Étale theta functions, mono-theta environments, and [IUTchI] §1-§3, I

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We start from : an initial Θ-datum

An **initial Θ-datum** (initial Θ-data in the original paper) is a 7-tuple

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

satisfying the following 7 conditions.
Condition (1). F : a number field s.t. $\sqrt{-1} \in F$, and $\overline{F}$: an alg. closure of F.

Condition (2)

$X_F = E_F \setminus \{0\}$, E_F: ell. curve $/F$ s.t.

- F is Galois over $F_{\mathrm{mod}} := \mathbb{Q}(j(E_F))$.
- E_F has good or semistable red. at any $v \nmid \infty$.
- $E_F[6](\overline{F}) = E_F[6](F)$.
  ($\Longrightarrow E_F$ has good or split semistable red. at any $v \nmid \infty$.)

Condition (3)

ℓ: a prime number ≥ 5, s.t. the image of

$$\rho_{E_F,\ell} : G_F := \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{F}_\ell}(E[\ell](\overline{F})) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$$

contains $\mathrm{SL}_2(\mathbb{F}_\ell)$. Set

$$\mathrm{Ker}\, \rho_{E_F,\ell} =: \mathrm{Gal}(\overline{F}/K).$$

($\Longrightarrow K/F$: Galois). We moreover assume:

- $C_K := (X_F/\pm 1) \otimes_F K$ is a K-core, i.e. any étale morphism over K between finite étale coverings of C_K is over C_K.

Condition (4)

$\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$: a finite set of places of F_{mod} s.t.

- $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \neq \emptyset$
- $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \Longrightarrow v \nmid 2\ell\infty$
- $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \Longrightarrow j(E_F) \notin \mathcal{O}_{F_{\mathrm{mod},v}}$
  ($\Longrightarrow X_F$: bad red. at any $w|v$.)

We do not assume that $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ contains all the bad primes.

Condition (5)

$\underline{C}_K = \underline{X}_K/\{\pm 1\}$, where
$\underline{X}_K \to X_K = X_F \otimes_F K$: cyclic cov. of deg. ℓ, unramified at the boundary whose points at the boundary are K-rational.

$-1 : \underline{X}_K \xrightarrow{\cong} \underline{X}_K$: a lift of $-1 : X_K \xrightarrow{\cong} X_K$
(a lift $-1 \longleftrightarrow$ a cusp $\underline{0}$ of $\underline{X}_K$).

So $(\underline{X}_K, \underline{0})$ comes from an isogeny $\underline{E}_K \to E_K$ of elliptic curves.
$(\underline{X}_K, \underline{0})$: unique up to isom. $/X_F$ since $\mathrm{Image}\,\rho_{E_F,\ell}$: large.

$\rightsquigarrow \underline{\underline{X}}_{\overline{F}}, \underline{\underline{C}}_{\overline{F}}$ over $\overline{F}$.

$\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\Theta} \twoheadrightarrow \overline{\Delta}_X$, $\sharp \overline{\Delta}_X = \ell^3$.

$$1 \to \overline{\Delta}_{\Theta} \to \overline{\Delta}_X \to \overline{\Delta}_X^{\mathrm{ell}} \to 1,$$

$\overline{\Delta}_{\Theta} \cong \mathbb{Z}/\ell\mathbb{Z}$, $\overline{\Delta}_X^{\mathrm{ell}} \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2}$.

$\underline{X}_{\overline{F}} \to X_{\overline{F}}$: degree $\ell \longleftrightarrow \overline{\Delta}_{\underline{X}} \subset \overline{\Delta}_X$: index ℓ.

$$1 \to \overline{\Delta}_{\Theta} \to \overline{\Delta}_{\underline{X}} \to \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \to 1,$$

$\overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \subset \overline{\Delta}_X^{\mathrm{ell}}$, $\overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \cong \mathbb{Z}/\ell\mathbb{Z}$.

Recall: Construction of $\underline{\underline{X}}_{\overline{F}}$ (continued)

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{\Delta}_{\Theta} & \longrightarrow & \overline{\Delta}_{X} & \longrightarrow & \overline{\Delta}_{X}^{\mathrm{ell}} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \overline{\Delta}_{\Theta} & \longrightarrow & \overline{\Delta}_{\underline{X}} & \longrightarrow & \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} & \longrightarrow & 1.
\end{array}
$$

$\overline{\Delta}_{\underline{X}}$: abelian, $\{\pm 1\} \circlearrowleft \overline{\Delta}_{\underline{X}}$

$\rightsquigarrow \overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\Theta} \times \overline{\Delta}_{\underline{X}}^{\mathrm{ell}}$.

$\underline{\underline{X}}_{\overline{F}} \to \underline{X}_{\overline{F}}$: degree $\ell \longleftrightarrow \overline{\Delta}_{\underline{\underline{X}}} = \overline{\Delta}_{\underline{X}}^{\mathrm{ell}} \subset \overline{\Delta}_{\Theta}$ index ℓ.

Condition (6). $\underline{\epsilon}$: a non-zero cusp of $\underline{C}_K$.

$\rightsquigarrow$ $\underset{\rightarrow}{X}_K$, $\underset{\rightarrow}{C}_K$ as follows:

${}^{\exists!}f$: rational function on $\underline{C}_K$ whose values at $0, \underline{\epsilon}$, "$[2](\underline{\epsilon})$" are $\infty, 0, 1$, respectively and which has a simple zero at $\underline{\epsilon}$.

$\underset{\rightarrow}{X}_K$, $\underset{\rightarrow}{C}_K$: the (orbi)curve obtained from $\underline{X}_K$, $\underline{C}_K$ by adjoining a ℓ-th root of f.

Condition (7). $\underline{\mathbb{V}}$: a set of places of K s.t.

- The following composite is bijective:

$$\underline{\mathbb{V}} \hookrightarrow (\text{the places of } K) \to (\text{the places of } F_{\text{mod}}) =: \mathbb{V}_{\text{mod}}$$

- Set $\underline{\mathbb{V}}^{\text{bad}} := \{\underline{v} \in \underline{\mathbb{V}} \mid \underline{v}|^{\exists} v \in \mathbb{V}^{\text{bad}}_{\text{mod}}\}$.
  Then for any $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, $\underline{C}_{\underline{v}} = \underline{C}_K \otimes_K K_{\underline{v}}$: type $(1, \mathbb{Z}/\ell\mathbb{Z})_{\pm}$, i.e, $\underline{X}_{\underline{v}}$ comes from $\mathbb{G}_m^{\text{an}}/q_{\underline{v}}^{\ell\mathbb{Z}} \twoheadrightarrow \mathbb{G}_m^{\text{an}}/q_{\underline{v}}^{\mathbb{Z}} = E_{K_{\underline{v}}}$. Moreover $\underline{\epsilon}$ specializes to a point in the unique component of the special fiber adjacent to the component of the origin.

An Aim

For an initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

and for $\underline{v} \in \underline{\mathbb{V}}$, we will construct

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}, \mathcal{C}_{\underline{v}}, \mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}, \mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}}$$

where ...

An Aim (continued)

where

- $\underline{\underline{\mathcal{F}}}_{\underline{v}}$: a Frobenioid when $\underline{v} \nmid \infty$
- $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$: $p_{\underline{v}}$-adic (resp. archimedean) Frobenioids if $\underline{v} \nmid \infty$ (resp. $\underline{v} | \infty$) (so its divisor monoid is monoprime).
- $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$: characteristic splittings ($\doteqdot$ splitting of the inclusion of functors "$\mathcal{O}^{\times} \subset \mathcal{O}^{\rhd}$") of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$

such that when $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$ and $(\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ are naturally isomorphic.

On Next Slides

Construction of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is hard when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. In the next slides we will focus on this case.

Bad Local Situation

K : a finite ext. of $\mathbb{Q}_p$.
E : a Tate elliptic curve / K s.t.

- $E[2](\overline{K}) = E[2](K)$
- $X^{\log} = (E, 0)$: not K-arithmetic

$\mathfrak{X}^{\log}$: stable model of $X^{\log}$ over $\mathfrak{S}^{\log} = (\mathrm{Spec}\, \mathcal{O}_K, \text{the closed point})$.
(so the special fiber of $\mathfrak{X}^{\log}$ is irreducible with one singular point, at which the formal completion is of the form $\mathcal{O}_K[[u, v]]/(uv - q_E)$)
$Y^{\log} \to X^{\log}$: the topological $\underline{\mathbb{Z}}$-covering

Fix an odd integer $\ell \geq 5$ satisfying $p_{\underline{v}} \nmid \ell$ and $E[\ell](\overline{K}) = E[\ell](K)$.
$Y^{\log} \to \underline{X}^{\log} \to X^{\log}$, $\deg(\underline{X}) = \ell$.
$\rightsquigarrow \underline{\underline{X}}^{\log}, \underline{\underline{Y}}^{\log}$.
We have a cartesian diagram:

$$\begin{array}{ccccc} \underline{\underline{Y}}^{\log} & \longrightarrow & \underline{\underline{X}}^{\log} & & \\ \big\downarrow & & \big\downarrow & & \\ Y^{\log} & \longrightarrow & \underline{X}^{\log} & \longrightarrow & X^{\log}. \end{array}$$

Choose $\dot{X}^{\log} \to X$, degree 2, unramified at 0.
Taking composite with $\dot{X}$, we have

$$\begin{array}{ccccc}
\underline{\underline{\ddot{Y}}}^{\log} & \longrightarrow & \underline{\underline{\dot{X}}}^{\log} & & \\
\downarrow & & \downarrow & & \\
\ddot{Y}^{\log} & \longrightarrow & \underline{\dot{X}}^{\log} & \longrightarrow & \dot{X}^{\log}.
\end{array}$$

Action of ± 1

We can take a lift of $-1 : X \to X$ to an involution of

$$\begin{array}{ccccc}
\dot{\underline{\underline{X}}}^{\log} & \longrightarrow & \dot{\underline{X}}^{\log} & \longrightarrow & \dot{X}^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\underline{X}}^{\log} & \longrightarrow & \underline{X}^{\log} & \longrightarrow & X^{\log}
\end{array}$$

Passing to the quotient by $\{\pm 1\}$ we have

$$\begin{array}{ccccc}
\underline{\underline{\dot{C}}}^{\log} & \longrightarrow & \underline{\dot{C}}^{\log} & \longrightarrow & \dot{C}^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\underline{C}}^{\log} & \longrightarrow & \underline{C}^{\log} & \longrightarrow & C^{\log}.
\end{array}$$

The Next Slides

In the bad local situation, we will construct a Frobenioid

$$\underline{\underline{\mathcal{F}}}.$$

Recall: a **Frobenioid** is a category with some additional structures. (In our case, it turns out that the additional structures can be recovered from the underlying category. We often regard $\underline{\underline{\mathcal{F}}}$ just as a category.)

Frobenioid

Recall: a Frobenioid is a quadruple $(\mathcal{F}, \mathcal{D}, \Phi, \mathcal{F} \to \mathbb{F}_\Phi)$, where

- $\mathcal{F}$: a category,
- $\mathcal{D}$: a connected, totally epimorphic cat. (=: *E*-cat.)
- Φ : a divisorial monoid on $\mathcal{D}$ ($\rightsquigarrow \mathbb{F}_\Phi$ the associated category)
- $\mathcal{F} \to \mathbb{F}_\Phi$: a covariant functor,

that satisfies a lot of technical conditions. The underlying category is $\mathcal{F}$. The category $\mathcal{D}$ is called the **base category**, and Φ the **divisor monoid**.

A Typical Example

$\mathcal{D}$: a cat. of connected regular noeth. schemes.
Assume $\mathcal{D}$: E-cat.
Φ: the monoid on $\mathcal{D}$ given by

$$\Phi(X) = (\text{effective divisors on } X).$$

$\rightsquigarrow$ a Frobenioid $\mathcal{F}$ defined as follows:

A Typical Example (continued)

$\mathcal{F}$: category of pairs $(X, \mathcal{L})$ where

- X : an object of $\mathcal{D}$
- $\mathcal{L}$ an invertible $\mathcal{O}_X$-module

A morphism $\phi : (X, \mathcal{L}) \to (Y, \mathcal{L}')$ is a triple $(\phi_{\mathcal{D}}, n_\phi, \iota_\phi)$ where

- $\phi_{\mathcal{D}} : X \to Y$: a morphism of $\mathcal{D}$
- n_ϕ : an integer ≥ 1
- $\iota_\phi : \mathcal{L}^{\otimes n_\phi} \hookrightarrow \phi_{\mathcal{D}}^* \mathcal{L}'$ an injection

This gives an example of model Frobenioids.

Model Frobenioids

To a quadruple $(\mathcal{D}, \Phi, \mathbb{B}, \mathrm{Div}_{\mathbb{B}})$ where

- $\mathcal{D}$: E-category
- Φ : a divisorial monoid on $\mathcal{D}$ ($\rightsquigarrow \Phi^{\mathrm{gp}}$ is also a monoid on $\mathcal{D}$)
- $\mathbb{B}$: a group-like monoid on $\mathcal{D}$
- $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$: homomorphism,

we can associate the following Frobenioid $\mathcal{F}$.

Model Frobenioids (continued)

... we can associate the following Frobenioid $\mathcal{F}$.

Objects: pairs (A, α) of $A \in \mathrm{Obj}(\mathcal{D})$, $\alpha \in \Phi^{\mathrm{gp}}(A)$

Morphisms: a morphism $(A, \alpha) \to (B, \beta)$ is a quadruple $(\phi_{\mathcal{D}}, Z_\phi, n_\phi, u_\phi)$, where $(\phi_{\mathcal{D}}, Z_\phi, n_\phi)$ is a morphism of $\mathbb{F}_\Phi$ and $u_\phi \in \mathbb{B}(A)$, such that

$$n_\phi \alpha + Z_\phi = \phi_{\mathcal{D}}^* \beta + \mathrm{Div}_{\mathbb{B}}(u_\phi)$$

in $\Phi(A)^{\mathrm{gp}}$.

A Frobenioid constructed in this way is called a **model Frobenioid**.

The Frobenioid $\underline{\underline{\mathcal{F}}}$ that we would like to construct in the bad local situation is a model Frobenioid.

In the next slides we will construct a datum $(\mathcal{D}, \Phi, \mathbb{B}, \mathrm{Div}_{\mathbb{B}})$ producing $\underline{\underline{\mathcal{F}}}$.

Construction of $\underline{\underline{\mathcal{F}}}$ (1) : the base category

We construct the base category $\mathcal{D}$.

Let $\mathcal{D} = \mathbb{B}^{\mathrm{tp}}(\underline{\underline{X}})^{\circ}$: cat. of connected tempered coverings of $\underline{\underline{X}}^{\log}$.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) : the divisor monoid

We construct the divisor monoid Φ on $\mathcal{D}$.

For any object B of $\mathcal{D}$, let

B^{ell} : the maximal subcovering of the composite $B \to \underline{\underline{X}}^{\log} \to X^{\log}$ s.t. B^{ell} is unramified at the cusp of $X^{\log}$.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) (continued)

The divisor monoid is roughly

$$B \mapsto \Phi(B) := \text{“}\mathrm{DIV}_+(\mathfrak{B}^{\mathrm{ell}})^{\mathrm{pf}}\text{”},$$

where

$\mathfrak{B}^{\mathrm{ell}}$: the stable model of B^{ell}.

DIV_+ : the effective divisors supported on the union of the special fiber and the cusps

pf : perfection (e.g., $(\mathbb{Z}_{\geq 0})^{\mathrm{pf}} = \mathbb{Q}_{\geq 0}$)

N.B. The actual definition is more complicated.

Construction of $\underline{\underline{\mathcal{F}}}$ (2) (remark)

We regard $\Phi(B)$ as a submonoid of Φ_0, defined roughly as

$$\Phi_0(B) = \text{“DIV}_+(\mathfrak{B})^{\mathrm{pf}}\text{”}.$$

when B admits a suitable stable model $\mathfrak{B}$.

N.B. The actual definition is more complicated, and is given by introducing the notion of divisors on universal combinatorial coverings and then by doing some “sheafification” process.

Construction of $\underline{\underline{\mathcal{F}}}$ (3) : the remaining structure

We construct the group-like monoid $\mathbb{B}$ on $\mathcal{D}$.

$$\mathbb{B}_0 : B \mapsto \text{“}\{f : \text{log-merom. on } \mathfrak{B}\}\text{”}.$$

when B admits a suitable stable model $\mathfrak{B}$.

N.B. The actual definition is more complicated,

Here log-merom. = mero. func. f on B s.t. for $^{\forall}N$,

f admits a N-th root in a tempered covering of B.

$$\mathbb{B}(B) = \{f \in \mathbb{B}_0(B) \mid \mathrm{div}(f) \in \Phi(B)\}.$$

$\mathrm{Div}_{\mathbb{B}}$: the restriction of div.

Properties of $\underline{\underline{\mathcal{F}}}$

$(\mathcal{D}, \Phi, \mathbb{B}, \mathrm{Div}_{\mathbb{B}}) \rightsquigarrow$ a model Frobenioid $\underline{\underline{\mathcal{F}}}$.

The Frobenioid $\underline{\underline{\mathcal{F}}}$ has the following properties:

- $\mathcal{D}$: slim, of FSMFF type
- Φ : perfect, perf-factorial, non-dilating, cuspidally pure, rational
- $\underline{\underline{\mathcal{F}}}$: of unit-profinite type, of isotropic type, of model type, of sub-quasi-Frobenius trivial type, not of group-like type, of standard type, of rationally standard type

(I will not explain the terminology appeared here.)

Consequence

As a consequence, we can reconstruct

$$\mathcal{D}, \Phi, \text{ and } \underline{\underline{\mathcal{F}}} \to \mathbb{F}_{\Phi} \to \mathcal{D}$$

category theoretically from $\underline{\underline{\mathcal{F}}}$.

The Base-Field-Theoretic Hull $\mathcal{C}$

$\mathbb{F}_0 \subset \mathbb{B}_0$; submonoid of constant functions.
$\Longrightarrow \mathbb{F}_0 \subset \mathbb{B} \subset \mathbb{B}_0$.

$$\Phi^{\mathrm{bs}}\text{-}\mathrm{fld} := \mathbb{Q}_{>0} \cdot \mathrm{Image}(\mathbb{F}_0 \to \Phi_0^{\mathrm{gp}}) \cap \Phi.$$

$(\mathcal{D}, \Phi^{\mathrm{bs}}\text{-}\mathrm{fld}, \mathbb{F}_0, \mathbb{F}_0 \to (\Phi^{\mathrm{bs}}\text{-}\mathrm{fld})^{\mathrm{gp}})$
$\rightsquigarrow$ the model Frobenioid $\mathcal{C}$ (called the **base-field-theoretic hull** of $\underline{\underline{\mathcal{F}}}$).

One can reconstruct $\mathcal{C}$ category theoretically from $\underline{\underline{\mathcal{F}}}$.

Next Slides

We will go back to the global situation.

In later pages, we will come back again to the bad local situation and do further study.

Let us go back to our first setting, i.e.

An initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}, \underline{\epsilon})$$

is given.

For every $\underline{v} \in \underline{\mathbb{V}}$, we will construct $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (which is a Frobenioid when $\underline{v} \nmid \infty$), $p_{\underline{v}}$-adic or archimedean Frobenioids $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$ and characteristic splittings $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$ of $\mathcal{C}^{\vdash}_{\underline{v}}$ and $\mathcal{C}^{\Theta}_{\underline{v}}$ such that $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$ and $(\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ are naturally isomorphic.

We divide the situation into the following three cases

- $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$
- $\underline{v} \notin \underline{\mathbb{V}}^{\mathrm{bad}}$, $\underline{v} \nmid \infty$
- $\underline{v} | \infty$

Construction of the Frobenioids (1) : when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$\underline{\underline{\mathcal{F}}}_{\underline{v}}$: the Frobenioid $\underline{\underline{\mathcal{F}}}$ in the bad local situation. In particular its base category is $\mathcal{D}_{\underline{v}} = B^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}})^{\circ}$.
$\mathcal{C}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}$: the base-field theoretic hull.
Set $\mathcal{D}^{\vdash}_{\underline{v}} := B(\mathrm{Spec}\, K_{\underline{v}})^{\circ}$.

We have an adjunction

$$\mathcal{D}_{\underline{v}} \leftrightarrows \mathcal{D}^{\vdash}_{\underline{v}}$$

Next Slides

We will construct a Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$. We need preliminaries.

Recall: $\mathcal{O}^{\triangleright}(\ )$ and $\mathcal{O}^{\times}(\ )$

For a Frobenioid $\mathcal{F}$ and an object A of $\mathcal{C}$, we set

$$\mathcal{O}^{\triangleright}(A) := \{f : A \to A \mid f \mapsto (\mathrm{id}, *, 1) \text{ under } \mathcal{F} \to \mathbb{F}_{\Phi}\},$$

$$\mathcal{O}^{\times}(A) = (\mathcal{O}^{\triangleright}(A))^{\times}, \text{ and}$$
$$\mu_N(A) = \mathrm{Ker}(N : \mathcal{O}^{\times}(A) \to \mathcal{O}^{\times}(A)).$$

Notation

For $W \in \mathrm{Obj}(\mathcal{D}_{\underline{v}})$, let $\mathbb{T}_W = (W, 0)$ denote the Frobenius trivial object lying over W.

We use the subscript $\underline{v}$ to denote objects of $\mathrm{Obj}(\mathcal{D}_{\underline{v}})$ introduced in the bad local situation, when $X^{\log} = (E_{K_{\underline{v}}}, 0)$.

$\underline{\underline{q}}_{\underline{v}} := q_{\underline{v}}^{1/2\ell}$

$q_{\underline{v}}$: the q-parameter of $E_{K_{\underline{v}}}/K_{\underline{v}}$. We regard $q_{\underline{v}}$ as an element of $\mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$.

The assumption on $E_F[2]$ and the definition of K $\Longrightarrow q_{\underline{v}}$ admits a 2ℓ-th root $\underline{\underline{q}}_{\underline{v}} := q_{\underline{v}}^{1/2\ell}$ in $\mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$.

$\mathcal{C}^{\vdash}_{\underline{v}}$ and $\tau^{\vdash}_{\underline{v}}$

$\underline{\underline{q}}_{\underline{v}}$ defines the constant section $\mathbb{N}_{\mathcal{D}_{\underline{v}}} \hookrightarrow \mathcal{C}_{\underline{v}}$ of $\Phi_{\mathcal{C}_{\underline{v}}}$: the divisorial monoid for $\mathcal{C}_{\underline{v}}$. Denote this section by $\log(\underline{\underline{q}}_{\underline{v}})$.

Set

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} = \mathbb{N} \cdot \log(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}} \subset \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}.$$

$\rightsquigarrow$ ($p_{\underline{v}}$-adic) $\mathcal{C}^{\vdash}_{\underline{v}}$ whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$.

$\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}} \rightsquigarrow \underline{\underline{q}}_{\underline{v}}$ defines a characteristic splitting $\tau^{\vdash}_{\underline{v}}$ modulo $\mu_{2\ell}$.

Next Slides

We will construct a Frobenioid $\mathcal{C}_{\underline{v}}^{\Theta}$ and its base category is $\mathcal{D}_{\underline{v}}^{\Theta}$. We again need preliminaries.

$\underline{\underline{\Theta}}_{\underline{v}}$

Let us regard $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ as an object of $\mathcal{D}_{\underline{v}}$ via $\underline{\underline{\ddot{Y}}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$.

$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$: the inverse of the Frobenioid theoretic ℓ-th root of theta function. Here the superscript $\div$ means the biratioalization (i.e., localization with respect to the pre-steps).

Remark. Relation of $\underline{\underline{\Theta}}_{\underline{v}}$ with $\underline{\underline{q}}$

We have $\underline{\underline{\Theta}}_{\underline{v}}(\sqrt{-q_{\underline{v}}}) = \underline{\underline{q}}_{\underline{v}}$.
(Note. Both $\underline{\underline{\Theta}}_{\underline{v}}$ and $\underline{\underline{q}}_{\underline{v}}$ are determined only up to $\mu_{2\ell}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$.)

The base category $\mathcal{D}^{\Theta}_{\underline{v}}$

$\mathcal{D}^{\Theta}_{\underline{v}} \subset (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$ the full subcat. whose obj. are the products of objects of $\mathcal{D}^{\vdash}_{\underline{v}}$ and $\underline{\underline{\ddot{Y}}}_{\underline{v}}$.

$\Longrightarrow \mathcal{D}^{\vdash}_{\underline{v}} \cong \mathcal{D}^{\Theta}_{\underline{v}}$: equivalence.

Define $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}$: monoid on $\mathcal{D}^{\Theta}_{\underline{v}}$ as

$$A^{\Theta} \mapsto \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}})(\underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}})^{\mathbb{N}} \subset \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}}).$$

$$\Longrightarrow \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \cong \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) := \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)^{\times}.$$

The Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}}$

$$\Longrightarrow \mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \cong \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-),\ \underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}.$$

$\rightsquigarrow$ $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}}$ whose base cat. is $\mathcal{D}^{\Theta}_{\underline{v}}$ and a characteristic splitting $\tau^{\Theta}_{\underline{v}}$ modulo $\mu_{2\ell}$ such that

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) \cong \mathcal{F}^{\Theta}_{\underline{v}} := (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}}).$$

Theorem.

We can reconstruct the followings category theoretically from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$:

- $\mathcal{D}_{\underline{v}}$, $\mathcal{D}^{\vdash}_{\underline{v}}$, $\mathcal{D}^{\Theta}_{\underline{v}}$
- $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$
- $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$