Étale theta functions, mono-theta environments, and [IUTchI] §1-§3, II

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Construction of the Frobenioids (2) : when $\underline{v} \notin \underline{\mathbb{V}}^{\mathrm{bad}}$ and $\underline{v} \nmid \infty$

In this case, $\underline{\underline{\mathcal{F}}}_{\underline{v}}$: $p_{\underline{v}}$-adic Frobenioid whose base category is $\mathcal{D}_{\underline{v}} := B(\underrightarrow{X}_{\underline{v}})^{\circ}$ and whose divisor monoid is given by the composite

$$\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}} = B(\mathrm{Spec}\, K_{\underline{v}})^{\circ} \to \mathfrak{Mon}$$

where the last functor is $\mathrm{Spec}\, L \mapsto (\mathcal{O}^{\rhd}_{L}/\mathcal{O}^{\times}_{L})^{\mathrm{pf}}$.

The Frobenioids $\mathcal{C}_{\underline{v}}$ and $\mathcal{C}^{\vdash}_{\underline{v}}$

Set $\mathcal{C}_{\underline{v}} := \underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Define the submonoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of the monoid $\operatorname{Spec} L \mapsto (\mathcal{O}^{\rhd}_{L}/\mathcal{O}^{\times}_{L})^{\mathrm{pf}}$ on $\mathcal{D}^{\vdash}_{\underline{v}}$ as $\operatorname{Spec} L \mapsto \mathbb{Z}^{\rhd}_{p_{\underline{v}}}/\mathbb{Z}^{\times}_{p_{\underline{v}}}$.
$\rightsquigarrow p_{\underline{v}}$-adic Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ whose base category is $\mathcal{D}^{\vdash}_{\underline{v}}$.

$p_{\underline{v}}$ gives a characteristic splitting $\tau^{\vdash}_{\underline{v}}$. Set $\mathcal{F}^{\vdash}_{\underline{v}} = (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$.

The Frobenioid $\mathcal{C}_{\underline{v}}^{\Theta}$

Construct $\mathcal{F}_{\underline{v}}^{\Theta} = (\mathcal{C}_{\underline{v}}^{\Theta}, \tau_{\underline{v}}^{\Theta})$ just by adjoining the formal symbol "$\log \underline{\underline{\Theta}}$" to $\mathcal{F}_{\underline{v}}^{\vdash}$.

We have an isom. $\mathcal{F}_{\underline{v}}^{\vdash} \cong \mathcal{F}_{\underline{v}}^{\Theta}$.

Reconstructibility

We can reconstruct

$$\mathcal{D}_{\underline{v}}, \mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{F}^{\vdash}_{\underline{v}}, \mathcal{C}^{\Theta}_{\underline{v}}, \mathcal{F}^{\Theta}_{\underline{v}}$$

category theoretically from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Construction of the Frobenioids (3) : when $\underline{v}|\infty$

$$\underset{\rightarrow}{X}_{\underline{v}} = \underset{\rightarrow}{X}_K \otimes_K K_{\underline{v}} \rightsquigarrow \text{Aut-holomorphic space } \underset{\rightarrow}{\mathbb{X}}_{\underline{v}}.$$

By definition, $\underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$ is a pair

$$\underset{\rightarrow}{\mathbb{X}}_{\underline{v}} = (\underset{\rightarrow}{X}(K_{\underline{v}}), (\mathrm{Aut}_{\mathrm{hol}}(U))_U)$$

where U runs over the connected open subsets of $\underset{\rightarrow}{X}(K_{\underline{v}})$.

$\overline{\mathcal{A}}_{\underrightarrow{\mathbb{X}}_{\underline{v}}}$

Here

- we regard $\underrightarrow{X}(K_{\underline{v}})$ just as a topological space
- we regard $\mathrm{Aut}_{\mathrm{hol}}(U)$ as a subgroup of $\mathrm{Aut}_{\mathrm{top}}(U)$

$\underrightarrow{\mathbb{X}}_{\underline{v}} \rightsquigarrow \overline{\mathcal{A}}_{\underrightarrow{\mathbb{X}}_{\underline{v}}}$: complex archimedean top. field ($\cong \mathbb{C}$).

Construction of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, $\mathcal{C}_{\underline{v}}$

We set $\underline{\underline{\mathcal{F}}}_{\underline{v}} := (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$, where

- $\mathcal{C}_{\underline{v}}$: the archimedean Frobenioid given by circular sectors centered at 0 in $K_{\underline{v}}$, whose base category is one-morphism cat. $\mathrm{Spec}(K_{\underline{v}})$.
- $\mathcal{D}_{\underline{v}} = \underrightarrow{\mathbb{X}}_{\underline{v}}$: Aut-hol. space.
- $\kappa_{\underline{v}} : \mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})(\cong \mathcal{O}^{\triangleright}_{K_{\underline{v}}}) \hookrightarrow \mathcal{A}_{\mathcal{D}_{\underline{v}}} := \overline{\mathcal{A}}_{\underrightarrow{\mathbb{X}}_{\underline{v}}} \setminus \{0\}$.

Construction of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\tau^{\vdash}_{\underline{v}}$

We set $\mathcal{C}^{\vdash}_{\underline{v}} := \mathcal{C}_{\underline{v}}$
$\mathbb{R}_{>0} \subset \mathbb{C}^{\times} \rightsquigarrow \tau^{\vdash}_{\underline{v}}$: characteristic splitting of $\mathcal{C}^{\vdash}_{\underline{v}}$.

$\mathcal{D}^{\vdash}_{\underline{v}}$: the "split topological monoid" determined by $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$.

$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$.

Construction of $\mathcal{C}_{\underline{v}}^{\Theta}$, $\tau_{\underline{v}}^{\Theta}$ and Reconstructibility

Construct $\mathcal{F}_{\underline{v}}^{\Theta} = (\mathcal{C}_{\underline{v}}^{\Theta}, \mathcal{D}_{\underline{v}}^{\Theta}, \tau_{\underline{v}}^{\Theta})$ just by adjoining the formal symbol "$\log \Theta$" (Here $\mathcal{D}_{\underline{v}}^{\Theta} = \mathcal{D}_{\underline{v}}^{\vdash}$).

One can reconstruct $\mathcal{F}_{\underline{v}}^{\vdash}$ and $\mathcal{F}_{\underline{v}}^{\Theta}$ algorithmically from the datum $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Next Slides

Let us go back to the bad local situation.

The Model mod. N Mono Θ-environment

Recall: in the bad local situation, for an integer $N \geq 1$, the **model mod. N Θ-environment**

$$\mathbb{M}^{\mp} = (\Pi^{\mp}, \mathcal{D}^{\mp}, s^{\Theta}_{\Pi^{\mp}})$$

is a triple of a topological group $\Pi^{\mp}$, a subgroup $\mathcal{D}^{\mp}$ of $\mathrm{Out}(\Pi^{\mp})$, and a set $s^{\Theta}_{\Pi^{\mp}}$ of subgroups of $\Pi^{\mp}$ given as follows:

The model mod. N mono Θ-environment (continued)

- $\Pi^{\mathfrak{E}} := \underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}} \ltimes \mu_N$, where $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}} \twoheadrightarrow G_K \circlearrowleft \mu_N$.
- $\mathcal{D}^{\mathfrak{E}} \subset \mathrm{Out}(\Pi^{\mathfrak{E}})$: the subgroup generated by the image of $K^\times$ (under the composite $K^\times \to H^1(K, \mu_N) \to H^1(\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}}, \mu_N) \to \mathrm{Out}(\Pi^{\mathfrak{E}})$) and $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \cong \ell\underline{\mathbb{Z}}$.
- $s^{\Theta}_{\Pi^{\mathfrak{E}}}$: the μ_N-conj. class of subgroups of $\Pi^{\mathfrak{E}}$ given by the images of **mod.** N **theta sections**.

Mod. N-Theta Sections

Here mod. N-theta sections are homomorphisms $\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \hookrightarrow \Pi^{\text{モ}}$ obtained from the composite

$$\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \overset{\subset}{\to} \underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{Y}}} \xrightarrow{\text{tautological}} \Pi^{\text{モ}}$$

by twisting by the orbit of étale theta classes $\underline{\underline{\ddot{\eta}}}^{\Theta,\ell\underline{\mathbb{Z}}\times\mu_2} \subset H^1(\underline{\Pi}^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, \ell\cdot\underline{\Delta}_{\Theta})$.

Main Bad Local Theorems

Main Bad Local Theorem 1. We can reconstruct category theoretically the canonical isomorphism

$$\mu_N \cong \underline{\Delta}_{\Theta} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z}$$

from $\underline{\underline{\mathcal{F}}}$.

Main Bad Local Theorem 2. We can reconstruct $\mathbb{M}^{\mathfrak{E}}$ (up to isom.) category theoretically from $\underline{\underline{\mathcal{F}}}$.

Geometry of the Divisor Monoid of $\underline{\underline{\mathcal{F}}}$

Let (B, α) be an object of $\underline{\underline{\mathcal{F}}}$. $M = \Phi(B)$.

M is a perfection of the monoid of effective divisors on $\mathfrak{B}^{\mathrm{ell}}$ supported on the union of the cusps and the special fiber.
$\Longrightarrow M$ is decomposed as the direct sum of the cuspidal part and the non-cuspidal part, i.e., the part supported on the special fiber.

Next Slides

For example, when $B = \ddot{Y}$, the divisor of zeros of $\ddot{\Theta}$ belongs to the cuspidal part, and the divisor of poles of $\ddot{\Theta}$ belongs to the non-cuspidal part

One can describe this decomposition monoid-theoretically using $(M, \Phi^{\mathrm{bs-fld}}(B))$. A main idea is the following observation: an effective divisor with finite support is non-cuspidal iff it is bounded by the divisor of a costant function. A detailed explanation will be given in the next slides.

Cuspidal and Non-cuspidal Elements

$a \in M$: primary element.

- a is non-cuspidal iff $a + b = c$, ${}^{\exists}b \in M$, ${}^{\exists}c \in \Phi^{\mathrm{bs-fld}}(B)$.
- a is cuspidal iff b is not non-cuspidal.

$a \in M$: an element.

- a is non-cuspidal (resp. cuspidal) iff any primary $b \in M$ satisfying $na = b + c$ for some $n \geq 1$ and $c \in M$ is non-cuspidal (resp. cuspidal).

Cupsidal Pre-steps

Recall: a morphism f of $\underline{\underline{\mathcal{F}}}$ is a pre-step iff the functor $\underline{\underline{\mathcal{F}}} \to \mathbb{F}_\Phi$ maps f to $(\cong, *, 1)$.

A pre-step f of $\underline{\underline{\mathcal{F}}}$ is called **non-cuspidal** (resp. **cuspidal**) iff $f \mapsto (\cong, *, 1)$ with $*$: cuspidal (resp. non-cuspidal).

Category Theoreticity

One can reconstruct the non-cuspidal (resp. cuspidal) pre-steps category theoretically from $\underline{\underline{\mathcal{F}}}$.

For any object (B, α) of $\underline{\underline{\mathcal{F}}}$, one can reconstruct the non-cuspidal (resp. cuspidal) elements of $\Phi(M)$ category theoretically from $(\underline{\underline{\mathcal{F}}}, (B, \alpha))$.

Category Theoreticity of $A_{\circledcirc} = (\underline{\underline{\ddot{Y}}}, 0)$

Set $A_{\circledcirc} = (A_{\circledcirc}^{\mathrm{bs}}, 0) \in \mathrm{Obj}(\underline{\underline{\mathcal{F}}})$, where $A_{\circledcirc} = \underline{\underline{\ddot{Y}}}$.

One can reconstruct $A_{\circledcirc}^{\mathrm{bs}}$ (up to isom.) category theoretically from $\underline{\underline{\mathcal{F}}}$.

Theorem One can reconstruct $A_{\circledcirc}$ (up to isom.) category theoretically from $\underline{\underline{\mathcal{F}}}$.

$\Psi^{\Phi}_{A_{\circledcirc}}$

$\Psi : \underline{\underline{\mathcal{F}}} \to \underline{\underline{\mathcal{F}}}$: an auto-equivalence of cat.

Category theoreticity of $A_{\circledcirc} \Longrightarrow \Psi(A_{\circledcirc})^{\exists} \cong A_{\circledcirc}$.
Category theoreticity of $\Phi \Longrightarrow \Phi(A_{\circledcirc})^{\exists} \cong \Phi(\Psi(A_{\circledcirc}))$.

Composition $\Longrightarrow \Psi^{\Phi}_{A_{\circledcirc}} : \Phi(A_{\circledcirc}) \xrightarrow{\cong} \Phi(A_{\circledcirc})$.

Category Theoreticity of the Geometry of Divisors

Proposition. $\Psi_{A_{\circledcirc}}^{\Phi}$ preserves the followings:

- the cuspidal elements
- the surjection $\mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{cusp}} \twoheadrightarrow \mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{ncsp}}$
- $\mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{ncsp}} \cong \mathbb{Z}$ up to translates, ± 1
- $\mathrm{div}\ddot{\Theta} \in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ up to translates, ± 1

Main Idea of Proof

Consider the rational funcctions such as

$$\frac{(t - q_E)(t - q_E^{-1})}{(t-1)^2}$$

$\rightsquigarrow$they provide divisors ~ 0 with small supports.

Intersection theory $\Longrightarrow$non-existence of a non-zero divisor ~ 0 with smaller support.

Main Idea of Proof (continued)

$\rightsquigarrow$The Frobenioid structure can recover the map $\mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{cusp}} \twoheadrightarrow \mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{ncsp}}$ and the pairs of two adjacent non-cuspidal components.

Étale Theta Function in the Frobenioid-theoretic setting

Let N : an integer ≥ 1.
Recall: we constructed $Y_{\ell N}$, $\ddot{Y}_{\ell N}$, $Z_{\ell N}$.
A Description over $\overline{K}$:

- $Y_{\ell N, \overline{K}}$: the composite of $Y_{\overline{K}} \to X_{\overline{K}}$ and $[\ell N] : X_{\overline{K}} \to X_{\overline{K}}$
- $\ddot{Y}_{\ell N, \overline{K}} = Y_{2\ell N, \overline{K}}$
- $Z_{\ell N, \overline{K}}$: obtained from $Y_{\ell N, \overline{K}}$ by killing the $\underline{\Delta}_{\Theta} \otimes_{\mathbb{Z}} \mathbb{Z}/\ell N\mathbb{Z}$-part from the fundamental group of $Y_{\ell_N, \overline{K}}$

From now on assume N: odd.
$\mathfrak{Y}_{\ell N}, \ddot{\mathfrak{Y}}_{\ell N}, \mathfrak{Z}_{\ell N}$: stable models of $Y_{\ell N}, \ddot{Y}_{\ell N}, Z_{\ell N}$.
Recall: we constructed

- $s_{\ell N} \in \Gamma(\mathfrak{Z}_{\ell N}, \mathcal{L}_{\ell N}|_{\mathfrak{Z}_{\ell N}})$: "the zeros $\Theta^{1/\ell N}$"
- $\tau_{\ell N} \in \Gamma(\ddot{\mathfrak{Y}}, \mathcal{L}_{\ell N}|_{\ddot{\mathfrak{Y}}_{\ell N}})$: "the poles of $\Theta^{1/\ell N}$"

$\ddot{Z}_{\ell N}$: the composite of $Z_{\ell N}$ and $\ddot{Y}_{\ell N} \rightsquigarrow$

$$s_{\ell N}|_{\ddot{\mathfrak{Z}}_{\ell N}}, \tau_{\ell N}|_{\ddot{\mathfrak{Z}}_{\ell N}} : \mathcal{O}_{\ddot{\mathfrak{Z}}_{\ell N}} \hookrightarrow \mathcal{L}_{\ell N}|_{\ddot{\mathfrak{Z}}_{\ell N}}.$$

We regard these as objects of $\underline{\underline{\mathcal{F}}}$ and denote them by "$s_{\ell N}$", "$\tau_{\ell N}$". They are pre-steps.
We will deal with ("$s_{\ell N}$", "$\tau_{\ell N}$") by constructing a theory called bi-Kummer theory.

Kummer theory vs. bi-Kummer theory

We would like to introduce Kummer theory into the world of Frobenioids.

An example. X: a connected regular noetherian scheme, $1/N \in \Gamma(X, \mathcal{O}_X)$. f : a meromorphic function on X, invertible on $U \subset X$.

Usual Kummer theory: $f \rightsquigarrow \pi_1^{\mathrm{et}}(U) \circlearrowleft f^{1/N}$ $\rightsquigarrow \kappa_f \in H^1(\pi_1^{\mathrm{et}}(U), \mu_N)$.

bi-Kummer theory: decompose $\operatorname{div} f$ as

$$\operatorname{div} f = D_+ - D_-,$$

with D_+, D_-: effective, having disjoint supports ($\Longrightarrow D_+$ and D_- are linearly equiv).

Interpret f as a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\subset} & \mathcal{O}_X(D_+) \\ \| & & \cong \downarrow \times f \\ \mathcal{O}_X & \xrightarrow{\subset} & \mathcal{O}_X(D_-). \end{array}$$

bi-Kummer theory describes κ_f in this framework.

General bi-Kummer theory

Let $\mathcal{F} \to \mathbb{F}_{\Phi}$ be a Frobenioid satisfying certain technical conditions, such that $\mathcal{D} = B^{\mathrm{tp}}(\Pi)^{\circ}[D]$ for some Π, D. Let $A_{\circledcirc}$ be a "Frobenius-trivial" object s.t. $A_{\circledcirc}^{\mathrm{bs}} = \mathrm{Base}(A_{\circledcirc})$ is "Galois".
Note. Our $(\underline{\underline{\mathcal{F}}}, A_{\circledcirc})$ satisfies the conditions.

$\mathcal{F}^{\div}$: the localization of $\mathcal{F}$ w.r.t. pre-steps.
$\Longrightarrow \mathcal{F}^{\div}$ has a natural structure of Frobenioid.

Fraction Pair

$A \in \mathrm{Obj}(\mathcal{F})$, $f \in \mathcal{O}^\times(A^{\div})$.
(right) fraction pair for f is a pair

$$s^{\sqcap}, s^{\sqcup} : A \to B$$

of pre-steps s.t.

- $\mathrm{Base}(s^{\sqcap}) = \mathrm{Base}(s^{\sqcup})$
- $s^{\sqcap} \circ (s^{\sqcup})^{-1} = f$
- $\mathrm{Div}(s^{\sqcap})$ and $\mathrm{Div}(s^{\sqcup})$ have disjoint supports

$\mathrm{Div}(s^{\sqcap})$: "divisor of zeros of f",
$\mathrm{Div}(s^{\sqcap})$: "divisor of poles of f".

Terminology

$A \in \mathrm{Obj}(\mathcal{F})$ is **Galois** iff $A^{\mathrm{bs}} \in \mathrm{Obj}(\mathcal{D})$ is Galois.

When A: Galois,
$H_{A^{\mathrm{bs}}} := \mathrm{Image}(H_{\circledcirc} \hookrightarrow \Pi \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}}))$, where
$H_{\circledcirc} \lhd \Pi$: the normal subgroup corresponding to $A_{\circledcirc}$.
$H_A = H_{A^{\mathrm{bs}}} \cap \mathrm{Image}(\mathrm{Aut}_{\mathcal{F}}(A) \to \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}}))$.

A : $H_{\circledcirc}$-**ample** iff $H_A = H_{A^{\mathrm{bs}}}$.

$(N, H_{\circledcirc}, f)$-saturated object

For

- A : a Galois and H_A-ample object of $\mathcal{F}$
- $N \geq 1$: an integer
- $f \in \mathcal{O}^{\times}(A^{\div})^{H_A}$

One can define the notion "$(N, H_{\circledcirc}, f)$-**saturated**" which is a property of A.

Let us consider a fraction pair when $A = A_{\circledcirc}$:

- $f \in \mathcal{O}^{\times}(A_{\circledcirc}^{\div})$,
- $s^{\sqcap}, s^{\sqcup} : A_{\circledcirc} \to B_{\circledcirc}$: fraction pair for f.

Then

- $(s^{\sqcap}, s^{\sqcup})$ is unique up to isomorphisms
- For any $N \geq 1$, one can construct an "N-th root of $(s^{\sqcap}, s^{\sqcup})$"
- N-th root of $(s^{\sqcap}, s^{\sqcup})$ is unique up to "isomorphisms"

N-th root of $(s^{\sqcap}, s^{\sqcup})$

N-th root of $(s^{\sqcap}, s^{\sqcup})$ is a pair $(s^{\sqcap}_N, s^{\sqcup}_N)$ of pre-steps

$$s^{\sqcap}_N, s^{\sqcup}_N : A_N \to B_N$$

satisfying

- $\mathrm{Base}(s^{\sqcap}_N) = \mathrm{Base}(s^{\sqcup}_N)$
- A_N : $(N, H_{\circledcirc}, f|_{A_N})$-saturated

and ...

... and

- $^{\exists}\alpha : A_N \to A$, $^{\exists}\beta : B_N \to B$ isometries of Frobenius degree N s.t. α is "of base Frobenius type" and

$$\begin{array}{ccc} A_N & \xrightarrow{s_N^{\sqcap}} & B_N \\ \alpha \downarrow & & \downarrow \beta \\ A_{\circledcirc} & \xrightarrow{s_N^{\sqcap}} & B_{\circledcirc} \end{array} \quad \text{and} \quad \begin{array}{ccc} A_N & \xrightarrow{s_N^{\sqcup}} & B_N \\ \alpha \downarrow & & \downarrow \beta \\ A_{\circledcirc} & \xrightarrow{s_N^{\sqcup}} & B_{\circledcirc} \end{array}$$

are commutative

Bi-Kummer Roots

Definition of $(s_N^{\sqcap}, s_N^{\sqcup})$ implies that A_N is Frobenius-trivial.
$\rightsquigarrow$Can lift $H_{A_N} \subset \mathrm{Aut}_{\mathcal{C}}(A_N)/\mathcal{O}^{\times}(A_N)$ to an action of H_{A_N} on A_N.

This actions is transported via $s_N^{\sqcap}$, $s_N^{\sqcup}$ to the actions $s_N^{\sqcap \mathrm{gp}}$, $s_N^{\sqcup \mathrm{gp}}$ of H_{B_N} on B_N.

$(s_N^{\sqcap \mathrm{gp}}, s_N^{\sqcup \mathrm{gp}})$ is called a **bi-Kummer** N**-th root** of $(s^{\sqcap}, s^{\sqcup})$.

Relation with the Kummer Class

f has a N-th root in $\mathcal{O}^\times(A_N^{\div})$
$\rightsquigarrow \kappa_f \in H^1(H_{A_N}, \mu_N(A_N))$: the **Kummer class of** f.

$s_N^{\sqcap\mathrm{gp}} \cdot (s_N^{\sqcup\mathrm{gp}})^{-1}$ gives an element in $\kappa_f|_{B_N} \in H^1(H_{B_N}, \mu_N(B_N))$ is equal to the Kummer class of f.

Category theoreticity of bi-Kummer Data

Theorem

(1) For $(\mathcal{F} \to \mathcal{D}, A_{\circledcirc}^{\mathrm{bs}}, A, f \in \mathcal{O}^{\times}(A^{\div}))$, the followings are categorical:

- the property "A is $(N, H_{\circledcirc}, f)$-saturated",
- the Kummer class $\kappa_f \in H^1(H_A, \mu_N(A))$ when A is $(N, N_{\circledcirc}, f)$-saturated

(2) For $(\mathcal{F} \to \mathcal{D}, A_{\circledcirc}^{\mathrm{bs}}, B_N)$, the bi-Kummer N-th root $(s_N^{\sqcap \mathrm{gp}}, s_N^{\sqcup \mathrm{gp}})$ is categorical up to diagonal conjugation by $\mathcal{O}^{\times}(B_N)$ and the conjugation by $\mu_N(B_N) \times \mu_N(B_N)$.

When $\mathcal{F} = \underline{\underline{\mathcal{F}}}$

Proposition. Suppose that $\mathcal{F} = \underline{\underline{\mathcal{F}}}$. Then

- ("$s_{\ell N}$", "$\tau_{\ell N}$") is a ℓN-th root of a fraction pair of "$\ddot{\Theta}$".
- The group actions "$s_{\ell N}$","$\tau_{\ell N}$" are the same as the action given by the theory bi-Kummer roots.
- The Kummer class given by the bi-Kummer ℓN-th roots corresponds to the class of $\underline{\underline{\ddot{\eta}}}^{\Theta}$ modulo ℓN via the natural isomorphism $\mu_{\ell N} \cong \underline{\underline{\Delta}}_{\Theta} \otimes \mathbb{Z}/\ell N\mathbb{Z}$.

(ℓ, N)-theta Saturated Object

Let $S \in \mathrm{Obj}(\underline{\underline{\mathcal{F}}}) \rightsquigarrow (\ell\underline{\Delta}_{\Theta})_S$: a subquotient of $\mathrm{Aut}_{\mathcal{D}}(S^{\mathrm{bs}})$.

We say that S is (ℓ, N)-**theta saturated** if

$$\sharp\mu_{\ell N}(S) = \ell N \text{ and } \sharp(\ell\underline{\Delta}_{\Theta})_S \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} = N.$$

S : (ℓ, N)-theta saturated $\Longrightarrow S \to \underline{\underline{X}}$ is of sufficently high level to produce the following isomorphism:

The Kummer Class of $\ddot{\Theta}$ Relates Two Cyclotomes of Diffrent Origins

S : (ℓ, N)-theta saturated
$\Longrightarrow$the Kummer class in the above proposition gives an isom.

$$(\ell \cdot \underline{\Delta}_{\Theta}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \overset{\cong}{\to} \mu_N(S) = \ell \cdot \mu_{\ell N}(S). \quad (0.1)$$

Main Bad Local Theorems

Main Theorem 1. One can reconstruct the (ℓ, N)-theta saturated objects category theoretically from $\underline{\underline{\mathcal{F}}}$.
Moreover, for any given (ℓ, N)-theta object S of $\underline{\underline{\mathcal{F}}}$, one can reconstruct the isomorphism (0.1) above.

Theorem. One can reconstruct, up to μ_ℓ and translations by elements in $\underline{\mathbb{Z}}$, the ℓ-th roots of Θ-function of standard type.

Main Theorem 2. We can reconstruct an isomorph of $\mathbb{M}^{\text{Ŧ}}$ category theoretically from $\underline{\underline{\mathcal{F}}}$.