Near Miss *abc*-Triples in Compactly Bounded Subsets

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Agenda of today's talk

Today's talk is divided into two sections:

1. In §1, we state the background and the statement of the main result of today's talk.
2. In §2, we survey the outline of the proof of the main result.

§1 : Background and Statement of the Main Result

Definition of *abc*-triples

First, we review the definition of *abc*-triples.

Definition

Let $a, b, c \in \mathbb{Z}$ such that

$$a + b + c = 0,$$
$$a \neq 0, \quad b \neq 0, \quad c \neq 0,$$
$$(a, b) = 1.$$

Then we shall say that (a, b, c) is an *abc*-triple.

Definition of N and λ

Next, we introduce two important parameters associated to a given *abc*-triple.

Definition

Let (a, b, c) be an *abc*-triple. Then we define $N_{(a,b,c)}$ and $\lambda_{(a,b,c)}$ associated to a given *abc*-triple as follows:

$$N_{(a,b,c)} := \prod_{\substack{p \mid abc,\\ p \in \mathfrak{Primes}}} p, \quad \lambda_{(a,b,c)} := -\frac{b}{a}.$$

Examples of how to calculate N

We show some examples of how to calculate N.

$$(2,3,-5) \Rightarrow N_{(2,3,-5)} = 2 \cdot 3 \cdot 5 = 30,$$
$$(4,9,-13) \Rightarrow N_{(4,9,-13)} = 2 \cdot 3 \cdot 13 = 78,$$
$$(32,27,-59) \Rightarrow N_{(32,27,-59)} = 2 \cdot 3 \cdot 59 = 354,$$

λ determines a natural bijection

Write M for the set of *abc*-triples. We define an equivalence relation on M as follows:

$$
\begin{aligned}
& (a, b, c) \sim (a', b', c') \\
\overset{\text{def}}{\Leftrightarrow}\ & (a, b, c) = (\epsilon a', \epsilon b', \epsilon c') \text{ for some } \epsilon \in \{1, -1\}.
\end{aligned}
$$

By $M \ni (a, b, c) \mapsto \lambda_{(a,b,c)} \in \mathbb{Q} \setminus \{0, 1\}$, we obtain a natural bijection

$$M/\sim\ \overset{\cong}{\to}\ \mathbb{Q} \setminus \{0, 1\}.$$

λ coincides with the quantity "λ" of a Frey curve

An *abc*-triple (a, b, c) determines an elliptic curve over $\mathbb{Q}$ as follows:

$$(a, b, c) \Rightarrow y^2 = x(x + a)(x - b).$$

This is called a Frey curve.

$\lambda_{(a,b,c)} = -\frac{b}{a}$ coincides with the quantity "λ" which appears in the Legendre form of this elliptic curve. The "λ"'s which appear in the Legendre form are not uniquely determined, and, actually, the indeterminacy corresponds to permutations of (a, b, c).

Review of the *abc* Conjecture

We review the *abc* Conjecture.

Theorem (*abc* Conjecture)

For $\gamma \in \mathbb{R}_{>0}$, there exists a $C_\gamma \in \mathbb{R}_{>0}$ such that, for every abc-triple (a, b, c), the following inequality holds:

$$\max\{|a|, |b|, |c|\} < C_\gamma N_{(a,b,c)}^{1+\gamma}.$$

Statement of Masser's theorem

Does the "$\gamma = 0$" version of the *abc* Conjecture hold?
Masser's theorem asserts that the answer is NO.

Theorem (Masser)

Let $\delta \in (0, \frac{1}{2})$, $N_0 \in \mathbb{R}_{>0}$. *Then there exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0,$$
$$|abc| \geq N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right).$$

Masser's result implies that the "$\gamma = 0$" version of the *abc* Conjecture does not hold

Let us observe that Masser's Theorem implies that the "$\gamma = 0$" version of the *abc* Conjecture does not hold.

Suppose that the "$\gamma = 0$" version of the *abc* Conjecture holds. Then, there exists a $C_0 \in \mathbb{R}_{>0}$ such that, for every *abc*-triple (a, b, c),

$$\max\{|a|, |b|, |c|\} < C_0 N_{(a,b,c)}. \tag{1}$$

In particular, for every *abc*-triple (a, b, c),

$$|abc| < C_0^3 N_{(a,b,c)}^3. \tag{2}$$

Fix δ, N_0 such that

$$\exp((\log N_0)^{\frac{1}{2}-\delta}) > C_0^3. \tag{3}$$

Masser's Theorem asserts that there exists an *abc*-triple (a', b', c') such that

$$|a'b'c'| \geq N_{(a',b',c')}^3 \exp((\log N_0)^{\frac{1}{2}-\delta}). \tag{4}$$

It follows from the above inequalities that

$$|a'b'c'| \geq C_0^3 N_{(a',b',c')}^3. \tag{5}$$

This is a contradiction.

Since any infinite collection of *abc*-triples as in Masser's theorem for $N_0 \to \infty$ yields a counterexample to the "$\gamma = 0$" version of the *abc* Conjecture, we shall refer to such *abc*-triples as *near miss abc-triples*.

Definition of compactly bounded subsets

Before mentioning the Main Theorem, we define compactly bounded subsets.

Definition

Let $r \in \mathbb{Q}$, $\varepsilon \in \mathbb{R}_{>0}$, and Σ a finite subset of the set of valuations of $\mathbb{Q}$ which contains the unique archimedean valuation $\{\infty\}$ on $\mathbb{Q}$. Then we shall write

$$K_{r,\varepsilon,\Sigma} := \{r' \in \mathbb{Q} \mid \|r' - r\|_v \leq \varepsilon \text{ for every } v \in \Sigma\}.$$

We shall say that $K_{r,\varepsilon,\Sigma}$ is an (r, ε, Σ)-compactly bounded subset.

Statement of the Main Theorem

Finally, we state the Main Theorem.

Theorem (W)

Let $\delta \in (0, \frac{1}{2})$, $N_0 \in \mathbb{R}_{>0}$ *and* $K_{r,\varepsilon,\Sigma}$ *a given compactly bounded subset. Then there exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0,$$

$$|abc| \geq N_{(a,b,c)}^3 \exp\left(\left(\log\log N_{(a,b,c)}\right)^{\frac{1}{2}-\delta}\right),$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}.$$

By a similar argument to the argument given above, we observe that the *abc*-triples of the Main Theorem may be thought of as near miss *abc*-triples.

Comparison with Masser's theorem

Theorem (Masser)

Let $\delta \in (0, \frac{1}{2})$, $N_0 \in \mathbb{R}_{>0}$. *Then there exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0,$$
$$|abc| \geq N_{(a,b,c)}^3 \exp\left((\log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right).$$

§2 : Outline of the proof of the Main Theorem

Definition of functions related to prime numbers

We define various functions related to prime numbers.

Definition

Let $n \in \mathbb{Z} \setminus \{0\}$, $x, y \in \mathbb{R}_{\geq 2}$.

- If $n \neq \pm 1$, then we denote the largest prime number dividing n by $\mathrm{LPN}(n)$. If $n = \pm 1$, then we set $\mathrm{LPN}(n) := 1$.
- $\pi(x) := \sharp\{x' \in \mathfrak{Primes} \mid x' \leq x\}$.
- $\theta(x) := \sum_{\mathfrak{Primes} \ni p \leq x} \log p$.
- $\Psi(x, y) := \sharp\{x' \in \mathbb{Z} \mid 2 \leq x' \leq x, \ \ \mathrm{LPN}(x') \leq y\}$.

The function $\Psi(x, y)$ plays a key roll in the proof of the Main Theorem.

Estimates of $\pi(x)$ and $\theta(x)$

These estimates are well known consequences of the Prime Number Theorem.

Lemma (Prime Number Theorem)

Let $x \in \mathbb{R}_{\geq 2}$. Then the following estimates hold:

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right),$$
$$\theta(x) = x + O\left(\frac{x}{(\log x)^2}\right).$$

Estimate of $\Psi(x, y)$

By estimating the number of lattice points in a region of Euclidean space by the volume of the region, Ennola obtained the following lemma.

Lemma (Ennola)

Let $i \in \mathbb{Z}_{\geq 1}$. We denote the i-th smallest prime number by p_i.
Let $x \in \mathbb{R}_{>0}, y \in \mathbb{R}_{\geq 2}$. Then the following inequality holds:

$$\frac{(\log x)^{\pi(y)}}{\pi(y)! \cdot \left(\prod_{i=1}^{\pi(y)} \log p_i\right)} < \Psi(x, y) + 1$$

$$\leq \frac{(\log x)^{\pi(y)}}{\pi(y)! \cdot \left(\prod_{i=1}^{\pi(y)} \log p_i\right)} \left(1 + \sum_{i=1}^{\pi(y)} \frac{\log p_i}{\log x}\right)^{\pi(y)}.$$

When $y = (\log x)^{\frac{1}{2}}$, by applying the preceding lemmas, we obtain a better estimate of $\Psi(x, y)$.

Lemma

Let $x \in \mathbb{R}_{>\exp(4)}$. Then the following estimate holds:

$$\Psi(x, (\log x)^{\frac{1}{2}}) = \exp\left((\log x)^{\frac{1}{2}} + 4\frac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right).$$

This estimate plays a key roll in the proof of the Main Theorem.

Proof of the Main Theorem: notation

- $\Sigma_f := \Sigma \setminus \{\infty\}$
- q: a prime number such that $q > N_0$ and $\|q\|_v = 1$ for $v \in \Sigma_f$
- x: a (sufficiently large) positive real number
- $y := (\log x)^{\frac{1}{2}}$
- l: a positive integer such that $q^l \approx \Psi(x, y)$

Proof of the Main Theorem: plan

First, we shall prove that we obtain a pair of positive integers (d, e) such that

- $\mathrm{LPN}(d), \ \ \mathrm{LPN}(e) \leq y$,
- $d - e$ is divided by the l-th power of q.

Next, we define an *abc*-triple $(a, b, c) := (d, -e, -(d - e))$. Then

$$N_{(a,b,c)} \leq \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p \leq y}} p \right) \frac{c}{q^{l-1}}.$$

Now we prove the Main Theorem. The proof of the Main Theorem is divided into 4 steps.

Step 1

It follows from the Box Principle that there exists a pair of positive integers (d_1, e_1) such that

$$\mathrm{LPN}(d_1) \leq y, \quad \mathrm{LPN}(e_1) \leq y,$$

$$q^l \mid d_1 - e_1,$$

$$d_1 \leq x, \quad e_1 \leq x,$$

$$\frac{e_1}{d_1} \in K_{1,\varepsilon,\Sigma}.$$

Step 2

It follows from the Chinese Remainder Theorem that there exists a pair of positive integers (d_2, e_2) such that

$$\mathrm{LPN}(d_2) \leq y, \quad \mathrm{LPN}(e_2) \leq y,$$

$$q^l \mid d_2 - e_2,$$

$$d_2 \leq x, \quad e_2 \leq \exp\left(\exp\left(3(\log x)^{\frac{1}{2}}\right)\right),$$

$$\frac{e_2}{d_2} \in K_{r,\varepsilon,\Sigma_f}.$$

Step 3

By using pairs of positive integers (d_1, e_1) and (d_2, e_2) obtained in Step 1 and Step 2, we shall create a new pair of positive integers (d_3, e_3) such that

$$\mathrm{LPN}(d_3) \le y, \quad \mathrm{LPN}(e_3) \le y,$$
$$q^l \mid d_3 - e_3,$$
$$d_3 \le x^{x^{1+\delta}}, \quad e_3 \le x^{x^{1+\delta}}$$
$$\frac{e_3}{d_3} \in K_{r,\varepsilon,\Sigma}.$$

Outline of the proof of Step 3

There exists an $\alpha \in \mathbb{Z}$ such that

$$\frac{e_2}{d_2}\left(\frac{e_1}{d_1}\right)^{\alpha} \in (r-\varepsilon, r+\varepsilon).$$

This property, together with certain estimates at the nonarchimedean valuations, that

$$\frac{e_2}{d_2}\left(\frac{e_1}{d_1}\right)^{\alpha} \in K_{r,\varepsilon,\Sigma}.$$

We define $(d_3, e_3) := (d_2 d_1^{\alpha}, e_2 e_1^{\alpha})$. The pair of positive integers (d_3, e_3) satisfies the conditions of the preceding slide.

Step 4

We define an *abc*-triple $(a, b, c) := (d_3, -e_3, -(d_3 - e_3))$. Then it follows that

$$N_{(a,b,c)} > N_0,$$
$$|abc| \geq N_{(a,b,c)}^3 \exp\left((\log\log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right),$$
$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}.$$

Now we survey the outline of the proof that the second condition holds.

Outline of the estimate $N_{(a,b,c)}$

$$N_{(a,b,c)} = \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p|ab}} p \right) \left(\prod_{\substack{p \in \mathfrak{Primes}, \\ p|c}} p \right) \tag{1}$$

$$\leq \exp(\theta(y)) \frac{|c|}{q^{l-1}}, \tag{2}$$

that is,

$$|c| \geq \frac{N_{(a,b,c)}}{q} q^{l} \exp(-\theta(y)). \tag{3}$$

Let us estimate $q^l \exp(-\theta(y))$.

$$
\begin{aligned}
q^l \exp(-\theta(y)) \approx& \Psi(x, (\log x)^{\frac{1}{2}}) \exp(-\theta((\log x)^{\frac{1}{2}}) && (4)\\
=& \exp\left((\log x)^{\frac{1}{2}} + 4\frac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right)\\
&\cdot \exp\left(-(\log x)^{\frac{1}{2}} + O\left(\frac{(\log x)^{\frac{1}{2}}}{(\log\log x)^2}\right)\right) && (5)\\
=& \exp\left(4\frac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{\log\log x}\right)\right). && (6)
\end{aligned}
$$

Here we apply the estimates of the previous lemmas

$$
\Psi(x, (\log x)^{\frac{1}{2}}) = \exp\left((\log x)^{\frac{1}{2}} + 4\frac{(\log x)^{\frac{1}{2}}}{\log\log x} + O\left(\frac{(\log x)^{\frac{1}{2}}}{\log\log x}\right)\right), \qquad (7)
$$

$$
\theta(x) = x + O\left(\frac{x}{(\log x)^2}\right). \qquad (8)
$$

Since $d_3 \leq x^{x^{1+\delta}}, \quad e_3 \leq x^{x^{1+\delta}}$, it follows that

$$N_{(a,b,c)} \leq |abc| \leq 2x^{3x^{1+\delta}}, \tag{9}$$

that is,

$$\log \log N_{(a,b,c)} = O(\log x). \tag{10}$$

Thus, it follows that

$$|c| \geq N_{(a,b,c)} \exp\left((\log \log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right), \tag{11}$$

which easily implies that

$$|abc| \geq N_{(a,b,c)}^3 \exp\left((\log \log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right), \tag{12}$$

Thank you for your attention!

Review of the Main Theorem and Masser's result

Theorem (W)

Let $\delta \in (0, \frac{1}{2})$, $N_0 \in \mathbb{R}_{>0}$ *and* $K_{r,\varepsilon,\Sigma}$ *a given compactly bounded subset. Then there exists an abc-triple* (a, b, c) *such that*

$$N_{(a,b,c)} > N_0,$$

$$|abc| \geq N^3_{(a,b,c)} \exp\left((\log \log N_{(a,b,c)})^{\frac{1}{2}-\delta} \right),$$

$$\lambda_{(a,b,c)} \in K_{r,\varepsilon,\Sigma}.$$

Theorem (Masser)

Let $\delta \in (0, \frac{1}{2})$, $N_0 \in \mathbb{R}_{>0}$. *Then there exists an abc-triple* (a, b, c) *such that*

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