Étale theta functions, mono-theta environments, and [IUTchI] §1-§3, III

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Summary of the First and the Second Talk (1)

For an given initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}, \underline{\epsilon})$$

and for $\underline{v} \in \underline{\mathbb{V}}$, we constructed

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}, \mathcal{C}_{\underline{v}}, \mathcal{C}_{\underline{v}}^{\vdash}, \tau_{\underline{v}}^{\vdash}, \mathcal{C}_{\underline{v}}^{\Theta}, \tau_{\underline{v}}^{\Theta}$$

where ...

Summary of the First and the Second Talk (2)

where

- $\underline{\underline{\mathcal{F}}}_{\underline{v}}$: a Frobenioid when $\underline{v} \nmid \infty$
- $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$: $p_{\underline{v}}$-adic (resp. archimedean) Frobenioids if $\underline{v} \nmid \infty$ (resp. $\underline{v} | \infty$) (so its divisor monoid is monoprime).
- $\tau^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$: characteristic splittings ($\doteqdot$ splitting of the inclusion of functors "$\mathcal{O}^{\times} \subset \mathcal{O}^{\rhd}$") of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{C}^{\Theta}_{\underline{v}}$

Summary of the First and the Second Talk (3)

Main Bad Local Theorem 1. The canonical isomorphisms

$$(\ell \cdot \underline{\Delta}_{\Theta}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \xrightarrow{\cong} \mu_N(S) = \ell \cdot \mu_{\ell N}(S).$$

for (ℓ, N)-theta saturated objects S of $\underline{\underline{\mathcal{F}}}$ are category theoretical with respect to $\underline{\underline{\mathcal{F}}}$.

Main Theorem 2. The mono-Θ enviroment $\mathbb{M}^{\mathfrak{E}}$ is, up to isomorphisms, category theoretical with respect to $\underline{\underline{\mathcal{F}}}$.

Global realified Frobenioid $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$

$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$: the realification of the arithmetic Frobenioid given by F_{mod} and its trivial Galois ext. $F_{\mathrm{mod}}/F_{\mathrm{mod}}$. By definition,

- the base category of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ is the one-morphism cat. $\mathrm{Spec}\, F_{\mathrm{mod}}$.
- $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} = \bigoplus_{v \in \mathbb{V}_{\mathrm{mod}}} \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$ where $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} = (\mathcal{O}^{\rhd}_{F_{\mathrm{mod},v}}/\mathcal{O}^{\times}_{F_{\mathrm{mod},v}})^{\mathrm{pf}} \otimes \mathbb{R}_{\geq 0}$
- $\mathbb{B}_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} = \mathbb{R} \cdot \mathrm{Image}\, F^{\times}_{\mathrm{mod}} \subset \Phi^{\mathrm{gp}}_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$

For every $v \in \mathbb{V}_{\mathrm{mod}}$, we have a canonical isomorphism

$$\Phi_{\mathcal{C}_{\mathrm{mod}},\Vdash,v} \cong \mathbb{R}_{\geq 0},\ \log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto 1,$$

where p_v is the residue characteristic at v (resp. π) if $v \nmid \infty$ (resp. $v | \infty$).

Recall $\underline{\mathbb{V}} \to \mathbb{V}_{\mathrm{mod}}$ is bijective. Let $\underline{v} \in \underline{\mathbb{V}}$ the unique element that is mapped to v.

The restriction functor $\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{mod}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$ is equal to the functor induced by

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}, v} \xrightarrow{\cong} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}},$$

$$\log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \log_{\Phi}(p_{\underline{v}}).$$

Variant with Θ

$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}} = \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \cdot \log(\underline{\underline{\Theta}})$, where $\log(\underline{\underline{\Theta}})$: formal symbol.
$\Longrightarrow$a Frobenioid $\mathcal{C}^{\Vdash}_{\mathrm{tht}}$ with a natural equivalence $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \cong \mathcal{C}^{\Vdash}_{\mathrm{tht}}$.
We have "the natural restriction functor" $\mathcal{C}_{\rho^{\Theta}_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{tht}} \to (\mathcal{C}^{\Theta}_{\underline{v}})^{\mathrm{rlf}}$ which is equal to the functor induced by $\rho^{\Theta}_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht},v}} \to \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$,

$$\log^{\Vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}})$$

$$\mapsto \begin{cases} \frac{1}{[K_{\underline{v}}:F_{\mathrm{mod},v}]} \cdot \log_{\Phi}(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) & \text{if } \underline{v} : \text{ good} \\ \frac{1}{[K_{\underline{v}}:F_{\mathrm{mod},v}]} \cdot \log_{\Phi}(p_{\underline{v}}) \cdot \frac{\log(\underline{\underline{\Theta}}_{\underline{v}})}{\log(\underline{\underline{q}}_{\underline{v}})} & \text{if } \underline{v} \text{ bad} \end{cases}$$

$\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ and $\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$

Set

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} = (\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \overset{\cong}{\to} \underline{\mathbb{V}}, \{\mathfrak{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

and

$$\mathfrak{F}^{\Vdash}_{\mathrm{tht}} = (\mathcal{C}^{\Vdash}_{\mathrm{tht}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \overset{\cong}{\to} \underline{\mathbb{V}}, \{\mathfrak{F}^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}).$$

Then we have a natural isomorphism

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \cong \mathfrak{F}^{\Vdash}_{\mathrm{tht}}.$$

$\mathcal{D}$-version

$$\mathfrak{F}^{\Vdash}_{\mathcal{D}} = (\mathcal{D}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \cong \underline{\mathbb{V}}, \{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho^{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

Here

- $\mathcal{D}^{\Vdash}_{\mathrm{mod}}$: a copy of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$
- $(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$: $\mathbb{R}_{\geq 0}$ constructed from $\mathcal{D}^{\vdash}_{\underline{v}}$
- $\rho^{\mathcal{D}}_{\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod},v}} \to (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$,
  $\log^{\mathcal{D}}_{\mathrm{mod}}(p_{\underline{v}}) \mapsto \frac{1}{[K_{\underline{v}}:F_{\mathrm{mod},v}]} \log_{\Phi}(p_{\underline{v}})$.

Reconstructibility

We have an algorithm reconstructing $\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ and $\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ from $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$.

Θ-Hodge Theater ${}^{\dagger}\mathcal{HT}^{\Theta}$

Let

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

be an initial Θ-datum.

$\Longrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}$: Θ-Hodge theater:

$${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$$

Here ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ and ${}^{\dagger}\mathfrak{F}_{\mathrm{mod}}$ is as follows:

On ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$

${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is

- $\underline{v} \nmid \infty \Longrightarrow$ a cat. equiv. to $\underline{\underline{\mathcal{F}}}_{\underline{v}}$
- $\underline{v} | \infty \Longrightarrow$ a triple $({}^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$, where
    - ${}^{\dagger}\mathcal{C}_{\underline{v}}$: a category isomorphic to $\mathcal{C}_{\underline{v}}$
    - ${}^{\dagger}\mathcal{D}_{\underline{v}}$: an Aut-hol. space isom. to $\underrightarrow{\mathbb{X}}$
    - ${}^{\dagger}\kappa_{\underline{v}}$: Kummer structure $\mathcal{O}^{\triangleright}({}^{\dagger}\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{{}^{\dagger}\mathcal{D}_{\underline{v}}}$

On ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$

${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is a tuple

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \cong \underline{\mathbb{V}}, \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{{}^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

such that

- ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$: a cat. eqiv. to $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$
- $\mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \cong \underline{\mathbb{V}}$: bijection
- From ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ we construct ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $\Phi^{\mathrm{rlf}}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$
- ${}^{\dagger}\rho_{\underline{v}} : \Phi_{{}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod},v}} \xrightarrow{\cong} \Phi^{\mathrm{rlf}}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$: isom. of top. monoids

and that

${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ and $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ are isomorphic.

Reconstructability

We have an algorithm reconstructing ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ and ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ from ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$.

Θ-link

Let ${}^{\dagger}\mathcal{HT}^{\Theta}$, ${}^{\ddagger}\mathcal{HT}^{\Theta}$: two Θ-Hodge theaters

Then there exists an isomorphism ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\cong} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$.
So the full poly-isomorphism is non-empty.

The inverse of this full poly-isomorphism is called a Θ-**link** from ${}^{\dagger}\mathcal{HT}^{\Theta}$ to ${}^{\ddagger}\mathcal{HT}^{\Theta}$.

Θ-link poly-isomorphism

The poly-isomorphism

$$^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \underset{\text{poly}}{\cong} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$$

given by the composite

$$^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \underset{\text{poly}}{\cong} {}^{\ddagger}\mathcal{D}^{\Theta}_{\underline{v}} \cong {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}$$

is called the Θ-**link poly-isomorphism**.

Θ-link poly-isomorphism

The poly-isomorphism

$$\mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \underset{\text{poly}}{\cong} \mathcal{O}^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$$

given by the composite

$$\mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \cong \mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}}} \underset{\text{poly}}{\cong} \mathcal{O}^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$$

is called the Θ-**link poly-isomorphism**.

Next Slides

In the next slides, we will introduce several terminology concerning special kinds of rational functions on curves related to $C_{F_{\mathrm{mod}}}$.

We assume that an initial Θ-datum

$$(\overline{F}/F, X_F, \ell, \underline{C}_K, \underline{\mathbb{V}}, \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \underline{\epsilon})$$

is given.

κ-coric Functions and Kummer Theory

Let us consider $C_{F_{\mathrm{mod}}}$ over F_{mod}.

Let

- L : either F_{mod} or $F_{\mathrm{mod},v}$ for some $v \in \mathbb{V}_{\mathrm{mod}}$
- L_C : the function field of $C_L = C_{F_{\mathrm{mod}}} \otimes_{F_{\mathrm{mod}}} L$
- $\overline{L}_C$: an algebraic closure of L_C

An element of L_C or $\overline{L}_C$ is called a function.

Critical Points

$M \subset \overline{L}_C$: finite subextension of L_C
$\Longleftrightarrow Z_M \to |C_L|^*(\cong \mathbb{P}^1_L)$
($|C_L|^*$: the compactification of the coarse scheme of C_L).

We say that a closed point of Z_M is **critical** (resp. **strictly critical**) if its image in $|C_L|^*$ comes from a 2-torsion point (resp. a non-zero 2-torsion point) of E_F.

κ-coric functions

Let $f \in L_C$.
We say that f is κ-**coric** if either f is a root of unity or the following conditions are satisfied:

- f has a (possibly multiple) pole at an only one point and zeros at least two points
- f does not have a pole or zero at the critical points
- The values of f at the strict critical points are roots of unity

Some variants

Let $f \in \overline{L}_C$. We say that

- f : ${}_{\infty}\kappa$-**coric** if $f^n \in L_C$ and f^n: κ-coric for some $n \geq 1$.
- f : ${}_{\infty}\kappa\times$-**coric** if cf: ${}_{\infty}\kappa$-coric for some $c \in \overline{L}^{\times}$ (resp. $c \in \mathcal{O}_L^{\times}$) when $L = F_{\mathrm{mod}}$ (resp. when $L = F_{\mathrm{mod},v}$).

$L_C(\kappa\text{-sol})$

Suppose $L = F_{\mathrm{mod}}$. F_{sol} : the maximal solvable extension of F_{mod} inside $\overline{L}_C$.

$f \in \overline{L}_C$ is called κ-**solvable** if cf is ${}_\infty\kappa$-coric for some $c \in F_{\mathrm{sol}}^\times$.

- $L_C(\kappa\text{-sol}) \subset \overline{L}_C$: subfield gen. by L_C, and κ-solvable elements, i.e., the subfield gen. by L_C, F_{sol}, and the power roots of κ-coric elements.
- $L_C(\underline{C}_K) \subset \overline{L}_C$: subfield gen. by L_C and the images of $F(\mu_\ell) \cdot L_C$-linear embeddings of the function field of $\underline{C}_K$ into $\overline{L}_C$.

Action of $\mathrm{Gal}(L_C(\underline{C}_K)/F(\mu_\ell)L_C)$ on $L_C(\underline{C}_K) \cdot L_C(\kappa\text{-sol})$

Let $\mathrm{Gal}(L_C(\underline{C}_K)/F(\mu_\ell)L_C)$ act on $L_C(\underline{C}_K) \cdot L_C(\kappa\text{-sol})$ via the isomorphism

$$\begin{aligned}&\mathrm{Gal}(L_C(\underline{C}_K)/F(\mu_\ell)L_C)\\ &\cong \mathrm{Gal}(L_C(\underline{C}_K) \cdot L_C(\kappa\text{-sol})/F(\mu_\ell) \cdot L_C(\kappa\text{-sol}))\end{aligned}$$

Note: $\overline{L}_C$ and $F(\mu_\ell) \cdot L_C(\kappa\text{-sol})$ are lin. disj. over $F(\mu_\ell) \cdot L_C$.

κ-solvable Open Subgroup

$H \subset \mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$: a subgroup.
We say that H is κ-**solvable open subgroup** if

- $H \lhd \mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$: open and normal
- $^{\exists}\widetilde{H} \lhd \mathrm{Gal}(\overline{L}_C/L_C)$ open and normal s.t. $H = \widetilde{H} \cap \mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$

$\mathrm{Aut}^{\kappa\text{-sol}}$, $\mathrm{Out}^{\kappa\text{-sol}}$

$\mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \subset \mathrm{Aut}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ subgroup of automorphisms fixing every κ-solvable open subgroups.

$$\mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) := \frac{\mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))}{\mathrm{Inn}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))} \subset \mathrm{Out}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))$$

Discrete analogue of the proposition in the next topic
$\Longrightarrow \mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ is center-free.
$\Longrightarrow$the diagram

$$\begin{array}{ccc} \mathrm{Gal}(\overline{L}_C/L_C) & \longrightarrow & \mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(LC(\kappa\text{-sol})/L_C) & \longrightarrow & \mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \end{array}$$

is cartesian.

Next Slides

In the next slides, we will give a survey of §2 of [IUT-I]. The theme is

> profinite conjugates vs. tempered (or discrete) conjugates

for abstract or fundamental groups.

Commensurably Terminal

For

- G : Hausdorff topological group
- $H \subset G$: closed subgroup

set

$$C_G(H) = \{g \in G \mid gHg^{-1} \cap H \text{ is of finite index both in } H$$

and call it the **commensurator** of H in G. This is a subgroup of G.

We say that H is **commensurably terminal** in G if $C_G(H) = H$.

Notation

Now let K : a CDVF of mixed char. $(0, p)$, with a finite residue field k.

We use the following (standard) notation:

- $\mathcal{O}_K$: the ring of integers
- $\overline{K}$: an alg. closure of K
- $\overline{k}$: the residue field of $\overline{K}$
- $G_K := \mathrm{Gal}(\overline{K}/K)$

Notation

Suppose that we are given

X/K : a hyperbolic curve with stable red. over $\mathcal{O}_K$.

We use the following notation

- $\mathcal{X}$: the stable model of X
- $\mathcal{X}_k$: the special fiber
- $\mathcal{X}_{\overline{k}} := \mathcal{X}_k \otimes_k \overline{k}$

Let $\widehat{\Sigma}$: a non-empty set of prime numbers (e.g., $\widehat{\Sigma} = \mathfrak{Primes}$: the set of prime numbers).

Fundamental groups

- $\widehat{\Pi}_X$: the pro-$\widehat{\Sigma}$ fundamental group of X
- $\widehat{\Delta}_X$: the pro-$\widehat{\Sigma}$ fundamental group of $X_{\overline{k}}$
- Π_X^{tp} : $\widehat{\Sigma}$-tempered quotient of $\pi_1^{\mathrm{tp}}(X)$
- Δ_X^{tp} : $\widehat{\Sigma}$-tempered quotient of $\pi_1^{\mathrm{tp}}(X_{\overline{k}})$

We have $\Pi_X^{\mathrm{tp}} \hookrightarrow \widehat{\Pi}_X$, $\Delta_X^{\mathrm{tp}} \hookrightarrow \widehat{\Delta}_X$ which give isomorphisms from the pro-$\widehat{\Sigma}$ completions of the domains to the codomains.

Dual Semi-graph

Semi-graph : a generalization of the notion of (unoriented) graph
The only difference : a semi-graph allows open edges

$\mathcal{X} \rightsquigarrow$the **dual semi-graph** $\mathbb{G}_X$
$\mathbb{G}_X$ is obtained from the usual dual graph of $\mathcal{X}_{\overline{k}}$ by adjoining the open edges corresponding to the boundary points (=: the cusps)

The Category Associated to a Semi-graph

To a semi-graph $\mathbb{G}$, one can associate the small category $\mathrm{Cat}(\mathbb{G})$ as follows:

Objects : the vertices and the edges of $\mathbb{G}$,

Non-id. morphisms : $e \to v$ when v is an endpoint of e

Anabelioid

Connected anabelioid : a category equivalent to $B(G)$ for some profinite group G.

For two connected anabelioids $\mathcal{G}$, $\mathcal{H}$, a morphism (resp. an isomorphism) $\mathcal{G} \to \mathcal{H}$ is an exact functor (resp. an equivalence of categories) $\mathcal{H} \to \mathcal{G}$.

Semi-graph of anabelioids

A **semi-graph of anabelioids** is a pair of a semi-graph $\mathbb{G}$ and a functor from $\mathrm{Cat}(\mathbb{G})$ to the category of anabelioids.

In concrete terms, a **semi-graph of anabelioids** is a quadruple $\mathcal{G} = (\mathbb{G}, (\mathcal{G}_v)_v, (\mathcal{G}_e)_e, (\mathcal{G}_e \to \mathcal{G}_v)_{(e,v)})$ that consists of the following data:

Content of $\mathbb{G}$

- $\mathbb{G}$: a semi-graph
- $(\mathcal{G}_v)_v$: a family of conn. anabelioids indexed by the vertices v of $\mathbb{G}$
- $(\mathcal{G}_e)_e$: a family of conn. anabelioids indexed by the edges e of $\mathbb{G}$
- a morphism $\mathcal{G}_e \to \mathcal{G}_v$ of connected anabelioids is given for each pair (e, v) s.t. v is an endpoint of e

The Semi-graph of Anabelioids Associated to X

Let $\Sigma \subset \widehat{\Sigma}$: a non-empty subset with $p \notin \Sigma$.

$\mathcal{X}$ (and Σ) $\rightsquigarrow$a semi-graph of anabelioids $\mathcal{G}$ (well-defined up to "isomorphisms").

$$\mathcal{G} = (\mathbb{G}, (\mathcal{G}_v)_v, (\mathcal{G}_e)_e, (\mathcal{G}_e \to \mathcal{G}_v)_{(e,v)}), \textit{ where}$$

$\mathbb{G}$: the dual semi-graph of $\mathcal{X}$.
To define the second component let
U_v : the component of $\mathcal{X}_{\overline{k}}$ corresponding to v, with nodes and cusps removed.

Semi-graph of Anabelioids of Pro-Σ PSC-type

- $\mathcal{G}_v$: the connected anabelioid of finite étale Σ-cover of U_v, tamely ramified at the boundaries.
- $\mathcal{G}_e$: a copy of the ababelioid of finite étale Σ-cover of $\mathrm{Spec}\,\overline{k}((t))$, tamely ramified at $t = 0$.
- $\mathcal{G}_e \to \mathcal{G}_v$: given by the inertia subgroup at e

A semi-graph of anabelioid obtained in this way (for some K and a hyperbolic X) is called **pro-Σ PSC-type**.

Category $B(\mathcal{G})$

Let $\mathcal{G}$: a semi-graph of anabelioids whose underlying graph is non-empty and connected.
$\rightsquigarrow$the category $B(\mathcal{G})$ is constructed as follows:

Pro-$\widehat{\Sigma}$ Fundamental Group of $\mathcal{G}$

Regard $\mathcal{G}$ as a functor $\mathrm{Cat}(\mathbb{G}) \to$ (anabelioids).
$\mathcal{E}$: the category of pairs (x, S) of $x \in \mathrm{Obj}(\mathrm{Cat}(\mathbb{G}))$ and $S \in \mathrm{Obj}(\mathcal{G}_x)$. A morphism from (x, S) to (y, T) is pair of $f : x \to y$ and an isomorphism $S \cong f^* T$ in $B(\mathcal{G}_x)$.

Then $B(\mathcal{G})$ is a category of sections to the natural projection $\mathcal{E} \to \mathrm{Cat}(\mathbb{G})$.
$\Longrightarrow B(\mathcal{G})$ is a conn. anabelioid. Using this we can define a profinite fundamental group $\widehat{\pi}_1(\mathcal{G})$ and a pro-$\widehat{\Sigma}$ fundamental group $\widehat{\Pi}_{\mathcal{G}}$.

Tempered Fundamental Group of $\mathcal{G}$

If moreover $\mathcal{G}$ satsifies certain additional properties, one can define a tempered fundamental group $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ with a natural inclusion $\Pi_{\mathcal{G}}^{\mathrm{tp}} \hookrightarrow \widehat{\Pi}_{\mathcal{G}}$ by constructing a certain category $B^{\mathrm{tp}}(\mathcal{G})$ with $B(\mathcal{G}) \subset B^{\mathrm{tp}}(\mathcal{G})$.

This is the case when $\mathcal{C}$ is associated with $\mathcal{X}$, and we have natural surjections $\widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}$ and $\Delta_X^{\mathrm{tp}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{tp}}$.

Proposition.

Let

- $\mathcal{G}$: a semi-graph of anabelioids of pro-Σ PSC-type
- $\Lambda \subset \Pi_{\mathcal{G}}^{\mathrm{tp}}$: compact subgroup, $\Lambda \neq \{1\}$

Then any $\gamma \in \widehat{\Pi}_{\mathcal{G}}$ satisfying

$$\gamma \Lambda \gamma^{-1} \subset \Pi_{\mathcal{G}}^{\mathrm{tp}}$$

is in fact an element of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$.

A Consequence

Suppose that $\mathcal{G}$ is a semi-graph of anabelioids associated with $\mathcal{X}$ and Σ. Let

- $\mathbb{G}$: the underlying semi-graph
- $\mathbb{H} \subset \mathbb{G}$: a sub semi-graph
- $\mathcal{H}$: the semi-graph of anabelioids obtained by restricting $\mathcal{G}$ to $\mathcal{H}$

We set

$$\Delta_{X,\mathbb{H}}^{\mathrm{tp}} := \Delta_X^{\mathrm{tp}} \times_{\Pi_{\mathcal{G}}^{\mathrm{tp}}} \Pi_{\mathcal{H}}^{\mathrm{tp}}, \ \widehat{\Delta}_{X,\mathbb{H}} \widehat{\Delta}_X \times_{\widehat{\Pi}_{\mathcal{G}}} \widehat{\Pi}_{\mathcal{H}}.$$

Facts

- The closure of $\Delta^{\mathrm{tp}}_{X,\mathbb{H}}$ in $\widehat{\Delta}_X$ is equal to $\widehat{\Delta}_{X,\mathbb{H}}$
- We have $\widehat{\Delta}_{X,\mathbb{H}} \cap \Delta^{\mathrm{tp}}_X = \Delta^{\mathrm{tp}}_{X,\mathbb{H}}$
- Suppose $\Sigma \subsetneqq \widehat{\Sigma} \setminus \{p\}$ or $\widehat{\Sigma} = \mathfrak{Primes}$.
$\Longrightarrow \widehat{\Delta}_X$ is slim, and using this one can construct $\Pi^{\mathrm{tp}}_{X,\mathbb{H}}$ and $\widehat{\Pi}_{X,\mathbb{H}}$ from $\Delta^{\mathrm{tp}}_{X,\mathbb{H}}$, $\widehat{\Delta}_{X,\mathbb{H}}$.

Corollary.

Let the notation be as above. Then

- The subgroup $\Delta^{\mathrm{tp}}_{X,\mathbb{H}} \subset \Delta^{\mathrm{tp}}_{X}$ and the subgroup $\widehat{\Delta}_{X,\mathbb{H}} \subset \widehat{\Delta}_{X}$ are commensurably terminal.
- Suppose $\Sigma \subsetneqq \widehat{\Sigma} \setminus \{p\}$ or $\widehat{\Sigma} = \mathfrak{Primes}$ ($\Longrightarrow \Pi^{\mathrm{tp}}_{X,\mathbb{H}}$ and $\widehat{\Pi}_{X,\mathbb{H}}$). Then the subgroup $\Pi^{\mathrm{tp}}_{X,\mathbb{H}} \subset \Pi^{\mathrm{tp}}_{X}$ and the subgroup $\widehat{\Pi}_{X,\mathbb{H}} \subset \widehat{\Pi}_{X}$ are commensurably terminal.

Applications of the Proposition

Using the proposition above, one can prove several important propeties of the inclusions

$$\begin{array}{ccc} \Delta_X^{\mathrm{tp}} & \stackrel{\subset}{\longrightarrow} & \widehat{\Delta}_X \\ \downarrow & & \downarrow \\ \Pi_X^{\mathrm{tp}} & \stackrel{\subset}{\longrightarrow} & \widehat{\Pi}_X . \end{array}$$

Properties (1) (2)

Property (1). $\Lambda \subset \Delta_X^{\mathrm{tp}}$: a pro-Σ compact subgroup, $\Lambda \neq \{1\}$. Then any $\gamma \in \widehat{\Pi}_X$ satisfying

$$\gamma \Lambda \gamma^{-1} \subset \Pi_X^{\mathrm{tp}}$$

is in fact an element of Π_X^{tp}.

Property (2). Δ_X^{tp} is commensurably terminal in $\widehat{\Delta}_X$, and Π_X^{tp} is commensurably terminal in $\widehat{\Pi}_X$.

Property (3)

In this page we assume that $\widehat{\Sigma} = \mathfrak{Primes}$.

Property (3) Let x be a closed point or a cusp of X, $D_x \subset \Pi_X^{\mathrm{tp}}$ a decomposition group at x ($\Longrightarrow D_x$ is also a decomposition group at x in $\widehat{\Pi}_X$). Then

- any $\widehat{\Pi}_X$-conjugate of D_x contained in Π_X^{tp} is a decomposition group at x in Π_X^{tp}
- any $\widehat{\Pi}_X$-conjugate of Π_X^{tp} containing D_x is equal to Π_X^{tp}

Property (4)

In this page we assume that $\widehat{\Sigma} = \mathfrak{Primes}$.

Property (4) Let x be a cusp of X, $I_x \subset \Pi_X^{\mathrm{tp}}$ an inertia group at x ($\Longrightarrow I_x$ is also an inertia group at x in $\widehat{\Pi}_X$). Then

- any $\widehat{\Pi}_X$-conjugate of I_x contained in Π_X^{tp} is an inertia group at x in Π_X^{tp}.
- any $\widehat{\Pi}_X$-conjugate of Π_X^{tp} containing D_x is equal to Π_X^{tp}

Discrete Analogue of the Proposition

Theorem F : a group, a $H \subset F$: subgroup s.t.

- $F \cong \pi_1(Z(\mathbb{C}))$ for some hyperbolic curve $Z/\mathbb{C}$
- H : non-abelian

($\Longrightarrow F \subset \widehat{F}$: the profinite completion). Then any $\gamma \in \widehat{F}$ satisfying

$$\gamma H \gamma^{-1} \subset F$$

is in fact an element of F.

Discrete Analogue of the Proposition (continued)

One can generalize the statement to the case when $H \neq \{1\}$ by replacing the conclusion "$\gamma \in F$" with the following weaker statement:

$\gamma \in F \cdot N_{\widehat{F}}(H \cap G)$ for any subgroup $G \subset F$ of finite index.