NF}}} & {}^{\ddagger}\mathcal{D}^{\circledcirc} \end{array}$$

- $$\mathrm{Isom}({}^{\dagger}\phi_{※}^{\mathrm{NF}}, {}^{\ddagger}\phi_{※}^{\mathrm{NF}}) \text{ forms an } \mathbb{F}_{\ell}^{⋇}\text{-torsor.}$$

- A $\mathcal{D}$-Θ-bridge is a poly-morphism ${}^{\dagger}\phi^{\Theta}_{\divideontimes} : {}^{\dagger}\mathfrak{D}_J = \{{}^{\dagger}\mathcal{D}_j\}_{j \in J} \to {}^{\dagger}\mathfrak{D}_{>}$ from a capsule of $\mathcal{D}$-prime-strips to a $\mathcal{D}$-prime-strip, which fits into the following commutative diagram:

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{D}_J & \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\divideontimes}} & {}^{\dagger}\mathfrak{D}_{>} \\ \uparrow \exists\simeq & & \uparrow \exists\simeq \\ \mathfrak{D}_{\divideontimes} & \xrightarrow{\phi^{\Theta}_{\divideontimes}} & \mathfrak{D}_{>} \end{array}$$

- A morphism of $\mathcal{D}$-Θ-bridges is a pair of poly-morphisms fitting into the following diagram:

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{D}_J & \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\divideontimes}} & {}^{\dagger}\mathfrak{D}_{>} \\ \text{capsule-full poly-iso.} \downarrow & & \downarrow \text{full poly-iso.} \\ {}^{\ddagger}\mathfrak{D}_{J'} & \xrightarrow{{}^{\ddagger}\phi^{\Theta}_{\divideontimes}} & {}^{\ddagger}\mathfrak{D}_{>} \end{array}$$

- $$\mathrm{Isom}({}^{\dagger}\phi^{\Theta}_{\divideontimes}, {}^{\ddagger}\phi^{\Theta}_{\divideontimes}) = \{*\}.$$

Base-ΘNF-Hodge theaters

A $\mathcal{D}$-ΘNF-Hodge theater is a collection of data

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = \left({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_J \overset{{}^{\dagger}\phi_{\circledast}^{\Theta}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>}\right) $$

fitting into the following commutative diagram

$$ \begin{array}{ccccc} {}^{\dagger}\mathcal{D}^{\circledcirc} & \xleftarrow{\;{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}\;} & {}^{\dagger}\mathfrak{D}_J & \xrightarrow{\;{}^{\dagger}\phi_{\circledast}^{\Theta}\;} & {}^{\dagger}\mathfrak{D}_{>} \\ \uparrow \exists\simeq & & \uparrow \exists\simeq & & \uparrow \exists\simeq \\ \mathcal{D}^{\circledcirc} & \xleftarrow{\;\phi_{\circledast}^{\mathrm{NF}}\;} & \mathfrak{D}_{\circledast} & \xrightarrow{\;\phi_{\circledast}^{\Theta}\;} & \mathfrak{D}_{>} \end{array} $$

A morphism of $\mathcal{D}$-ΘNF-Hodge theaters is a pair of morphisms between the respective associated $\mathcal{D}$-bridges fitting into the following diagram:

$$ \begin{array}{ccccc} {}^{\dagger}\mathcal{D}^{\circledcirc} & \xleftarrow{\;{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}\;} & {}^{\dagger}\mathfrak{D}_J & \xrightarrow{\;{}^{\dagger}\phi_{\circledast}^{\Theta}\;} & {}^{\dagger}\mathfrak{D}_{>} \\ \downarrow & & \downarrow & & \downarrow \\ {}^{\ddagger}\mathcal{D}^{\circledcirc} & \xleftarrow{\;{}^{\ddagger}\phi_{\circledast}^{\mathrm{NF}}\;} & {}^{\ddagger}\mathfrak{D}_{J'} & \xrightarrow{\;{}^{\ddagger}\phi_{\circledast}^{\Theta}\;} & {}^{\ddagger}\mathfrak{D}_{>} \end{array} $$

- ${}^{\dagger}\chi : \pi_0({}^{\dagger}\mathfrak{D}_J) = J \xrightarrow{\sim} \mathbb{F}_{\ell}^{\circledast}$.
- $\forall j \in J = \mathbb{F}_{\ell}^{\circledast}, \quad {}^{\dagger}\phi_j^{\mathrm{LC}} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_{>})$.
- $\exists! [{}^{\dagger}\underline{\epsilon}] \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ s.t. under $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_{\ell}^{\circledast}$,

$$ {}^{\dagger}\phi_j^{\mathrm{LC}}([{}^{\dagger}\underline{\epsilon}]) = {}^{\dagger}\phi_1^{\mathrm{LC}}({}^{\dagger}\chi(j) \cdot [{}^{\dagger}\underline{\epsilon}]) \mapsto {}^{\dagger}\chi(j). $$

$\rightsquigarrow$ **synchronized indeterminacy**:

$$ \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J. $$

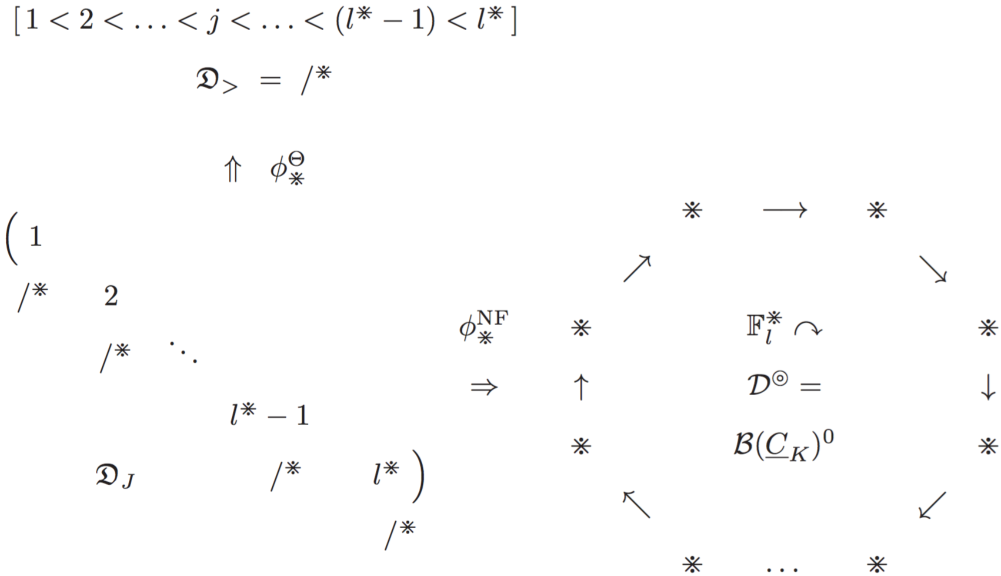

Figure: $\mathcal{D}$-ΘNF-Hodge theater (Fig. 4.4 of [IUT-I])

- For the forgetful functor

$$\begin{array}{rcl} \{\mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\} & \to & \{\mathcal{D}\text{-NF-bridges}\} \\ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi_{*}^{\mathrm{NF}}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{J} \overset{{}^{\dagger}\phi_{*}^{\Theta}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>}) & \mapsto & ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi_{*}^{\mathrm{NF}}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{J}) \end{array}$$

  the output data has $\mathbb{F}_{\ell}^{*}$-**symmetry**. ($J \xrightarrow{\sim} \mathbb{F}_{\ell}^{*}$)
- For the forgetful functor

$$\begin{array}{rcl} \{\mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\} & \to & \left\{\ell^{*}\text{-capsules of } \mathcal{D}\text{-(resp. } \mathcal{D}^{\vdash}\text{-)prime-strips}\right\} \\ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} & \mapsto & {}^{\dagger}\mathfrak{D}_{J}\,(\text{resp. } {}^{\dagger}\mathfrak{D}_{J}^{\vdash}) \end{array}$$

  the output data has $\mathfrak{S}_{\ell^{*}}$-symmetry. (forgetting $J \xrightarrow{\sim} \mathbb{F}_{\ell}^{*}$)
- Reduce $(\ell^{*})^{\ell^{*}}$-indeterminacy to $\ell^{*}!$-indeterminacy:

$$({}^{\dagger}\phi_{*}^{\Theta} \mapsto {}^{\dagger}\mathfrak{D}_{J}) \rightsquigarrow ({}^{\dagger}\phi_{*}^{\Theta} \mapsto \mathrm{Proc}({}^{\dagger}\mathfrak{D}_{J})), \quad ({}^{\dagger}\phi_{*}^{\Theta} \mapsto \mathrm{Proc}({}^{\dagger}\mathfrak{D}_{J}^{\vdash})).$$

  (A procession in a category is a diagram $P_1 \hookrightarrow P_2 \hookrightarrow \cdots \hookrightarrow P_n$ with P_j a j-capsule of objects and $\hookrightarrow$ the collection of all capsule-full poly-morphisms. A morphism of processions is an order-preserving injection $\iota : \{1, \cdots, n\} \hookrightarrow \{1, \cdots, m\}$ plus the capsule-full poly-morphisms $P_j \hookrightarrow Q_{\iota(j)}$.)

The $\underline{\mathcal{D}\text{-NF-link}}$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}} \xrightarrow{\mathcal{D}} {^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}}}$$

between two $\mathcal{D}$-ΘNF-Hodge theaters is the (induced) full poly-isomorphism

$$^{\dagger}\mathfrak{D}^{\vdash}_{>} \xrightarrow{\sim} {^{\ddagger}\mathfrak{D}^{\vdash}_{>}}. \quad \textbf{(mono-analytic core)}$$

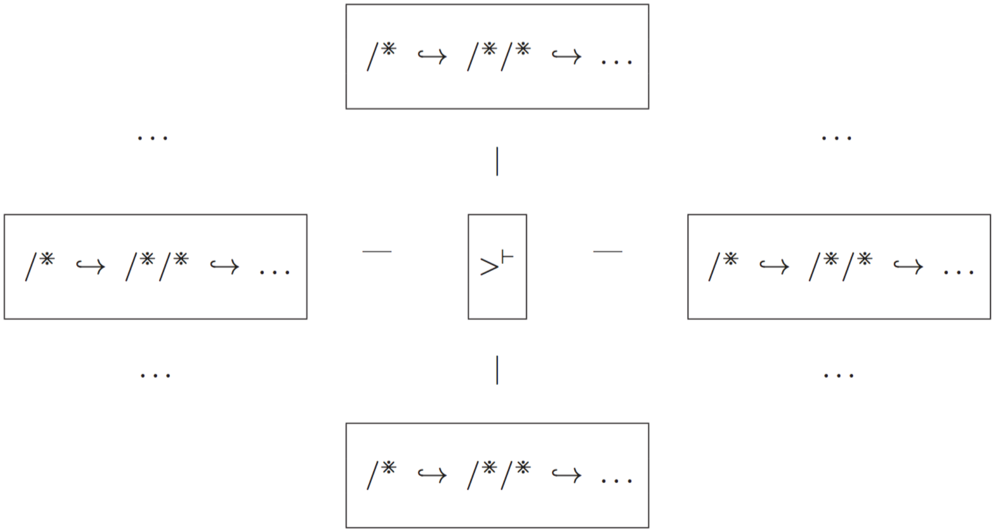

Figure: Étale-picture of $\mathcal{D}$-ΘNF-Hodge theaters (Fig. 4.7 of [IUT-I])

Let ${}^\dagger\mathcal{D}^\circledcirc \simeq \mathcal{D}^\circledcirc$ be a category. The Θ-approach to $\pi_1({}^\dagger\mathcal{D}^\circledcirc) \rightsquigarrow$

- $$\mathbb{M}^\circledast({}^\dagger\mathcal{D}^\circledcirc) \simeq \overline{F}^\times, \quad \overline{\mathbb{M}}^\circledast({}^\dagger\mathcal{D}^\circledcirc) \simeq \overline{F}.$$
- $\pi_1({}^\dagger\mathcal{D}^\circledcirc) \hookrightarrow \pi_1({}^\dagger\mathcal{D}^\circledast)$ $(\leftrightarrow \underline{C}_K \to C_{F_{\mathrm{mod}}})$, ${}^\dagger\mathcal{D}^\circledcirc \to {}^\dagger\mathcal{D}^\circledast := \mathcal{B}(\pi_1({}^\dagger\mathcal{D}^\circledast))^\circ$.
- $\pi_1({}^\dagger\mathcal{D}^\circledast)$-invariants $\rightsquigarrow$
$$\mathbb{M}^\circledast_{\mathrm{mod}}({}^\dagger\mathcal{D}^\circledcirc)(\simeq (F^\times_{\mathrm{mod}}), \quad \overline{\mathbb{M}}^\circledast_{\mathrm{mod}}({}^\dagger\mathcal{D}^\circledcirc)(\simeq F_{\mathrm{mod}}).$$
- Belyi cuspidalization $\rightsquigarrow$ $(G_{K_{C_{F_{\mathrm{mod}}}}} \simeq)\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \twoheadrightarrow \pi_1({}^\dagger\mathcal{D}^\circledast)$ (well-defined up to inner action of kernel), pseudo-monoids
$$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright \mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc), \quad \mathbb{M}^\circledast_{\kappa}({}^\dagger\mathcal{D}^\circledcirc)(= \mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc)^{\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)}), \quad \mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc).$$
- $\overline{\mathbb{M}}^\circledast({}^\dagger\mathcal{D}^\circledcirc) \rightsquigarrow \overline{\mathbb{V}}({}^\dagger\mathcal{D}^\circledcirc)(\simeq \underline{\mathbb{V}}(\overline{F})) \rightsquigarrow$
$$\Phi^\circledast({}^\dagger\mathcal{D}^\circledcirc) : \mathrm{Ob}({}^\dagger\mathcal{D}^\circledast) \ni A \mapsto \text{monoid of arithmetic divisors on } \overline{\mathbb{M}}^\circledast({}^\dagger\mathcal{D}^\circledcirc)^A$$
$$\rightsquigarrow \text{model Frobenioid } \mathcal{F}^\circledast({}^\dagger\mathcal{D}^\circledcirc) \text{ over } {}^\dagger\mathcal{D}^\circledast.$$

Let ${}^\dagger\mathcal{F}^\circledast \simeq \mathcal{F}^\circledast({}^\dagger\mathcal{D}^\circledcirc)$ be a category ($\rightsquigarrow$ Frobenioid structure on it). Suppose we are given ${}^\dagger\mathcal{D}^\circledcirc \to \mathrm{base}({}^\dagger\mathcal{F}^\circledast)$ isomorphic to ${}^\dagger\mathcal{D}^\circledcirc \to {}^\dagger\mathcal{D}^\circledast$. Then identify (by F-cority of C_F)
$$\mathrm{base}({}^\dagger\mathcal{F}^\circledast) = {}^\dagger\mathcal{D}^\circledast.$$

- Define
$${}^\dagger\mathcal{F}^\circledcirc := {}^\dagger\mathcal{F}^\circledast|_{{}^\dagger\mathcal{D}^\circledcirc},$$
$${}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}} := {}^\dagger\mathcal{F}^\circledast|_{\text{terminal objects of } {}^\dagger\mathcal{D}^\circledast},$$
$$(\text{“}=\text{”Frobenioid of arithmetic line bundles on } [\mathrm{Spec}\,\mathcal{O}_K/\,\mathrm{Gal}(K/F_{\mathrm{mod}})]).$$

- $A \in \mathrm{Ob}({}^\dagger\mathcal{F}^\circledast) \rightsquigarrow \mathcal{O}^\times(A^{\mathrm{birat}})$ (= mult. group of the finite extension of F_{mod} corresponding to A).
- Thus varying Frobenius-trivial objects $A \in \mathrm{Ob}({}^\dagger\mathcal{F}^\circledast)$ over Galois objects of ${}^\dagger\mathcal{D}^\circledast$, $\rightsquigarrow$

$$\pi_1({}^\dagger\mathcal{D}^\circledast) \curvearrowright {}^\dagger\mathbb{M}^\circledast.$$

- $\forall \mathfrak{p} \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^\circledast}(A)) \rightsquigarrow \mathcal{O}^\triangleright_\mathfrak{p} := \left(\mathcal{O}^\times(A^{\mathrm{birat}}) \to \Phi_{{}^\dagger\mathcal{F}^\circledast}(A)^{\mathrm{gp}}\right)^{-1}(\mathfrak{p} \cup \{0\})$.
- For A_0 lying over a *terminal* object of ${}^\dagger\mathcal{D}^\circledast$ and $\mathfrak{p}_0 \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^\circledast}(A_0))$: Consider the elements of $\mathrm{Aut}_{{}^\dagger\mathcal{F}^\circledast}(A)$ fixing $\mathcal{O}^\triangleright_\mathfrak{p}$ for a system of $\mathfrak{p}|\mathfrak{p}_0$, $\rightsquigarrow$ the closed subgroup (well-defined up to conjugation) $\Pi_{\mathfrak{p}_0} \subset \pi_1({}^\dagger\mathcal{D}^\circledast)$.
- Look at a pair

$$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright {}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times} \text{ isomorphic to } \pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright \mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc):$$

$$(\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\} \rightsquigarrow) \quad \exists! \quad \mu^\Theta_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^\circledast)) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}({}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times}) \text{ s.t.}$$

$$\begin{array}{ccc}
\mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc) & \hookrightarrow & \varinjlim_{H\subset\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)\text{ open}} H^1(H, \mu^\Theta_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^\circledast))) \\
\simeq\downarrow & & \simeq\downarrow \\
{}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times} & \hookrightarrow & \varinjlim_{H\subset\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)\text{ open}} H^1(H, \mu_{\widehat{\mathbb{Z}}}({}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times}))
\end{array}$$

- Similarly have canonical isomorphism

$$\mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc) \xrightarrow{\sim} {}^\dagger\mathbb{M}^\circledast_{\infty\kappa},$$

and canonical isomorphism *compatible with the integral submonoids* $\mathcal{O}^\triangleright_\mathfrak{p}$

$$\mathbb{M}^\circledast({}^\dagger\mathcal{D}^\circledcirc) \xrightarrow{\sim} {}^\dagger\mathbb{M}^\circledast.$$

- So, ${}^{\dagger}\mathcal{F}^{\circledast}$ carries natural structures

$$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} = ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})^{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})}.$$

  - $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times} \rightsquigarrow \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$.
    (Consider the subset of elements for which the Kummer class restricted to some subgroup of $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$ corresponding to an open subgroup of the decomposition group of some strictly critical point of $C_{F_{\mathrm{mod}}}$ is a root of unity.)
  - $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \rightsquigarrow {}^{\dagger}\mathbb{M}^{\circledast}$ plus the field structure on ${}^{\dagger}\mathbb{M}^{\circledast} \cup \{0\}$.
    (Consider Kummer classes arising from ${}^{\dagger}\mathbb{M}^{\circledast}_{\kappa}$ restricted to subgroups of $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$ corresponding to decomposition groups of non-critical $\overline{F}$-valued points of $C_{F_{\mathrm{mod}}}$.)

- In particular, ${}^{\dagger}\mathcal{F}^{\circledast} \rightsquigarrow$

$$\mathrm{Prime}({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})/\pi_1({}^{\dagger}\mathcal{D}^{\circledast}).$$

- $\rightsquigarrow \mathfrak{p}|\mathfrak{p}_0$ (assumed nonarchimedean) determines a valuation on $\mathcal{O}^{\times}(A^{\mathrm{birat}}) \cup \{0\}$, $\rightsquigarrow$ $\mathcal{O}^{\triangleright}_{\widehat{\mathfrak{p}}}$ (= mult. monoid of nonzero integral elements of the completion at $\mathfrak{p}$ of the number field corresponding to A).
- Varying $A \rightsquigarrow$ (considered up to conjugation by $\Pi_{\mathfrak{p}_0}$)

$$\Pi_{\mathfrak{p}_0} \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}_{\widehat{\mathfrak{p}_0}} \quad \text{(ind-topological monoid)}$$

("MLF-Galois $\mathbb{TM}$-pair of strictly Belyi type")

$\underline{\mathcal{F}\text{-prime-strip}}$:

$$^{\ddagger}\mathfrak{F} = \{^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

$$\text{s.t. at } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}, {}^{\ddagger}\mathcal{F}_{\underline{v}} = {}^{\ddagger}\mathcal{C}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}_{\underline{v}}; \text{ at } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}, \ldots$$

$$\rightsquigarrow \text{ associated } \mathcal{D}\text{-prime-strip } {}^{\ddagger}\mathfrak{D} = \{^{\ddagger}\mathcal{D}_{\underline{v}}\}.$$

At $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}, \pi_1(^{\ddagger}\mathcal{D}_{\underline{v}}) \rightsquigarrow$

- $^{\ddagger}\mathcal{D}_{\underline{v}} \to {}^{\ddagger}\mathcal{D}_{v}$ $(\leftrightarrow \underline{\underline{X}}_{\underline{v}}, \underset{\to}{X}_{\underline{v}} \to C_v, v \in \mathbb{V}_{\mathrm{mod}})$.
- $\pi_1^{\mathrm{rat}}(^{\ddagger}\mathcal{D}_v) \twoheadrightarrow \pi_1(^{\ddagger}\mathcal{D}_v)$.
- $$\pi_1(^{\ddagger}\mathcal{D}_v) \curvearrowright \mathbb{M}_v(^{\ddagger}\mathcal{D}_{\underline{v}})(\simeq \mathcal{O}^{\triangleright}_{F_v}),$$
  $$\pi_1^{\mathrm{rat}}(^{\ddagger}\mathcal{D}_v) \curvearrowright \mathbb{M}_{\kappa v}(^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa v}(^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa\times v}(^{\ddagger}\mathcal{D}_{\underline{v}}).$$
- For an isomorph $^{\ddagger}\mathbb{M} = {}^{\ddagger}\mathbb{M}_v, {}^{\ddagger}\mathbb{M}_{\infty\kappa v}, {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$ of $\mathbb{M}_v(^{\ddagger}\mathcal{D}_{\underline{v}})$, $\mathbb{M}_{\infty\kappa v}(^{\ddagger}\mathcal{D}_{\underline{v}})$, $\mathbb{M}_{\infty\kappa\times v}(^{\ddagger}\mathcal{D}_{\underline{v}})$ respectively:
  $$\exists! \quad \mu^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1(^{\ddagger}\mathcal{D}_{\underline{v}})) \xrightarrow{\sim} \mu_{\widehat{\mathbb{Z}}}(^{\ddagger}\mathbb{M}), \text{ s.t. } {}^{\ddagger}\mathbb{M}(^{\ddagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}.$$
- Thus, $^{\ddagger}\mathcal{F}_{\underline{v}}$ carries natural structures:
  $$\pi_1(^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_v, \quad \pi_1^{\mathrm{rat}}(^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa v}, {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}, {}^{\ddagger}\mathbb{M}_{\kappa v} := {}^{\ddagger}\mathbb{M}_{\infty\kappa v}^{\pi_1^{\mathrm{rat}}(^{\ddagger}\mathcal{D}_v)}.$$
- $$^{\ddagger}\mathbb{M}_{\infty\kappa\times v} \rightsquigarrow {}^{\ddagger}\mathbb{M}_{\infty\kappa v} \rightsquigarrow {}^{\ddagger}\mathbb{M}_{\kappa v} \rightsquigarrow {}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}, \quad (^{\ddagger}\mathbb{M}_v^{\mathrm{gp}})^{\pi_1(^{\ddagger}\mathcal{D}_v)},$$
  plus the field structures on $^{\ddagger}\mathbb{M}_v^{\mathrm{gp}} \cup \{0\}, (^{\ddagger}\mathbb{M}_v^{\mathrm{gp}})^{\pi_1(^{\ddagger}\mathcal{D}_v)} \cup \{0\}$.

(Analogue at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.)

$$({}^{\ddagger}\mathfrak{F} \rightsquigarrow)$$

- $\underline{\mathcal{F}^{\vdash}\text{-prime-strip}}$:

$${}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \text{ with}$$

$${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\vdash}_{\underline{v}} \text{ (splitting Frobenioid)}, \underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}; \quad {}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} = \ldots, \underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}.$$

- A morphism of $\mathcal{F}$-prime-strips is a collection of isomorphisms indexed by $\underline{\mathbb{V}}$. Similarly for $\mathcal{F}^{\vdash}$-prime-strips.
- Globally realified mono-analytic Frobenioid-prime-strip:

$${}^{\ddagger}\mathfrak{F}^{\Vdash} = \left({}^{\ddagger}\mathcal{C}^{\Vdash}, \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\ddagger}\mathfrak{F}^{\vdash}, \{{}^{\ddagger}\rho_{\underline{v}} : \Phi_{{}^{\ddagger}\mathcal{C}^{\Vdash},\underline{v}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

$$:= \text{collection of data} \simeq \mathfrak{F}^{\Vdash}_{\mathrm{mod}}.$$

$$\mathrm{Isom}({}^{1}\mathcal{F}^{\circledast}, {}^{2}\mathcal{F}^{\circledast}) \xrightarrow{\sim} \mathrm{Isom}(\mathrm{base}({}^{1}\mathcal{F}^{\circledast}), \mathrm{base}({}^{2}\mathcal{F}^{\circledast})),$$

$$\mathrm{Isom}({}^{1}\mathcal{F}^{\circledcirc}, {}^{2}\mathcal{F}^{\circledcirc}) \xrightarrow{\sim} \mathrm{Isom}(\mathrm{base}({}^{1}\mathcal{F}^{\circledcirc}), \mathrm{base}({}^{2}\mathcal{F}^{\circledcirc})),$$

$$\mathrm{Isom}({}^{1}\mathfrak{F}, {}^{2}\mathfrak{F}) \xrightarrow{\sim} \mathrm{Isom}({}^{1}\mathfrak{D}, {}^{2}\mathfrak{D}),$$

$$\mathrm{Isom}({}^{1}\mathfrak{F}^{\vdash}, {}^{2}\mathfrak{F}^{\vdash}) \twoheadrightarrow \mathrm{Isom}({}^{1}\mathfrak{D}^{\vdash}, {}^{2}\mathfrak{D}^{\vdash}).$$

Recall

$$^\dagger\mathcal{HT}^{\Theta} = (\{^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{mod}}), \quad {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^\dagger\mathcal{D}^{\circledcirc} \overset{{}^\dagger\phi^{\mathrm{NF}}_{\divideontimes}}{\longleftarrow} {}^\dagger\mathfrak{D}_J \overset{{}^\dagger\phi^{\Theta}_{\divideontimes}}{\longrightarrow} {}^\dagger\mathfrak{D}_{>}).$$

Assume the $\mathcal{D}$-prime-strip associated to $^\dagger\mathcal{HT}^{\Theta}$ is equal to $^\dagger\mathfrak{D}_{>}$.
Thus the $\mathcal{F}$-prime-strip $^\dagger\mathfrak{F}_{>}$ of $^\dagger\mathcal{HT}^{\Theta}$ has $^\dagger\mathfrak{D}_{>}$ as its associated $\mathcal{D}$-prime-strip.

- $\forall j \in J$, $\mathcal{F}$-prime-strip $^\dagger\mathfrak{F}_j = \{^\dagger\mathcal{F}_{\underline{v}_j}\}_{\underline{v}_j\in\underline{\mathbb{V}}_j}$ with associated $\mathcal{D}$-prime-strip $^\dagger\mathfrak{D}_j$, $^\dagger\phi^{\Theta}_j : {}^\dagger\mathfrak{D}_j \to {}^\dagger\mathfrak{D}_{>} \rightsquigarrow$ (unique)

$$^\dagger\psi^{\Theta}_j : {}^\dagger\mathfrak{F}_j \longrightarrow {}^\dagger\mathfrak{F}_{>}, \quad {}^\dagger\psi^{\Theta}_{\divideontimes} : {}^\dagger\mathfrak{F}_J \longrightarrow {}^\dagger\mathfrak{F}_{>}.$$

(The associated $\mathcal{D}$-prime-strip of $({}^\dagger\phi^{\Theta}_j)^*({}^\dagger\mathfrak{F}_{>})$ is equal to $^\dagger\mathfrak{D}_j$, $\rightsquigarrow {}^\dagger\mathfrak{F}_j \simeq ({}^\dagger\phi^{\Theta}_j)^*({}^\dagger\mathfrak{F}_{>})$.)

For any $\delta \in \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc})$, $\exists!$ $\mathrm{Aut}_{\underline{\epsilon}}({}^\dagger\mathcal{D}^{\circledcirc})$-orbit of isomorphisms $^\dagger\mathcal{D}^{\circledcirc} \xrightarrow{\sim} \mathcal{D}^{\circledcirc}$ mapping δ to $[\underline{\epsilon}]$.

- A $\underline{\delta\text{-valuation}}$ of $\mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc})$ is an element mapping to an element of $\underline{\mathbb{V}}^{\pm\mathrm{un}} := \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cdot \underline{\mathbb{V}}$ via this $\mathrm{Aut}_{\underline{\epsilon}}({}^\dagger\mathcal{D}^{\circledcirc})$-orbit of isomorphisms.
- At a δ-valuation $\underline{v} \in \mathbb{V}({}^\dagger\mathcal{D}^{\circledcirc})$,

$$\Pi_{\mathfrak{p}_0} \curvearrowright \widetilde{\mathcal{O}}^{\rhd}_{\widehat{\mathfrak{p}}_0}|_{\text{open subgps. of } \Pi_{\mathfrak{p}_0} \cap \pi_1^{(\mathrm{tp})}({}^\dagger\mathcal{D}^{\circledcirc}) \text{ corresponding to } \underline{\underline{X}}, \underset{\to}{X}, \text{ determined by } \delta}$$

$$\rightsquigarrow p_{\underline{v}}\text{-adic Frobenioids.}$$

(Analogue at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.)

- $\rightsquigarrow$ $\mathcal{F}$-prime-strip

$$^\dagger\mathcal{F}^{\circledcirc}|_{\delta} \quad (\curvearrowleft \pi_1({}^\dagger\mathcal{D}^{\circledcirc})).$$

(only well-defined up to isomorphism, because $\underline{\mathbb{V}}^{\pm\mathrm{un}} \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$ is not injective.)

- For an $\mathcal{F}$-prime-strip ${}^{\ddagger}\mathfrak{F}$, a poly-morphism

$$ {}^{\ddagger}\mathfrak{F} \longrightarrow {}^{\dagger}\mathcal{F}^{\circledcirc} := \text{a full poly-iso. } {}^{\ddagger}\mathfrak{F} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta} \text{ for some } \delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}). $$

  (Such ${}^{\ddagger}\mathfrak{F} \longrightarrow {}^{\dagger}\mathcal{F}^{\circledcirc}$ is fixed by composition with elements in $\mathrm{Aut}({}^{\ddagger}\mathfrak{F})$ or $\mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{F}^{\circledcirc})$.)

- Over a given ${}^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\dagger}\mathfrak{D}_J \to {}^{\dagger}\mathcal{D}^{\circledcirc}$, $\exists!$ poly-morphism

$$ {}^{\dagger}\psi_{\divideontimes}^{\mathrm{NF}} : {}^{\dagger}\mathfrak{F}_J \to {}^{\dagger}\mathcal{F}^{\circledcirc}. $$

-

$$ \underline{\mathbb{V}}_{\langle J \rangle} \overset{\text{diag.}}{\hookrightarrow} \underline{\mathbb{V}}_J = \prod \underline{\mathbb{V}}_j, \quad {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J \rangle} = \{ {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}, \underline{\mathbb{V}}_{\langle J \rangle} \xrightarrow{\sim} \mathrm{Prime}({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}) \}, $$

$$ {}^{\dagger}\mathcal{F}^{\circledast}_{j} = \{ {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}, \underline{\mathbb{V}}_{j} \xrightarrow{\sim} \mathrm{Prime}({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}) \}. $$

- Any poly-morphism ${}^{\dagger}\mathfrak{F}_{\langle J \rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ induces an isomorphism class of functors

$$ ({}^{\dagger}\mathcal{F}^{\circledast} \supset) \quad {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J \rangle} \xrightarrow{\text{res.}} {}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J \rangle}}, \quad \forall \underline{v}_{\langle J \rangle} \in \underline{\mathbb{V}}_{\langle J \rangle}, $$

  (independent of choice of ${}^{\dagger}\mathfrak{F}_{\langle J \rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ among its $\mathbb{F}_{\ell}^{\divideontimes}$-conjugates) hence isomorphism classes of restriction functors

$$ {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J \rangle} \to {}^{\dagger}\mathfrak{F}_{\langle J \rangle}. $$

- Similarly, have collections of isomorphism classes of restriction functors

$$ {}^{\dagger}\mathcal{F}^{\circledast}_{J} \to {}^{\dagger}\mathfrak{F}_J, \quad {}^{\dagger}\mathcal{F}^{\circledast}_{j} \to {}^{\dagger}\mathfrak{F}_j. $$

- An NF-bridge:=

$$({}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi^{\mathrm{NF}}_{\circledast}} {}^{\ddagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\ddagger}\mathcal{F}^{\circledast}):$$

  - ${}^{\ddagger}\mathfrak{F}_J$ a capsule of $\mathcal{F}$-prime-strips. (${}^{\ddagger}\mathfrak{D}_J$:= associated capsule of $\mathcal{D}$-prime-strips.)
  - ${}^{\ddagger}\mathcal{F}^{\circledcirc} \simeq {}^{\dagger}\mathcal{F}^{\circledcirc}, {}^{\ddagger}\mathcal{F}^{\circledast} \simeq {}^{\dagger}\mathcal{F}^{\circledast}$ categories. (${}^{\ddagger}\mathcal{D}^{\circledcirc}, {}^{\ddagger}\mathcal{D}^{\circledast}$=bases.)
  - ${}^{\ddagger}\psi^{\mathrm{NF}}_{\circledast}$ the poly-morphism lifting a $\mathcal{D}$-NF-bridge ${}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathcal{D}^{\circledcirc}$.
  - ${}^{\ddagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\ddagger}\mathcal{F}^{\circledast}$ a morphism ${}^{\ddagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledast}$ (abstractly) equivalent to ${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$ plus an isomorphism ${}^{\ddagger}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast}|_{{}^{\ddagger}\mathcal{D}^{\circledcirc}}$.
- A morphism of NF-bridges

$$({}^{1}\mathfrak{F}_{J_1} \longrightarrow {}^{1}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{1}\mathcal{F}^{\circledast}) \to ({}^{2}\mathfrak{F}_{J_2} \longrightarrow {}^{2}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{2}\mathcal{F}^{\circledast})$$

  consists of (capsule-full poly-)isomorphisms

$${}^{1}\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{J_2}, \quad {}^{1}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledcirc}, \quad {}^{1}\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledast}$$

  compatible with the ($\mathcal{D}$-)NF-bridges.
- A Θ-bridge:=

$$({}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi^{\Theta}_{\circledast}} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta}):$$

  - ${}^{\ddagger}\mathfrak{F}_J$ as above.
  - ${}^{\ddagger}\mathcal{HT}^{\Theta}$ a Θ-Hodge theater.
  - ${}^{\ddagger}\mathfrak{F}_{>}$ the $\mathcal{F}$-prime-strip associated to ${}^{\ddagger}\mathcal{HT}^{\Theta}$. (${}^{\ddagger}\mathfrak{D}_{>}$=associated $\mathcal{D}$-prime-strip.)
  - ${}^{\ddagger}\psi^{\Theta}_{\circledast}$ a poly-morphism lifting a $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi^{\Theta}_{\circledast} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathfrak{D}_{>}$.
- A morphism of Θ-bridges is defined similarly.

Constructions above $\rightsquigarrow$

$$^{\ddagger}\mathcal{HT}^{\Theta\mathrm{NF}} = ({}^{\ddagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\ddagger}\mathcal{F}^{\circledcirc} \overset{{}^{\ddagger}\psi^{\mathrm{NF}}_{*}}{\longleftarrow} {}^{\ddagger}\mathfrak{F}_J \overset{{}^{\ddagger}\psi^{\Theta}_{*}}{\longrightarrow} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta})$$

- $$\forall \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}, \quad \mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}_{\underline{v}}).$$

  (Consider the rational function and divisor monoids of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.)
- $\rightsquigarrow$
  $$\mathrm{Isom}(\Theta\text{-Hodge theaters}) \xrightarrow{\sim} \mathrm{Isom}(\text{associated } \mathcal{D}\text{-prime-strips})$$

  (The global data $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ admits no nontrivial automorphisms.)
- $$\mathrm{Isom}(\text{NF-bridges}) \xrightarrow{\sim} \mathrm{Isom}(\text{associated } \mathcal{D}\text{-NF-bridges}),$$
  $$\mathrm{Isom}(\Theta\text{-bridges}) \xrightarrow{\sim} \mathrm{Isom}(\text{associated } \mathcal{D}\text{-}\Theta\text{-bridges}),$$
  $$\mathrm{Isom}(\Theta\text{NF-Hodge theaters}) \xrightarrow{\sim} \mathrm{Isom}(\text{associated } \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}).$$
- Given an NF-bridge $^{\ddagger}\psi^{\mathrm{NF}}_{*}$ and a Θ-bridge $^{\ddagger}\psi^{\Theta}_{*}$,
  $$\left\{\text{capsule-full poly-isomorphisms } {}^{\ddagger}\mathfrak{F}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_J \text{ gluing them into } {}^{\ddagger}\mathcal{HT}^{\Theta\mathrm{NF}}\right\} \simeq \mathbb{F}^{*}_{\ell}.$$

- $\mathbb{F}_\ell^{\rtimes\pm} := \mathbb{F}_\ell \rtimes \{\pm 1\}$. $(\{\pm1\} \hookrightarrow \mathbb{F}_\ell^\times)$
- $\mathbb{F}_\ell^{\pm}$-group:=set E plus $\{\pm1\}$-orbits of bijections $E \simeq \mathbb{F}_\ell$.
- $\mathbb{F}_\ell^{\pm}$-torsor:=set T plus $\mathbb{F}_\ell^{\rtimes\pm}$-orbits of bijections $T \simeq \mathbb{F}_\ell$.
$$(\mathbb{F}_\ell^{\rtimes\pm} \curvearrowright \mathbb{F}_\ell, z \mapsto \pm z + \lambda)$$
- For ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^\dagger\mathcal{D}_{\underline{v}} \rightsquigarrow {}^\dagger\mathcal{D}_{\underline{v}}^{\pm}$ ($\leftrightarrow \underline{X}_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, ...for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)
$$\rightsquigarrow {}^\dagger\underline{\mathfrak{D}}^{\pm} = \{{}^\dagger\mathcal{D}_{\underline{v}}^{\pm}\}_{\underline{v}\in\underline{\mathbb{V}}}.$$
- A <u>±-label class of cusps</u> of ${}^\dagger\mathcal{D}_{\underline{v}}$=
$$\{\text{cusps of } {}^\dagger\mathcal{D}_{\underline{v}} \text{ lying over a single cusp of } {}^\dagger\mathcal{D}_{\underline{v}}^{\pm}\} \quad (\leftrightarrow \text{elements of } Q).$$
$$\{\mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}})\backslash{}^\dagger\eta_{\underline{v}}^{0}\}/\{\pm1\} \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})(\xrightarrow{\sim} \mathbb{F}_\ell^{*})$$
$${}^\dagger\eta_{\underline{v}}^{\pm} \mapsto {}^\dagger\eta_{\underline{v}}$$
- $\rightsquigarrow$
$$\mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathbb{F}_\ell \text{ as an } \mathbb{F}_\ell^{\pm}\text{-group}, \rightsquigarrow$$
$$1 \to \mathrm{Aut}_+({}^\dagger\mathcal{D}_{\underline{v}}) \to \mathrm{Aut}({}^\dagger\mathcal{D}_{\underline{v}}) \to \{\pm1\} \to 1.$$
- For $\alpha \in \{\pm1\}^{\underline{\mathbb{V}}}$, have $\mathrm{Aut}_\alpha({}^\dagger\mathfrak{D}) \subset \mathrm{Aut}({}^\dagger\mathfrak{D})$ of α-signed automorphisms.

Given another $\mathcal{D}$-prime-strip ${}^\ddagger\mathfrak{D} = \{{}^\ddagger\mathcal{D}_{\underline{v}}\}$:

- A <u>+-full poly-isomorphism</u> ${}^\dagger\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^\ddagger\mathcal{D}_{\underline{v}}$:= an $\mathrm{Aut}_+({}^\dagger\mathcal{D}_{\underline{v}})$-orbit of an isomorphism ${}^\dagger\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^\ddagger\mathcal{D}_{\underline{v}}$.
- A <u>+-full poly-isomorphism</u> ${}^\dagger\mathfrak{D} \xrightarrow{\sim} {}^\ddagger\mathfrak{D}$:= an $\mathrm{Aut}_+({}^\dagger\mathfrak{D})$-orbit of an isomorphism ${}^\dagger\mathfrak{D} \xrightarrow{\sim} {}^\ddagger\mathfrak{D}$. (If ${}^\dagger\mathfrak{D} = {}^\ddagger\mathfrak{D}$, these poly-isomorphism $\leftrightarrow \{\pm1\}^{\underline{\mathbb{V}}}$.)

$$\mathcal{D}^{\circledcirc\pm} = \mathcal{B}(\underline{X}_K)^\circ.$$

- Outer homomorphism

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K) \to \mathrm{GL}_2(\mathbb{F}_\ell) \to \mathrm{GL}_2(\mathbb{F}_\ell)/\{\pm 1\}$$

(adapted to $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_\ell \twoheadrightarrow Q$) has image containing a Borel of $\mathrm{SL}_2(\mathbb{F}_\ell)/\{\pm 1\} \rightsquigarrow$

$$1 \to \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm}) \to \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \to \mathbb{F}_\ell^{\circledast} \to 1.$$

($\rightsquigarrow$ crucial $\mathbb{F}_\ell^{\circledast}$-rigidity for the Hodge-Arakelov-theoretic evaluation of étale theta function.)

- $(\mathrm{Aut}_K(\underline{X}_K) \subset) \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm}) \curvearrowright^{\text{transitively}} \{\text{cusps of } \underline{X}_K\}$.
- $\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) :=$ automorphisms which fix the cusps of $\underline{X}_K$.
- $\mathrm{Aut}_+(\mathcal{D}^{\circledcirc\pm}) \subset \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm}) :=$ unique index 2 subgroup $\supset \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$.
- Choice of the cusp $\underline{\epsilon}$ of $\underline{C}_K \rightsquigarrow$

$$(\mathrm{Aut}_K(\underline{X}_K/X_K) \rtimes \{\pm 1\} \simeq) \, \mathrm{Aut}_K(\underline{X}_K) \xrightarrow{\sim} \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm}) / \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \overset{\epsilon}{\xrightarrow{\sim}} \mathbb{F}_\ell^{\rtimes\pm},$$

thus

$$(\mathrm{Aut}_K(\underline{X}_K/X_K) \simeq) \, \mathrm{Aut}_+(\mathcal{D}^{\circledcirc\pm}) / \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_\ell, \quad \text{as an } \mathbb{F}_\ell^{\pm}\text{-group;}$$

$$\{\text{cusps of } \mathcal{D}^{\circledcirc\pm}\} =: \mathrm{LabCusp}^\pm(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_\ell, \quad \text{as an } \mathbb{F}_\ell^{\pm}\text{-torsor.}$$

(Fix this from now on.)

- Copies of $\mathcal{D}$-prime-strips:

$$\mathfrak{D}_{\succ} = \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \quad \mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}\in\underline{\mathbb{V}}}, t \in \mathbb{F}_\ell \text{ (as } \mathbb{F}_\ell^{\pm}\text{-group)}.$$

- Positive $+$-full poly-isomorphisms (tautological):

$$\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t} \xrightarrow{\sim} \mathcal{D}_{\succ,\underline{v}}, \quad \phi^{\Theta^{\pm}}_{t} : \mathfrak{D}_t \xrightarrow{\sim} \mathfrak{D}_{\succ}.$$

- $\rightsquigarrow$

$$\phi^{\Theta^{\pm}}_{\pm} = \{\phi^{\Theta^{\pm}}_{t}\}_{t\in\mathbb{F}_\ell} : \mathfrak{D}_{\pm} = \{\mathfrak{D}_t\}_{t\in\mathbb{F}_\ell} \xrightarrow{\sim} \mathfrak{D}_{\succ}$$

$$(\phi^{\Theta^{\pm}}_{\pm} \curvearrowleft -1|_{\mathbb{F}_\ell} : \mathfrak{D}_t \mapsto \mathfrak{D}_{-t}, \mathfrak{D}_{\succ} \xrightarrow{\sim} \mathfrak{D}_{\succ} \text{ +-full poly-iso. with } -1\text{-sign at all } \underline{v})$$

$$(\forall \alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}, \phi^{\Theta^{\pm}}_{\pm} \curvearrowleft \alpha^{\Theta^{\pm}} : \mathfrak{D}_t = \mathfrak{D}_t, \mathfrak{D}_{\succ} \xrightarrow{\sim} \mathfrak{D}_{\succ} \text{ } \alpha\text{-signed +-full poly-iso.})$$

- Let T be an $\mathbb{F}_\ell^{\pm}$-group. $|T| := T/\{\pm 1\}$, $T^{*} := |T|\backslash\{0\} \rightsquigarrow$

$$\mathfrak{D}_{|T|}, \mathfrak{D}_{T^{*}}.$$

- A <u>$\mathcal{D}$-$\Theta^{\pm}$-bridge</u> ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$ is a poly-morphism fitting into the following diagram:

$$\begin{array}{ccc}
{}^{\dagger}\mathfrak{D}_T & \xrightarrow{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} & {}^{\dagger}\mathfrak{D}_{\succ} \\
\exists \simeq \text{ inducing } \mathbb{F}_\ell \xrightarrow{\sim} T \uparrow & & \uparrow \exists\simeq \\
\mathfrak{D}_{\pm} & \xrightarrow{\phi^{\Theta^{\pm}}_{\pm}} & \mathfrak{D}_{\succ}
\end{array}$$

- A morphism of $\mathcal{D}$-$\Theta^{\pm}$-bridges is a pair of poly-morphisms fitting into:

$$\begin{array}{ccc}
{}^{\dagger}\mathfrak{D}_T & \xrightarrow{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} & {}^{\dagger}\mathfrak{D}_{\succ} \\
\text{capsule-+-full poly-iso. inducing } T \simeq T' \downarrow & & \downarrow \text{+-full poly-iso.} \\
{}^{\ddagger}\mathfrak{D}_{T'} & \xrightarrow{{}^{\ddagger}\phi^{\Theta^{\pm}}_{\pm}} & {}^{\ddagger}\mathfrak{D}_{\succ}
\end{array}$$

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$: (Look at the cover $\underline{\underline{X}}_{v} \to \underline{X}_{v} \to \underline{X}_{K}$ or $\underline{\overrightarrow{X}}_{v} \to \underline{X}_{v} \to \underline{X}_{K}$.)

$$\begin{array}{ccc}
\mathcal{D}_{\underline{v}} & \xrightarrow{\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}}} & \mathcal{D}^{\circledcirc\pm} \\
\uparrow{\scriptstyle \alpha\in\mathrm{Aut}_{+}(\mathcal{D}_{\underline{v}})} & & \downarrow{\scriptstyle \beta\in\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})} \\
\mathcal{D}_{\underline{v}} & \xrightarrow{\beta\circ\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}}\circ\alpha} & \mathcal{D}^{\circledcirc\pm}
\end{array}$$

(analogue for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

- $$\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}} = \{\beta \circ \phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} \circ \alpha\}_{\alpha,\beta},$$
  $$\phi^{\Theta^{\mathrm{ell}}}_{t} : \mathfrak{D}_{t} = \{\mathcal{D}_{\underline{v}_t=(\underline{v},t)}\} \xrightarrow{\{\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}}\}} \mathcal{D}^{\circledcirc\pm} \xrightarrow{\cdot t} \mathcal{D}^{\circledcirc\pm}, \quad \forall t \in \mathbb{F}_{\ell} \text{ as } \mathbb{F}_{\ell}^{\pm}\text{-torsor},$$
- $$\phi^{\Theta^{\mathrm{ell}}}_{\pm} = \{\phi^{\Theta^{\mathrm{ell}}}_{t}\}_{t\in\mathbb{F}_{\ell}} : \mathfrak{D}_{\pm} = \{\mathfrak{D}_{t}\}_{t\in\mathbb{F}_{\ell}} \to \mathcal{D}^{\circledcirc\pm}.$$
  ($\phi^{\Theta^{\mathrm{ell}}}_{\pm} \curvearrowleft \gamma \in \mathbb{F}_{\ell}^{\rtimes\pm} : \mathfrak{D}_{t} \mapsto \mathfrak{D}_{\gamma(t)}$ $+$-full poly-iso. with sign $\mathrm{sign}(\gamma)$ at all $\underline{v}$)
- Let T be an $\mathbb{F}_{\ell}^{\pm}$-torsor. A $\underline{\mathcal{D}\text{-}\Theta^{\mathrm{ell}}\text{-bridge}}$ ${}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}$ is a poly-morphism s.t.

$$\begin{array}{ccc}
{}^{\dagger}\mathfrak{D}_{T} & \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} & {}^{\dagger}\mathcal{D}^{\circledcirc\pm} \\
\uparrow{\scriptstyle \exists \simeq \text{ inducing } \mathbb{F}_{\ell} \xrightarrow{\sim} T} & & \uparrow{\scriptstyle \exists\simeq} \\
\mathfrak{D}_{\pm} & \xrightarrow{\phi^{\Theta^{\mathrm{ell}}}_{\pm}} & \mathcal{D}^{\circledcirc\pm}
\end{array}$$

- A morphism of $\mathcal{D}$-Θ^{ell}-bridges is a pair of poly-morphisms s.t.

$$\begin{array}{ccc}
{}^{\dagger}\mathfrak{D}_{T} & \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} & {}^{\dagger}\mathcal{D}^{\circledcirc\pm} \\
\downarrow{\scriptstyle \text{capsule-}+\text{-full poly-iso. inducing } T \simeq T'} & & \downarrow{\scriptstyle \mathrm{Aut}_{\mathrm{csp}}({}^{\ddagger}\mathcal{D}^{\circledcirc\pm})\text{-orbit of iso.}} \\
{}^{\ddagger}\mathfrak{D}_{T'} & \xrightarrow{{}^{\ddagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} & {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}
\end{array}$$

Transport of $\pm$-cusp labels via base-bridges

- Bijection compatible with $\mathbb{F}_\ell^{\pm}$-torsor structures:

$$\mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{{}^\dagger\phi_{\pm}^{\Theta^{\mathrm{ell}}}} \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}^{\circledcirc\pm}).$$

- $\rightsquigarrow$ identification of $\mathbb{F}_\ell^{\pm}$-groups:

$$\color{blue}{\mathrm{LabCusp}^{\pm}({}^\dagger\mathfrak{D}_t)} := \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}_t}) = \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{w}_t}).$$

Thus natural bijection compatible with $\mathbb{F}_\ell^{\pm}$-torsor structures:

$${}^\dagger\zeta_t^{\Theta^{\mathrm{ell}}} : \mathrm{LabCusp}^{\pm}({}^\dagger\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}^{\circledcirc\pm}).$$

$\rightsquigarrow$ **synchronized indeterminacy**:

$$\begin{array}{ccc} T & \xrightarrow{\sim} & \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}^{\circledcirc\pm}) \\ t & \mapsto & {}^\dagger\zeta_t^{\Theta^{\mathrm{ell}}}(0) \end{array}$$

- Bijection of $\mathbb{F}_\ell^{\pm}$-groups:

$$\mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}} \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\succ,\underline{v}}).$$

- $\rightsquigarrow$ identification of $\mathbb{F}_\ell^{\pm}$-groups:

$$\color{blue}{\mathrm{LabCusp}^{\pm}({}^\dagger\mathfrak{D}_{\succ})} := \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\succ,\underline{v}}) = \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\succ,\underline{w}}).$$

Thus natural bijection of $\mathbb{F}_\ell^{\pm}$-groups:

$$\mathrm{LabCusp}^{\pm}({}^\dagger\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^\dagger\mathfrak{D}_{\succ}).$$

A $\underline{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theater}}$ is a collection of data

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} := \left({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xleftarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{D}_{\succ}\right)$$

fitting into the following commutative diagram

$$\begin{array}{ccccc} {}^{\dagger}\mathcal{D}^{\circledcirc} & \xleftarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} & {}^{\dagger}\mathfrak{D}_{T} & \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} & {}^{\dagger}\mathfrak{D}_{\succ} \\ \uparrow \exists\simeq & & \uparrow \exists\simeq & & \uparrow \exists\simeq \\ \mathcal{D}^{\circledcirc\pm} & \xleftarrow{\phi_{\pm}^{\Theta^{\mathrm{ell}}}} & \mathfrak{D}_{\pm} & \xrightarrow{\phi_{\pm}^{\Theta^{\pm}}} & \mathfrak{D}_{\succ} \end{array}$$

(A morphism of $\mathcal{D}$-$\Theta\mathrm{NF}$-Hodge theaters is defined in the obvious way as before.)

- $$\mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}) \text{ forms a } (\{\pm1\} \times \{\pm1\}^{\underline{\mathbb{V}}})\text{-torsor.}$$
  $$\mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\mathrm{ell}}\text{-bridges}) \text{ forms an } \mathbb{F}_{\ell}^{\pm}\text{-torsor.}$$
  $$\mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}) \text{ forms a } \{\pm1\}\text{-torsor.}$$
- Given a $\mathcal{D}$-$\Theta^{\pm}$-bridge and a $\mathcal{D}$-Θ^{ell}-bridge,

$$\left\{\text{capsule-}+\text{-full poly-isomorphisms } {}^{\dagger}\mathfrak{D}_{T} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{T} \text{ gluing them into } {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}\right\}$$

$$\text{forms a } (\{\pm1\}^{\underline{\mathbb{V}}} \times \mathbb{F}_{\ell}^{\rtimes\pm})\text{-torsor.}$$

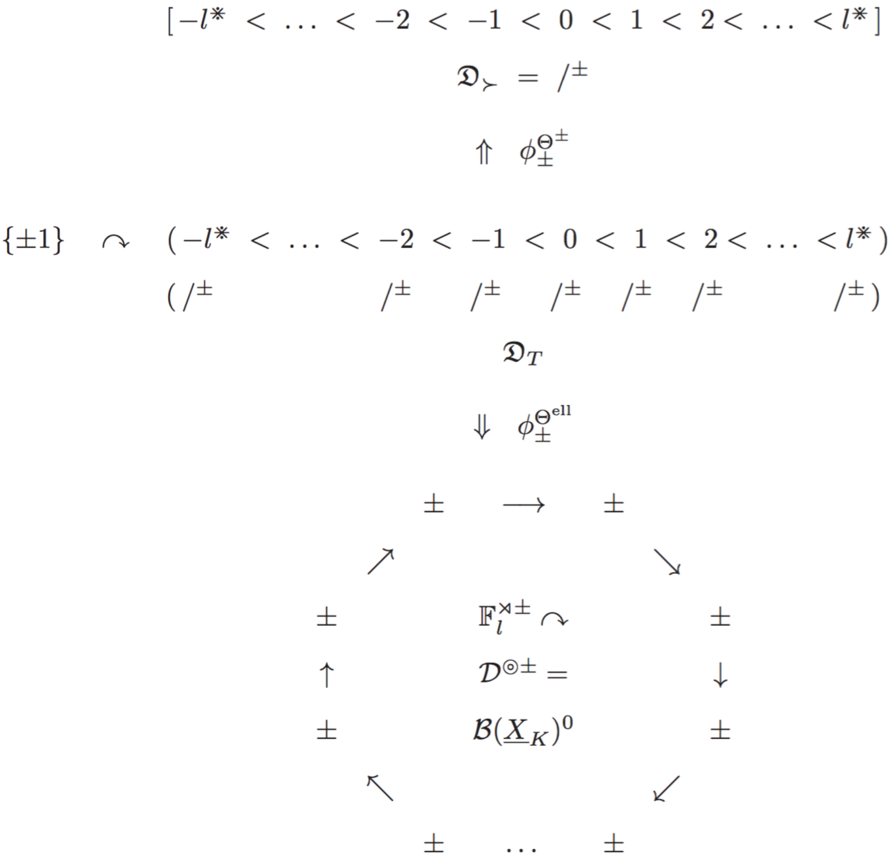

Figure: $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater (Fig. 6.1 of [IUT-I])

- For the forgetful functor

$$\{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\} \to \{\mathcal{D}\text{-}\Theta^{\mathrm{ell}}\text{-bridges}\}$$
$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \xleftarrow{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} {}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \mapsto ({}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

  the output data has $\mathbb{F}_{\ell}^{\rtimes\pm}$-symmetry.
- For the forgetful functor

$$\{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\} \to \{\ell^{\pm}\text{-capsules of } \mathcal{D}\text{-(resp. } \mathcal{D}^{\vdash}\text{-)prime-strips}\}$$
$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \mapsto {}^{\dagger}\mathfrak{D}_{|T|}\ (\text{resp. } {}^{\dagger}\mathfrak{D}^{\vdash}_{|T|})$$

  the output data has $\mathfrak{S}_{\ell^{\pm}}$-symmetry.
- Reduce $(\ell^{\pm})^{\ell^{\pm}}$-indeterminacy to $\ell^{\pm}!$-indeterminacy:

$$({}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto {}^{\dagger}\mathfrak{D}_{T}) \rightsquigarrow ({}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto \mathrm{Proc}({}^{\dagger}\mathfrak{D}_{T})), \quad ({}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto \mathrm{Proc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T})).$$

- Compatibility:
  For $j \in \{1, \cdots, \ell^{*}\}$ and $t = j + 1$, the inclusion

$$\{1, \cdots, j\} \hookrightarrow \{0, 1, \cdots, t-1\}$$

  determines natural transformations

$$^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto \left(\mathrm{Proc}({}^{\dagger}\mathfrak{D}_{T^{*}}) \hookrightarrow \mathrm{Proc}({}^{\dagger}\mathfrak{D}_{T})\right),$$

$$^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto \left(\mathrm{Proc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T^{*}}) \hookrightarrow \mathrm{Proc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T})\right).$$

- $\mathcal{D}$-$\Theta^{\pm}$-bridge $\rightsquigarrow$ $\mathcal{D}$-Θ-bridge:

$$\begin{array}{ccc} ({}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathfrak{D}_{\succ}) & \rightsquigarrow & ({}^{\dagger}\phi^{\Theta}_{\divideontimes} : {}^{\dagger}\mathfrak{D}_{T^*} \to {}^{\dagger}\mathfrak{D}_{>}) \\ {}^{\dagger}\mathfrak{D}_T|_{T\setminus\{0\}} & \mapsto & {}^{\dagger}\mathfrak{D}_{T^*}, \\ {}^{\dagger}\mathfrak{D}_0, {}^{\dagger}\mathfrak{D}_{\succ} & \mapsto & {}^{\dagger}\mathfrak{D}_{>}. \end{array}$$

- The $\underline{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-link}}$

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \xrightarrow{\mathcal{D}} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$$

is the full poly-isomorphism

$${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\succ}. \quad \textbf{(mono-analytic core)}$$

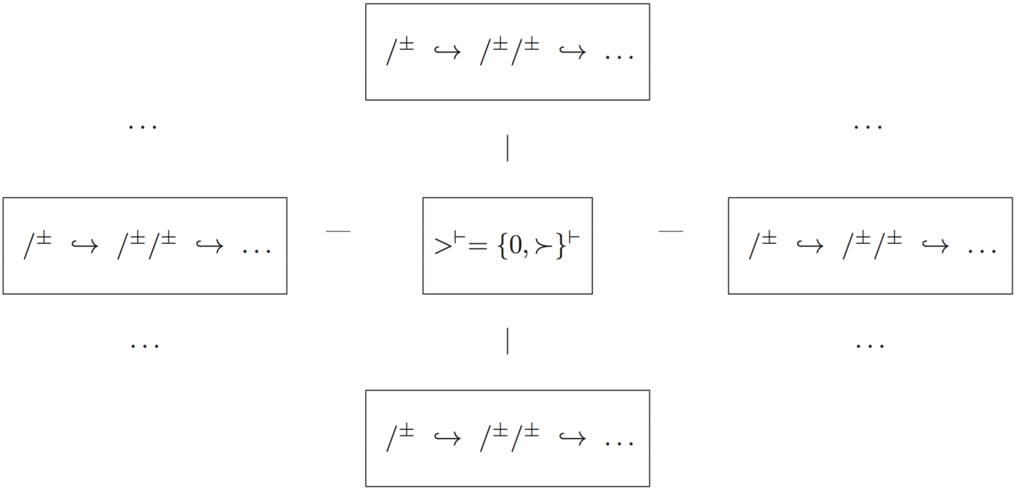

Figure: Étale-picture of $\mathcal{D}$-ΘNF-Hodge theaters (Fig. 6.3 of [IUT-I])

- A $\Theta^{\pm}$-bridge is a poly-morphism

$$^{\dagger}\mathfrak{F}_T \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{F}_{\succ}$$

  between a capsule of $\mathcal{F}$-prime-strips indexed by an $\mathbb{F}_{\ell}^{\pm}$-group T and a $\mathcal{F}$-prime-strip, which lifts a $\mathcal{D}$-$\Theta^{\pm}$-bridge $^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathfrak{D}_{\succ}$.
- A Θ^{ell}-bridge is a poly-morphism

$$^{\dagger}\mathfrak{F}_T \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

  between a capsule of $\mathcal{F}$-prime-strips indexed by an $\mathbb{F}_{\ell}^{\pm}$-torsor T and a category equivalent to $\mathcal{D}^{\circledcirc\pm}$, which lifts a $\mathcal{D}$-Θ^{ell}-bridge $^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$.
- A morphism of bridges is defined to be a pair of poly-isomorphisms on the domains and codomains, which lifts a morphism of the associated base-bridges.
-

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}} := ({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xleftarrow{^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathfrak{F}_T \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{F}_{\succ}).$$

-

$$\mathrm{Isom}(\Theta^{\pm}\text{-bridges}) \xrightarrow{\sim} \mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}),$$
$$\mathrm{Isom}(\Theta^{\mathrm{ell}}\text{-bridges}) \xrightarrow{\sim} \mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\mathrm{ell}}\text{-bridges}),$$
$$\mathrm{Isom}(\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}) \xrightarrow{\sim} \mathrm{Isom}(\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}).$$

- Given a $\Theta^{\pm}$-bridge and a Θ^{ell}-bridge,

$$\left\{\text{capsule-}+\text{-full poly-isomorphisms gluing them into } {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}\right\}$$

  forms a $(\{\pm 1\}^{\underline{\mathbb{V}}} \times \mathbb{F}_{\ell}^{\rtimes\pm})$-torsor.

Recall

- $$ {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \overset{{}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longleftarrow} {}^{\dagger}\mathfrak{F}_{T} \overset{{}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{\succ}), $$
  $$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \overset{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{\succ}). $$
- $$ {}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\dagger}\mathcal{F}^{\circledcirc} \overset{{}^{\dagger}\psi^{\mathrm{NF}}_{\circledast}}{\longleftarrow} {}^{\dagger}\mathfrak{F}_{J} \overset{{}^{\dagger}\psi^{\Theta}_{\circledast}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}), $$
  $$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_{J} \overset{{}^{\dagger}\phi^{\Theta}_{\circledast}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>}). $$
- $\mathcal{D}$-$\Theta^{\pm}$-bridge $\rightsquigarrow$ $\mathcal{D}$-Θ-bridge:
  $$ ({}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} : {}^{\dagger}\mathfrak{D}_{T} \to {}^{\dagger}\mathfrak{D}_{\succ}) \rightsquigarrow ({}^{\dagger}\phi^{\Theta}_{\circledast} : {}^{\dagger}\mathfrak{D}_{T^{\circledast}} \to {}^{\dagger}\mathfrak{D}_{>}). $$

Assuming $\mathcal{D}$-Θ-bridge ${}^{\dagger}\mathfrak{D}_{T^{\circledast}} \overset{{}^{\dagger}\phi^{\Theta}_{\circledast}}{\longrightarrow} {}^{\dagger}\mathfrak{D}_{>}$ is associated to the Θ-bridge ${}^{\dagger}\mathfrak{F}_{J} \overset{{}^{\dagger}\psi^{\Theta}_{\circledast}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}$, we glue the Θ-bridge to the $\Theta^{\pm}$-bridges ${}^{\dagger}\mathfrak{F}_{T} \overset{{}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} {}^{\dagger}\mathfrak{F}_{\succ})$. Such a gluing is unique because

$$ \mathrm{Isom}({}^{\dagger}\phi^{\Theta}_{\circledast}, {}^{\ddagger}\phi^{\Theta}_{\circledast}) = \{*\}, $$

$$ \mathrm{Isom}(\Theta\text{-bridges}) \overset{\sim}{\longrightarrow} \mathrm{Isom}(\text{associated } \mathcal{D}\text{-}\Theta\text{-bridges}). $$

$\rightsquigarrow$ $\Theta^{\pm\mathrm{ell}}$NF-Hodge-theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

Summary

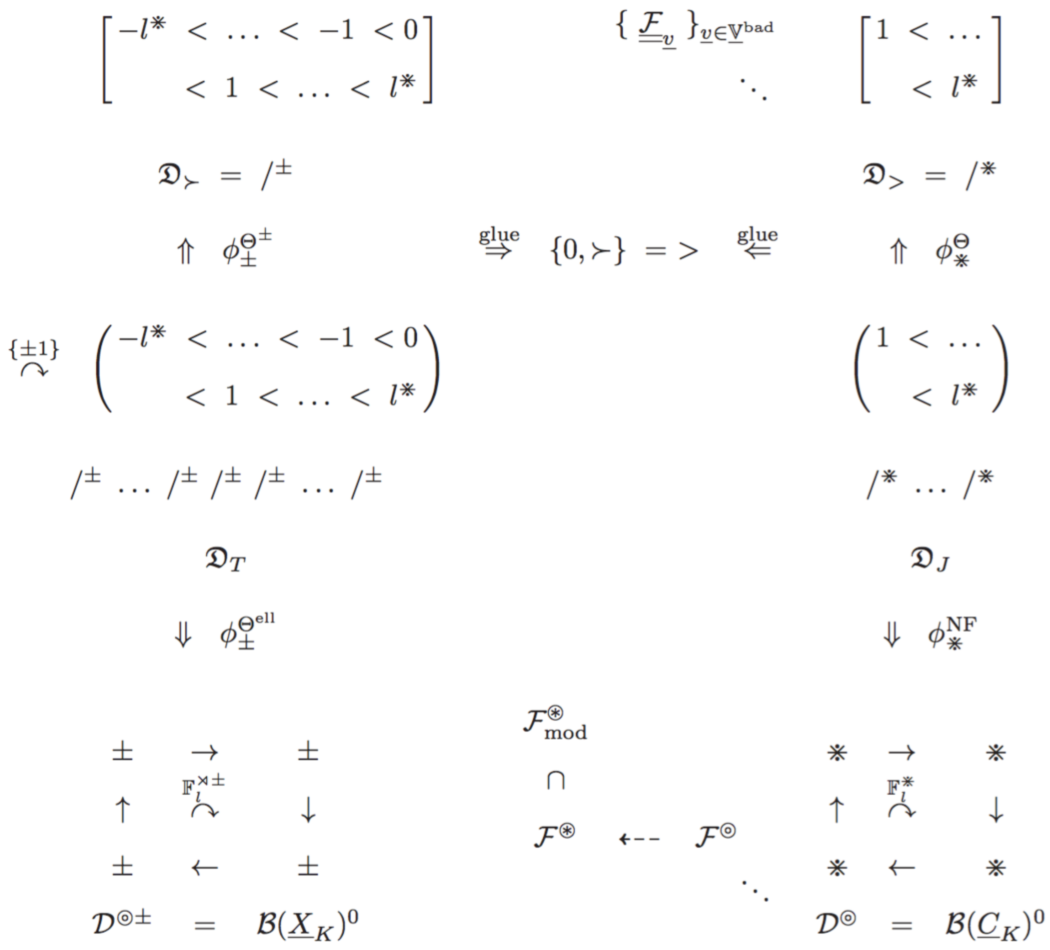

Figure: $\Theta^{\pm\mathrm{ell}}$NF-Hodge-theater (Fig. 6.5 of [IUT-I])