COMMENTS ON "THE GEOMETRY OF FROBENIOIDS I"

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April 2020

(1.) In Definition A.1 of the Appendix: The phrase "*isomorphism classes* of morphisms" in line 4 should read "*isomorphism classes* of 1-morphisms". The phrase "*coarsification* of $\mathcal{C}$" in line 5 should read "*coarsification* of $\mathcal{D}$".

(2.) The hypothesis that the Frobenioids under consideration be of *"unit-profinite type"* in Proposition 5.6 — hence also in Corollary 5.7, (iii) — may be *removed.* Indeed, if, in the notation of the proof of Proposition 5.6, one writes $\phi'_p = c_p \cdot \phi_p$, where $c_p \in \mathcal{O}^\times(A)$, for $p \in \mathfrak{Primes}$, then one has

$$\begin{aligned} c_2 \cdot c_p^2 \cdot \phi_2 \cdot \phi_p = c_2 \cdot \phi_2 \cdot c_p \cdot \phi_p = \phi'_2 \cdot \phi'_p = \phi'_p \cdot \phi'_2 \\ = c_p \cdot \phi_p \cdot c_2 \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_p \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_2 \cdot \phi_p \end{aligned}$$

— so $c_2 \cdot c_p^2 = c_p \cdot c_2^p$, i.e., $c_p = c_2^{p-1}$, for $p \in \mathfrak{Primes}$. Thus, $\phi'_p = c_2^{-1} \cdot \phi_p \cdot c_2$, so by taking $u \stackrel{\mathrm{def}}{=} c_2^{-1}$, one may *eliminate the final two paragraphs* of the proof of Proposition 5.6.

(3.) In the second to last sentence of Definition 1.1, (ii),

Φ^{pf}"

should read as follows:

"Φ^{pf}"

(4.) The phrase "If M is a $\mathbb{Q}$-monoprime monoid" toward the end of the discussion entitled "Numbers" in §0 should read "If M is a $\mathbb{Q}$- or $\mathbb{R}$-monoprime monoid".

(5.) In the proof of Theorem 3.4, (iv), the phrase "α arises as the endomorphism of A" should read "β arises as the endomorphism of B"; also, in the same sentence, the notation "$(\mathcal{P}_i)_A$" should read "$(\mathcal{P}_i)_B$".

(6.) The phrase "in which α, β are *primary* with zero divisor in $\mathfrak{p}$;" immediately following the final display of the proof of Theorem 4.9 should read "in which α, β are *primary*;".

(7.) In the proof of Theorem 3.4, (i), the phrase "for each $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{istr}})$ that" should read "for each $A \in \mathrm{Ob}(\mathcal{C}_1^{\mathrm{istr}})$, that".

(8.) In the proof of Theorem 4.2, (i), the phrase "[cf. also Theorem 3.4, (ii)]" should read "[cf. also Theorem 3.4, (ii), (iii)]".

(9.) In the fourth paragraph of the proof of Theorem 5.1, the notation "$\psi : B' \to C$" should read "$\psi' : B' \to C$".

(10.) In the first display of the proof of Theorem 5.2, (iv), the notation "$(B \to A, A \to C)$" should read "$(B \to A, B \to C)$".

(11.) In Example 6.1, the phrase "may be identified with the *group of Cartier divisors* on $V[L]$, and" should read "may be identified with the *group of Cartier divisors* on $V[L]$ with support in $\mathbb{D}_L$, and".

(12.) In the proof of Lemma 6.5, (ii), the phrase "Indeed, since the ..." should read "Indeed, suppose that there exist $\lambda_1, \lambda_2 \in \mathbb{Q}_{>0}$ as in the statement of assertion (ii). Then since the ..."

(13.) In the discussion of §0 entitled "Numbers" the phrase "Also, we shall refer to ..." should read "Here, we regard the elements of the set $\{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ as being equipped with the ordering $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$. Also, we shall refer to ...".

(14.) In §I4, the phrase "that as appear as" should read "that appear as".

(15.) In the second paragraph of the proof of Theorem 4.9, the phrase "this subset maps the subset" should read "this isomorphism maps the subset".

(16.) In the discussion following the first display of the statement of Theorem 3.4, (iii), the notation "$\Psi^{\mathbb{N}}_{\geq 1}$" should read "$\Psi^{\mathbb{N}_{\geq 1}}$".

(17.) In the first paragraph of the proof of Theorem 3.4, (iv), the phrase "existence of a a" should read "existence of a".

(18.) In Example 6.1 and the statement of Theorem 6.2, the phrase "[possibly subvarieties of codimension ≥ 1]" (which is logically correct, but misleading) should be *deleted*.

(19.) In the second sentence following the display of Remark 3.1.2, the "$\mathbb{Z}_{\geq 0}$" is to be understood as the image of $\mathbb{Z}_{\geq 0}$ in $\mathbb{F}$ via the natural inclusion $\mathbb{Z}_{\geq 0} \xrightarrow{\sim} \mathbb{Z}_{\geq 0} \times \{1\} \hookrightarrow \mathbb{F} = \mathbb{Z}_{\geq 0} \times \mathbb{N}_{\geq 1}$ (cf. the final portion of Definition 1.1, (iii)) into the first factor of the product determined by $1 \in \mathbb{N}_{\geq 1}$.

(20.) In the statement of Theorem 5.2, (i), (b), the phrase "**projection** *to* $\mathcal{D}$ *to* ϕ" should read "**projection** *to* $\mathcal{D}$ *of* ϕ".

(21.) In the statement of Proposition 1.6, (v), (vi), the phrase "A object" should read "An object".

(22.) In the statement of Corollary 4.11, (iii), the phrase "Ψ^{Base} of (i)" should read "Ψ^{Base} of (ii)".

(23.) In the second sentence of the proof of Theorem 5.1, the notation "$\psi : B' \to C'$" should read "$\psi' : B' \to C'$".

(24.) In the explanation immediately following the first display of the statement of Theorem 5.2, (i), the notation "$A \stackrel{\text{def}}{=} (A_{\mathcal{D}}, \alpha)$," should be inserted immediately following the word "set".

(25.) In the statement of Theorem 5.2, (i), the notation "$\underline{\Phi}(\text{Base}(\phi))$" (2 instances) should read "$\underline{\Phi}(\text{Base}(\phi))^{\text{gp}}$".

(26.) In the statement of Proposition 1.9, (v), the phrase "restriction to $\mathcal{C}$" should read "restriction to "$\mathcal{C}^{\text{istr}}$".

(27.) The first sentence of the statement of Theorem 5.2, (iv), should read as follows:

> *Suppose that* $\mathcal{C}$ *is of* **isotropic** *and* **model** *type;* $\Phi = \underline{\Phi}$*;* $\mathbb{B}$ *is the* **rational function monoid** *on* $\mathcal{D}$ *associated to the Frobenioid* $\mathcal{C}$ *[cf. Proposition 4.4, (ii)];* $\text{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\text{gp}}$ *is the natural homomorphism* $\mathcal{O}^{\times}(-) \to \Phi^{\text{gp}} = \underline{\Phi}^{\text{gp}}$ *[cf. Proposition 4.4, (iii)].*

(28.) In §0, *condition (b)* of the definition of a category *of FSMFF-type* should read as follows:

> for every $A \in \text{Ob}(\mathcal{C})$, there exists a natural number N such that for every composite
>
> $$\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_2 \circ \phi_1$$
>
> of some morphism ϕ_1 whose domain is equal to A with FSMI-morphisms $\phi_2, \ldots, \phi_n$, it holds that $n \leq N$.

The statement of Proposition 1.14, (iii), should read as follows:

> *Suppose that ϕ is* **irreducible**. *Then ϕ is a* **non-pre-step** *if and only if the following condition holds: If ϕ is an* **FSM-morphism**, *then there exists an $N \in \mathbb{N}_{\geq 1}$ such that for every equality of composites in $\mathcal{C}$*
>
> $$\alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_2 \circ \alpha_1 = \psi \circ \phi$$
>
> *— where α_1 and ψ are irreducible morphisms, $n \in \mathbb{N}_{\geq 1}$, and $\alpha_2, \ldots, \alpha_n$, ψ are* **FSMI-morphisms** *[cf. §0] — it holds that $n \leq N$.*

The proof of Proposition 1.14, (iii), should read as follows:

> Next, we consider assertion (iii). By assertion (i), it suffices to show that assertion (iii) holds for each of the three types of morphisms "(a), (b), (c)" discussed in assertion (i). If ϕ is an *irreducible pre-step* [hence also, by assertion (ii), an *FSM-morphism*], then it follows immediately — by taking ψ to be a prime-Frobenius morphism of *increasingly large* Frobenius degree [cf. assertion (i); Definition 1.3, (iii), (d); Proposition 1.4, (i); Proposition 1.10, (ii)] — that the condition in the statement of assertion (iii) is *false* [as desired]. On the other hand, if ϕ is a *non-pre-step*, then it is an *isometry*. Now if the condition in the statement of assertion (iii) is *false*, then ϕ is an *FSM-morphism*, and, moreover, there exist equalities
>
> $$\alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_2 \circ \alpha_1 = \psi \circ \phi$$
>
> where α_1 and ψ are irreducible morphisms, $n \in \mathbb{N}_{\geq 1}$ is *arbitrarily large*, and $\alpha_2, \ldots, \alpha_n$, ψ are *FSMI-morphisms*. Next, observe that since ϕ is an isometry, it follows from the fact that ψ is *irreducible* [cf. also assertion (i); Definition 1.1, (ii), (b); Remark 1.1.1; Proposition 1.11, (vi)] that $\mathrm{Div}(\psi \circ \phi)$ is either zero or irreducible; since, moreover, $\deg_{\mathrm{Fr}}(\psi \circ \phi)$ always divides a product of two prime numbers [cf. assertion (i); the irreducibility of ϕ, ψ], it thus follows that in *any* factorization of $\psi \circ \phi$ by irreducible morphisms, all but *three* [i.e., corresponding to *two* possible prime factors of the Frobenius degree, plus *one* possible irreducible factor of the zero divisor] of the factorizing irreducible morphisms are *pull-back morphisms* [cf. assertion (i)]. On the other hand, this implies that factorizations of arbitrarily large length determine *chains of morphisms, all but the first of which are FSMI-morphisms* [cf. assertion (i); Proposition 1.11, (vi)], originating from the projection to $\mathcal{D}$ of the domain of ϕ which are also of *arbitrarily large length*, in contradiction to condition (b) of the definition of a "category of FSMFF-type" in §0. This completes the proof of assertion (iii).

The above modifications to the definition of the term "category of FSMFF-type" and to the statement and proof of Proposition 1.14, (iii), have *no effect* on the remainder of the present paper or on subsequent papers, except that *minor formal changes* are necessary in the proof of [FrdII], Proposition 3.4, (viii).

(29.) The following modifications concerning the *birationalization* of a Frobenioid should be made to §4:

(i) The statement of Proposition 4.4, (ii), should read as follows:

> *The functor $\mathcal{C}^{\text{birat}} \to \mathbb{F}_{0_{\mathcal{D}}}$ of (i) determines a structure of* **pre-Frobenioid of group-like type** *on $\mathcal{C}^{\text{birat}}$. Moreover, the functor $\mathcal{C} \to \mathcal{C}^{\text{birat}}$ is* **faithful**. *In particular, for every $A \in \text{Ob}(\mathcal{C})$ with image A^{birat} in $\mathcal{C}^{\text{birat}}$, the functor $\mathcal{C} \to \mathcal{C}^{\text{birat}}$ determines an* **injection** *of groups $\mathcal{O}^{\rhd}(A)^{\text{gp}} \hookrightarrow \mathcal{O}^{\times}(A^{\text{birat}})$. Suppose, further, that $\mathcal{C}$ is* **of birationally Frobenius-normalized type** *[cf. Definition 4.5, (i), below]. Then the pre-Frobenioid $\mathcal{C}^{\text{birat}}$ is a* **Frobenioid***; we shall refer to the functor "$\mathcal{O}^{\times}(-)$" on $\mathcal{D}$ associated to the Frobenioid $\mathcal{C}^{\text{birat}}$ [cf. Proposition 2.2, (ii), (iii)] as the* **rational function monoid** *of the Frobenioid $\mathcal{C}$.*

(ii) The final paragraph of the proof of Proposition 4.4 should read as follows:

> In the context of assertion (ii), we observe that it is immediate from the definitions [and the *total epimorphicity* of $\mathcal{C}$] that the functor $\mathcal{C} \to \mathcal{C}^{\text{birat}}$ is *faithful* and determines an injection $\mathcal{O}^{\rhd}(A)^{\text{gp}} \hookrightarrow \mathcal{O}^{\times}(A^{\text{birat}})$, for $A \in \text{Ob}(\mathcal{C})$. In light of the *"dictionary"* provided by assertion (iv) [cf. also Proposition 1.4, (iv); the equivalence of categories of Proposition 1.9, (ii)], it is now a routine exercise to check, whenever $\mathcal{C}$ is *of birationally Frobenius-normalized type* [cf. Definition 4.5, (i), below], that $\mathcal{C}^{\text{birat}}$ is, in fact, a *Frobenioid of group-like type*. [Here, in the context of the verification of Definition 1.3, (iii), (c), we observe that the natural functor $\mathcal{C}^{\text{birat}} \to (\mathcal{C}^{\text{istr}})^{\text{birat}}$ is *faithful* [cf. Definition 1.3, (iii), (d); Definition 1.3, (v), (a); Proposition 1.9, (v); the *total epimorphicity* of $\mathcal{C}$], that every object of $(\mathcal{C}^{\text{istr}})^{\text{birat}}$ is *Frobenius-trivial* [cf. Definition 1.3, (i), (a), (b); Proposition 1.4, (i)], and that any group G such $(\alpha \cdot \beta)^2 = \alpha^2 \cdot \beta^2$ for all $\alpha, \beta \in G$ is *abelian*.] This completes the proof of assertion (ii). Now assertion (iii) follows immediately from the existence of the functor $\mathcal{C}^{\text{birat}} \to \mathbb{F}_{\Phi^{\text{gp}}}$ of assertion (i) [cf. also Proposition 1.5, (ii)]; here, we note that the computation of the *kernel* of the surjection of assertion (iii) follows from Definition 1.3, (vi).

(iii) In Proposition 4.8, (ii): the phrase "*of* **perfect** *and* **isotropic** *type*" should read "*of* **perfect**, **isotropic**, *and* **birationally Frobenius-normalized** *type*".

(iv) In Proposition 4.8, (iv): the phrase "**pre-model** *type*" should read "**model** *type*".

(v) The portion preceding the display of the first sentence in the proof of Corollary 4.11, (ii), should read as follows: "First, we observe that, by assertion (i), we may assume without loss of generality that $\mathcal{C}$ is of *unit-trivial*, hence also [cf. Proposition 4.4, (iii)] *birationally Frobenius-normalized* type; moreover, we have a 1-commutative diagram".

The above modifications have *no effect* on the remainder of the present paper or on subsequent papers, except that *minor formal changes* are necessary in [FrdII], Definition 5.3, (v); [FrdII], Proposition 5.4; the statement and proof of [FrdII], Theorem 5.5, (iv).

(30.) In the discussion of the "*new functor*" in the proof of Theorem 3.4, (v), the following misprints should be corrected:

· "$(\mathcal{P}_i)_{D_\mathcal{D}}$" should read "$(\mathcal{P}_i)_D$";
· "$(\mathcal{P}_i)_{E_\mathcal{D}}$" should read "$(\mathcal{P}_i)_E$";
· "$(\mathcal{P}_i)_{A_\mathcal{D}}$" should read "$(\mathcal{P}_i)_A$";
· "$(\mathcal{P}_i)_{A'_\mathcal{D}}$" should read "$(\mathcal{P}_i)_{A'}$".

Bibliography

[FrdII] S. Mochizuki, The Geometry of Frobenioids II: Poly-Frobenioids, *Kyushu J. Math.* **62** (2008), pp. 401-460.