THE GEOMETRY OF FROBENIOIDS I: THE GENERAL THEORY

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Abstract. We develop the theory of *Frobenioids*, which may be regarded as a category-theoretic abstraction of the theory of divisors and line bundles on models of finite separable extensions of a given function field or number field. This sort of abstraction is analogous to the role of *Galois categories* in Galois theory or *monoids* in the geometry of log schemes. This abstract category-theoretic framework preserves many of the important features of the classical theory of divisors and line bundles on models of finite separable extensions of a function field or number field such as the *global degree* of an arithmetic line bundle over a number field, but also exhibits interesting new phenomena, such as a *"Frobenius endomorphism"* of the Frobenioid associated to a number field.

Introduction

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§I1. Technical Summary

In the present paper, we introduce the notion of a *Frobenioid*. The simplest kind of Frobenioid "$\mathbb{F}_M$" is the non-commutative monoid given by forming the "semi-direct product monoid" of a given commutative monoid M with the multiplicative monoid of positive integers $\mathbb{N}_{\geq 1}$ [cf. §0], where $n \in \mathbb{N}_{\geq 1}$ acts on M by multiplication by n; that is to say, the underlying set of $\mathbb{F}_M$ is the product

$$M \times \mathbb{N}_{\geq 1}$$

equipped with the monoid structure is given as follows: if $a_1, a_2 \in M$, $n_1, n_2 \in \mathbb{N}_{\geq 1}$, then $(a_1, n_1) \cdot (a_2, n_2) = (a_1 + n_1 \cdot a_2, n_1 \cdot n_2)$ [cf. Definition 1.1, (iii)]. For instance, when M is taken to be the additive monoid of nonnegative integers $\mathbb{Z}_{\geq 0}$ [cf. §0], we shall write $\mathbb{F} \stackrel{\text{def}}{=} \mathbb{F}_{\mathbb{Z}_{\geq 0}}$ and refer to $\mathbb{F}$ as the *standard Frobenioid*. Note that in general, any monoid [such as $\mathbb{F}_M$, for instance] may be thought of as a *category*, i.e., the category with precisely one object whose monoid of endomorphisms is the given monoid.

More generally, one may start with a *"family of commutative monoids"* Φ on a *"base category"* $\mathcal{D}$ [where Φ, $\mathcal{D}$ satisfy certain properties] and form the associated *elementary Frobenioid* $\mathbb{F}_\Phi$ by taking the "semi-direct product" of $\mathbb{N}_{\geq 1}$ with Φ [cf. Definition 1.1, (iii), for more details]. Here, $\mathbb{F}_\Phi$ is a *category*.

In general, a *Frobenioid* $\mathcal{C}$ is a category equipped with a functor $\mathcal{C} \to \mathbb{F}_\Phi$ to an elementary Frobenioid $\mathbb{F}_\Phi$ satisfying certain properties [cf. Definition 1.3 for more details] to the effect that the structure of $\mathcal{C}$ is *"substantially reflected"* in this functor $\mathcal{C} \to \mathbb{F}_\Phi$. From the point of view of conventional arithmetic geometry, a Frobenioid may be thought of as a sort of a *category-theoretic abstraction of the theory of divisors and line bundles on models of finite separable extensions of a given function field or number field*. That is to say, the base category $\mathcal{D}$ corresponds to the category of models of finite separable extensions of a given function field or number field; the functor Φ corresponds to the divisors on such models; the "$\mathbb{N}_{\geq 1}$ portion" of $\mathbb{F}_\Phi$ corresponds to the operation of multiplying a divisor by an element $n \in \mathbb{N}_{\geq 1}$ [or, if one considers the line bundle associated to such a divisor, to the operation of forming the n-th tensor power of the line bundle].

In some sense, the *main result* of the theory of present paper is the following:

> Under various technical conditions, the *functor* $\mathcal{C} \to \mathbb{F}_\Phi$ that determines the structure of $\mathcal{C}$ as a *Frobenioid* may be reconstructed purely *category-theoretically*, i.e., from the structure of $\mathcal{C}$ as a category [cf. Corollary 4.11].

These technical conditions are typically satisfied by Frobenioids that arise naturally from arithmetic geometry [cf. Theorems 6.2, 6.4]. Also, we observe that these technical conditions appear unlikely to be superfluous. Indeed, we also give various examples, involving Frobenioids which do *not* satisfy various of these technical conditions, of equivalences of categories with respect to which various portions of the functor $\mathcal{C} \to \mathbb{F}_\Phi$ are *not* preserved [cf. Examples 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 4.3].

Perhaps the most *fundamental example* of this phenomenon of "the intrinsic category-theoretic reconstruction of $\mathcal{C} \to \mathbb{F}_\Phi$ from $\mathcal{C}$" is the following. The prototype of a *base category* $\mathcal{D}$ is given by [the subcategory of connected objects of] a *Galois category*, i.e., a category in which the monoids of endomorphisms of objects have the structure of *finite groups*. On the other hand, the prototype of the *"non-base category portion"* of a Frobenioid, i.e., the "relative structure of $\mathcal{C}$ over $\mathcal{D}$", is given by the monoid "$\mathbb{F}$" [or, more generally, the monoids "$\mathbb{F}_M$"] discussed above. Then one central aspect of the phenomenon that "the relative structure of $\mathcal{C}$ over $\mathcal{D}$ is *never confused* with the structure of $\mathcal{D}$" is illustrated by the following easily verified observation:

> If G is a finite group, then any homomorphism of monoids $\mathbb{F} \to G$ factors through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$.

[We refer to Remark 3.1.2 for more details.] Note that this property *fails* to hold if, for instance, one replaces $\mathbb{F} = \mathbb{F}_{\mathbb{Z}_{\geq 0}}$ by $\mathbb{Z}_{\geq 0}$ [and the surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$ by the surjection $\mathbb{Z}_{\geq 0} \twoheadrightarrow \{0\}$]. Put another way, this property may be thought of as a consequence of the *non-abelian nature* of $\mathbb{F}$. In particular, if one thinks of the category-theoretic reconstructibility of the functor "$\mathcal{C} \to \mathbb{F}_\Phi$" as a sort of *rigidity*, then this property is vaguely reminiscent of the *"extraordinary rigidity"* asserted by Grothendieck in descriptions of his anabelian philosophy.

After defining and examining the first properties of Frobenioids in §1, we proceed to discuss, in §2, various versions of *"Frobenius functors"* on Frobenioids, which are intended as category-theoretic abstractions of the Frobenius morphism in positive characteristic algebraic geometry [cf. Remark 6.2.1]. In §3, we begin the category-theoretic reconstruction of the functor "$\mathcal{C} \to \mathbb{F}_\Phi$" by showing that, under certain conditions, the *base category* and *"Frobenius degree"* [i.e., in effect, the "$\mathbb{N}_{\geq 1}$ portion of $\mathbb{F}_\Phi$"] may be reconstructed category-theoretically [cf. Theorem 3.4]. In the theory of §3, we apply a certain purely category-theoretic technique, which we shall refer to as *"slim exponentiation"*; this technique is entirely independent of the theory of Frobenioids and is discussed in detail in the Appendix. In §4, we then complete the category-theoretic reconstruction of the functor "$\mathcal{C} \to \mathbb{F}_\Phi$" by showing that, under certain conditions, the *divisor monoid* Φ may also be reconstructed category-theoretically [cf. Theorem 4.9]. In §5, we study the extent to which, under certain conditions, one may write down *"explicit models"* of fairly general Frobenioids, in a fashion reminiscent of the explicit description of the elementary Frobenioid $\mathbb{F}_\Phi$ [cf. Theorem 5.2]. This study leads naturally to the investigation of various auxiliary structures on a Frobenioid, namely, *base sections* and *base-Frobenius pairs*, that may be used to relate a given Frobenioid satisfying certain conditions to such a *"model Frobenioid"* [cf. Theorem 5.2, (iv)].

One important technique in the theory of §3, §4, §5 is the operation of passing from a Frobenioid to the *perfection* or *realification* of the Frobenioid. Roughly speaking, from the point of view of the monoid $\mathbb{F} = \mathbb{F}_{\mathbb{Z}_{\geq 0}}$ introduced above, these operations correspond, respectively, to passing from "$\mathbb{Z}_{\geq 0}$" to the monoids "$\mathbb{Q}_{\geq 0}$" [in the case of the perfection] or "$\mathbb{R}_{\geq 0}$" [in the case of the realification]. Another important technique in this theory is the operation of passing to the *birationalization* of a Frobenioid. This may be thought of as a category-theoretic abstraction of the notion of "working with rational functions" in algebraic geometry; alternatively, from the point of view of the monoid $\mathbb{F} = \mathbb{F}_{\mathbb{Z}_{\geq 0}}$ introduced above, it may be thought of as corresponding to the operation of passing from "$\mathbb{Z}_{\geq 0}$" to the *groupification* "$\mathbb{Z}$" of $\mathbb{Z}_{\geq 0}$.

operation	*effect on* $\mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$
perfection	$\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{Q}_{\geq 0}$
realification	$\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{R}_{\geq 0}$
birationalization	$\mathbb{Z}_{\geq 0} \rightsquigarrow \mathbb{Z}$

Finally, in §6, we consider the main motivating examples of Frobenioids that arise from number fields and function fields. In particular, we observe in passing that this "Frobenioid-theoretic formulation of the elementary arithmetic of number fields" also gives rise to some interesting "Frobenioid-theoretic interpretations" of such classical results in number theory as the *Dirichlet unit theorem* and *Tchebotarev's density theorem*, as well as a result in transcendence theory due to *Lang* [cf. Theorem 6.4, (i), (iii), (iv)].

§I2. Abstract Combinatorialization of Arithmetic Geometry

From a somewhat more conceptual point of view, one *central theme* of the present paper is the goal of **"abstract combinatorialization of scheme-theoretic arithmetic geometry"**. Classical examples of this phenomenon of "abstract combinatorialization" may be seen in the theory of *Galois categories* or the theory of *monoids* in the geometry of log schemes [or, more classically, toric varieties]. That is to say, even if one starts by considering various finite étale coverings of schemes, the associated *Galois category* is a purely "abstract combinatorial" mathematical object that captures the "Galois structure" of the various coverings involved in a fashion that is entirely *independent of scheme theory.* In a similar vein, although a monoid of the sort that appears in log geometry arises as a submonoid of the multiplicative monoid determined by some commutative ring, the "abstract combinatorial" structure of such a monoid is sufficient to capture various essential properties [such as normality, etc.] of the ring structure of the ambient ring in a fashion that is entirely *independent of ring/scheme theory.*

A somewhat less classical example of this phenomenon of "abstract combinatorialization of scheme theory" is given by the theory of [Mzk8], where it is shown that very general *locally noetherian log schemes* may be "represented" by *categories*, in the sense that equivalences between such categories arise from uniquely determined isomorphisms of log schemes. The theory of [Mzk8] is generalized in [Mzk9] so as to take into account the *archimedean primes* of log schemes which are locally of finite type over a Zariski localization of [the ring of rational integers] $\mathbb{Z}$. As is discussed in the introduction to [Mzk8], this kind of result is motivated partly by the *anabelian philosophy* of Grothendieck, but perhaps more essentially by the idea that instead of working with *set-theoretic objects*, such as *schemes* or *log schemes*, one should regard *categories* — which may be thought of as "abstract combinatorial" mathematical objects constituted by some *abstract collection of arrows* — as the *"fundamental, primitive objects" of mathematics discourse*. Thus, Grothendieck's anabelian philosophy may be regarded as a *"special case"* of this point of view, i.e., the case where the categories in question are *Galois categories*

$$\begin{pmatrix} \text{abstract,} \\ \text{combinatorial} \\ \text{mathematical} \\ \text{objects} \end{pmatrix} \supseteq \begin{pmatrix} \text{categories} \\ \text{— i.e., abstract} \\ \text{collections} \\ \text{of arrows} \end{pmatrix} \supseteq \begin{pmatrix} \text{Galois} \\ \text{categories} \end{pmatrix}$$

— cf. the *"absolute anabelian geometry"* developed in [Mzk5], [Mzk6], [Mzk7], [Mzk10], [Mzk11], [Mzk12], [Mzk14].

One important drawback of the *"anabelian branch"* of this category-theoretic approach to mathematics is that although it is very well-suited to capturing essential aspects of the geometry of schemes at *nonarchimedean primes*, it is *ill-suited* to capturing the *archimedean aspects* of the geometry of schemes, and, in particular, those aspects of the *global geometry* of schemes over number fields — such as *heights* — that are of interest in Diophantine geometry. Thus, from this point of view, the extension given in [Mzk9] of the theory of [Mzk8] has the virtue, relative to anabelian geometry, of providing a natural way to incorporate such archimedean and global phenomena as the *global degree* of an arithmetic line bundle over a number field [cf. [Mzk9], Example 5.1] into the above-mentioned *category-theoretic approach to mathematics*.

The approach of [Mzk9], however, has the following *fundamental drawback*: The categories of [Mzk9] are quite *"large"* and *"complicated"* by comparison to Galois categories, in the sense that they include a *very diverse* collection of arithmetic schemes, by comparison to the finite étale coverings of a fixed scheme. This makes it relatively easy to reconstruct the original arithmetic log scheme from the category. On the other hand, this relative ease of reconstruction is a reflection, in essence, of the fact that the geometry of such categories is really *not so different from the conventional geometry of arithmetic log schemes*. Thus, in other words, one doesn't gain very much in the way of *essentially new geometric phenomena* by working with such categories, relative to the conventional geometry of arithmetic log schemes.

By contrast, the relatively simple structure of Galois categories [cf. also the categories of [Mzk13]] makes it much more difficult to reconstruct the scheme from the category — indeed, such a reconstruction is only possible in the case of very special *"anabelian"* schemes — but, on the other hand, this difficulty of reconstruction may be regarded as a reflection of the fact that there is indeed *some interesting new geometry* that arises from working with Galois categories that does not exist in the conventional geometry of schemes. Perhaps the most fundamental example of this phenomenon is the well-known fact that *the absolute Galois groups of non-isomorphic finite fields are isomorphic*. Another less elementary example of this phenomenon is the well-known fact that the Galois category associated to a *nonarchimedean mixed-characteristic local field* [i.e., a finite extension of the p-adic number field] admits self-equivalences [i.e., the associated absolute Galois group admits automorphisms] that do *not* arise from scheme theory [i.e., from an isomorphism of fields — cf., e.g., [NSW], p. 674].

Put another way, the difference between the *"geometry of categories"* — i.e., the approach to arithmetic geometry constituted by working with the strictly category-theoretic properties of categories — and the classical approach to arithmetic geometry constituted by working with *set-theoretic objects equipped with various complicated auxiliary structures* may be regarded as analogous to the difference between working with the notion of an *abstract group* and working with *groups of explicit matrices*. That is to say, working with strictly group-theoretic properties of abstract groups allows one to contemplate various structures that are *common* to various distinct groups of explicit matrices, but which are not so evident if one happens to be ignorant of the notion of an "abstract group" and hence obliged to restrict oneself to manipulations involving explicit matrices.

This state of affairs prompts the following *question*:

> Can one perhaps represent certain special arithmetic log schemes of interest by categories whose *"level of complexity" is closer to Galois categories* [i.e., substantially *simpler* than the categories of [Mzk9]] — thus allowing one to hope that the geometry of such categories exhibits *fundamentally new phenomena that do not appear in the conventional geometry of arithmetic log schemes* — on the one hand, but which nevertheless allow one *to work naturally with archimedean primes and heights* on the other?

This sort of question constituted one of the *principal motivations* for the author to develop the theory discussed in the present paper.

The answer to the above question constituted by the theory of present paper is, in a word, the *notion of a Frobenioid*. From the point of view of the question posed above:

> Frobenioids provide a *single framework* [cf. the notion of a "Galois category"; the role of monoids in log geometry] that allows one to capture the essential aspects of *both* the *Galois* and the *divisor* theory of *number fields*, on the one hand, and *function fields*, on the other, in such a way that one may continue to work with, for instance, *global degrees* of arithmetic line bundles on a number field, but which also exhibits the new phenomenon [not present in the classical theory of number fields] of a *"Frobenius endomorphism"* of the Frobenioid associated to a number field.

Here, we remark that the *base category* $\mathcal{D}$ is typically a category that is of a *level of "simplicity"* [cf. the above discussion] that is reminiscent of a *Galois category* [cf. also the *"temperoids"* of [Mzk11]; the *categories of Riemann surfaces* discussed in [Mzk13], §2]. Indeed, in the examples of §6, the base category is [the subcategory of connected objects of] a Galois category. From this point of view, the *main ingredients* of a Frobenioid — that is to say, roughly speaking, *"Galois"* [i.e., the base category $\mathcal{D}$], *"Frobenius"* [i.e., "$\mathbb{N}_{\geq 1}$"], and *"metrics/integral structures"* [i.e., the family of monoids Φ] — are reminiscent of the theory of the *"ring of p-adic periods"* $\mathbb{B}_{\mathrm{crys}}$ of p-adic Hodge theory.

§I3. Frobenius Endomorphisms of a Number Field

From a somewhat less conceptual point of view, one of the main motivations for the author in developing the theory of Frobenioids came from the long-term goal of developing a sort of *arithmetic Teichmüller theory* for *number fields equipped with an elliptic curve*, in a fashion that is analogous to the p-adic Teichmüller theory of [Mzk1], [Mzk2]. That is to say, here one wishes to regard number fields as corresponding to hyperbolic curves over finite fields and elliptic curves [over a number field] as corresponding to the nilpotent ordinary indigenous bundles [on a hyperbolic curve over a finite field] of [Mzk1], [Mzk2].

In the p-adic Teichmüller theory of [Mzk1], [Mzk2], certain *canonical Frobenius liftings* play a central role. Thus, since Frobenius liftings are, literally, liftings of the Frobenius morphism in positive characteristic, in order to develop an "arithmetic Teichmüller theory" for number fields equipped with an elliptic curve, one must first have an *analogue for number fields* of the Frobenius morphism in positive characteristic scheme theory. If one starts to consider such an analogue from a completely naive point of view, then one must contend with the fact that, if, for instance, $n \geq 2$ is a integer, then the morphism

$$p \mapsto p^n$$

[where p is a prime number] clearly does not extend to a ring homomorphism $\mathbb{Z} \to \mathbb{Z}$! That is to say, it is difficult to see how to accommodate such a "Frobenius morphism for number fields" within the framework of scheme theory.

On the other hand, if one works with *monoids* as in the theory of log schemes, then such a morphism "$p \mapsto p^n$" does indeed make sense. Moreover, even if, for instance, one considers roots π of p, the mapping $\pi \mapsto \pi^n$ is *Galois-equivariant*. Thus, in summary:

> One important motivation for the author in developing the theory of Frobenioids was the goal of developing a geometric framework — i.e., roughly speaking, a geometry built up solely from *"Galois theory"* and *"monoids"* — in which a *"Frobenius morphism on number fields"* may be constructed.

Once one has constructed such a "Frobenius morphism on number fields", the next step to realizing an "arithmetic Teichmüller theory" consists of constructing a *"canonical Frobenius lifting"*. Although the construction of such "canonical Frobenius liftings" lies [well!] beyond the scope of the present paper, we remark that the ideas that lie behind such a construction are motivated by the [scheme-theoretic!] *Hodge-Arakelov theory of elliptic curves* surveyed in [Mzk3], [Mzk4], a theory in which the *theta function on a Tate curve* plays a central role. In particular, in order to construct "canonical Frobenius liftings", it is necessary to *extract* the essential *"abstract, combinatorial content"* of the scheme-theoretically formulated Hodge-Arakelov theory of [Mzk3], [Mzk4]. In fact, certain aspects of such an "extraction process" are achieved precisely by applying the theory of Frobenioids, as is done in a certain sequel to the present paper and [Mzk15] — namely, [Mzk16].

Here, we pause to observe that to pass from the geometry of schemes to, say, the geometry of Frobenioids amounts to a certain *"partial dismantling of scheme theory"*, i.e., to *"forgetting"* a certain portion of scheme theory. As discussed above, one wants to execute such a "partial dismantling of scheme theory" precisely in order to *allow* the construction of such objects as a *"Frobenius morphism on number fields"* which are not possible within the framework of scheme theory. On the other hand, if the dismantling process that one executes is *too drastic*, then there is a *danger of destroying so much of the geometry of scheme theory* that one is not left with a geometry that is sufficiently rich so as to allow the further development of the theory. From this point of view, one of the main themes of the present paper [and [Mzk15]] consists of verifying that:

> The geometry of Frobenioids *retains a substantial portion of the geometry of scheme theory* and, in particular, is sufficiently rich so as to permit the execution of many geometric constructions and arguments familiar from scheme theory.

The *centerpiece* of this verification process is the *reconstruction of the functor "$\mathcal{C} \to \mathbb{F}_\Phi$"*, as discussed in §I1. Another aspect of this verification process, which may be seen throughout the theory of the present paper, is the *step-by-step translation* of various scheme-theoretic terms and constructions that appear in the theory of divisors and line bundles on models of finite separable extensions of a given function field or number field into *purely category-theoretic language*. For instance, one important example of this "step-by-step translation" is the theory of *base sections* and *base-Frobenius pairs* developed in §5, which may be thought of as a sort of *category-theoretic translation* of the notion of the *tautological section of a trivial line bundle* [cf. Remark 5.6.1].

§I4. Étale-like vs. Frobenius-like Structures

Finally, let us return to the *"main result"* discussed in §I1, i.e., the reconstruction of the functor "$\mathcal{C} \to \mathbb{F}_\Phi$". One way to think about this result is that it is a statement to the effect that:

> The structure of a ["permissible"] *base category* $\mathcal{D}$ [e.g., the subcategory of connected objects of a *Galois category*] is **fundamentally combinatorially different** — indeed, different in a *category-theoretically distinguishable fashion* — from the structure of the *"Frobenius portion"* $\mathbb{F}$ of a Frobenioid.

This phenomenon may be thought of as a sort of *fundamental dichotomy between* **types of combinatorial structures** — i.e., between *"étale-like"* structures which are *"indifferent to order"* [cf. the finite groups that as appear as Galois groups in a Galois category] and *"Frobenius-like"* structures which are *"order-conscious"* [cf. the monoids "$\mathbb{Z}_{\geq 0}$", "$\mathbb{N}_{\geq 1}$" that constitute the standard Frobenioid $\mathbb{F}$]. One may also think of "étale-like" structures as *"descent-compatible"* structures, whereas "Frobenius-like" structures are *"descent-incompatible"*, in the sense that compatibility with "descent" may be thought of as a sort of *violation* of the "order" constituted by "the object *upstairs*" in the descent operation and the "the object *downstairs*". Relative to the theme of "abstract combinatorialization" discussed in §I2, the point here is that the difference between "étale-like" and "Frobenius-like" structures is an **intrinsic structural difference**, not just a matter of *"arbitrarily imposed labels motivated by scheme theory"* [such as "base category", "divisor monoid", "Frobenius degree", etc.]! For more on this fundamental dichotomy between *"étale-like"* and *"Frobenius-like"* categorical structures, we refer to Remark 3.1.3.

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Section 0: Notations and Conventions

Sets:

If E is a *partially ordered set*, then we shall denote by

$$\mathrm{Order}(E)$$

the category whose *objects* are elements $e \in E$, and whose *morphisms* $e_1 \to e_2$ [where $e_1, e_2 \in E$] are the relations $e_1 \leq e_2$.

Numbers:

We denote by

$$\mathbb{N}_{\geq 1}$$

the *[discrete] multiplicative monoid* of rational integers ≥ 1 and by

$$\mathfrak{Primes}$$

the set of *prime numbers*. Thus, one may think of $\mathbb{N}_{\geq 1}$ as the *free commutative monoid* generated by $\mathfrak{Primes}$.

We shall write:

$$\mathbb{R}_{>0} \stackrel{\mathrm{def}}{=} \{a \in \mathbb{R} \mid a > 0\} \subseteq \mathbb{R}_{\geq 0} \stackrel{\mathrm{def}}{=} \{a \in \mathbb{R} \mid a \geq 0\} \subseteq \mathbb{R}$$

We shall refer to an element

$$\Lambda \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$$

as a *monoid type* and write $\Lambda_{>0} \stackrel{\mathrm{def}}{=} \Lambda \bigcap \mathbb{R}_{>0} \subseteq \mathbb{R}$, $\Lambda_{\geq 0} \stackrel{\mathrm{def}}{=} \Lambda \bigcap \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$, $\mathbb{N} \stackrel{\mathrm{def}}{=} \mathbb{Z}_{\geq 0}$. Also, we shall refer to a monoid isomorphic to [the *additive* monoid] $\Lambda_{\geq 0}$ as a Λ-*monoprime monoid* and to a monoid which is a Λ-monoprime monoid for some Λ as *monoprime*. If M is a $\mathbb{Q}$-monoprime monoid, then we shall write

$$M \otimes \mathbb{R}_{\geq 0}$$

for the $\mathbb{R}$-monoprime monoid obtained by *completing* M relative to the topology defined by the ordering on the monoid M.

We shall refer to as a *number field* any finite extension of the field of rational numbers.

Monoids:

Observe that any [not necessarily commutative!] *monoid* M may be thought of as a special type of *category*, i.e., the category with precisely one object whose endomorphisms are given by the monoid M.

Write $\mathfrak{Mon}$ for the *category of commutative monoids* [relative to some universe fixed throughout the discussion]. Let M be an object of $\mathfrak{Mon}$; the monoid operation of M will be written *additively*. We shall denote by

$$M^{\pm} \subseteq M$$

the submonoid [which, in fact, forms a group] of *invertible elements* of M, by

$$M \twoheadrightarrow M^{\text{char}} \stackrel{\text{def}}{=} M/M^{\pm}$$

the quotient monoid of M by $M^{\pm}$, which we shall refer to as the *characteristic* of M, and by

$$M \to M^{\text{gp}}$$

the natural homomorphism from M to its *groupification* M^{gp}. Thus, M^{gp} is the monoid [which is, in fact, a group] given by the set of equivalence classes of pairs $(a, b) \in M \times M$, where two such pairs (a_1, b_1); (a_2, b_2) are considered equivalent if $a_1 + b_2 + c = b_1 + a_2 + c$, for some $c \in M$, and the monoid operation on this set is the monoid operation induced by the monoid operation of M. We shall say that M is *torsion-free* if M has no torsion elements; we shall say that M is *sharp* if $M^{\pm} = 0$; we shall say that M is *integral* if the natural map $M \to M^{\text{gp}}$ is injective; we shall say that M is *saturated* if every $a \in M^{\text{gp}}$ for which $n \cdot a$ lies in the image of M for some $n \in \mathbb{N}_{\geq 1}$ lies in the image of M.

Denote by

$$M^{\text{pf}}$$

the *perfection* of M, that is to say, the *inductive limit* of the inductive system I_* of monoids

$$\ldots \to M \xrightarrow{n\cdot} M \to \ldots$$

given by assigning to each element of $a \in \mathbb{N}_{\geq 1}$ a copy of M, which we denote by I_a, and to every two elements $a, b \in M$ such that a divides b the morphism $I_a = M \to I_b = M$ given by multiplication by $n \stackrel{\text{def}}{=} b/a$. Thus, the object I_1 of the inductive system I_* determines a *natural morphism*

$$M \to M^{\text{pf}}$$

which is *injective* if M is *torsion-free*, *integral*, and *saturated*, hence, in particular, if M is *sharp*, *integral*, and *saturated*. We shall say that M is *perfect* if multiplication by any element of $\mathbb{N}_{\geq 1}$ on M is bijective. Thus, M^{pf} is always perfect; M is perfect if and only if the natural map $M \to M^{\text{pf}}$ is an isomorphism.

Note that M is *saturated* if and only if M^{char} is. We shall say that M is *of characteristic type* if the fibers of the natural map $M \to M^{\text{char}}$ are torsors over $M^{\pm}$. Note that if M is of characteristic type, then M is *integral* if and only if M^{char} is. If $\phi : M_1 \to M_2$ is a morphism of $\mathfrak{Mon}$, then we shall say that ϕ is *characteristically injective* if ϕ is injective, and, moreover, the morphism $M_1^{\text{char}} \to M_2^{\text{char}}$ induced by ϕ is injective.

Now suppose that M is *sharp*, *integral*, and *saturated*. If $a, b \in M$, then we shall write

$$a \leq b$$

if $\exists\ c \in M$ such that $a + c = b$ and

$$a \preccurlyeq b$$

if $\exists n \in \mathbb{N}_{\geq 1}$ such that $a \leq n \cdot b$. If a subset $S \subseteq M$ satisfies the property that there exists a $b \in M$ such that $a \leq b$ for all $a \in S$, then we shall say that S is *bounded* [by b]. If $S \subseteq M$ is a subset and $b \in M$, then we shall write

$$\mathrm{Bound}_S(b) \stackrel{\mathrm{def}}{=} \{a \in S \mid a \leq b\}$$

[i.e., $\mathrm{Bound}_S(b)$ is the maximal subset of S that is bounded by b]. Observe that if M is $\mathbb{R}$-monoprime, then every bounded subset $S \subseteq M$ possesses a [unique] *supremum*

$$\sup(S) \in M$$

[i.e., S is bounded by b if and only if $b \geq \sup(S)$]. We shall say that $0 \neq a \in M$ is *irreducible* if any equation $a = b + c$ in M, where $b, c \in M$, implies that $b = 0$ or $c = 0$. We shall say that $0 \neq a \in M$ is *primary* if for any $M \ni b \preccurlyeq a$, where $b \neq 0$, it holds that $a \preccurlyeq b$. Denote by $\mathrm{Primary}(M)$ the set of primary elements of M. One verifies immediately that the relation "$a \preccurlyeq b$" [where $a, b \in \mathrm{Primary}(M)$] determines an *equivalence relation* on $\mathrm{Primary}(M)$. A $\preccurlyeq$-equivalence class of elements of $\mathrm{Primary}(M)$ will be referred to as a *prime* of M. [Note that this notion of a "prime" differs from the conventional notion of a "prime ideal" of M.] Denote by

$$\mathrm{Prime}(M)$$

the *set of primes* of M. If $\mathfrak{p} \in \mathrm{Prime}(M)$, then we shall denote by

$$M_{\mathfrak{p}} \subseteq M$$

the submonoid generated by the elements contained in the subset $\mathfrak{p} \subseteq M$. Note that each subset $\mathfrak{p} \subseteq M$, where $\mathfrak{p} \in \mathrm{Prime}(M)$, is closed under multiplication by elements of $\mathbb{N}_{\geq 1}$, and that

$$\mathrm{Primary}(M^{\mathrm{pf}}) = \{a \in M^{\mathrm{pf}} \mid \exists n \in \mathbb{N}_{\geq 1} \text{ such that } n \cdot a \in \mathrm{Primary}(M)\}$$

$$\mathrm{Primary}(M) = \mathrm{Primary}(M^{\mathrm{pf}}) \bigcap M$$

$$\mathrm{Prime}(M) \xrightarrow{\sim} \mathrm{Prime}(M^{\mathrm{pf}})$$

[where we regard M as a subset of M^{pf} via the natural inclusion]. Finally, we observe that the relation "$\leq$" on elements of M determines a *category*

$$\mathrm{Order}(M)$$

[via the *partially ordered set* structure on M determined by "$\leq$"].

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G.

If Π is a *profinite group*, then we shall write

$$\mathcal{B}(\Pi)$$

for the *category* whose *objects* are *finite* sets equipped with a continuous Π-action and whose *morphisms* are morphisms of Π-sets. Thus, $\mathcal{B}(\Pi)$ is a *Galois category*, or, in the terminology of [Mzk7], a *connected anabelioid*. If $Z_\Pi(H) = \{1\}$ for every open subgroup $H \subseteq \Pi$, then we shall say that Π is *slim*.

Categories:

Let $\mathcal{C}$ be a *category*. We shall denote the collection of *objects* (respectively, *arrows*) of $\mathcal{C}$ by:

$$\mathrm{Ob}(\mathcal{C}) \text{ (respectively, } \mathrm{Arr}(\mathcal{C}))$$

The *opposite category* to $\mathcal{C}$ will be denoted by $\mathcal{C}^{\text{opp}}$. A category with precisely one object will be referred to as a *one-object category*; a category with precisely one morphism [which is necessarily the identity morphism of the unique object of such a category] will be referred to as a *one-morphism category*. Thus, a one-morphism category is always a one-object category.

If $A \in \mathrm{Ob}(\mathcal{C})$ is an *object* of $\mathcal{C}$, then we shall denote by

$$\mathcal{C}_A$$

the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose *morphisms* [from an object $B_1 \to A$ to an object $B_2 \to A$] are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$ and by

$$_A\mathcal{C}$$

the category whose *objects* are morphisms $A \to B$ of $\mathcal{C}$ and whose morphisms (from an object $A \to B_1$ to an object $A \to B_2$) are morphisms $B_1 \to B_2$ in $\mathcal{C}$ that are compatible with the given arrows $A \to B_1$, $A \to B_2$. Thus, we have a *natural functor*

$$(j_A)_! : \mathcal{C}_A \to \mathcal{C}$$

[given by forgetting the structure morphism to A]. Similarly, if $f : A \to B$ is a *morphism* in $\mathcal{C}$, then f defines a *natural functor*

$$f_! : \mathcal{C}_A \to \mathcal{C}_B$$

by mapping an arrow [i.e., an object of $\mathcal{C}_A$] $C \to A$ to the object of $\mathcal{C}_B$ given by the composite $C \to A \to B$ with f. We shall call an object $A \in \text{Ob}(\mathcal{C})$ *terminal* (respectively, *pseudo-terminal*) if for every object $B \in \text{Ob}(\mathcal{C})$, there exists a unique arrow (respectively, there exists a [not necessarily unique!] arrow) $B \to A$ in $\mathcal{C}$.

We shall say that two arrows of a category are *co-objective* if their domains and codomains coincide.

We shall say that an arrow $\beta : B \to A$ of a category $\mathcal{C}$ is *fiberwise-surjective* if, for every arrow $\gamma : C \to A$ of $\mathcal{C}$, there exist arrows $\delta_B : D \to B$, $\delta_C : D \to C$ such that $\beta \circ \delta_B = \gamma \circ \delta_C$. An arrow of a category which is a fiberwise-surjective monomorphism will be referred to as an *FSM-morphism*. One verifies immediately that every composite of FSM-morphisms is again an FSM-morphism. A category $\mathcal{C}$ which satisfies the property that every FSM-morphism of $\mathcal{C}$ is, in fact, an isomorphism will be referred to as a *category of FSM-type*.

Let $\mathcal{C}$ be a category; $A \in \text{Ob}(\mathcal{C})$. Write

$$\text{End}_{\mathcal{C}}(A); \quad \text{Aut}_{\mathcal{C}}(A)$$

for the monoids of endomorphisms and automorphisms of A in $\mathcal{C}$, respectively. We shall say that an endomorphism $\alpha \in \text{End}_{\mathcal{C}}(A)$ of $\mathcal{C}$ is a *sub-automorphism* if there exists an arrow $\phi : B \to A$ of $\mathcal{C}$ and an automorphism $\beta \in \text{Aut}_{\mathcal{C}}(B)$ such that $\phi \circ \beta = \alpha \circ \phi$; write

$$(\text{Aut}_{\mathcal{C}}(A) \subseteq)\ \text{Aut}^{\text{sub}}_{\mathcal{C}}(A) \subseteq \text{End}_{\mathcal{C}}(A)$$

for the subset of $\text{End}_{\mathcal{C}}(A)$ determined by the sub-automorphisms of A. We shall say that A is Aut-*saturated* (respectively, Aut^{sub}-*saturated*; *of* Aut-*type*) if $\text{Aut}_{\mathcal{C}}(A) = \text{Aut}^{\text{sub}}_{\mathcal{C}}(A)$ (respectively, $\text{Aut}^{\text{sub}}_{\mathcal{C}}(A) = \text{End}_{\mathcal{C}}(A)$; $\text{Aut}_{\mathcal{C}}(A) = \text{End}_{\mathcal{C}}(A)$). If every object of $\mathcal{C}$ is Aut-saturated (respectively, Aut^{sub}-saturated; of Aut-type), then we shall say that $\mathcal{C}$ is Aut-*saturated* (respectively, Aut^{sub}-*saturated*; *of* Aut-*type*). We shall say that an arrow $A \to B$ of $\mathcal{C}$ is an End-*equivalence* if there exists an arrow $B \to A$ in $\mathcal{C}$.

We shall refer to a *natural transformation* between functors all of whose component morphisms are *isomorphisms* as an *isomorphism between the functors* in question. If $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor between categories $\mathcal{C}_1$, $\mathcal{C}_2$, then we shall denote by $\text{Aut}(\phi)$ — or, when there is no fear of confusion,

$$\text{Aut}(\mathcal{C}_1 \to \mathcal{C}_2)$$

— the group of automorphisms of the functor ϕ, and by $\text{End}(\phi)$ — or, when there is no fear of confusion,

$$\text{End}(\mathcal{C}_1 \to \mathcal{C}_2)$$

— the monoid of natural transformations from the functor ϕ to itself. We shall say that ϕ is *rigid* if $\text{Aut}(\phi)$ is trivial. A category $\mathcal{C}$ will be called *slim* if the natural functor $\mathcal{C}_A \to \mathcal{C}$ is *rigid*, for every $A \in \text{Ob}(\mathcal{C})$. We recall that if Π is a profinite group, then Π is *slim* if and only if the category $\mathcal{B}(\Pi)$ is slim [cf. [Mzk7], Corollary 1.1.6].

A diagram of functors between categories will be called 1-*commutative* if the various composite functors in question are *isomorphic*. When such a diagram "commutes in the literal sense" we shall say that it 0-*commutes*. Note that when a diagram in which the various composite functors are all *rigid* "1-commutes", it follows from the *rigidity* hypothesis that any isomorphism between the composite functors in question is necessarily *unique*. Thus, to state that such a diagram 1-commutes does not result in any "loss of information" by comparison to the datum of a *specific isomorphism* between the various composites in question.

A category $\mathcal{C}$ will be called a *skeleton* if any two isomorphic objects of $\mathcal{C}$ are, in fact, equal. A *skeletal subcategory* of a category $\mathcal{C}$ is a full subcategory $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{S}$ is a skeleton, and, moreover, the inclusion functor $\mathcal{S} \hookrightarrow \mathcal{C}$ is an *equivalence of categories*.

We shall say that a nonempty [i.e., non-initial] object $A \in \mathrm{Ob}(\mathcal{C})$ is *connected* if it is not isomorphic to the coproduct of two nonempty objects of $\mathcal{C}$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *mobile* if there exists an object $B \in \mathrm{Ob}(\mathcal{C})$ such that the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has *cardinality* ≥ 2 [i.e., the diagonal from this set to the product of this set with itself is not bijective]. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *quasi-connected* if it is either *immobile* [i.e., not mobile] or *connected*. Thus, connected objects are always quasi-connected. We shall say that a category $\mathcal{C}$ is *totally* (respectively, *almost totally*) *epimorphic* if every morphism in $\mathcal{C}$ whose domain is *arbitrary* (respectively, *nonempty*) and whose codomain is *arbitrary* (respectively, *connected*) is an *epimorphism*.

We shall say that $\mathcal{C}$ is *of finitely (respectively, countably) connected type* if it is closed under formation of finite (respectively, countable) coproducts; every object of $\mathcal{C}$ is a coproduct of a finite (respectively, countable) collection of connected objects; and, moreover, all finite (respectively, countable) coproducts $\coprod A_i$ in the category satisfy the condition that the natural map

$$\coprod \mathrm{Hom}_{\mathcal{C}}(B, A_i) \to \mathrm{Hom}_{\mathcal{C}}(B, \coprod A_i)$$

is *bijective*, for all connected $B \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is *of finitely or countably connected type*, then every nonempty object of $\mathcal{C}$ is *mobile*; in particular, a nonempty object of $\mathcal{C}$ is connected *if and only if* it is quasi-connected.

If a *mobile* object $A \in \mathrm{Ob}(\mathcal{C})$ satisfies the condition that every morphism in $\mathcal{C}$ whose domain is nonempty and whose codomain is A is an *epimorphism*, then A is *connected*. [Indeed, $C_1 \coprod C_2 \xrightarrow{\sim} A$, where C_1, C_2 are *nonempty*, implies that the composite map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(A, B) &\hookrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, B) \times \mathrm{Hom}_{\mathcal{C}}(C_2, B) \\ = \mathrm{Hom}_{\mathcal{C}}(C_1 \coprod C_2, B) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, B)\end{aligned}$$

is *bijective*, for all $B \in \mathrm{Ob}(\mathcal{C})$.] In particular, it follows that if $\mathcal{C}$ is a *totally epimorphic category*, then every object of $\mathcal{C}$ is *quasi-connected*.

If $\mathcal{C}$ is a *category of finitely or countably connected type*, then we shall write

$$\mathcal{C}^0 \subseteq \mathcal{C}$$

for the *full subcategory* of connected objects. [Note, however, that in general, objects of $\mathcal{C}^0$ are *not necessarily connected* — or even *quasi-connected* — as objects of $\mathcal{C}^0$!] On the other hand, if, in addition, $\mathcal{C}$ is *almost totally epimorphic*, then $\mathcal{C}^0$ is *totally epimorphic* [so every object of $\mathcal{C}$ is *quasi-connected*].

If $\mathcal{C}$ is a *category*, then we shall write

$$\mathcal{C}^{\perp} \text{ (respectively, } \mathcal{C}^{\top})$$

for the category formed from $\mathcal{C}$ by taking *arbitrary "formal" [possibly empty] finite (respectively, countable) coproducts* of objects in $\mathcal{C}$. That is to say, we define the "Hom" of $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) by the following formula:

$$\mathrm{Hom}(\coprod_i A_i, \coprod_j B_j) \stackrel{\mathrm{def}}{=} \prod_i \coprod_j \mathrm{Hom}_{\mathcal{C}}(A_i, B_j)$$

[where the A_i, B_j are objects of $\mathcal{C}$]. Thus, $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) is a *category of finitely (respectively, countably) connected type*. Note that objects of $\mathcal{C}$ define *connected* objects of $\mathcal{C}^{\perp}$ or $\mathcal{C}^{\top}$. Moreover, there are natural [up to isomorphism] *equivalences of categories*

$$(\mathcal{C}^{\perp})^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{C}^{\top})^0 \xrightarrow{\sim} \mathcal{C}; \quad (\mathcal{D}^0)^{\perp} \xrightarrow{\sim} \mathcal{D}; \quad (\mathcal{E}^0)^{\top} \xrightarrow{\sim} \mathcal{E}$$

for $\mathcal{D}$ (respectively, $\mathcal{E}$) a *category of finitely connected type* (respectively, *category of countably connected type*). If $\mathcal{C}$ is a *totally epimorphic category*, then $\mathcal{C}^{\perp}$ (respectively, $\mathcal{C}^{\top}$) is an *almost totally epimorphic category of finitely (respectively, countably) connected type.*

In particular, the operations "0", "$\perp$" (respectively, "$\top$") define *one-to-one correspondences* [up to equivalence] between the *totally epimorphic categories* and the *almost totally epimorphic categories of finitely (respectively, countably) connected type.*

We observe in passing that if $\mathcal{C}$ is a *totally epimorphic category*, and $\alpha \circ \beta$ [where $\alpha, \beta \in \mathrm{Arr}(\mathcal{C})$] is an isomorphism, then α, β are *isomorphisms.*

If $\mathcal{C}$ is a *[small] category*, then we shall write $\mathbb{G}(\mathcal{C})$ for the *graph associated to* $\mathcal{C}$. This graph is the graph with precisely one *vertex* for each object of $\mathcal{C}$ and precisely one *edge* for each arrow of $\mathcal{C}$ [joining the vertices corresponding to the domain and codomain of the arrow]. We shall refer to the full subcategories of $\mathcal{C}$ determined by the objects and arrows that compose a connected component of the graph $\mathbb{G}(\mathcal{C})$ as a *connected component* of $\mathcal{C}$. In particular, we shall say that $\mathcal{C}$ is *connected* if $\mathbb{G}(\mathcal{C})$ is connected. [Note that by working with respect to some "sufficiently large" enveloping *universe*, it makes sense to speak of a category which is not necessarily small at being *connected.*]

Given two arrows $f_i : A_i \to B_i$ (where $i = 1, 2$) in a category $\mathcal{C}$, we shall refer to a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\sim} & A_2 \\ \downarrow f_1 & & \downarrow f_2 \\ B_1 & \xrightarrow{\sim} & B_2 \end{array}$$

— where the horizontal arrows are isomorphisms in $\mathcal{C}$ — as an *abstract equivalence* from f_1 to f_2. If there exists an abstract equivalence from f_1 to f_2, then we shall say that f_1, f_2 are *abstractly equivalent*.

If $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ are *categories*, and

$$\Phi_1 : \mathcal{C}_1 \to \mathcal{D}; \quad \Phi_2 : \mathcal{C}_2 \to \mathcal{D}$$

are *functors*, then we define the *"CFP"* — i.e., *"categorical fiber product"* —

$$\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2$$

of $\mathcal{C}_1$*,* $\mathcal{C}_2$ *over* $\mathcal{D}$ to be the *category* whose *objects* are triples

$$(A_1, A_2, \alpha : \Phi_1(A_1) \xrightarrow{\sim} \Phi_2(A_2))$$

where $A_i \in \mathrm{Ob}(\mathcal{C}_i)$ (for $i = 1, 2$); α is an isomorphism of $\mathcal{D}$; and whose *morphisms*

$$(A_1, A_2, \alpha : \Phi_1(A_1) \xrightarrow{\sim} \Phi_2(A_2)) \to (B_1, B_2, \beta : \Phi_1(B_1) \xrightarrow{\sim} \Phi_2(B_2))$$

are pairs of morphisms $\gamma_i : A_i \to B_i$ [in $\mathcal{C}_i$, for $i = 1, 2$] such that $\beta \circ \Phi_1(\gamma_1) = \Phi_2(\gamma_2) \circ \alpha$. One verifies easily that if Φ_2 is an *equivalence*, then the *natural projection functor*

$$\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \to \mathcal{C}_1$$

is also an *equivalence*.

Let $\mathcal{C}$ be a *category*; $\mathcal{S}$ a *collection of arrows in* $\mathcal{C}$; $\phi \in \mathrm{Arr}(\mathcal{C})$. Then we shall say that ϕ is *minimal-adjoint to* $\mathcal{S}$ (respectively, *minimal-coadjoint to* $\mathcal{S}$; *mid-adjoint to* $\mathcal{S}$) if every factorization $\phi = \alpha \circ \beta$ (respectively, $\phi = \beta \circ \alpha$; $\phi = \alpha \circ \beta \circ \gamma$) of ϕ in $\mathcal{C}$ such that β lies in $\mathcal{S}$ satisfies the property that β is, in fact, an *isomorphism*. If ϕ admits a factorization $\phi = \alpha \circ \beta \circ \gamma$ in $\mathcal{C}$, then we shall say that β is *subordinate* to ϕ. If ϕ is not an isomorphism, but, for every factorization $\phi = \alpha \circ \beta$ in $\mathcal{C}$, it holds that either α or β is an isomorphism, then we shall say that ϕ is *irreducible*. We shall refer to an FSM-morphism which is irreducible as an *FSMI-morphism*. Thus, a category of FSM-type does not contain any FSMI-morphisms.

We shall say that a category $\mathcal{C}$ is *of FSMFF-type* [i.e., "FSM-finitely factorizable type"] if the following two conditions hold: (a) every FSM-morphism of $\mathcal{C}$ which is not an isomorphism factors as a composite of finitely many FSMI-morphisms; (b) for every $A \in \mathrm{Ob}(\mathcal{C})$, there exists a natural number N such that for every composite

$$\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_2 \circ \phi_1$$

of FSMI-morphisms $\phi_1, \ldots, \phi_n$ such that the domain of ϕ_1 is equal to A, it holds that $n \leq N$. Thus, if $\mathcal{C}$ is of FSM-type, then it is of FSMFF-type. Also, we observe that [by condition (b)] no *endomorphism* of an object of a category of FSMFF-type is an FSMI-morphism.

If $\mathcal{C}$ is a *totally epimorphic category*, $A \in \mathrm{Ob}(\mathcal{C})$, and $G \subseteq \mathrm{Aut}_{\mathcal{C}}(A)$ is a *subgroup*, then we shall say that an arrow $\phi : A \to B$ of $\mathcal{C}$ is a *categorical quotient of* A *by* G if the following conditions hold: (a) $\phi \circ \gamma = \phi$, for all $\gamma \in G$; (b) for every morphism $\psi : A \to C$ such that $\psi \circ \gamma = \psi$ for all $\gamma \in G$, there exists a unique morphism $\psi' : B \to C$ such that $\psi = \psi' \circ \phi$. If $\phi : A \to B$ is a categorical quotient of A by G, then we shall say that $A \to B$ is *mono-minimal* if the following condition holds: For every factorization $\phi = \phi' \circ \zeta$, where $\zeta : A \to A'$ is a *monomorphism* such that there exists a *subgroup* $G' \subseteq \mathrm{Aut}_{\mathcal{C}}(A')$, together with an isomorphism $G \xrightarrow{\sim} G'$ that is *compatible*, relative to ζ, with the respective actions of G, G' on A, A' [which implies, by *total epimorphicity*, that $\phi' : A' \to B$ is a categorical quotient of A' by G'], it holds that ζ is an isomorphism. Thus, [by total epimorphicity] it follows that an isomorphism is always a *mono-minimal categorical quotient* of its domain by the trivial group.

If $\mathcal{C}$ is a *category*, then we shall say that $A \in \mathrm{Ob}(\mathcal{C})$ is an *anchor* if there only exist finitely many isomorphism classes of objects of ${}_A\mathcal{C}$ that arise from irreducible arrows $A \to B$. We shall say that $A \in \mathrm{Ob}(\mathcal{C})$ is a *subanchor* if there exists an arrow $A \to B$, where B is an anchor. If $\mathcal{C}$ is a *totally epimorphic category*, then we shall say that $A \in \mathrm{Ob}(\mathcal{C})$ is an *iso-subanchor* if there exist a subanchor $B \in \mathrm{Ob}(\mathcal{C})$, a subgroup $G \subseteq \mathrm{Aut}_{\mathcal{C}}(B)$, and a morphism $B \to A$ [in $\mathcal{C}$] which is a *mono-minimal categorical quotient* of B by G.

Section 1: Definitions and First Properties

In the present §1, we discuss the notion of a *Frobenioid*, which may be thought of as a category whose internal structure behaves roughly like that of an *"elementary Frobenioid"*. An "elementary Frobenioid" is, in essence, a sort of semi-direct product of the multiplicative monoid $\mathbb{N}_{\geq 1}$ [which is to be thought of as a *"Frobenius action"*] with a system of *monoids* [which are roughly of the sort that appear in the theory of log structures] on a *"base category"* [a category which behaves roughly like a *Galois category*].

We begin by introducing the fundamental notions of *"elementary Frobenioids"* and *"pre-Frobenioids"*.

Definition 1.1.

(i) We shall say that $M \in \mathrm{Ob}(\mathfrak{Mon})$ is *pre-divisorial* if it is *integral* [cf. §0], *saturated* [cf. §0], and *of characteristic type* [cf. §0]. Suppose that M is pre-divisorial. Then we shall say that M is *group-like* if M^{char} is zero; we shall say that M is *divisorial* if M is sharp [cf. §0]. [Thus, if M is pre-divisorial, then M^{char} is divisorial.] If α is an endomorphism of a pre-divisorial monoid $M \in \mathrm{Ob}(\mathfrak{Mon})$, then we shall say that α is *non-dilating* if the endomorphism α^{char} of M^{char} induced by α is the identity endomorphism of M^{char} whenever $\alpha^{\mathrm{char}}(a) \preccurlyeq a$ for all *primary* [cf. §0] $a \in M^{\mathrm{char}}$.

(ii) Let $\mathcal{D}$ be a *category*. Then we shall refer to a contravariant functor

$$\Phi : \mathcal{D} \to \mathfrak{Mon}$$

as a *monoid on* $\mathcal{D}$ if the following conditions are satisfied: (a) every morphism of monoids $\alpha^* : \Phi(A) \to \Phi(B)$ induced by a morphism $\alpha : B \to A$ of $\mathcal{D}$ is *characteristically injective* [cf. §0]; (b) if α is an *FSM-morphism* [cf. §0] of $\mathcal{D}$, then $\alpha^* : \Phi(A) \to \Phi(B)$ is an *isomorphism* of monoids. If, moreover, *every* monoid $\Phi(A)$ [as A ranges over the objects of $\mathcal{D}$] (respectively, *some* monoid $\Phi(A)$ [where $A \in \mathrm{Ob}(\mathcal{D})$]) satisfies some property of monoids [e.g., is pre-divisorial, sharp, etc.], then we shall say that Φ (respectively, A) satisfies this property. Note that if Φ is a monoid on $\mathcal{D}$, then Φ determines monoids "Φ^{char}", "Φ^{gp}", Φ^{pf}" on $\mathcal{D}$ [i.e., by assigning $A \mapsto \Phi(A)^{\mathrm{char}}$, $A \mapsto \Phi(A)^{\mathrm{gp}}$, $A \mapsto \Phi(A)^{\mathrm{pf}}$], which we shall refer to, respectively, as the *characteristic*, *groupification*, and *perfection* of Φ. If Φ is pre-divisorial, then we shall say that Φ is *non-dilating* if the endomorphisms of $\Phi(A)$, where $A \in \mathrm{Ob}(\mathcal{D})$, induced by endomorphisms $\in \mathrm{End}_{\mathcal{D}}(A)$ are non-dilating.

(iii) Let Φ be a *monoid* on a category $\mathcal{D}$. Then we shall refer to as the *elementary Frobenioid associated to* Φ the category

$$\mathbb{F}_\Phi$$

defined as follows: The *objects* of $\mathbb{F}_\Phi$ are the objects of $\mathcal{D}$. If $A, B \in \text{Ob}(\mathbb{F}_\Phi)$, whose respective images in $\mathcal{D}$ we denote by $A_\mathcal{D}, B_\mathcal{D} \in \text{Ob}(\mathcal{D})$, then a *morphism* $\phi : A \to B$ of $\mathbb{F}_\Phi$ is defined to be a collection of data

$$(\phi_\mathcal{D}, Z_\phi, n_\phi)$$

where $\phi_\mathcal{D} : A_\mathcal{D} \to B_\mathcal{D}$ is a morphism of $\mathcal{D}$; $Z_\phi \in \Phi(A_\mathcal{D})$; $n_\phi \in \mathbb{N}_{\geq 1}$. Here, $\phi_\mathcal{D}$ (respectively, $A_\mathcal{D}$) will be referred to as the *projection* $\text{Base}(\phi)$ (respectively, $\text{Base}(A)$) of ϕ (respectively, A) to $\mathcal{D}$; Z_ϕ as the *zero divisor* $\text{Div}(\phi)$ of ϕ; and n_ϕ as the *Frobenius degree* $\deg_{\text{Fr}}(\phi)$ of ϕ. If $C_\mathcal{D} \stackrel{\text{def}}{=} \text{Base}(C) \in \text{Ob}(\mathcal{D})$, then the *composite* of two morphisms

$$\phi = (\phi_\mathcal{D}, Z_\phi, n_\phi) : A \to B; \quad \psi = (\psi_\mathcal{D}, Z_\psi, n_\psi) : B \to C$$

is given as follows:

$$\psi \circ \phi = (\psi_\mathcal{D} \circ \phi_\mathcal{D}, \phi_\mathcal{D}^*(Z_\psi) + n_\psi \cdot Z_\phi, n_\psi \cdot n_\phi) : A \to C$$

Observe that the assignment $\Phi \mapsto \mathbb{F}_\Phi$ is *functorial* with respect to *homomorphisms of functors* [on $\mathcal{D}$] *valued in monoids* $\Phi \to \Phi'$; also, we have a *natural projection functor*:

$$\mathbb{F}_\Phi \to \mathcal{D}$$

We shall refer to the $\mathcal{D}$ as the *base category* of $\mathbb{F}_\Phi$. If $M \in \text{Ob}(\mathfrak{Mon})$, then observe that the elementary Frobenioid $\mathbb{F}_{\Phi_M}$ associated to the functor Φ_M on any one-morphism [cf. §0] category that assigns to the unique object of the category the monoid M is itself a one-object [cf. §0] category, whose endomorphism monoid we shall denote by $\mathbb{F}_M$ and refer to as the *elementary Frobenioid associated to* M. [Thus, the notation "$\mathbb{F}_\square$" denotes a category (respectively, monoid) when the subscript "$\square$" is a functor (respectively, monoid).] More explicitly, the underlying set of $\mathbb{F}_M$ is the product

$$M \times \mathbb{N}_{\geq 1}$$

equipped with the monoid structure is given as follows: if $a_1, a_2 \in M$, $n_1, n_2 \in \mathbb{N}_{\geq 1}$, then $(a_1, n_1) \cdot (a_2, n_2) = (a_1 + n_1 \cdot a_2, n_1 \cdot n_2)$. Also, we shall write $\mathbb{F} \stackrel{\text{def}}{=} \mathbb{F}_{\mathbb{Z}_{\geq 0}}$ and refer to $\mathbb{F}$ as the *standard Frobenioid.*

(iv) Let $\mathcal{D}$, Φ, $\mathbb{F}_\Phi$ be as in (iii); $\mathcal{C}$ a *category.* Assume further that Φ is *divisorial*, and that $\mathcal{C}$, $\mathcal{D}$ are *connected, totally epimorphic categories* [cf. §0]. Then we shall refer to a [covariant] functor

$$\mathcal{C} \to \mathbb{F}_\Phi$$

as a *pre-Frobenioid structure* on $\mathcal{C}$. The natural projection functor $\mathbb{F}_\Phi \to \mathcal{D}$ thus restricts to a *natural projection functor*

$$\mathcal{C} \to \mathcal{D}$$

on $\mathcal{C}$; similarly, the operations "$\text{Base}(-)$", "$\text{Div}(-)$", "$\deg_{\text{Fr}}(-)$" on $\mathbb{F}_\Phi$ restrict to operations on $\mathcal{C}$ which [by abuse of notation] we shall denote by the same notation.

We shall refer to the $\mathcal{D}$ as the *base category* of $\mathcal{C}$. By abuse of notation, we shall often regard Φ as a *functor on* $\mathcal{C}$ [i.e., by composing the original functor Φ with the natural projection functor $\mathcal{C} \to \mathcal{D}$] and apply similar terminology to objects of $\mathcal{C}$ and "Φ as a functor on $\mathcal{C}$" to the terminology applied to objects of $\mathcal{D}$ and "Φ as a functor on $\mathcal{D}$" [cf. (ii)]. We shall refer to a category $\mathcal{C}$ equipped with a pre-Frobenioid structure $\mathcal{C} \to \mathbb{F}_\Phi$ as a *pre-Frobenioid* and to the monoid Φ as the *divisor monoid* of the pre-Frobenioid.

Remark 1.1.1. If $\phi \circ \psi$ is a composite of morphisms ϕ, ψ of a *pre-Frobenioid*, then the *operations* "Base$(-)$", "Div$(-)$", "$\deg_{\mathrm{Fr}}(-)$" behave in the following way under *composition*:

$$\mathrm{Base}(\phi \circ \psi) = \mathrm{Base}(\phi) \circ \mathrm{Base}(\psi)$$

$$\mathrm{Div}(\phi \circ \psi) = (\mathrm{Base}(\psi))^*(\mathrm{Div}(\phi)) + \deg_{\mathrm{Fr}}(\phi) \cdot \mathrm{Div}(\psi)$$

$$\deg_{\mathrm{Fr}}(\phi \circ \psi) = \deg_{\mathrm{Fr}}(\phi) \cdot \deg_{\mathrm{Fr}}(\psi)$$

Indeed, this follows immediately from the definition of an *elementary Frobenioid* in Definition 1.1, (iii).

Next, we introduce various terms to describe *types of morphisms and objects* in a pre-Frobenioid.

Definition 1.2. Let Φ be a *divisorial monoid* on a connected, totally epimorphic category $\mathcal{D}$; $\mathcal{C} \to \mathbb{F}_\Phi$ a *pre-Frobenioid*; $\phi \in \mathrm{Arr}(\mathcal{C})$. Write $\phi : A \to B$ [where $A, B \in \mathrm{Ob}(\mathcal{C})$]; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$, $B_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(B) \in \mathrm{Ob}(\mathcal{D})$. Then:

(i) We shall say that ϕ is *linear* if $\deg_{\mathrm{Fr}}(\phi) = 1$. We shall say that ϕ is *isometric*, or, alternatively, an *isometry*, if $\mathrm{Div}(\phi) = 0$ [cf. Definition 1.1, (iii)]. If $\psi \in \mathrm{Arr}(\mathcal{C})$ is *co-objective* with ϕ [cf. §0], then we shall say that ϕ, ψ are *metrically equivalent* if $\mathrm{Div}(\phi) = \mathrm{Div}(\psi)$.

(ii) We shall refer to ϕ as a *base-isomorphism* (respectively, *base-FSM-morphism*) if $\mathrm{Base}(\phi)$ is an isomorphism (respectively, FSM-morphism [cf. §0]) in $\mathcal{D}$. We shall refer to two objects of $\mathcal{C}$ that map to isomorphic objects of $\mathcal{D}$ as *base-isomorphic*. We shall refer to ϕ as a *pull-back morphism* if the natural transformation of contravariant functors on $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, B) \times_{\mathrm{Hom}_{\mathcal{D}}(-, B_{\mathcal{D}})|_{\mathcal{C}}} (\mathrm{Hom}_{\mathcal{D}}(-, A_{\mathcal{D}})|_{\mathcal{C}})$$

[where "$|_{\mathcal{C}}$" denotes the restriction of a functor on $\mathcal{D}$ to a functor on $\mathcal{C}$ via the natural projection functor $\mathcal{C} \to \mathcal{D}$] induced by ϕ is an *isomorphism*. If $\psi \in \mathrm{Arr}(\mathcal{C})$ is *co-objective* with ϕ [cf. §0], then we shall say that ϕ, ψ are *base-equivalent* (respectively, Div-*equivalent*) if $\mathrm{Base}(\phi) = \mathrm{Base}(\psi)$ (respectively, $\Phi(\phi) = \Phi(\psi)$). If $A = B$ [i.e., ϕ is an *endomorphism*], then we shall say that ϕ is a *base-identity* (respectively, Div-*identity*) endomorphism if it is base-equivalent (respectively, Div-equivalent) to the identity endomorphism of A. Write

$$\mathcal{O}^{\times}(A) \subseteq \mathrm{Aut}_{\mathcal{C}}(A); \quad \mathcal{O}^{\triangleright}(A) \subseteq \mathrm{End}_{\mathcal{C}}(A)$$

for the submonoids of base-identity linear endomorphisms.

(iii) We shall say that ϕ is a *pre-step* [a term motivated by the point of view that the only possibly non-isomorphic portion of such a morphism is the "step" constituted by a non-zero zero divisor] if it is a linear base-isomorphism. If ϕ is a pre-step, then we shall say that it is a *step* (respectively, a *primary* pre-step) if ϕ is not an isomorphism (respectively, if the zero divisor $\mathrm{Div}(\phi) \in \Phi(A)$ of ϕ is a primary [cf. §0] element of the monoid $\Phi(A)$). We shall say that ϕ is *co-angular* [a term that arises from a certain *"coincidence of angles"* that occurs for co-angular morphisms in the case of Frobenioids that arise in an *archimedean* context — cf. [Mzk15], Definition 3.1, (iii)] if, for any factorization $\phi = \alpha \circ \beta \circ \gamma$ in $\mathcal{C}$, where α is *linear*, β is an *isometric pre-step*, and either α or γ is a *base-isomorphism*, it follows that β is an *isomorphism*. We shall say that ϕ is *LB-invertible* [i.e., *"line bundle-invertible"* — a term motivated by the isomorphism induced by such a morphism between the "image line bundle of the domain" and "the line bundle portion of the codomain" in the case of various Frobenioids that arise from arithmetic geometry] if it is *co-angular* and *isometric*. We shall say that ϕ is a *morphism of Frobenius type* [a term motivated by the fact that, in the case of Frobenioids that arise from arithmetic geometry, such a morphism corresponds to simply "raising to the n-th tensor power" for some $n \in \mathbb{N}_{\geq 1}$] if ϕ is an *LB-invertible base-isomorphism*. We shall say that ϕ is a *prime-Frobenius morphism*, or, alternatively, a $\deg_{\mathrm{Fr}}(\phi)$-*Frobenius morphism*, if it is a morphism of Frobenius type such that $\deg_{\mathrm{Fr}}(\phi) \in \mathfrak{Primes}$ [cf. §0].

(iv) A *Frobenius-ample object* of $\mathcal{C}$ is defined to be an object C such that for any $n \in \mathbb{N}_{\geq 1}$, C admits an endomorphism of Frobenius degree n. A *Frobenius-trivial object* of $\mathcal{C}$ is defined to be an object C such that there exists a homomorphism of monoids $\zeta : \mathbb{N}_{\geq 1} \to \mathrm{End}_{\mathcal{C}}(C)$ which satisfies the following properties: (a) the composite of ζ with the map to $\mathbb{N}_{\geq 1}$ given by the Frobenius degree is the identity on $\mathbb{N}_{\geq 1}$; (b) the endomorphisms in the image of ζ are base-identity endomorphisms of Frobenius type. A Div-*Frobenius-trivial object* of $\mathcal{C}$ is defined to be an object C such that there exists a homomorphism of monoids $\zeta : \mathbb{N}_{\geq 1} \to \mathrm{End}_{\mathcal{C}}(C)$ which satisfies the following properties: (a) the composite of ζ with the map to $\mathbb{N}_{\geq 1}$ given by the Frobenius degree is the identity on $\mathbb{N}_{\geq 1}$; (b) the endomorphisms in the image of ζ are Div-identity endomorphisms of Frobenius type. A *universally* Div-*Frobenius-trivial object* of $\mathcal{C}$ is defined to be an object C such that for every pull-back morphism $C' \to C$ of $\mathcal{C}$, it follows that C' is a Div-Frobenius-trivial object. A *quasi-Frobenius-trivial object* of $\mathcal{C}$ is defined to be an object C such that for any $n \in \mathbb{N}_{\geq 1}$, C admits a base-identity endomorphism [which is *not necessarily* of Frobenius type!] of Frobenius degree n. A *sub-quasi-Frobenius-trivial object* of $\mathcal{C}$ is defined to be an object C such that there exists a co-angular pre-step $D \to C$ in $\mathcal{C}$ such that D is quasi-Frobenius trivial. A *metrically trivial object* of $\mathcal{C}$ is defined to be an object C such that for any co-angular pre-step $C \to D$, it holds that

D is isomorphic to C. A *base-trivial object* of $\mathcal{C}$ is defined to be an object C such that any object $D \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Base}(C) \cong \mathrm{Base}(D)$ [in $\mathcal{D}$] is, in fact, isomorphic to C. An Aut-*ample* (respectively, $\mathrm{Aut}^{\mathrm{sub}}$-*ample*; End-*ample*) *object* of $\mathcal{C}$ is defined to be an object C such that, if we write $C_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(C)$, then the natural map $\mathrm{Aut}_{\mathcal{C}}(C) \to \mathrm{Aut}_{\mathcal{D}}(C_{\mathcal{D}})$ (respectively, $\mathrm{Aut}^{\mathrm{sub}}_{\mathcal{C}}(C) \to \mathrm{Aut}^{\mathrm{sub}}_{\mathcal{D}}(C_{\mathcal{D}})$; $\mathrm{End}_{\mathcal{C}}(C) \to \mathrm{End}_{\mathcal{D}}(C_{\mathcal{D}})$) is surjective. A *perfect object* of $\mathcal{C}$ is defined to be an object C such that for every $n \in \mathbb{N}_{\geq 1}$, it holds that every $B \in \mathrm{Ob}(\mathcal{C})$ base-isomorphic to C appears as the codomain of a morphism of Frobenius type of Frobenius degree n, and, moreover, for every pair of morphisms of Frobenius type $\phi_1 : B_1 \to B_1'$, $\phi_2 : B_2 \to B_2'$ of Frobenius degree n, where B_1, B_2 are base-isomorphic to C, and every pre-step $\psi' : B_1' \to B_2'$, there exists a unique pre-step $\psi : B_1 \to B_2$ such that $\psi' \circ \phi_1 = \phi_2 \circ \psi$. A *group-like object* of $\mathcal{C}$ is defined to be an object C such that $\Phi(C) = 0$ [or, equivalently, $\Phi(C)$ is group-like — cf. the conventions of Definition 1.1, (i), (ii), (iv)]. A *Frobenius-compact object* of $\mathcal{C}$ is defined to be an object C such that $\mathcal{O}^{\times}(C)$ is *commutative*, $\mathcal{O}^{\times}(C)^{\mathrm{pf}} \neq 0$, and every element of $\mathrm{Aut}_{\mathcal{C}}(C)$ that acts on $\mathcal{O}^{\times}(C)^{\mathrm{pf}}$ via multiplication by an element $\in \mathbb{Q}_{>0}$ in fact acts *trivially* on $\mathcal{O}^{\times}(C)^{\mathrm{pf}}$. A *Frobenius-normalized object* of $\mathcal{C}$ is defined to be an object C such that if $\phi \in \mathrm{End}_{\mathcal{C}}(C)$ is a base-identity endomorphism of Frobenius degree $d \in \mathbb{N}_{\geq 1}$, and $\alpha \in \mathcal{O}^{\rhd}(C)$, then $\alpha^d \circ \phi = \phi \circ \alpha$. A *unit-trivial object* of $\mathcal{C}$ is defined to be an object C such that $\mathcal{O}^{\times}(C) = \{1\}$. An *isotropic object* [a term motivated by the *archimedean* case — cf. [Mzk15], Definition 3.1, (iii)] of $\mathcal{C}$ is defined to be an object C such that any isometric pre-step $C \to D$ in $\mathcal{C}$ is, in fact, an *isomorphism.* We shall write

$$\mathcal{C}^{\mathrm{istr}} \subseteq \mathcal{C}$$

for the *full subcategory of isotropic objects* and

$$\mathcal{C}^{\mathrm{lin}} \subseteq \mathcal{C}; \quad \mathcal{C}^{\text{bs-iso}} \subseteq \mathcal{C}; \quad \mathcal{C}^{\text{pl-bk}} \subseteq \mathcal{C}$$

for the *subcategories* determined, respectively, by the *linear morphisms*, *base-isomorphisms*, and *pull-back morphisms.* We shall say that $\phi : A \to B$ is an *isotropic hull* [of A] if ϕ is an isometric pre-step, B is isotropic, and for every morphism $\gamma : A \to C$, where C is isotropic, there exists a unique morphism $\beta : B \to C$ such that $\gamma = \beta \circ \phi$. A *Frobenius-isotropic object* of $\mathcal{C}$ is defined to be an object C such that there exists a morphism of Frobenius type $C \to D$ such that D is isotropic.

(v) If every object of $\mathcal{C}$ is Frobenius-ample (respectively, Frobenius-trivial; Div-Frobenius-trivial; universally Div-Frobenius-trivial; quasi-Frobenius-trivial; sub-quasi-Frobenius-trivial; metrically trivial; base-trivial; Aut-ample; $\mathrm{Aut}^{\mathrm{sub}}$-ample; End-ample; perfect; group-like; Frobenius-compact; Frobenius-normalized; unit-trivial; isotropic; Frobenius-isotropic), then we shall say that the pre-Frobenioid $\mathcal{C} \to \mathbb{F}_{\Phi}$ is *of Frobenius-ample type* (respectively, *of Frobenius-trivial type*; *of* Div-*Frobenius-trivial type*; *of universally* Div-*Frobenius-trivial type*; *of quasi-Frobenius-trivial type*; *of sub-quasi-Frobenius-trivial type*; *of metrically trivial type*; *of base-trivial type*; *of* Aut-*ample type*; *of* $\mathrm{Aut}^{\mathrm{sub}}$-*ample type*; *of* End-*ample type*; *of perfect type*; *of group-like type*; *of Frobenius-compact type*; *of Frobenius-normalized type*; *of unit-trivial type*; *of isotropic type*; *of Frobenius-isotropic type*).

Remark 1.2.1. The following implications follow formally from the definitions:

pull-back morphism which is a base-isomorphism $\iff$ isomorphism

base-trivial $\implies$ metrically trivial

base-identity $\implies$ Div-identity

universally Div-Frobenius-trivial $\implies$ Div-Frobenius-trivial

We are now ready to define the notion of a *"Frobenioid"*.

Definition 1.3. Let $\mathcal{D}$, Φ, $\mathcal{C} \to \mathbb{F}_\Phi$ be as in Definition 1.2. Then we shall say that the pre-Frobenioid $\mathcal{C} \to \mathbb{F}_\Phi$ [i.e., $\mathcal{C}$ equipped with this functor] is a *Frobenioid* if the following conditions are satisfied:

(i) *(Surjectivity to the Base Category via Pull-back Morphisms)* (a) Every isomorphism class of $\mathcal{D}$ arises as the image via the natural projection functor $\mathcal{C} \to \mathcal{D}$ of an isomorphism class of a *Frobenius-trivial* object of $\mathcal{C}$. (b) If $A, B \in \mathrm{Ob}(\mathcal{C})$, $A_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$, $B_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(B)$, and $\alpha : A_\mathcal{D} \xrightarrow{\sim} B_\mathcal{D}$ is an isomorphism, then there exist *pre-steps* $\phi : C \to A$, $\psi : C \to B$ such that $\alpha = \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi)^{-1}$. (c) For every $A \in \mathrm{Ob}(\mathcal{C})$, the *fully faithful* [cf. the isomorphism of functors appearing in the definition of a "pull-back morphism" given in Definition 1.2, (ii)] functor

$$\mathcal{C}_A^{\text{pl-bk}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\text{pl-bk}})_A \to \mathcal{D}_{A_\mathcal{D}}$$

[where $A_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$] determined by the natural projection functor $\mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* [cf. §0].

(ii) *(Surjectivity to $\mathbb{N}_{\geq 1}$ via Morphisms of Frobenius Type)* For every $A \in \mathrm{Ob}(\mathcal{C})$, $n \in \mathbb{N}_{\geq 1}$, there exists a *morphism of Frobenius type* $\phi : A \to B$ in $\mathcal{C}$ of Frobenius degree n; moreover, if $\psi : A \to C$ is any other morphism of Frobenius type in $\mathcal{C}$ of Frobenius degree n, then there exists a(n) [unique — since $\mathcal{C}$ is *totally epimorphic*] isomorphism $\beta : B \xrightarrow{\sim} C$ such that $\beta \circ \phi = \psi$.

(iii) *(Surjectivity to the Divisor Monoid via Co-angular Morphisms)* (a) The *co-angular* morphisms of $\mathcal{C}$ are closed under composition. (b) If $A' \to A$ is a co-angular pre-step of $\mathcal{C}$, then any morphism $A' \to A$ is *co-angular*. (c) Given any *co-angular pre-step* $\phi : A \to B$, there exists a [uniquely determined] *bijection* of monoids

$$\mathcal{O}^\triangleright(A) \xrightarrow{\sim} \mathcal{O}^\triangleright(B)$$

such that $\mathcal{O}^\triangleright(A) \ni \alpha \mapsto \beta \in \mathcal{O}^\triangleright(B)$ implies $\beta \circ \phi = \phi \circ \alpha$; moreover, this bijection depends only [among the bijections induced by the various co-angular pre-steps $A \to B$] on $\mathrm{Base}(\phi)$. (d) Denote by $\mathcal{C}^{\text{coa-pre}} \subseteq \mathcal{C}$ the subcategory determined by the *co-angular pre-steps*. Then the natural functors

$${}_A\mathcal{C}^{\text{coa-pre}} \stackrel{\mathrm{def}}{=} {}_A(\mathcal{C}^{\text{coa-pre}}) \to \mathrm{Order}(\Phi(A)); \quad \mathcal{C}_A^{\text{coa-pre}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\text{coa-pre}})_A \to \mathrm{Order}(\Phi(A))^{\mathrm{opp}}$$

[obtained by assigning to an arrow $\phi : A \to B$ the element $\text{Div}(\phi) \in \Phi(A)$ and to an arrow $\psi : B \to A$ the element $(\psi^*)^{-1}(\text{Div}(\psi)) \in \Phi(A)$ [since $\psi^* : \Phi(A) \xrightarrow{\sim} \Phi(B)$ is a *bijection* — cf. the fact that ψ is a *base-isomorphism*!] are *equivalences of categories*.

(iv) *(Factorization of Arbitrary Morphisms)* Let $\phi : A \to B$ be a morphism of $\mathcal{C}$. Then: (a) ϕ admits a *factorization*

$$\phi = \alpha \circ \beta \circ \gamma$$

where α is an *pull-back morphism*, β is a *pre-step*, and γ is a *morphism of Frobenius type*; this factorization is *unique*, up to replacing the triple (α, β, γ) by a triple of the form $(\alpha \circ \delta, \delta^{-1} \circ \beta \circ \epsilon, \epsilon^{-1} \circ \gamma)$, where δ, ϵ are isomorphisms of $\mathcal{C}$. (b) Every pull-back morphism of $\mathcal{C}$ is *LB-invertible* and *linear*.

(v) *(Factorization of Pre-steps)* Let $\phi : A \to B$ be a *pre-step* of $\mathcal{C}$. Then: (a) ϕ is a *monomorphism*. (b) ϕ admits a *factorization*

$$\phi = \alpha \circ \beta$$

where α is an *isometric* pre-step, and β is a *co-angular* pre-step; this factorization is *unique*, up to replacing the pair (α, β) by a pair of the form $(\alpha \circ \gamma, \gamma^{-1} \circ \beta)$, where γ is an isomorphism of $\mathcal{C}$. (c) ϕ admits a *factorization* $\phi = \alpha' \circ \beta'$, where α' is a *co-angular* pre-step, and β' is an *isometric* pre-step; this factorization is *unique*, up to replacing the pair (α', β') by a pair of the form $(\alpha' \circ \gamma', (\gamma')^{-1} \circ \beta')$, where γ' is an isomorphism of $\mathcal{C}$.

(vi) *(Faithfulness up to Units)* Let $\phi, \psi : A \to B$ be *base-equivalent*, *metrically equivalent* co-angular pre-steps of $\mathcal{C}$. Then there exists a [necessarily unique] $\alpha \in \mathcal{O}^\times(B)$ such that $\phi = \alpha \circ \psi$.

(vii) *(Isotropic Objects)* (a) For every $A \in \text{Ob}(\mathcal{C})$, there exists a [necessarily unique, up to unique isomorphism] *isotropic hull* $A \to B$. (b) If $A \in \text{Ob}(\mathcal{C})$ is isotropic, and $A \to C$ is a morphism of $\mathcal{C}$, then C is also *isotropic*.

Remark 1.3.1. Note that it follows from Definition 1.3, (iii), (b), (c), that if $\mathcal{C}$ is a Frobenioid, then the monoid $\mathcal{O}^\rhd(A)$ is *commutative*, for all $A \in \text{Ob}(\mathcal{C})$.

Proposition 1.4. **(Co-angular and LB-invertible Morphisms)** *Let Φ be a* **divisorial monoid** *on a connected, totally epimorphic category $\mathcal{D}$; $\mathcal{C} \to \mathbb{F}_\Phi$ a* **pre-Frobenioid**; *$\phi : A \to B$ a morphism of $\mathcal{C}$. Then:*

(i) Suppose that the codomain of any arrow of $\mathcal{C}$ whose domain is equal to A is **isotropic**. *Then ϕ is* **co-angular**. *In particular, ϕ is a* **morphism of Frobenius type** *if and only if it is an isometric base-isomorphism.*

(ii) Suppose that $\mathcal{C}$ is a **Frobenioid**. *Then ϕ is a* **pull-back morphism** *if and only if it is an* **LB-invertible linear morphism** *[i.e., a* **co-angular linear isometry***].*

(iii) Suppose that $\mathcal{C}$ is a **Frobenioid**. *Then every* **LB-invertible pre-step** *is an isomorphism.*

(iv) Suppose that $\mathcal{C}$ is a **Frobenioid**. *Then a morphism ϕ of $\mathcal{C}$ is* **co-angular** *if and only if, in the factorization $\phi = \alpha \circ \beta \circ \gamma$ of Definition 1.3, (iv), (a), the pre-step β is* **co-angular**.

(v) Suppose that $\mathcal{C}$ is a **Frobenioid**. *Then a morphism ϕ of $\mathcal{C}$ is* **LB-invertible** *if and only if it is of the form $\alpha \circ \beta$, where α is a* **pull-back morphism**, *and β is a* **morphism of Frobenius type**.

Proof. Assertion (i) follows formally from the definitions of the terms "isotropic", "isometric pre-step", "co-angular", and "morphism of Frobenius type" [cf. Definition 1.2, (i), (iii), (iv)]. As for assertion (ii), if ϕ is a *pull-back morphism*, then it follows from Definition 1.3, (iv), (b), that ϕ is an *LB-invertible linear morphism.* Now suppose that ϕ is *LB-invertible* and *linear*. Then by applying Remark 1.1.1 to the *factorization* of Definition 1.3, (iv), (a), the fact that ϕ is a linear isometry implies that ϕ may be written in the form $\alpha \circ \beta$, where α is a *pull-back morphism*, and β is an *isometric pre-step.* On the other hand, since ϕ is *co-angular*, it follows that β is an *isomorphism*, hence that ϕ is a pull-back morphism, as desired. Assertion (iii) follows from either the *uniqueness* of the *factorization of pre-steps* of Definition 1.3, (v), (b), or the *essential uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)].

Next, we consider assertion (iv). If β is *co-angular*, then since α, γ are *co-angular* [cf. assertion (ii); Definition 1.2, (iii)], it follows from Definition 1.3, (iii), (a), that ϕ is *co-angular.* Conversely, if ϕ is *co-angular*, and $\beta = \beta_1 \circ \beta_2 \circ \beta_3$, where β_2 is an *isometric pre-step*, then by applying Remark 1.1.1, together with the fact that $\mathcal{D}$ is *totally epimorphic* [cf. the discussion of §0] to this factorization of β, we conclude that β_1, β_3 are *pre-steps*, hence that $\alpha \circ \beta_1$ is *linear*, and that $\beta_3 \circ \gamma$ is a *base-isomorphism*; thus, the *co-angularity* of $\phi = (\alpha \circ \beta_1) \circ \beta_2 \circ (\beta_3 \circ \gamma)$ implies that β_2 is an *isomorphism*, hence that β is *co-angular*, as desired.

Finally, we consider assertion (v). If $\phi = \alpha \circ \beta$, where α is a *pull-back morphism*, and β is a *morphism of Frobenius type*, then [since α, β are *LB-invertible* — cf. assertion (ii); Definition 1.2, (iii)] it follows from Remark 1.1.1 that ϕ is *isometric* and from Definition 1.3, (iii), (a), that ϕ is *co-angular*, hence *LB-invertible.* Now suppose that ϕ is *LB-invertible*, and that we have a *factorization* $\phi = \alpha \circ \beta \circ \gamma$, where α, β, and γ are as in Definition 1.3, (iv), (a). By assertion (iv), β is *co-angular*; by Remark 1.1.1, β is *isometric.* Thus, β is an LB-invertible pre-step, hence [cf. assertion (iii)] an isomorphism, as desired. This completes the proof of assertion (v). ○

Remark 1.4.1. We refer to the *Chart of Types of Morphisms in a Frobenioid* given at the end of the present paper for a summary of the properties of the base category projections, zero divisors, and Frobenius degrees satisfied by various types of morphisms in a Frobenioid.

Proposition 1.5. **(Elementary Frobenioids are Frobenioids)** *Let* Φ *be a* **pre-divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*. Then:*

(i) $\mathbb{F}_\Phi$*, equipped with the natural functor* $\mathbb{F}_\Phi \to \mathbb{F}_{\Phi^{\text{char}}}$*, is a* **Frobenioid** *of* Aut-**ample***,* Aut^{sub}**-ample***,* End-**ample***,* **base-trivial***,* **Frobenius-trivial***,* **Frobenius-normalized***, and* **isotropic** *type.*

(ii) There is a **natural, functorial isomorphism**

$$\mathcal{O}^{\rhd}(A) \xrightarrow{\sim} \Phi(A)$$

[so $\mathcal{O}^\times(A) \xrightarrow{\sim} \Phi(A)^\pm$*] for objects* $A \in \text{Ob}(\mathbb{F}_\Phi)$*.*

(iii) If all of the monoids in the image of Φ *are* **perfect** *(respectively,* **group-like***), then* $\mathbb{F}_\Phi$ *is of* **perfect** *(respectively,* **group-like***) type.*

Proof. Since $\mathcal{D}$ is a *connected, totally epimorphic category*, the fact that $\mathbb{F}_\Phi$ is as well follows immediately from the definition of the morphisms of $\mathbb{F}_\Phi$ in Definition 1.1, (iii); the fact that a *pre-divisorial monoid* is *integral* [cf. Definition 1.1, (i)]; and the *injectivity* condition of Definition 1.1, (ii), (a). Thus, $\mathbb{F}_\Phi$ is a *pre-Frobenioid*. It is immediate from the definitions that assertion (ii) holds, and that all objects of $\mathbb{F}_\Phi$ are Aut-ample, Aut^{sub}-ample, End-ample, base-trivial, Frobenius-trivial, Frobenius-normalized, and isotropic. Also, one verifies immediately [cf. the definition of the category $\mathbb{F}_\Phi$ in Definition 1.1, (iii)] that a morphism of $\mathbb{F}_\Phi$ is a *pull-back morphism* if and only if it is a *linear isometry*. The fact that $\mathbb{F}_\Phi$ satisfies the conditions of Definition 1.3 now follows immediately from the definition of the category $\mathbb{F}_\Phi$ in Definition 1.1, (iii), together with assertion (ii) and the *"explicit description"* of *co-angular morphisms* and *morphisms of Frobenius type* in Proposition 1.4, (i) [which is applicable to all morphisms of $\mathbb{F}_\Phi$ since $\mathbb{F}_\Phi$ is *of isotropic type*]. This completes the proof of assertion (i). Assertion (iii) is immediate from the definitions and assertion (i). ◯

One important technique for constructing *new Frobenioids* is given by the following result.

Proposition 1.6. **(Categorical Fiber Products)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid***. Let* $\mathcal{D}'$ *be a* **connected, totally epimorphic category***;* $\mathcal{D}' \to \mathcal{D}$ *a functor that maps FSM-morphisms to FSM-morphisms. Denote by* $\Phi' : \mathcal{D}' \to \mathfrak{Mon}$ *the divisorial monoid obtained by restricting* Φ *to* $\mathcal{D}'$*. Then:*

(i) There is a **natural equivalence of categories**

$$\mathbb{F}_{\Phi'} \xrightarrow{\sim} \mathbb{F}_\Phi \times_{\mathcal{D}} \mathcal{D}'$$

[where the latter category is the **categorical fiber product** *of* §*0].*

(ii) The **categorical fiber product** *[cf. §0]*

$$\mathcal{C}' \stackrel{\mathrm{def}}{=} \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$$

equipped with the functor $\mathcal{C}' \to \mathbb{F}_{\Phi'}$ *[obtained by applying "*$(-) \times_{\mathcal{D}} \mathcal{D}'$*" to the functor* $\mathcal{C} \to \mathbb{F}_{\Phi}$*] is a* **Frobenioid***.*

(iii) A morphism of $\mathcal{C}'$ *is a(n)* **isometry** *(respectively, morphism of a given* **Frobenius degree***;* **co-angular morphism***;* **LB-invertible morphism***;* **pull-back morphism***) if and only if its projection to* $\mathcal{C}$ *is.*

(iv) A base-isomorphism of $\mathcal{C}'$ *is a* **morphism of Frobenius type** *(respectively,* **pre-step***;* **step***) if and only if its projection to* $\mathcal{C}$ *is. Moreover, the projection functor* $\mathcal{C}' \to \mathcal{C}$ *determines a* **bijection of monoids** $\mathcal{O}^{\triangleright}(A') \xrightarrow{\sim} \mathcal{O}^{\triangleright}(A)$*, for every* $A' \in \mathrm{Ob}(\mathcal{C}')$ *that projects to* $A \in \mathrm{Ob}(\mathcal{C})$*.*

(v) A object of $\mathcal{C}'$ *is* **Frobenius-trivial** *(respectively,* **quasi-Frobenius-trivial***;* **sub-quasi-Frobenius-trivial***;* **metrically trivial***;* **base-trivial***;* **perfect***;* **group-like***;* **unit-trivial***;* **Frobenius-normalized***;* **isotropic***;* **Frobenius-isotropic***) if and only if it projects to such an object of* $\mathcal{C}$*.*

(vi) A object of $\mathcal{C}'$ *is* Aut**-ample** *(respectively,* $\mathrm{Aut}^{\mathrm{sub}}$**-ample***;* End**-ample***) if it projects to such an object of* $\mathcal{C}$*.*

Proof. Assertion (i) follows formally from the definitions. Next, observe that the fact that $\mathcal{D}'$ is a *totally epimorphic category* implies immediately that $\mathcal{C}'$ is as well; similarly, [in light of the various properties of the natural projection functor $\mathcal{C} \to \mathcal{D}$ assumed in Definition 1.3, (i), (a), (b), (c)] the fact that $\mathcal{D}'$ is connected implies immediately that $\mathcal{C}'$ is also *connected*. Thus, $\mathcal{C}'$ [equipped with the functor $\mathcal{C}' \to \mathbb{F}_{\Phi'}$ obtained by applying "$(-) \times_{\mathcal{D}} \mathcal{D}'$" to the functor $\mathcal{C} \to \mathbb{F}_{\Phi}$] is a *pre-Frobenioid*. Now assertion (vi) follows immediately from the definitions; one checks immediately that the equivalences of assertions (iii), (iv), (v) hold. In light of these equivalences, the conditions of Definition 1.3 follow via a routine verification. Thus, $\mathcal{C}'$ is a *Frobenioid*. This completes the proof of assertion (ii). ○

Proposition 1.7. (Composites of Morphisms) *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_{\Phi}$ *a* **Frobenioid***. Then:*

(i) The following classes of morphisms are closed under **composition***:* **isometries***,* **base-isomorphisms***,* **base-FSM-morphisms***,* **pull-back morphisms***,* **linear morphisms***,* **pre-steps***,* **co-angular morphisms***,* **LB-invertible morphisms***,* **morphisms of Frobenius type***.*

(ii) A morphism of $\mathcal{C}$ *is a* **pull-back morphism** *if and only if it is* **minimal-adjoint** *to the* **base-isomorphisms** *of* $\mathcal{C}$*. A morphism of* $\mathcal{C}$ *is a* **base-isomorphism** *if and only if it is* **minimal-coadjoint** *to the* **pull-back morphisms** *of* $\mathcal{C}$*; alternatively, a morphism of* $\mathcal{C}$ *is a base-isomorphism if and only if it is may be written*

as a **composite** $\alpha \circ \beta$, *where* α *is a pre-step, and* β *is a morphism of Frobenius type.*

(iii) A morphism of $\mathcal{C}$ *is* **of Frobenius type** *if and only if it is* **minimal-coadjoint** *to the* **linear morphisms** *of* $\mathcal{C}$. *A morphism of* $\mathcal{C}$ *is* **linear** *if and only if it is* **minimal-adjoint** *to the* **morphisms of Frobenius type** *of* $\mathcal{C}$*; alternatively, a morphism of* $\mathcal{C}$ *is* **linear** *if and only if it is may be written as a* **composite** $\alpha \circ \beta$, *where* α *is a pull-back morphism, and* β *is a pre-step.*

(iv) A pre-step of $\mathcal{C}$ *is* **co-angular** *if and only if it is* **mid-adjoint** *[cf. §0] to the* **isometric pre-steps**.

(v) If a composite morphism $\phi = \alpha \circ \beta$ *of* $\mathcal{C}$ *is a(n)* **isomorphism** *(respectively,* **base-isomorphism***;* **linear morphism***;* **pre-step***;* **isometry***;* **co-angular pre-step***;* **co-angular linear morphism***;* **pull-back morphism***), then so are* α, β. *If, moreover, the domain of* ϕ *is* **isotropic***, then a similar statement holds for* **morphisms of Frobenius type**.

Proof. Assertion (i) follows immediately from the definitions for isometries, base-isomorphisms, base-FSM-morphisms, pull-back morphisms, linear morphisms, and pre-steps; from Definition 1.3, (iii), (a), for co-angular morphisms, hence also for LB-invertible morphisms and morphisms of Frobenius type. Next, the *sufficiency* of the various conditions given in assertions (ii), (iii) follows immediately from [definitions and] the [existence of the] *factorization* of Definition 1.3, (iv), (a). Moreover, in light of the existence of this factorization, the *necessity* of the various conditions given in assertions (ii), (iii) follows immediately for *pull-back morphisms* and *morphisms of Frobenius type* from the *essential uniqueness* of this factorization [and the total epimorphicity of $\mathcal{C}$]; for *base-isomorphisms* from the total epimorphicity of $\mathcal{D}$; and for *linear morphisms* from the well-known structure of the multiplicative monoid $\mathbb{N}_{\geq 1}$ and the essential uniqueness of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)].

In light of Remark 1.1.1, assertion (v) follows for isomorphisms (respectively, base-isomorphisms; linear morphisms; pre-steps; isometries) immediately from the fact that $\mathcal{C}$ is *totally epimorphic* (respectively, from the fact that $\mathcal{D}$ is *totally epimorphic*; from the well-known structure of the multiplicative monoid $\mathbb{N}_{\geq 1}$; from assertion (v) for base-isomorphisms and linear morphisms; from the fact that the monoid Φ on $\mathcal{D}$ is *sharp* [cf. Definition 1.1, (i)], together with the *characteristic injectivity* assumption of Definition 1.1, (ii), (a)). Now assertion (iv) follows formally from [the definitions and] assertion (v) for pre-steps [cf. the argument applied in the proof of Proposition 1.4, (iv)!]; assertion (v) for co-angular pre-steps follows from assertion (v) for pre-steps and assertion (iv). To prove assertion (v) for co-angular linear morphisms, suppose that ϕ is *co-angular* and *linear*. Then observe that by assertion (v) for linear morphisms, α, β are *linear*. Thus, by applying the factorization for linear morphisms of assertion (iii), together with the factorization of Definition 1.3, (v), (c) [cf. also Proposition 1.4, (ii); assertion (i) for co-angular linear morphisms], we may write $\alpha = \alpha_1 \circ \alpha_2$, $\beta = \beta_1 \circ \beta_2$, $\alpha_2 \circ \beta_1 = \gamma_1 \circ \gamma_2$, where α_1, β_1, γ_1 are *co-angular linear morphisms*, and α_2, β_2, γ_2 are *isometric pre-steps.*

Thus, $\phi = (\alpha_1 \circ \gamma_1) \circ (\gamma_2 \circ \beta_2)$, which [by the *co-angularity* of ϕ] implies that $\gamma_2 \circ \beta_2$ is an *isomorphism*, hence [by assertion (v) for isomorphisms] that β_2, γ_2 are isomorphisms. Thus, by the *co-angularity* of $\alpha_2 \circ \beta_1 = \gamma_1 \circ \gamma_2$, we conclude that α_2 is an isomorphism. In particular, it follows that α, β are *co-angular linear morphisms*, as desired. Now assertion (v) for *pull-back morphisms* follows from assertion (v) for *co-angular linear isometries* [cf. also Proposition 1.4, (ii)]. Finally, assertion (v) for morphisms of Frobenius type in $\mathcal{C}^{\text{istr}}$ [cf. Definition 1.3, (vii), (b)] follows from assertion (v) for isometric base-isomorphisms, since morphisms of $\mathcal{C}^{\text{istr}}$ are always *co-angular* [cf. Proposition 1.4, (i)]. This completes the proof of assertion (v). ◯

Proposition 1.8. **(Pre-steps)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid**. *Then:*

(i) If the natural projection functor $\mathcal{C} \to \mathcal{D}$ *is* **full**, *then every* **pre-step** *of* $\mathcal{C}$ *is a* **linear** End-**equivalence**. *If* $\mathcal{D}$ *is* **of** Aut-**type** *[cf. §0], then every linear* End-*equivalence of* $\mathcal{C}$ *is a pre-step.*

(ii) Suppose further that $\mathcal{C}$ *is of* **metrically trivial** *and* Aut-**ample** *type. Then a morphism of* $\mathcal{C}$ *is a* **co-angular pre-step** *if and only if it is* **abstractly equivalent** *[cf. §0] to a* **base-identity pre-step endomorphism** *of* $\mathcal{C}$.

(iii) An object $A \in \text{Ob}(\mathcal{C})$ *is* **non-group-like** *if and only if there exists a* **co-angular step** $A \to B$*; alternatively, an object* $A \in \text{Ob}(\mathcal{C})$ *is non-group-like if and only if there exists a co-angular step* $B \to A$. *Also, if* $A, B \in \text{Ob}(\mathcal{C})$ *are* **base-isomorphic** *objects, then* A *is group-like if and only if* B *is.*

Proof. First, we consider assertion (i). If $\phi \in \text{Arr}(\mathcal{C})$ is a *pre-step*, and the projection functor $\mathcal{C} \to \mathcal{D}$ is *full*, then the fact that it is a linear End-equivalence follows formally from the definition of a *"pre-step"* [cf. Definition 1.2, (iii)]; the fullness assumption on $\mathcal{C} \to \mathcal{D}$. On the other hand, if $\phi \in \text{Arr}(\mathcal{C})$ is a *linear* End-*equivalence*, and $\mathcal{D}$ is *of* Aut-*type*, then it follows formally that ϕ is a *base-isomorphism*, hence a pre-step, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). If $\phi \in \text{Arr}(\mathcal{C})$ is a *co-angular pre-step*, then it follows formally from the assumption that $\mathcal{C}$ is of *metrically trivial* and Aut-*ample* type that ϕ is abstractly equivalent to a base-identity pre-step endomorphism of $\mathcal{C}$. On the other hand, if $\phi \in \text{Arr}(\mathcal{C})$ is *abstractly equivalent* to a *base-identity pre-step endomorphism of* $\mathcal{C}$ [hence *co-angular*, by Definition 1.3, (iii), (b)], then it follows formally that ϕ is a *co-angular linear base-isomorphism*, hence that ϕ is a co-angular pre-step, as desired. This completes the proof of assertion (ii). Finally, we observe that the various equivalences of assertion (iii) follow formally from the definitions and the equivalences of categories of Definition 1.3, (iii), (d). ◯

Proposition 1.9. **(Isotropic Objects and Isometries)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid**.

Write $\mathcal{C}^{\text{imtr-pre}} \subseteq \mathcal{C}$ *for the subcategory determined by the* **isometric pre-steps** *and*

$$\mathcal{C}_A^{\text{imtr-pre}} \stackrel{\text{def}}{=} (\mathcal{C}^{\text{imtr-pre}})_A$$

for $A \in \text{Ob}(\mathcal{C})$. *Then:*

(i) Any **base-isomorphism** $\phi : A \to B$ *of* $\mathcal{C}$ *admits a factorization*

$$\phi = \alpha \circ \beta$$

where α *is an* **isometric** *pre-step, and* β *is a* **co-angular** *base-isomorphism; this factorization is* **unique**, *up to replacing the pair* (α, β) *by a pair of the form* $(\alpha \circ \gamma, \gamma^{-1} \circ \beta)$, *where* γ *is an isomorphism of* $\mathcal{C}$. *Here,* ϕ *is* **isometric** *if and only if* β *is a* **morphism of Frobenius type**; ϕ *is* **co-angular** *if and only if* α *is an* **isomorphism**; ϕ *is a* **pull-back morphism** *if and only if* ϕ *is an* **isomorphism**.

(ii) Any **base-isomorphism** $\phi : A \to B$ *of* $\mathcal{C}$ *induces a* **functor** *[well-defined up to isomorphism]*

$$\phi_* : \mathcal{C}_A^{\text{imtr-pre}} \to \mathcal{C}_B^{\text{imtr-pre}}$$

that maps an isometric pre-step $C \to A$ *to the isometric pre-step* $D \to B$ *appearing in the factorization* $C \to D \to B$ *of (i) applied to the composite of the given pre-step* $C \to A$ *with* $\phi : A \to B$. *Moreover, if* ϕ *is a* **co-angular pre-step**, *then* ϕ_* *is an* **equivalence of categories**. *If* $u \in \mathcal{O}^\times(A)$, *then we shall denote by* $u^{\text{imtr-pre}}$ *the isomorphism class of the self-equivalence of the category* $\mathcal{C}_A^{\text{imtr-pre}}$ *induced by* u *and by*

$$\mathcal{O}^\times(A)^{\text{imtr-pre}} \subseteq \mathcal{O}^\times(A)$$

the subgroup of $v \in \mathcal{O}^\times(A)$ *for which* $v^{\text{imtr-pre}}$ *is the* **identity**.

(iii) Any **pull-back morphism** $\phi : A \to B$ *of* $\mathcal{C}$ *induces a* **functor** *[well-defined up to isomorphism]*

$$\phi^* : \mathcal{C}_B^{\text{imtr-pre}} \to \mathcal{C}_A^{\text{imtr-pre}}$$

that maps an isometric pre-step $\delta : D \to B$ *to the unique [up to isomorphism] isometric pre-step* $\gamma : C \to A$ *that fits into a commutative diagram*

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\ D & \xrightarrow{\delta} & B \end{array}$$

where ψ *is the pull-back morphism that arises by applying the equivalence of categories of Definition 1.3, (i), (c), to the arrow* $\text{Base}(\delta)^{-1} \circ \text{Base}(\phi)$, *and* γ *is the morphism that arises from the isomorphism of functors appearing in the definition of a "pull-back morphism" [cf. Definition 1.2, (ii)].*

(iv) Let $\phi : A \to B$ *be a* **co-angular linear morphism** *[e.g., a* **pull-back morphism** *— cf. Proposition 1.4, (ii)]. Then* A *is* **isotropic** *if and only if* B *is.*

(v) $\mathcal{C}^{\text{istr}}$ *[equipped with the restriction to* $\mathcal{C}$ *of the given functor* $\mathcal{C} \to \mathbb{F}_\Phi$*] is a* **Frobenioid**. *Moreover, the functor*

$$\mathcal{C} \to \mathcal{C}^{\text{istr}}$$

that assigns to an object $A \in \text{Ob}(\mathcal{C})$ *with isotropic hull* $A \to A^{\text{istr}}$ *the object* A^{istr} *and to a morphism of objects* $A \to B$ *with isotropic hulls* $A \to A^{\text{istr}}$, $B \to B^{\text{istr}}$ *the induced [i.e., by the definition of an* **"isotropic hull"**!*] morphism* $A^{\text{istr}} \to B^{\text{istr}}$ *forms a* **left adjoint** *to the inclusion functor* $\mathcal{C}^{\text{istr}} \hookrightarrow \mathcal{C}$*, through which the functor* $\mathcal{C} \to \mathbb{F}_\Phi$ **factors**. *We shall refer to this functor as the* **isotropification functor**. *The restriction of the isotropification functor to* $\mathcal{C}^{\text{istr}}$ *is isomorphic to the* **identity functor**. *Finally, the isotropification functor preserves* **morphisms of Frobenius type**, **Frobenius degrees**, **pre-steps**, **pull-back morphisms**, **base-isomorphisms**, **base-FSM-morphisms**, **base-identity endomorphisms**, Div**-identity endomorphisms**, **isometries**, **co-angular morphisms**, *and* **LB-invertible morphisms**; *moreover, all of these properties are* **compatible** *with the inclusion functor* $\mathcal{C}^{\text{istr}} \hookrightarrow \mathcal{C}$ *[in the sense that an arrow of* $\mathcal{C}^{\text{istr}}$ *satisfies one of these properties with respect to* $\mathcal{C}^{\text{istr}}$ *if and only if it does with respect to* $\mathcal{C}$*].*

(vi) A morphism of $\mathcal{C}$ *is an* **isotropic hull** *if and only if its codomain is* **isotropic**, *and, moreover, it is* **minimal-coadjoint** *to the morphisms with* **isotropic domain**.

(vii) A morphism $A \to B$ *of* $\mathcal{C}$ *is an* **isometric pre-step** *if and only if the composite of this morphism* $A \to B$ *with an isotropic hull* $B \to C$ *yields an* **isotropic hull** $A \to C$.

Proof. Since pull-backs which are base-isomorphisms are easily verified to be *isomorphisms* [cf. Remark 1.2.1], assertion (i) follows immediately from the (essentially) *unique factorization* of Definition 1.3, (iv), (a); the (essentially) *unique factorization of pre-steps* of Definition 1.3, (v), (b); the fact that *co-angular morphisms* are closed under *composition* [cf. Proposition 1.7, (i)]; the definition of *"co-angular"* [cf. Definition 1.2, (iii)]; the fact that $\mathcal{C}$ is *totally epimorphic*; the *essential uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)]; and Remark 1.1.1.

Next, we consider assertion (ii). The existence of the functor ϕ_* follows formally from the existence of the (essentially) *unique factorization* of assertion (i). Now suppose that ϕ is a *co-angular pre-step.* Then for any *isometric pre-step* $\beta : D \to B$, there exists a *co-angular pre-step* $\psi : C \to D$ such that

$$(\Phi(\beta \circ \psi))^{-1}(\text{Div}(\psi))) = (\Phi(\phi))^{-1}(\text{Div}(\phi))$$

[cf. the second equivalence of categories of Definition 1.3, (iii), (d)]. Thus, by applying the *factorization* of Definition 1.3, (v), (c), it follows that we may write $\beta \circ \psi = \phi' \circ \alpha'$, where $\alpha' : D \to A'$ is an isometric pre-step, and $\phi' : A' \to B$ is a co-angular pre-step. On the other hand, since $\mathrm{Div}(\beta \circ \psi) = \mathrm{Div}(\phi' \circ \alpha')$, and β, α' are *isometric*, it follows that

$$(\Phi(\phi))^{-1}(\mathrm{Div}(\phi)) = (\Phi(\phi'))^{-1}(\mathrm{Div}(\phi'))$$

— hence [by the second equivalence of categories of Definition 1.3, (iii), (d)] that there exists an isomorphism $\gamma : A' \xrightarrow{\sim} A$ such that $\phi \circ \gamma = \phi'$. Thus, if we take $\alpha \stackrel{\mathrm{def}}{=} \gamma \circ \alpha'$, then $\beta \circ \psi = \phi \circ \alpha$ — that is to say, ϕ_* is *essentially surjective*. Moreover, [by possibly replacing ϕ by ψ] this argument [i.e., the construction, given β, ϕ, of α, ψ such that $\beta \circ \psi = \phi \circ \alpha$] also implies that ϕ_* is *full*. Finally, since every pre-step is a *monomorphism* [cf. Definition 1.3, (v), (a)], it follows immediately that ϕ_* is *faithful*. This completes the proof of assertion (ii). Assertion (iii) follows formally from the definitions, together with the fact that pull-back morphisms are *linear isometries* [cf. Proposition 1.4, (ii)], which implies [cf. Remark 1.1.1] that γ is an *isometric pre-step*.

Next, we consider assertion (iv). Let $\phi : A \to B$ be a *co-angular linear morphism*. If A is *isotropic*, then so is B, by Definition 1.3, (vii), (b). Now suppose that B is *isotropic*. Thus, by the definition of an *isotropic hull*, it follows from the *existence of isotropic hulls* [cf. Definition 1.3, (vii), (a)] that there exists a factorization $\phi = \beta \circ \alpha$, where $\alpha : A \to A'$ is an isotropic hull [hence an isometric pre-step — cf. Definition 1.2, (iv)], and $\beta : A' \to B$ is *linear* [cf. Remark 1.1.1]. Thus, by the definition of *"co-angular"* [cf. Definition 1.2, (iii)], we conclude that α is an *isomorphism*, as desired. This completes the proof of assertion (iv).

Next, we consider assertion (v). By applying the definition of an *isotropic hull* [cf. Definition 1.2, (iv)], it follows immediately [from the fact that $\mathcal{C}$ is connected and totally epimorphic] that $\mathcal{C}^{\mathrm{istr}}$ is *connected* and *totally epimorphic*. Thus, $\mathcal{C}^{\mathrm{istr}}$ is a *pre-Frobenioid*. It is immediate from the definition of an isotropic hull that the isotropification functor is *left adjoint* to the inclusion functor $\mathcal{C}^{\mathrm{istr}} \hookrightarrow \mathcal{C}$; that the functor $\mathcal{C} \to \mathbb{F}_\Phi$ *factors* through the isotropification functor [cf. Remark 1.1.1]; that the restriction of the isotropification functor to $\mathcal{C}^{\mathrm{istr}}$ is isomorphic to the *identity functor*; and [cf. Remark 1.1.1] that the isotropification functor preserves *Frobenius degrees*, *pre-steps*, *base-isomorphisms*, *base-FSM-morphisms*, *base-identity endomorphisms*, Div-*identity endomorphisms*, *isometries*, and *co-angular morphisms* [cf. Proposition 1.4, (i)], hence also *LB-invertible morphisms* and *morphisms of Frobenius type* in a fashion that is *compatible* [cf. the statement of assertion (v)] with the inclusion $\mathcal{C}^{\mathrm{istr}} \hookrightarrow \mathcal{C}$. Since pull-back morphisms are *co-angular linear isometries* [cf. Proposition 1.4, (ii)], it follows immediately [in light of what we have shown so far] from Proposition 1.4, (ii), that the isotropification functor maps pull-back morphisms to morphisms which are pull-back morphisms *relative to* $\mathcal{C}$, hence a fortiori, pull-back morphisms *relative to* $\mathcal{C}^{\mathrm{istr}}$. Finally, in light of Proposition 1.4, (i); assertion (iv) [cf. also Definition 1.3, (vii), (b)], it follows immediately [from the fact that $\mathcal{C}$ is a *Frobenioid*!] that the pre-Frobenioid $\mathcal{C}^{\mathrm{istr}}$ satisfies the various conditions of Definition 1.3, hence that $\mathcal{C}^{\mathrm{istr}}$ is a *Frobenioid*, as desired. This completes the proof of assertion (v).

Finally, we observe that the *necessity* and *sufficiency* of the condition of assertion (vi) follow immediately from the definition of an *isotropic hull* [cf. Definition

1.2, (iv)], the *existence of isotropic hulls* [cf. Definition 1.3, (vii), (a)] and the *total epimorphicity* of $\mathcal{C}$; the *necessity* and *sufficiency* of the condition of assertion (vii) follow immediately from the *existence of isotropic hulls* [cf. Definition 1.3, (vii), (a)], the fact that *isometric pre-steps* between *isotropic* objects are *isomorphisms* [cf. Definition 1.3, (vii), (b); Proposition 1.4, (i), (iii)], and the following observation [which follows immediately from Proposition 1.7, (i), (v)]: Given morphisms α, β, γ of $\mathcal{C}$ such that $\gamma = \alpha \circ \beta$, if any two of the three morphisms α, β, γ is an *isometric pre-step*, then the same is true of the remaining morphism. ◯

Proposition 1.10. **(Morphisms of Frobenius Type)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid***. Then:*

(i) Let $\phi : A \to B$ *be an arbitrary morphism of* $\mathcal{C}$*. Suppose that* $\alpha : A \to A'$*,* $\beta : B \to B'$ *are* **morphisms of Frobenius type***, of Frobenius degree* $d \in \mathbb{N}_{\geq 1}$*. Then there exists a* **unique** *morphism* $\phi' : A' \to B'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow{\phi'} & B' \end{array}$$

In this situation, $\deg_{\mathrm{Fr}}(\phi) = \deg_{\mathrm{Fr}}(\phi')$*;* $\mathrm{Div}(\phi') = d \cdot \alpha_*(\mathrm{Div}(\phi))$ *[where we write* $\alpha_* : \Phi(A) \xrightarrow{\sim} \Phi(A')$ *for the bijection induced by applying the functor* Φ *to the base-isomorphism* α*]. Finally, if* ϕ *is a* **morphism of Frobenius type** *(respectively,* **pre-step***;* **pull-back morphism***;* **co-angular morphism***;* **base-isomorphism***;* **isometry***;* **LB-invertible morphism***), then the same is true of* ϕ'*.*

(ii) Any composite morphism $\beta \circ \alpha$ *of* $\mathcal{C}$*, where* α *is a* **pre-step***, and* β *is* **of Frobenius type***, may be written as a composite*

$$\alpha' \circ \beta' = \beta \circ \alpha$$

where α' *is a* **pre-step***, and* β' *is* **of Frobenius type** *such that:*

$$\deg_{\mathrm{Fr}}(\beta) = \deg_{\mathrm{Fr}}(\beta'); \quad \mathrm{Div}(\alpha') = \deg_{\mathrm{Fr}}(\beta) \cdot \beta'_*(\mathrm{Div}(\alpha))$$

[where we write β'_* *for the bijection induced by applying the functor* Φ *to the base-isomorphism* β'*].*

(iii) Suppose that $\mathcal{C}$ *is* **of perfect type***. Then the monoids in the image of* Φ *are* **perfect***. If, moreover,* $\mathcal{C}$ *is of* **isotropic** *and* **Frobenius-normalized** *type, then the monoids* $\mathcal{O}^{\rhd}(A)$ *and* $\mathcal{O}^{\times}(A)$ *are* **perfect***.*

(iv) A morphism of Frobenius type with **isotropic** *domain is a* **prime-Frobenius** *morphism if and only if it is* **irreducible** *[cf. §0]. In particular, if* $A \in \mathrm{Ob}(\mathcal{C})$ *is isotropic, then there exist* **infinitely many** *isomorphism classes of objects of* ${}_A\mathcal{C}$ *that arise from irreducible arrows with domain A.*

(v) A morphism of $\mathcal{C}$ is a morphism of Frobenius type if and only if it is a **composite of prime-Frobenius morphisms***.*

(vi) The Frobenioid $\mathcal{C}^{\mathrm{istr}}$ is of **sub-quasi-Frobenius-trivial** *type. Moreover, every* **group-like** *object $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{istr}})$ is* **Frobenius-trivial***.*

Proof. First, we consider assertion (i). Observe that *uniqueness* follows from the fact that $\mathcal{C}$ is *totally epimorphic*. Now it suffices to prove the *existence* of ϕ' as desired, first in the case where ϕ is a *morphism of Frobenius type*, then in the case where ϕ is a *pre-step*, and finally in the case where ϕ is a *pull-back morphism* [cf. the factorization of Definition 1.3, (iv), (a)]. In the first case, since morphisms of Frobenius type are *closed under composition*, with *multiplying Frobenius degrees* [cf. Proposition 1.7, (i); Remark 1.1.1], the existence of a morphism of Frobenius type ϕ' as desired follows immediately from the *existence* and (essential) *uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)]. In the case where ϕ is a pre-step, the existence of a pre-step ϕ' [which, moreover, is *co-angular* if ϕ is] as desired follows immediately from the *factorization* of Definition 1.3, (iv), (a) [cf. also Proposition 1.4, (iv)], together with the (essential) *uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)], and the fact that co-angular morphisms are closed under composition [cf. Proposition 1.7, (i)]. In a similar vein, since pull-back morphisms are LB-invertible [cf. Proposition 1.4, (ii)], and LB-invertible morphisms are closed under composition [cf. Proposition 1.7, (i)], the existence of a pull-back morphism ϕ' in the case where ϕ is a pull-back morphism follows immediately from the *factorization* of Proposition 1.4, (v), together with the (essential) *uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)]. The portion of assertion (i) concerning "$\deg_{\mathrm{Fr}}(-)$", "$\mathrm{Div}(-)$" then follows immediately from Remark 1.1.1. Finally, in light of what we have done so far, the fact that "if ϕ is a(n) *co-angular morphism* (respectively, *base-isomorphism*; *isometry*; *LB-invertible morphism*), then the same is true of ϕ'" follows immediately from the definitions; Remark 1.1.1; the factorization of co-angular morphisms given in Proposition 1.4, (iv); and the fact that co-angular morphisms are closed under composition [cf. Proposition 1.7, (i)]. This completes the proof of assertion (i). Now [in light of the *existence* of morphisms of Frobenius type of a given Frobenius degree — cf. Definition 1.3, (ii)] assertion (ii) follows formally from assertion (i).

Next, we consider assertion (iii). In light of the *existence* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)] and the *equivalences of categories* of Definition 1.3, (iii), (d), the fact that $\Phi(A)$ is *perfect* follows immediately [cf. Remark 1.1.1] from the fact that A is *perfect* [cf. Definition 1.2, (iv)]. Now suppose further that $\mathcal{C}$ is of *isotropic* [so all morphisms of $\mathcal{C}$ are co-angular — cf. Proposition 1.4, (i)] and *Frobenius-normalized* type. Then by the *existence of Frobenius-trivial objects* [cf. Definition 1.3, (i), (a), (b); the isomorphism of Definition 1.3, (iii), (c)], we may assume that A is *Frobenius-trivial*. Now the fact that the monoids $\mathcal{O}^{\triangleright}(A)$ and $\mathcal{O}^{\times}(A)$ are *perfect* follows immediately from the fact that A is *perfect* [cf. Definition 1.2, (iv), applied to the base-identity endomorphisms of Frobenius type of the Frobenius-trivial object A] and *Frobenius-normalized*. This completes the proof of assertion (iii).

Next, we observe that assertion (iv) follows immediately from Proposition 1.7, (v), and the well-known structure of the multiplicative monoid $\mathbb{N}_{\geq 1}$ [cf. also Definition 1.3, (ii)], and that assertion (v) follows immediately from Proposition 1.7, (i); Definition 1.3, (ii).

Finally, we consider assertion (vi). Let $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{istr}})$. Then by Definition 1.3, (i), (a), (b) [applied to the Frobenioid $\mathcal{C}^{\mathrm{istr}}$ — cf. Proposition 1.9, (v)], there exist *co-angular* [cf. Proposition 1.4, (i)] *pre-steps* $\alpha : B \to A$, $\gamma : B \to C$, where C is *Frobenius-trivial*. Thus, for $d \in \mathbb{N}_{\geq 1}$, there exists a base-identity endomorphism of Frobenius type $\phi_C \in \mathrm{End}_{\mathcal{C}}(C)$ such that $\deg_{\mathrm{Fr}}(\phi_C) = d$; by assertion (ii) [cf. also Proposition 1.4, (i)], we may write $\phi_C \circ \gamma = \gamma' \circ \psi$, where $\psi : B \to B'$ is a morphism of Frobenius type, and $\gamma' : B' \to C$ is a co-angular pre-step. Moreover, the portion of assertion (ii) concerning the relationship between $\mathrm{Div}(\gamma)$, $\mathrm{Div}(\gamma')$ implies, in light of the second equivalence of categories of Definition 1.3, (iii), (d), that γ' *factors* through γ, i.e., there exists a co-angular pre-step $\beta : B' \to B$ such that $\gamma \circ \beta = \gamma'$. Thus, if we set $\phi_B \stackrel{\mathrm{def}}{=} \beta \circ \psi \in \mathrm{End}_{\mathcal{C}}(B)$, then $\gamma \circ \phi_B = \phi_C \circ \gamma$. Moreover, since ϕ_C is a base-identity endomorphism of Frobenius degree d, and γ is a pre-step, it follows [cf. Remark 1.1.1] that ϕ_B is also a base-identity endomorphism of Frobenius degree d. Thus, we conclude that B is *quasi-Frobenius-trivial*, hence that A is *sub-quasi-Frobenius-trivial*, as desired. If, moreover, A is *group-like*, then [since $\mathcal{C}^{\mathrm{istr}}$ is a *Frobenioid* — cf. Proposition 1.9, (v)] it follows from Definition 1.3, (i), (a), (b), that there exist [*co-angular* — cf. Proposition 1.4, (i)] *pre-steps* $A' \to A$, $A' \to A''$, where A'' is *Frobenius-trivial*. But by Proposition 1.4, (iii), these pre-steps are *isomorphisms*, so A is Frobenius-trivial, as desired. This completes the proof of assertion (vi). ○

Proposition 1.11. **(Pull-back and Linear Morphisms)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid**. *Then:*

(i) Suppose further that $\mathcal{C}$ *is of* Aut-**ample** *and* **base-trivial** *type. Then the natural projection functor* $\mathcal{C}^{\text{pl-bk}} \to \mathcal{D}$ *is* **full**.

(ii) Suppose further that $\mathcal{C}$ *is* **of unit-trivial type**. *Then the natural projection functor* $\mathcal{C}^{\text{pl-bk}} \to \mathcal{D}$ *is* **faithful**.

(iii) Let $\phi : B \to A$ *be a* **pull-back morphism** *that projects to a morphism* $\phi_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\phi) : B_{\mathcal{D}} \to A_{\mathcal{D}}$ *of* $\mathcal{D}$ *. Then given any* $\alpha \in \mathrm{End}_{\mathcal{C}}(A)$, $\beta_{\mathcal{D}} \in \mathrm{End}_{\mathcal{D}}(B_{\mathcal{D}})$ *such that* $\mathrm{Base}(\alpha) \circ \phi_{\mathcal{D}} = \phi_{\mathcal{D}} \circ \beta_{\mathcal{D}}$*, there exists a* **unique** $\beta \in \mathrm{End}_{\mathcal{C}}(B)$ *such that* $\mathrm{Base}(\beta) = \beta_{\mathcal{D}}$*,* $\alpha \circ \phi = \beta \circ \phi$*.*

(iv) Every **co-angular linear morphism** $\phi : B \to A$ *determines an* **injection** *of monoids*

$$\mathcal{O}^{\triangleright}(A) \hookrightarrow \mathcal{O}^{\triangleright}(B)$$

which is uniquely determined by the condition that $\mathcal{O}^{\triangleright}(A) \ni \alpha \mapsto \beta \in \mathcal{O}^{\triangleright}(B)$ *implies* $\alpha \circ \phi = \phi \circ \beta$*.*

(v) The equivalences of categories of Definition 1.3, (iii), (d), are **"functorial"** *in the following sense: If $\phi : A \to B$ is an arbitrary morphism of $\mathcal{C}^{\mathrm{lin}}$, $\alpha : C \to A$ and $\beta : D \to B$ (respectively, $\alpha : A \to C$ and $\beta : B \to D$) are co-angular pre-steps such that $(\alpha^*)^{-1}(\mathrm{Div}(\alpha)) = \phi^*\{(\beta^*)^{-1}(\mathrm{Div}(\beta))\}$ (respectively, $\mathrm{Div}(\alpha) = \phi^*(\mathrm{Div}(\beta))$), then there exists a* **unique** *morphism $\psi : C \to D$ in $\mathcal{C}^{\mathrm{lin}}$ such that $\beta \circ \psi = \phi \circ \alpha$ (respectively, $\psi \circ \alpha = \beta \circ \phi$). Moreover, ϕ is a* **pull-back** *morphism if and only if ψ is.*

(vi) A pull-back morphism $\phi \in \mathrm{Arr}(\mathcal{C})$ is an **FSM-morphism** *(respectively,* **fiberwise-surjective morphism***;* **monomorphism***;* **irreducible morphism***) if and only if $\mathrm{Base}(\phi) \in \mathrm{Arr}(\mathcal{D})$ is.*

(vii) Let $\phi : A \to B$ be a **co-angular pre-step***; $\epsilon : C \to B$ a morphism. Then there exists a co-angular pre-step $\gamma : D \to C$ and a morphism $\alpha : D \to A$ such that $\epsilon \circ \gamma = \phi \circ \alpha$. In particular, every co-angular pre-step of $\mathcal{C}$ is an* **FSM-morphism***.*

Proof. First, we consider assertion (i). Let $A, B \in \mathrm{Ob}(\mathcal{C})$; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$; $B \stackrel{\mathrm{def}}{=} \mathrm{Base}(B)$; $\phi_{\mathcal{D}} : A_{\mathcal{D}} \to B_{\mathcal{D}}$ a morphism in $\mathcal{D}$. By the *equivalence of categories* of Definition 1.3, (i), (c), it follows that there exists a *pull-back morphism* $\psi : C \to B$ of $\mathcal{C}$ such that $\psi_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\psi) : C_{\mathcal{D}} \to B_{\mathcal{D}}$ of $\mathcal{D}$ defines an object of $\mathcal{D}_{B_{\mathcal{D}}}$ that is isomorphic to the object defined by $\phi_{\mathcal{D}}$. In particular, $C_{\mathcal{D}}$ is *isomorphic* to $A_{\mathcal{D}}$. Since $\mathcal{C}$ is of *base-trivial type*, it thus follows that A, C are *isomorphic*, so we may assume that $A = C$. Thus, ψ projects to a morphism $\psi_{\mathcal{D}} : A_{\mathcal{D}} \to B_{\mathcal{D}}$ of $\mathcal{D}$ such that $\phi_{\mathcal{D}} = \psi_{\mathcal{D}} \circ \delta$, for some $\delta \in \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}})$. Since $\mathcal{C}$ is of Aut-*ample type*, it thus follows that δ lifts to a $\gamma \in \mathrm{Aut}_{\mathcal{C}}(A)$. Thus, taking $\psi \circ \gamma : A \to B$ yields a morphism of $\mathcal{C}$ that projects to $\phi_{\mathcal{D}}$. This completes the proof of assertion (i).

Next, we consider assertion (ii). Let $A, B \in \mathrm{Ob}(\mathcal{C})$; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$; $B_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(B)$; $\phi, \psi : A \to B$ pull-back morphisms of $\mathcal{C}$ that project to the same morphism $A_{\mathcal{D}} \to B_{\mathcal{D}}$ of $\mathcal{D}$. By the definition of a *"pull-back morphism"* [cf. Definition 1.2, (ii)], it thus follows formally that there exist *base-identity endomorphisms* $\alpha, \beta \in \mathrm{End}_{\mathcal{C}}(A)$ such that $\psi = \phi \circ \alpha$, $\phi = \psi \circ \beta$. In particular, we obtain that $\psi = \psi \circ \beta \circ \alpha$, $\phi = \phi \circ \alpha \circ \beta$, hence [again by Definition 1.2, (ii)] that $\alpha \circ \beta$, $\beta \circ \alpha$ are both equal to the identity endomorphism of A, i.e., that $\alpha, \beta \in \mathrm{Aut}_{\mathcal{C}}(A)$. But this implies that $\alpha, \beta \in \mathcal{O}^{\times}(A) = \{1\}$, so $\phi = \psi$, as desired. This completes the proof of assertion (ii).

Next, we consider assertion (iii). The existence and uniqueness of β as asserted follows immediately from the isomorphism of functors appearing in the definition of a "pull-back morphism" [cf. Definition 1.2, (ii)]. This completes the proof of assertion (iii). Now since a *co-angular linear morphism* factors as the composite of a pull-back morphism with a co-angular pre-step [cf. Propositions 1.4, (iv); 1.7, (iii)], the existence of the map "$\hookrightarrow$" of assertion (iv) follows immediately [cf. Proposition 1.7, (iii)] from assertion (iii) and Definition 1.3, (iii), (c); the asserted *injectivity* of this map follows from the *total epimorphicity* of $\mathcal{C}$; the fact that this map is *uniquely determined* by the condition given in assertion (iii) follows from the fact that pre-steps are monomorphisms [cf. Definition 1.3, (v), (a)], and the definition of a "pull-back morphism" in Definition 1.2, (ii).

Next, we consider assertion (v). First, we observe that the *uniqueness* of ψ follows from the fact that β is a *monomorphism* [cf. Definition 1.3, (v), (a)] in the non-resp'd case and from the total epimorphicity of $\mathcal{C}$ applied to α in the resp'd case. When ϕ is a *pull-back morphism* [hence co-angular and linear — cf. Proposition 1.4, (ii)], the existence of a pull-back morphism ψ as desired follows immediately by applying the *equivalence of categories* induced by the projection functor in Definition 1.3, (i), (c); the definition of a "pull-back morphism" in Definition 1.2, (ii); Proposition 1.7, (i), (v) [applied to co-angular linear morphisms]; and the equivalences of categories of Definition 1.3, (iii), (d). When ϕ is an *isometric pre-step*, the existence of an isometric pre-step ψ as desired follows immediately from the *equivalence of categories* of Proposition 1.9, (ii) [in the "case of a co-angular pre-step"]. When ϕ is a *co-angular pre-step*, the existence of a co-angular pre-step ψ as desired follows formally from the equivalences of categories of Definition 1.3, (iii), (d). In light of the *factorizations* of Definition 1.3, (v), (b), (c); Proposition 1.7, (iii), this completes the proof of assertion (v).

Next, we observe that assertion (vi) follows formally from the isomorphism of functors appearing in the definition of a "pull-back morphism" [cf. Definition 1.2, (ii)], together with the *equivalence of categories* induced by the projection functor in Definition 1.3, (i), (c) [cf. also Proposition 1.7, (v), for pull-back morphisms].

Finally, we consider assertion (vii). By applying the *factorizations* of Definition 1.3, (iv), (a); Definition 1.3, (v), (b), it follows immediately that we may assume without loss of generality [from the point of view of showing the existence of γ, α with the desired properties] that ϵ is a *pull-back morphism*, an *isometric pre-step*, a *co-angular pre-step*, or a *morphism of Frobenius type*. If ϵ is a *pull-back morphism*, then it follows immediately [by "pulling back the zero divisor of ϕ via ϵ" — cf. assertion (v)] that there exist a pull-back morphism $\alpha : D \to A$ and a co-angular pre-step $\gamma : D \to C$ such that $\epsilon \circ \gamma = \phi \circ \alpha$. Next, observe that if ϵ is an *isometric pre-step*, then the existence of γ, α with the desired properties follows formally from the *equivalence of categories* of Proposition 1.9, (ii) [induced by ϕ]. Next, observe that if ϵ is a *co-angular pre-step*, then it follows immediately from the second equivalence of categories of Definition 1.3, (iii), (d), that there exist co-angular pre-steps $\alpha : D \to A$, $\gamma : D \to C$ such that $\epsilon \circ \gamma = \phi \circ \alpha$. Finally, we consider the case where ϵ is a *morphism of Frobenius type*. By applying the second equivalence of categories of Definition 1.3, (iii), (d), it follows that we may assume [by replacing ϕ by the composite of ϕ with an appropriate pre-step $A' \to A$] that $\mathrm{Div}(\phi) = \deg_{\mathrm{Fr}}(\epsilon) \cdot x$, for some $x \in \Phi(A)$. Thus, [by applying again the second equivalence of categories of Definition 1.3, (iii), (d)], it follows that there exist a morphism of Frobenius type $\alpha : D \to A$ and a *co-angular pre-step* $\gamma : D \to C$ such that $\epsilon \circ \gamma = \phi \circ \alpha$ [cf. also Proposition 1.10, (i)]. This completes the proof of the existence of γ, α with the desired properties. It thus follows formally that every co-angular pre-step of $\mathcal{C}$ is *fiberwise surjective*. On the other hand, by Definition 1.3, (v), (a), every pre-step is a *monomorphism*. Thus, we conclude that every co-angular pre-step of $\mathcal{C}$ is an FSM-morphism. This completes the proof of assertion (vii). ◯

Remark 1.11.1. Observe that in the situation of Proposition 1.11, (iii), if α is a *morphism of Frobenius type*, and $\beta_{\mathcal{D}}$ is an isomorphism, then β is a *morphism of Frobenius type*. [Indeed, then β is *co-angular* by Definition 1.3, (iii), (b), and *isometric* by Remark 1.1.1.] In particular, it follows [cf. Remark 1.2.1] that [at least in the case of *Frobenioids*] "Frobenius-trivial" implies "universally Div-Frobenius-trivial".

Proposition 1.12. **(Endomorphisms)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid***;* $A \in \mathrm{Ob}(\mathcal{C})$*;* $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$*. Then:*

(i) We have **natural exact sequences of monoids**

$$1 \to \mathcal{O}^\times(A) \to \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}})$$

$$1 \to \mathcal{O}^\rhd(A) \to \mathrm{End}_{\mathcal{C}}(A) \to \mathbb{N}_{\geq 1} \times \mathrm{End}_{\mathcal{D}}(A_{\mathcal{D}})$$

— where the second arrow in each sequence is the natural inclusion; the third arrow of the first sequence is determined by the natural projection functor to $\mathcal{D}$*; the third arrow of the second sequence is determined by the Frobenius degree and the natural projection functor to* $\mathcal{D}$*. If, moreover,* A *is* Aut**-ample** *(respectively,* End**-ample***;* **quasi-Frobenius-trivial***), then the map* $\mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}})$ *(respectively,* $\mathrm{End}_{\mathcal{C}}(A) \to \mathrm{End}_{\mathcal{D}}(A_{\mathcal{D}})$*;* $\mathrm{End}_{\mathcal{C}}(A) \to \mathbb{N}_{\geq 1}$*) is* **surjective***.*

(ii) An endomorphism of A *is a* **sub-automorphism** *[cf. §0] if and only if it is an* **isometric linear endomorphism** *that projects to a sub-automorphism of* $\mathcal{D}$*.*

(iii) A sub-automorphism of A *is an automorphism if and only if it is a* **base-isomorphism***.*

(iv) Suppose that A *is* $\mathrm{Aut}^{\mathrm{sub}}$**-ample***. Then* A *is* Aut**-saturated** *[cf. §0] if and only if* $A_{\mathcal{D}}$ *is.*

Proof. Assertion (i) is immediate from the definitions. The *necessity* of the conditions of assertion (ii), (iii) is immediate from Remark 1.1.1. To prove the *sufficiency* of the conditions of assertion (ii), (iii), it suffices, in light of the equivalence of categories [involving *pull-back morphisms*] of Definition 1.3, (i), (c) [cf. also Proposition 1.11, (iii)], and the fact that endomorphisms are always *co-angular* [cf. Definition 1.3, (iii), (b)], to observe that any LB-invertible linear base-isomorphism [i.e., LB-invertible pre-step] is, in fact, an *isomorphism* [cf. Proposition 1.4, (iii)]. Now assertion (iv) follows formally from assertions (ii), (iii) and the definitions. ◯

Proposition 1.13. **(Rigidity and Slimness)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid***;* $A \in \mathrm{Ob}(\mathcal{C})$*;* $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$*. Suppose further that the category* $\mathcal{D}$ *is* **slim** *[cf. §0]. Then:*

(i) The composite $\mathcal{C}_A \to \mathcal{D}$ of the natural functor $\mathcal{C}_A \to \mathcal{C}$ with the natural projection functor $\mathcal{C} \to \mathcal{D}$ is **rigid** *[cf. §0]. In particular, the functor $\mathcal{C} \to \mathcal{D}$ is rigid.*

(ii) The composite $\mathcal{C}_A \to \mathbb{F}_\Phi$ of the natural functor $\mathcal{C}_A \to \mathcal{C}$ with the functor $\mathcal{C} \to \mathbb{F}_\Phi$ is **rigid**. *In particular, the functor $\mathcal{C} \to \mathbb{F}_\Phi$ is rigid.*

(iii) Suppose, moreover, that every object $A \in \mathrm{Ob}(\mathcal{C})$ satisfies [at least] one of the following two conditions: (a) $\mathcal{O}^\times(A)^{\text{imtr-pre}} = \{1\}$ [cf. Proposition 1.9, (ii)]; (b) $\bigcap_{n\in\mathbb{N}_{\geq 1}} \{\mathcal{O}^\times(A)\}^n = \{1\}$, and, moreover, there exists a co-angular pre-step $B \to A$ [which, by Definition 1.3, (iii), (c), induces a bijection $\mathcal{O}^\times(B) \xrightarrow{\sim} \mathcal{O}^\times(A)$] such that B is **quasi-Frobenius-trivial** *and* **Frobenius-normalized**. *Then the category $\mathcal{C}$ is* **slim**.

Proof. First, we consider assertion (i). Any automorphism α of the functor $\mathcal{C}_A \to \mathcal{D}$ determines an automorphism of the composite functor $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C}_A \to \mathcal{D}$. On the other hand, this composite functor factors as a composite $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{D}_{A_\mathcal{D}} \to \mathcal{D}$, where the first functor $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{D}_{A_\mathcal{D}}$ is [by Definition 1.3, (i), (c)] an *equivalence of categories*. Thus, we conclude that α determines an automorphism of the natural functor $\mathcal{D}_{A_\mathcal{D}} \to \mathcal{D}$, which is necessarily *trivial*, since $\mathcal{D}$ is *slim*. Since A is *arbitrary*, we thus conclude that both $\mathcal{C}_A \to \mathcal{D}$ and $\mathcal{C} \to \mathcal{D}$ are *rigid*. This completes the proof of assertion (i).

Next, we consider assertion (ii). Let α be an automorphism of the functor $\mathcal{C}_A \to \mathbb{F}_\Phi$. By assertion (i), it follows that the automorphisms of objects of $\mathbb{F}_\Phi$ [which, by Proposition 1.5, (i), is itself a *Frobenioid*] induced by α are *base-identity automorphisms*. Since Φ is *divisorial*, hence, in particular, *sharp* [cf. Definition 1.1, (i), (ii)], it thus follows that all of these automorphisms are trivial, hence that α is trivial. Since A is *arbitrary*, we thus conclude that both $\mathcal{C}_A \to \mathbb{F}_\Phi$ and $\mathcal{C} \to \mathbb{F}_\Phi$ are *rigid*. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). Let α be an automorphism of the natural functor $\mathcal{C}_A \to \mathcal{C}$. By assertion (i), it follows that the automorphisms of objects of $\mathcal{C}$ induced by α are *base-identity automorphisms*, i.e., belong to "$\mathcal{O}^\times(-)$". Moreover, the *functoriality* of the automorphisms induced by α with respect to *isometric pre-steps* implies that these automorphisms belong to "$\mathcal{O}^\times(-)^{\text{imtr-pre}}$". Similarly, the *functoriality* of the automorphisms induced by α with respect to *base-identity endomorphisms* implies that, at least in the case of *quasi-Frobenius-trivial*, *Frobenius-normalized* objects — hence also [cf. Definition 1.3, (iii), (c)] objects as in (b) of the statement of assertion (iii) — these automorphisms belong to "$\bigcap_{n\in\mathbb{N}_{\geq 1}} \{\mathcal{O}^\times(-)\}^n$". Thus, we conclude that under either of the assumptions (a), (b) in the statement of assertion (iii), the automorphisms induced by α are *trivial*. This completes the proof of assertion (iii). ◯

Remark 1.13.1. Note that if the hypothesis of Proposition 1.13, (iii), fails to hold, then it is not necessarily the case that $\mathcal{C}$ is slim. Indeed, if M is a *perfect* *pre-divisorial monoid*, and $\mathcal{C}$ is a one-object category whose unique object has endomorphism monoid equal to the *elementary Frobenioid* $\mathbb{F}_M$ [so $\mathcal{C}$ equipped with the functor of one-object categories determined by the natural morphism of monoids $\mathbb{F}_M \to \mathbb{F}_{M^{\text{char}}}$ is a *Frobenioid*, by Proposition 1.5, (i)], then any collection of elements $\{\alpha_n\}_{n\in\mathbb{N}_{\geq 1}}$ of $M^{\pm}$ such that $\alpha_{nm} = m \cdot \alpha_n$ for all $n, m \in \mathbb{N}_{\geq 1}$ determines an automorphism of the natural functor $\mathcal{C}_A \to \mathcal{C}$ [which is *nontrivial* as soon as any of the α_n is nonzero] by assigning to an arrow $\phi : B \to A$ of $\mathcal{C}$ the automorphism $\alpha_{\deg_{\text{Fr}}(\phi)} \in \text{Aut}_{\mathcal{C}}(B)$.

One key result for analyzing the *category-theoretic structure of Frobenioids* [cf. §3] is the following:

Proposition 1.14. **(Irreducible Morphisms)** *Let* Φ *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}$*;* $\mathcal{C} \to \mathbb{F}_\Phi$ *a* **Frobenioid of isotropic type***;* $\phi \in \text{Arr}(\mathcal{C})$. *Suppose further that* $\mathcal{D}$ *is* **of FSMFF-type** *[cf. §0]. Then:*

(i) ϕ *is* **irreducible** *if and only if* ϕ *is one of the following: (a) a prime-Frobenius morphism; (b) a step such that* $\text{Div}(\phi)$ *is irreducible; (c) a pull-back morphism such that* $\text{Base}(\phi)$ *is an irreducible morphism of* $\mathcal{D}$.

(ii) ϕ *is a* **pre-step** *if and only if it is an* **FSM-morphism** *that is* **mid-adjoint** *[cf. §0] to the irreducible morphisms which are not pre-steps.*

(iii) Suppose that ϕ *is* **irreducible**. *Then* ϕ *is a* **non-pre-step** *if and only if the following condition holds: There exists an* $N \in \mathbb{N}_{\geq 1}$ *such that for every equality of composites in* $\mathcal{C}$

$$\alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_2 \circ \alpha_1 = \psi \circ \phi$$

— where $\alpha_1, \ldots, \alpha_n$*,* ψ *are* **FSMI-morphisms** *[cf. §0] — it holds that* $n \leq N$.

(iv) Let $\alpha \circ \beta = \beta' \circ \alpha'$ *be an equality of composites of* $\mathcal{C}$*, where* $\deg_{\text{Fr}}(\beta) = \deg_{\text{Fr}}(\beta')$*, and* α*,* α' *are* **irreducible**. *Then* α *is a* **prime-Frobenius morphism** *if and only if* α' *is; moreover,* $\deg_{\text{Fr}}(\alpha) = \deg_{\text{Fr}}(\alpha')$.

(v) Suppose further that Φ *is* **non-dilating***, and that* ϕ *is a* **non-pre-step irreducible endomorphism** *of a* **non-group-like** *object* $A \in \text{Ob}(\mathcal{C})$. *Then* ϕ *is a* Div**-identity prime-Frobenius endomorphism** *if and only if the following condition holds: For every* **step** $\alpha : A \to B$*, there exists a non-pre-step irreducible morphism* $\psi : B \to B'$ *and a step* $\beta : B \to B'$ *such that* $\psi \circ \alpha = \beta \circ \alpha \circ \phi$.

Proof. First, we consider assertion (i). The *sufficiency* of the condition of assertion (i) follows for morphisms as in (a) (respectively, (b); (c)) from Proposition 1.10, (iv) (respectively, the equivalences of categories of Definition 1.3, (iii), (d) [cf. also Propositions 1.4, (i); 1.7, (v)]; Proposition 1.11, (vi)). To verify the *necessity* of the condition of assertion (i), observe that it follows formally from the *factorization* of Definition 1.3, (iv), (a), that ϕ is either a morphism of Frobenius type, a step, or a pull-back morphism. Thus, by Propositions 1.7, (v); 1.10, (iv); 1.11, (vi), the *irreducibility* of ϕ implies immediately that ϕ is a morphism as in (a), (b), or (c).

Next, we consider assertion (ii). To verify the *sufficiency* of the condition of assertion (ii), observe first that by the *factorization* of Definition 1.3, (iv), (a), we may write $\phi = \alpha \circ \beta \circ \gamma$, where α is a pull-back morphism, β is a pre-step, and γ is a morphism of Frobenius type. By assertion (i) [cf. also Proposition 1.10, (v)], it follows that γ is an *isomorphism*; thus, we may assume without loss of generality that γ is the identity, i.e., $\phi = \alpha \circ \beta$. On the other hand, it follows formally from the fact that ϕ is an FSM-morphism that α is *fiberwise-surjective* [cf. §0]. Next, I *claim* that α is a *monomorphism*. Indeed, write $\phi : A \to B$, $\beta : A \to C$, $\alpha : C \to B$; let $\epsilon_1, \epsilon_2 : D \to C$ be such that $\alpha \circ \epsilon_1 = \alpha \circ \epsilon_2$. Then by Remark 1.1.1, it follows immediately that $\deg_{\mathrm{Fr}}(\epsilon_1) = \deg_{\mathrm{Fr}}(\epsilon_2)$, $\mathrm{Div}(\epsilon_1) = \mathrm{Div}(\epsilon_2)$, hence, by applying the *factorization* of Definition 1.3, (iv), (a) [and the *total epimorphicity* of $\mathcal{C}$; cf. also Definition 1.3, (ii), and the equivalences of categories of Definition 1.3, (iii), (d)], we may assume without loss of generality [from the point of view of showing that α is a monomorphism] that ϵ_1, ϵ_2 are *pull-back morphisms*. Now by "*adding* the pull-backs of $\beta_*(\mathrm{Div}(\beta))$ via ϵ_1, ϵ_2" [cf. Proposition 1.11, (v); the equivalences of categories of Definition 1.3, (iii), (d)], it follows that there exists a *pre-step* $\zeta : E \to D$ such that there exist $\gamma_1, \gamma_2 \in \mathrm{Arr}(\mathcal{C})$ satisfying $\epsilon_1 \circ \zeta = \beta \circ \gamma_1$, $\epsilon_2 \circ \zeta = \beta \circ \gamma_2$. Thus, we have: $\phi \circ \gamma_1 = \alpha \circ \beta \circ \gamma_1 = \alpha \circ \epsilon_1 \circ \zeta = \alpha \circ \epsilon_2 \circ \zeta = \alpha \circ \beta \circ \gamma_2 = \phi \circ \gamma_2$. But since ϕ is [an FSM-morphism, hence, in particular] a *monomorphism*, it follows that $\gamma_1 = \gamma_2$, hence [by the *total epimorphicity* of $\mathcal{C}$] that $\epsilon_1 = \epsilon_2$. This completes the proof of the *claim*. In particular, we conclude that α is an *FSM-morphism*.

Thus, it follows [cf. Proposition 1.11, (vi)] that $\mathrm{Base}(\alpha)$ is an FSM-morphism of $\mathcal{D}$. Since, however, we are operating under the assumption that $\mathcal{D}$ is *of FSMFF-type*, it follows that if α is *not* an isomorphism, then $\mathrm{Base}(\alpha)$ admits a *subordinate* [cf. condition (a) of the definition of a "category of FSMFF-type" in §0] FSMI-morphism, which implies [cf. Proposition 1.11, (vi)] that α admits a subordinate FSMI-morphism [which is also a pull-back morphism]. Since ϕ, however, is assumed to be *mid-adjoint* to the irreducible morphisms which are not pre-steps, we thus obtain a contradiction. Thus, α is an isomorphism, so ϕ is a *pre-step*. This completes the proof of the *sufficiency* of the condition of assertion (ii). Next, we consider the *necessity* of the condition of assertion (ii). Thus, suppose that ϕ is a *pre-step*. By Proposition 1.11, (vii), ϕ is an *FSM-morphism*; by Proposition 1.7, (v), ϕ is *mid-adjoint* to the non-pre-steps. This completes the proof of assertion (ii).

Next, we consider assertion (iii). By assertion (i), it suffices to show that assertion (iii) holds for each of the three types of morphisms "(a), (b), (c)" discussed in assertion (i). If ϕ is an *irreducible pre-step*, then it follows immediately — by taking ψ to be a prime-Frobenius morphism of *increasingly large* Frobenius degree [cf. Proposition 1.10, (ii)] — that the condition in the statement of assertion (iii) is *false* [as desired]. On the other hand, if ϕ is a *non-pre-step*, then it is an *isometry*. Now if the condition in the statement of assertion (iii) is *false*, then there exist equalities

$$\alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_2 \circ \alpha_1 = \psi \circ \phi$$

where $\alpha_1, \ldots, \alpha_n$, ψ are *FSMI-morphisms*, and n is *arbitrarily large*. Here, we note that since $\psi \circ \phi$ and ψ are FSM-morphisms, it thus follows formally that ϕ is also an *FSM-morphism*. Next, observe that since ϕ is an isometry, it follows from the fact that ψ is *irreducible* [cf. also assertion (i); Definition 1.1, (ii), (b); Remark 1.1.1] that $\text{Div}(\psi \circ \phi)$ is either zero or irreducible; since, moreover, $\deg_{\text{Fr}}(\psi \circ \phi)$ always divides a product of two prime numbers [cf. assertion (i); the irreducibility of ϕ, ψ], it thus follows that in *any* factorization of $\psi \circ \phi$ by FSMI-morphisms, all but *three* [i.e., corresponding to *two* possible prime factors of the Frobenius degree, plus *one* possible irreducible factor of the zero divisor] of the factorizing FSMI-morphisms are *pull-back morphisms* [cf. assertion (i)]. On the other hand, this implies that factorizations of arbitrarily large length determine chains of FSMI-morphisms [cf. assertion (i); Proposition 1.11, (vi)] originating from the projection to $\mathcal{D}$ of the domain of ϕ which are also of *arbitrarily large length*, a contradiction [cf. condition (b) of the definition of a "category of FSMFF-type" in §0]. This completes the proof of assertion (iii).

Next, we consider assertion (iv). Since $\deg_{\text{Fr}}(\beta) = \deg_{\text{Fr}}(\beta')$, it follows from Remark 1.1.1 that $\deg_{\text{Fr}}(\alpha) = \deg_{\text{Fr}}(\alpha')$, hence [since α, α' are *irreducible*], by assertion (i), that α is a *prime-Frobenius morphism* if and only if α' is. This completes the proof of assertion (iv).

Finally, we consider assertion (v). First, we observe that the *necessity* of the condition in the statement of assertion (v) [where we take ψ to be a prime-Frobenius morphism such that $\deg_{\text{Fr}}(\phi) = \deg_{\text{Fr}}(\psi)$] follows immediately from Proposition 1.10, (i) [cf. also Definition 1.3, (ii); assertion (i); the first equivalence of categories of Definition 1.3, (iii), (d)]. Next, we consider *sufficiency*. To show that ϕ is a *prime-Frobenius morphism*, it suffices [by assertion (i)] to show that it is *not* a pull-back morphism. Thus, suppose that ϕ is a pull-back morphism. Since A is *non-group-like*, it follows [cf. Proposition 1.4, (iii)] that there exists a step $\alpha : A \to B$, hence that there exist ψ, β as in the statement of assertion (v). By assertions (i), (iv), ψ is also a pull-back morphism. Write $x \stackrel{\text{def}}{=} \text{Div}(\alpha)$, $y \stackrel{\text{def}}{=} \alpha^*(\text{Div}(\beta))$ [where, for simplicity, we write α^* for $\Phi(\text{Base}(\alpha))$]. Then by Remark 1.1.1, it follows that $\phi^*(x+y) = x$, i.e., that $\phi^*(x) \leq x$. Since $x \neq 0$ is *arbitrary* [cf. the first equivalence of categories of Definition 1.3, (iii), (d)], it thus follows from our assumption that Φ is *non-dilating* that ϕ^* is the *identity morphism*. But this implies that $x + y = x$, i.e., [since Φ is *integral* — cf. Definition 1.1, (i)] that $y = 0$, in contradiction to our assumption that β is a *step* [i.e., not just a [necessarily co-angular!] pre-step — cf. Proposition 1.4, (i), (iii)]. Thus, we conclude [cf. assertion (iv)] that ϕ, ψ are *prime-Frobenius morphisms*, of the same Frobenius degree. In particular, if we write $x \stackrel{\text{def}}{=} \text{Div}(\alpha)$, $y \stackrel{\text{def}}{=} \alpha^*(\text{Div}(\beta))$, then it follows [cf. Remark 1.1.1] that $\phi^*(x + y) = \deg_{\text{Fr}}(\phi) \cdot x$, i.e., that $\phi^*(x) \preccurlyeq x$ [cf. §0], hence [by our assumption that Φ is *non-dilating*] that ϕ^* is the *identity morphism*. This completes the proof of assertion (v). ◯

Section 2: Frobenius Functors

In the present §2, we discuss various *functors* between Frobenioids that are intended to be reminiscent of the *Frobenius morphism* in positive characteristic scheme theory.

In the following discussion, we maintain the notation of §1. Also, we assume that we have been given a *divisorial monoid* Φ on a connected, totally epimorphic category $\mathcal{D}$ and a *Frobenioid* $\mathcal{C} \to \mathbb{F}_\Phi$.

Proposition 2.1. **(The Naive Frobenius Functor)** *Let $d \in \mathbb{N}_{\geq 1}$. Then:*

(i) The assignment

$$A \mapsto A'; \quad \phi \mapsto \phi'$$

— where $\phi : A \to B$ is an arbitrary morphism of $\mathcal{C}$; $\alpha : A \to A'$, $\beta : B \to B'$ are **morphisms of Frobenius type** *of Frobenius degree d; ϕ' is the unique morphism such that $\phi' \circ \alpha = \beta \circ \phi$ [cf. Proposition 1.10, (i)] — determines a* **functor**

$$\Psi : \mathcal{C} \to \mathcal{C}$$

[well-defined up to isomorphism of functors] which we shall refer to as the **naive Frobenius functor** *[of degree d] on $\mathcal{C}$. Finally, the* **composite** *of the naive Frobenius functor of degree $d_1 \in \mathbb{N}_{\geq 1}$ on $\mathcal{C}$ with the naive Frobenius functor of degree $d_2 \in \mathbb{N}_{\geq 1}$ on $\mathcal{C}$ is isomorphic to the* **naive Frobenius functor of degree** $d_1 \cdot d_2$ *on $\mathcal{C}$.*

(ii) The functor Ψ of (i) is **"1-compatible"***, relative to $\mathcal{C} \to \mathbb{F}_\Phi$, with the functor $\mathbb{F}_\Phi \to \mathbb{F}_\Phi$ — which we shall refer to as the* **Frobenius functor** *on $\mathbb{F}_\Phi$ — determined [cf. Definition 1.1, (iii)] by the endomorphism of the functor Φ given by* **multiplication by d***. Moreover, if, in the notation of (i), $A = A'$, A is* **Frobenius-normalized***, and the morphism $\alpha : A \to A'$ is taken to be a base-identity endomorphism, then the morphism of monoids $\mathcal{O}^\rhd(A) \to \mathcal{O}^\rhd(A')$ induced by Ψ is given by* **raising to the d-th power***.*

(iii) $\mathcal{C}$ is **of perfect type** *if and only if Ψ is an* **equivalence of categories***.*

Proof. Assertions (i), (ii) follow immediately from Definition 1.3, (ii); Proposition 1.10, (i) [cf. also Proposition 1.7, (i)]. Finally, we consider assertion (iii). The *sufficiency* of the condition of assertion (iii) follows immediately from the definition of "perfect" [cf. Definition 1.2, (iv); Remark 1.1.1]. To verify *necessity*, suppose that $\mathcal{C}$ is *of perfect type*. Then the *essential surjectivity* of Ψ follows immediately from the definition of "perfect" [cf. Definition 1.2, (iv)]. To verify that Ψ is *fully faithful*, we reason as follows: In light of the 1-*compatibility* of Ψ with the Frobenius functor on $\mathbb{F}_\Phi$ [cf. assertion (ii)], the *total epimorphicity* of $\mathcal{C}$, and the *factorization* of Definition 1.3, (iv), (a), it follows immediately that one may reduce to the case of *linear morphisms* by applying the *existence* and (essential) *uniqueness* of morphisms of Frobenius type of a given Frobenius degree [cf. Definition 1.3, (ii)]. Moreover, by applying the equivalence of categories [involving *pull-backs*] of Definition 1.3, (i), (c) [cf. also the isomorphism of functors appearing in the definition of a "pull-back morphism" in Definition 1.2, (ii)], one may reduce further to the case of *pre-steps*. But the case of pre-steps follows immediately from the definition of "perfect" [cf. Definition 1.2, (iv)]. This completes the proof of assertion (iii). ◯

Remark 2.1.1. If $\mathcal{C}$ is *of perfect type*, then for any $d = a/b \in \mathbb{Q}_{>0}$, where $a, b \in \mathbb{N}_{\geq 1}$, composing the *naive Frobenius functor of degree* a with some quasi-inverse functor to the *naive Frobenius functor of degree* b yields a *"naive Frobenius functor of degree* d*"*, which, by Proposition 2.1, (i), is independent of the choice of a, b.

Proposition 2.2. **(The Functor $\mathcal{O}^{\triangleright}(-)$)** *Write $\mathcal{D}^*$ for the category whose objects are the objects of $\mathcal{C}^{\text{istr}}$ and whose morphisms are given as follows:*

$$\text{Hom}_{\mathcal{D}^*}(A, B) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{D}}(A_{\mathcal{D}}, B_{\mathcal{D}})$$

[where $A, B \in \text{Ob}(\mathcal{C}^{\text{istr}})$; $A_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(A)$; $B_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(B)$]. Thus, the natural projection functor $\mathcal{C} \to \mathcal{D}$ determines natural functors $\mathcal{C}^{\text{istr}} \to \mathcal{D}^* \to \mathcal{D}$. Moreover:

(i) The functor $\mathcal{D}^* \to \mathcal{D}$ is an equivalence of categories.

(ii) There is a **unique contravariant functor**

$$\mathcal{D}^* \to \mathfrak{Mon}$$

$$\text{Ob}(\mathcal{C}^{\text{istr}}) = \text{Ob}(\mathcal{D}^*) \ni A \mapsto \mathcal{O}^{\triangleright}(A) \in \text{Ob}(\mathfrak{Mon})$$

such that for $\phi : A \to B$ in $\text{Arr}(\mathcal{C}^{\text{istr}})$, with image $\phi_{\mathcal{D}^*}$ in $\mathcal{D}^*$, the following properties are satisfied: (a) if ϕ is a [necessarily co-angular – cf. Proposition 1.4, (i)] **linear** *morphism, then $\mathcal{O}^{\triangleright}(\phi_{\mathcal{D}^*}) : \mathcal{O}^{\triangleright}(B) \to \mathcal{O}^{\triangleright}(A)$ is the inclusion of Proposition 1.11, (iv); (b) if ϕ is a [necessarily co-angular] pre-step, then $\mathcal{O}^{\triangleright}(\phi_{\mathcal{D}^*}) : \mathcal{O}^{\triangleright}(B) \to \mathcal{O}^{\triangleright}(A)$ is the bijection of Definition 1.3, (iii), (c). By abuse of notation, we shall also denote by "$\mathcal{O}^{\triangleright}(-)$" the restriction of this functor on $\mathcal{D}^*$ to $(\mathcal{C}^{\text{istr}})^{\text{lin}}$. Finally, by applying the equivalence of categories of (i), we obtain a contravariant functor $\mathcal{D} \to \mathfrak{Mon}$, which, by abuse of notation, we shall also denote by "$\mathcal{O}^{\triangleright}(-)$", and which is well-defined up to isomorphism.*

(iii) The assignment $\text{Ob}(\mathcal{C}^{\text{istr}}) \ni A \mapsto \mathcal{O}^{\times}(A)$ ($\subseteq \mathcal{O}^{\triangleright}(A)$) determines a **subfunctor** *of the functor of (ii) which is equal to the subfunctor $A \mapsto \mathcal{O}^{\triangleright}(A)^{\pm}$ [cf. the notation of §0]. Moreover, the operation "$\text{Div}(-)$" determines a* **functorial homomorphism**

$$\mathcal{O}^{\triangleright}(A) \to \Phi(A)$$

that induces an **inclusion** *$\mathcal{O}^{\triangleright}(A)^{\text{char}} = \mathcal{O}^{\triangleright}(A)/\mathcal{O}^{\times}(A) \hookrightarrow \Phi(A)$ [cf. the notation of §0].*

(iv) If $\phi : A \to A^{\text{istr}}$ is an **isotropic hull** *in $\mathcal{C}$, then ϕ determines a natural inclusion of monoids $\mathcal{O}^{\triangleright}(A) \hookrightarrow \mathcal{O}^{\triangleright}(A^{\text{istr}})$.*

Proof. As for assertion (i), *essential surjectivity* follows immediately from Definition 1.3, (i), (a) [i.e., applied to the Frobenioid $\mathcal{C}^{\text{istr}}$ — cf. Proposition 1.9, (v)], while *fully faithfulness* follows formally from the definition of the category $\mathcal{D}^*$. Next, we consider assertion (ii). Let $A, B \in \text{Ob}(\mathcal{C}^{\text{istr}})$; $A_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(A)$; $B_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(B)$. Now observe that any morphism $A_{\mathcal{D}} \to B_{\mathcal{D}}$ in $\mathcal{D}$ factors as the *composite* of an *isomorphism* $A_{\mathcal{D}} \stackrel{\sim}{\to} C_{\mathcal{D}}$, where $C \in \text{Ob}(\mathcal{C}^{\text{istr}})$, $C_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(C)$, with a morphism $C_{\mathcal{D}} \to B_{\mathcal{D}}$ which is the projection to $\mathcal{D}$ of a *pull-back morphism* $C \to B$ of $\mathcal{C}^{\text{istr}}$ [cf. Definition 1.3, (i), (c)]; moreover, this pull-back morphism is *uniquely determined*, as an object of $\mathcal{C}^{\text{istr}}_B$, up to isomorphism [cf. the isomorphism of functors appearing in the definition of a "pull-back morphism" in Definition 1.2, (ii)]. Thus, it follows that to construct the desired functor "$\mathcal{O}^{\rhd}(-)$" on $\mathcal{D}^*$, it suffices to construct, for each *isomorphism* $\phi_{\mathcal{D}} : A_{\mathcal{D}} \to B_{\mathcal{D}}$, a bijection $\mathcal{O}^{\rhd}(\phi_{\mathcal{D}}) : \mathcal{O}^{\rhd}(A) \stackrel{\sim}{\to} \mathcal{O}^{\rhd}(B)$ which is compatible with composition of isomorphisms. [Indeed, once one constructs "$\mathcal{O}^{\rhd}(-)$" in this fashion, the fact that this "$\mathcal{O}^{\rhd}(-)$" is *compatible with composites* of morphisms of $\mathcal{D}^*$ follows immediately from the *manifest functoriality* of the inclusion of Proposition 1.11, (iv).] This may be done by using *co-angular pre-steps* $\gamma_A : C \to A$, $\gamma_B : C \to B$ such that $\phi_{\mathcal{D}} = \text{Base}(\gamma_B) \circ \text{Base}(\gamma_A)^{-1}$ [cf. Definition 1.3, (i), (b)] and the bijections $\mathcal{O}^{\rhd}(\gamma_A) : \mathcal{O}^{\rhd}(A) \stackrel{\sim}{\to} \mathcal{O}^{\rhd}(C)$, $\mathcal{O}^{\rhd}(\gamma_B) : \mathcal{O}^{\rhd}(B) \stackrel{\sim}{\to} \mathcal{O}^{\rhd}(C)$ determined by γ_A, γ_B [cf. Definition 1.3, (iii), (c)]. Note, moreover, that the resulting bijection $\mathcal{O}^{\rhd}(\gamma_A)^{-1} \circ \mathcal{O}^{\rhd}(\gamma_B)$ is *independent* of the choice of γ_A, γ_B. [Indeed, if $\delta_A : D \to A$, $\delta_B : D \to B$ satisfy $\phi_{\mathcal{D}} = \text{Base}(\delta_B) \circ \text{Base}(\delta_A)^{-1}$, then there exist [cf. Definition 1.3, (i), (b)] co-angular pre-steps $\epsilon_C : E \to C$, $\epsilon_D : E \to D$ such that

$$\text{Base}(\gamma_A) \circ \text{Base}(\epsilon_C) = \text{Base}(\delta_A) \circ \text{Base}(\epsilon_D)$$

— which [since $\text{Base}(\delta_B) \circ \text{Base}(\delta_A)^{-1} = \text{Base}(\gamma_B) \circ \text{Base}(\gamma_A)^{-1}$] implies that

$$\text{Base}(\gamma_B) \circ \text{Base}(\epsilon_C) = \text{Base}(\delta_B) \circ \text{Base}(\epsilon_D)$$

hence that

$$\mathcal{O}^{\rhd}(\gamma_A \circ \epsilon_C) = \mathcal{O}^{\rhd}(\delta_A \circ \epsilon_D); \quad \mathcal{O}^{\rhd}(\gamma_B \circ \epsilon_C) = \mathcal{O}^{\rhd}(\delta_B \circ \epsilon_D)$$

[cf. Definition 1.3, (iii), (c)], i.e., that

$$\mathcal{O}^{\rhd}(\epsilon_C) \circ \mathcal{O}^{\rhd}(\gamma_A) = \mathcal{O}^{\rhd}(\epsilon_D) \circ \mathcal{O}^{\rhd}(\delta_A); \quad \mathcal{O}^{\rhd}(\epsilon_C) \circ \mathcal{O}^{\rhd}(\gamma_B) = \mathcal{O}^{\rhd}(\epsilon_D) \circ \mathcal{O}^{\rhd}(\delta_B)$$

— that is to say

$$\mathcal{O}^{\rhd}(\gamma_A)^{-1} \circ \mathcal{O}^{\rhd}(\gamma_B) = \mathcal{O}^{\rhd}(\delta_A)^{-1} \circ \mathcal{O}^{\rhd}(\delta_B)$$

as desired.] This completes the proof of assertion (ii). Assertion (iii) is immediate from the definitions [cf. also Definition 1.3, (iii), (b); Definition 1.3, (vi)]. Assertion (iv) follows immediately from the "universal property of an isotropic hull" [cf. Definition 1.2, (iv)] and the fact that an isotropic hull is always a *monomorphism* [cf. Definition 1.3, (v), (a)]. ◯

Definition 2.3. We shall refer to as a *characteristic splitting* on $\mathcal{C}$ a *subfunctor in monoids*

$$\tau : (\mathcal{C}^{\text{istr}})^{\text{lin}} \to \mathfrak{Mon}$$

of the functor $\mathcal{O}^{\rhd}(-) : (\mathcal{C}^{\text{istr}})^{\text{lin}} \to \mathfrak{Mon}$ of Proposition 2.2, (ii), such that the following properties hold: (a) for every $A \in \text{Ob}(\mathcal{C}^{\text{istr}})$, $\tau(A)$ maps bijectively onto $\mathcal{O}^{\rhd}(A)^{\text{char}}$, hence determines a *splitting of monoids*

$$\mathcal{O}^{\times}(A) \times \tau(A) \xrightarrow{\sim} \mathcal{O}^{\rhd}(A)$$

which is *functorial* in A; (b) for every *isotropic hull* $A \to A^{\text{istr}}$ of $\mathcal{C}$, $\tau(A^{\text{istr}}) \subseteq \mathcal{O}^{\rhd}(A^{\text{istr}})$ lies in the image of $\mathcal{O}^{\rhd}(A)$ via the natural injection of Proposition 2.2, (iv).

Definition 2.4.

(i) We shall say that $M \in \text{Ob}(\mathfrak{Mon})$ is *perf-factorial* if it satisfies the following conditions:

(a) M is *divisorial*.

(b) For every $\mathfrak{p} \in \text{Prime}(M)$ [cf. §0], the monoid $M_{\mathfrak{p}}$ is *monoprime* [cf. §0].

(c) The map

$$M^{\text{pf}} \to M^{\text{rlf}}_{\text{factor}} \stackrel{\text{def}}{=} \prod_{\mathfrak{p} \in \text{Prime}(M)} M^{\text{rlf}}_{\mathfrak{p}}$$
$$a \mapsto (\dots, \sup(\text{Bound}_{\mathfrak{p} \bigcup \{0\}}(a)), \dots)$$

[where we write $M^{\text{rlf}}_{\mathfrak{p}} \stackrel{\text{def}}{=} M^{\text{pf}}_{\mathfrak{p}} \otimes \mathbb{R}_{\geq 0}$; we refer to §0 for more on the notation "M^{pf}", "$M^{\text{pf}}_{\mathfrak{p}}$", "$\otimes\mathbb{R}_{\geq 0}$"; the "sup" at the index $\mathfrak{p}$ is taken in $M^{\text{rlf}}_{\mathfrak{p}}$] is a *well-defined* [i.e., the various $\text{Bound}_{\mathfrak{p} \bigcup \{0\}}(a) \subseteq M^{\text{rlf}}_{\mathfrak{p}}$ are bounded subsets] *injective homomorphism of monoids* whose image lies in $\prod_{\mathfrak{p} \in \text{Prime}(M)} M^{\text{pf}}_{\mathfrak{p}}$, hence determines an injective homomorphism

$$M^{\text{pf}} \hookrightarrow M^{\text{pf}}_{\text{factor}} \stackrel{\text{def}}{=} \prod_{\mathfrak{p} \in \text{Prime}(M)} M^{\text{pf}}_{\mathfrak{p}}$$

which we shall refer to as the *factorization homomorphism* of M^{pf}. We shall often use the factorization homomorphism to regard M^{pf} as a submonoid of $M^{\text{pf}}_{\text{factor}} \subseteq M^{\text{rlf}}_{\text{factor}}$.

(d) If $a \in M^{\text{rlf}}_{\text{factor}}$, then we shall write $\text{Supp}(a) \subseteq \text{Prime}(M)$ for the subset of $\mathfrak{p}$ for which the component of a at $\mathfrak{p}$ is nonzero and refer to $\text{Supp}(a)$ as the *support* of a. Then the submonoid $M^{\text{pf}} \subseteq M^{\text{pf}}_{\text{factor}}$ satisfies the following property: If $a \in M^{\text{pf}}_{\text{factor}}$ and $b \in M^{\text{pf}}$ satisfy $\text{Supp}(a) \subseteq \text{Supp}(b)$, then $a \in M^{\text{pf}}$. [Thus, in particular, if $a, b \in M^{\text{pf}}$, then an inequality "$a \leq b$" holds in M^{pf} if and only if it holds in $M^{\text{pf}}_{\text{factor}}$.]

Now suppose that M is *perf-factorial*. Then we shall refer to the [subset which is easily verified to be a] submonoid

$$M^{\rm rlf} \subseteq M^{\rm rlf}_{\rm factor}$$

of elements $a \in M^{\rm rlf}_{\rm factor}$ such that there exists a $b \in M^{\rm pf}$ satisfying $\mathrm{Supp}(a) \subseteq \mathrm{Supp}(b)$ as the *realification* of M. Thus, both the submonoid $M^{\rm pf} \subseteq M^{\rm pf}_{\rm factor}$ and the submonoid $M^{\rm rlf} \subseteq M^{\rm rlf}_{\rm factor}$ are completely determined by the collection of subsets $\mathrm{Supp}(a) \subseteq \mathrm{Prime}(M)$, as a ranges over the elements of $M^{\rm pf}$; if $a, b \in M^{\rm rlf}$, then an inequality "$a \leq b$" holds in $M^{\rm rlf}$ if and only if it holds in $M^{\rm rlf}_{\rm factor}$;

$$(M^{\rm rlf})^{\rm gp} \ \subseteq \ (M^{\rm rlf}_{\rm factor})^{\rm gp} \ = \prod_{\mathfrak{p} \in \mathrm{Prime}(M)} (M^{\rm rlf}_{\mathfrak{p}})^{\rm gp}$$

is an $\mathbb{R}$-vector space. Finally, one verifies immediately that $M^{\rm pf}$, $M^{\rm rlf}$ are also *perf-factorial*.

(ii) Let Λ be a *monoid type*. Then we shall say that Λ *supports* $M \in \mathrm{Ob}(\mathfrak{Mon})$ if any of the following conditions hold: (a) $\Lambda = \mathbb{Z}$; (b) $\Lambda = \mathbb{Q}$, and M is *perfect*; (c) $\Lambda = \mathbb{R}$, M is *perfect* and *perf-factorial*, and for every $\mathfrak{p} \in \mathrm{Prime}(M)$, the monoid $M_{\mathfrak{p}}$ is $\mathbb{R}$-monoprime. Note that if Λ supports M, then $\Lambda_{>0}$ acts naturally on M.

(iii) Let Λ be a *monoid type* that *supports* Φ [cf. Definition 1.1, (ii)]; $d \in \Lambda_{>0}$. Then we shall write

$$d \cdot \Phi(-) \subseteq \Phi(-)$$

for the subfunctor of Φ determined by the assignment $\mathrm{Ob}((\mathcal{C}^{\rm istr})^{\rm lin}) \ni A \mapsto d \cdot (\Phi(A))$ $(\subseteq \Phi(A))$ and

$$\mathcal{C}^{(d)} \subseteq \mathcal{C}$$

for the subcategory determined by the arrows whose zero divisor lies in $d \cdot \Phi(-) \subseteq \Phi(-)$. Finally, multiplication by d on $\Phi(-)$ determines a *"Frobenius functor"* [associated to d — cf. Proposition 2.1, (ii)]

$$\mathbb{F}_\Phi \to \mathbb{F}_\Phi$$

which is compatible with Frobenius degrees and the natural projection functor $\mathbb{F}_\Phi \to \mathcal{D}$.

Proposition 2.5. (The Unit-linear Frobenius Functor) *Let τ be a* **characteristic splitting** *on $\mathcal{C}$; Λ a* **monoid type** *that* **supports** Φ*; $d \in \Lambda_{>0}$. Suppose that the Frobenioid $\mathcal{C}$ is of* **Frobenius-normalized***,* **metrically trivial***, and* **Aut-ample** *type. Then:*

(i) The natural inclusion $\mathcal{O}^{\rhd}(A)^{\rm char} \hookrightarrow \Phi(A)$, where $A \in \mathrm{Ob}(\mathcal{C}^{\rm istr})$, of Proposition 2.2, (iii), is, in fact, a **bijection***.*

(ii) $\mathcal{C}^{\rm istr}$ is **of base-trivial type***. Moreover,* **every** *object of $\mathcal{C}^{\rm istr}$ is* **Frobenius-trivial***.*

(iii) There exists an **equivalence of categories**

$$\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{(d)}$$

— which we shall refer to as the **unit-linear Frobenius functor** *[associated to τ, d] — that satisfies the following properties: (a) Ψ acts as the* **identity** *on* **objects** *and* **isometries** *of $\mathcal{C}$; (b) Ψ is* 1-**compatible**, *relative to the functors*

$$\mathcal{C} \to \mathbb{F}_\Phi, \quad \mathcal{C}^{(d)} \to \mathbb{F}_{d\cdot\Phi} = (\mathbb{F}_\Phi)^{(d)} \subseteq \mathbb{F}_\Phi$$

with the **Frobenius functor** *associated to d on $\mathbb{F}_\Phi$ [which implies, in particular, that $\mathcal{C}^{(d)}$, equipped with the natural functor $\mathcal{C}^{(d)} \to \mathbb{F}_{d\cdot\Phi}$, is a* **Frobenioid***].*

Proof. First, we observe that, by applying either of the equivalences of categories of Definition 1.3, (iii), (d), assertion (i) follows formally from the fact that $\mathcal{C}$ is of *metrically trivial* and Aut-*ample* type. Next, we consider assertion (ii). Since $\mathcal{C}$ is *of metrically trivial type*, it follows from the existence of [necessarily co-angular — cf. Proposition 1.4, (i)] pre-steps relating base-isomorphic objects of $\mathcal{C}^{\text{istr}}$ [cf. Definition 1.3, (i), (b)], that the isomorphism class of an object of $\mathcal{C}^{\text{istr}}$ is *completely determined* by the isomorphism class of $\mathcal{D}$ to which it projects. In particular, it follows from the existence of Frobenius-trivial objects [cf. Definition 1.3, (i), (a)] that *every* object of $\mathcal{C}^{\text{istr}}$ is *Frobenius-trivial.* This completes the proof of assertion (ii).

Finally, we consider assertion (iii). By applying the *factorizations* of Definition 1.3, (iv), (a); (v), (c), together with the bijection of assertion (i) [cf. also the equivalences of categories of Definition 1.3, (iii), (d)], we conclude that every morphism ϕ of $\mathcal{C}$ admits a factorization

$$\phi = \alpha \circ \beta \circ \gamma \circ \delta$$

in $\mathcal{C}$, where α is a *pull-back morphism*; β is a *base-identity pre-step endomorphism* [hence is *co-angular*, by Definition 1.3, (iii), (b)]; γ is an *isometric pre-step*; δ is a *morphism of Frobenius type.* Moreover, this factorization is *unique* [cf. Definition 1.3, (iv), (a); (v), (c)], up to replacing $(\alpha, \beta, \gamma, \delta)$ by $(\alpha\circ\epsilon, \epsilon^{-1}\circ\beta\circ\zeta, \zeta^{-1}\circ\gamma\circ\theta, \theta^{-1}\circ\delta)$, where ϵ, θ are isomorphisms of $\mathcal{C}$, and $\zeta = \beta'\circ\epsilon$, for some *base-identity automorphism* β'. Suppose that $\beta \in \mathcal{O}^{\rhd}(A)$, for $A \in \text{Ob}(\mathcal{C})$. Thus, by applying the *characteristic splitting*

$$\mathcal{O}^{\times}(A) \times \tau(A) \xrightarrow{\sim} \mathcal{O}^{\rhd}(A)$$

[which applies even if A is *not isotropic* — cf. Definition 2.3, (a), (b)] to $\beta \in \mathcal{O}^{\rhd}(A)$, we obtain a factorization

$$\beta = \beta_0 \cdot \beta_1$$

[where $\beta_0 \in \mathcal{O}^{\times}(A)$, $\beta_1 \in \tau(A)$]. Now we set

$$\Psi(\beta) \stackrel{\text{def}}{=} \beta_0 \cdot \beta_1^d; \quad \Psi(\phi) \stackrel{\text{def}}{=} \alpha \circ \Psi(\beta) \circ \gamma \circ \delta$$

[where we note that the expression "β_1^d" makes sense for $d \in \Lambda_{>0}$, by assertion (i); Definition 2.4, (ii)]. Then it follows immediately from the *functoriality* of the characteristic splitting $\tau(-)$ that for any isomorphism $\epsilon : A' \xrightarrow{\sim} A$ in $\mathcal{C}$, $\beta' \in \mathcal{O}^\times(A)$, we have $\Psi(\epsilon^{-1} \circ \beta \circ \beta' \circ \epsilon) = \epsilon^{-1} \circ \Psi(\beta) \circ \beta' \circ \epsilon$. This implies immediately that $\Psi(\phi)$ is independent of the *choice of factorization* $\phi = \alpha \circ \beta \circ \gamma \circ \delta$.

Next, observe that by assertion (ii), it follows that if $\phi \in \mathrm{Arr}(\mathcal{C}^{\mathrm{istr}})$, then the morphism of Frobenius type δ may be taken to be a *base-identity endomorphism*. Thus, by the *functoriality* of τ with respect to morphisms of $(\mathcal{C}^{\mathrm{istr}})^{\mathrm{lin}}$, and our assumption that $\mathcal{C}$ is of *Frobenius-normalized* type — together with the elementary computation

$$\Psi(\beta^{d'}) = \Psi(\beta_0^{d'} \cdot \beta_1^{d'}) = \beta_0^{d'} \cdot (\beta_1^{d'})^d = (\beta_0 \cdot \beta_1^d)^{d'} = \Psi(\beta)^{d'}$$

for $d' \in \mathbb{N}_{\geq 1}$ — it follows that the assignment $\phi \mapsto \Psi(\phi)$ is *compatible with composites*, at least when $\phi \in \mathrm{Arr}(\mathcal{C}^{\mathrm{istr}})$. On the other hand, since isotropic hulls are *monomorphisms* [cf. Definition 1.3, (v), (a)], this implies [by relating an arbitrary $\phi \in \mathrm{Arr}(\mathcal{C})$ to the result of applying the *isotropification functor* of Proposition 1.9, (v), to ϕ] that the assignment $\phi \mapsto \Psi(\phi)$ is *compatible with composites*, for arbitrary $\phi \in \mathrm{Arr}(\mathcal{C})$. This completes the definition of a functor $\Psi : \mathcal{C} \to \mathcal{C}^{(d)}$ which satisfies the properties (a), (b) in the statement of Proposition 2.5, (iii). On the other hand, it is clear from the definition of Ψ, $\mathcal{C}^{(d)}$ that Ψ is *essentially surjective*, *faithful*, and *full* [cf. assertion (i)]. This completes the proof of Proposition 2.5. ○

Remark 2.5.1. If $\mathcal{C}$ is of *isotropic* and *unit-trivial* type, then the "unit-linear Frobenius functor" of Proposition 2.5, (iii), may be regarded as a sort of *generalization* of the "naive Frobenius functor" of Proposition 2.1, (i), to the case of $d \notin \mathbb{N}_{\geq 1}$.

Corollary 2.6. (Unit-wise Frobenius Functors I) *Let τ be a* **characteristic splitting** *on $\mathcal{C}$; $d \in \mathbb{N}_{\geq 1}$. Suppose that the Frobenioid $\mathcal{C}$ is of* **Frobenius-normalized, metrically trivial***, and* Aut-**ample** *type. Then there exists a* **functor**

$$\Psi : \mathcal{C} \to \mathcal{C}$$

— which we shall refer to as the **unit-wise Frobenius functor** *[associated to τ, d] — which satisfies the following properties:*

(a) Ψ is 1-**compatible***, relative to the functor $\mathcal{C} \to \mathbb{F}_\Phi$, with the* **identity functor** *on $\mathbb{F}_\Phi$.*

(b) Ψ maps an object (respectively, morphism of Frobenius type; pre-step; pull-back morphism) of $\mathcal{C}^{\mathrm{istr}}$ to an **isomorphic** *object (respectively,* **abstractly equivalent** *morphism;* **abstractly equivalent** *morphism;* **abstractly equivalent** *morphism) of $\mathcal{C}$.*

(c) If $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{istr}})$, then there exists an isomorphism $\Psi(A) \cong A$ such that the endomorphism of $\mathcal{O}^\times(A)$ induced by Ψ followed by conjugation by this isomorphism is given by **raising to the d-th power***.*

(d) If $\mathcal{C}$ is **of perfect type***, then Ψ is an* **equivalence of categories***. If $d = 1$ or $\mathcal{C}$ is of* **isotropic** *and* **unit-trivial** *type, then Ψ is isomorphic to the* **identity functor***.*

Proof. First, let us observe that the *naive Frobenius functor* $\mathcal{C} \to \mathcal{C}$ associated to d [cf. Proposition 2.1, (i)] *factors* naturally through the subcategory $\mathcal{C}^{(d)} \subseteq \mathcal{C}$ [cf. Proposition 2.1, (ii); Definition 2.4, (iii)]; write $\Psi_1 : \mathcal{C} \to \mathcal{C}^{(d)}$ for the resulting functor. Next, let us write $\Psi_2 : \mathcal{C}^{(d)} \to \mathcal{C}$ for some quasi-inverse functor to the *unit-linear Frobenius functor* [which is an equivalence of categories] associated to d [cf. Proposition 2.5, (iii)]. Set $\Psi \stackrel{\text{def}}{=} \Psi_2 \circ \Psi_1 : \mathcal{C} \to \mathcal{C}$. Then it follows immediately from Propositions 2.1, (ii); 2.5, (iii), (b), that Ψ satisfies property (a). Since [cf. Proposition 2.5, (ii)] the isomorphism class of an object of $\mathcal{C}^{\text{istr}}$ is *completely determined* by the isomorphism class of $\mathcal{D}$ to which it projects, it thus follows that Ψ preserves isomorphism classes of objects of $\mathcal{C}^{\text{istr}}$. Now the remainder of properties (b), (c), (d) follows immediately from the construction of Ψ_1, Ψ_2 in the proofs of Propositions 2.1, (i); 2.5, (iii) [cf. also Remark 2.5.1; Proposition 1.10, (i); Definition 1.3, (ii); equivalences of categories of Definition 1.3, (iii), (d); the definition of a "pull-back morphism" in Definition 1.2, (ii); Proposition 2.1, (iii)]. ○

Definition 2.7. Suppose that the Frobenioid $\mathcal{C}$ is of *isotropic* type.

(i) We shall refer to as a *base-section* of the Frobenioid $\mathcal{C}$ any subcategory $\mathcal{P} \subseteq \mathcal{C}^{\text{pl-bk}} \subseteq \mathcal{C}$ [where $\mathcal{C}^{\text{pl-bk}} \subseteq \mathcal{C}$ is as in Definition 1.3, (i), (c)] satisfying the following conditions: (a) $\mathcal{P}$ is a *skeleton* [cf. §0]; (b) every object of $\mathcal{P}$ is *Frobenius-trivial*; (c) the composite $\mathcal{P} \to \mathcal{D}$ of the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{C}$ with the natural projection functor $\mathcal{C} \to \mathcal{D}$ is an *equivalence of categories*. In this situation, we shall refer to the morphisms of $\mathcal{C}$ that lie in $\mathcal{P}$ as *$\mathcal{P}$-distinguished*.

(ii) Let $\mathcal{P} \subseteq \mathcal{C}$ be a *base-section*. Observe that since $\mathcal{D}$, hence also $\mathcal{P}$, is a *connected category*, it follows immediately that for any $\epsilon \in \text{End}(\mathcal{P} \hookrightarrow \mathcal{C})$, it makes sense to speak of the *Frobenius degree* $\deg_{\text{Fr}}(\epsilon) \in \mathbb{N}_{\geq 1}$ of ϵ — i.e., the Frobenius degree of the endomorphisms in $\mathcal{C}$ [of objects of $\mathcal{P}$] determined by ϵ [which, since $\mathcal{P}$ is *connected*, is easily seen to be *independent* of the choice of object of $\mathcal{P}$ — cf. Remark 1.1.1]. We shall refer to as a *[$\mathcal{P}$-]Frobenius-section* of the Frobenioid $\mathcal{C}$ any homomorphism of monoids

$$\mathcal{F} : \mathbb{N}_{\geq 1} \to \text{End}(\mathcal{P} \hookrightarrow \mathcal{C})$$

satisfying the following conditions: (a) the composite of $\mathcal{F}$ with the homomorphism $\text{End}(\mathcal{P} \hookrightarrow \mathcal{C}) \to \mathbb{N}_{\geq 1}$ determined by considering the *Frobenius degree* is the *identity* on $\mathbb{N}_{\geq 1}$; (b) the endomorphisms of objects of $\mathcal{C}$ determined by an element of $\text{End}(\mathcal{P} \hookrightarrow \mathcal{C})$ in the image of $\mathcal{F}$ are *base-identity endomorphisms of Frobenius type*. We shall refer to a Frobenius-section $\mathcal{F}$ which is regarded as being known only up to composition with automorphisms of the monoid $\mathbb{N}_{\geq 1}$ as a *quasi-Frobenius-section*. If $\mathcal{F}$ is a Frobenius-section, then we shall refer to the endomorphisms of $\mathcal{C}$ induced by elements of $\text{End}(\mathcal{P} \hookrightarrow \mathcal{C})$ in the image of $\mathcal{F}$ as *$\mathcal{F}$-distinguished*.

(iii) We shall refer to a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a base-section of $\mathcal{C}$, and $\mathcal{F}$ is a $\mathcal{P}$-Frobenius-section of $\mathcal{C}$, as a *base-Frobenius pair* of $\mathcal{C}$; when $\mathcal{F}$ is regarded as being known only up to composition with automorphisms of the monoid $\mathbb{N}_{\geq 1}$, we shall refer to such a pair as a *quasi-base-Frobenius pair.* If the Frobenioid $\mathcal{C}$ admits a base-Frobenius pair [or, equivalently, a quasi-base-Frobenius pair], then we shall say that $\mathcal{C}$ is *of pre-model type.*

Remark 2.7.1. The notions of a "base-section" and "Frobenius-section" are intended to be a sort of *"category-theoretic translation"* of the notion of a *"choice of trivialization of a trivial line bundle"*, which occurs naturally when $\mathcal{C}$ is a category of trivial line bundles [cf. Remark 5.6.1; Examples 6.1, 6.3 below].

Remark 2.7.2. Suppose that $\mathcal{C}$ is of *isotropic* type. Let $(\mathcal{P}, \mathcal{F})$ be a *base-Frobenius pair* of $\mathcal{C}$. Then the only arrows of $\mathcal{C}$ which are both *$\mathcal{F}$- and $\mathcal{P}$-distinguished* [hence base-identity automorphisms — cf., e.g., the factorization of Definition 1.3, (iv), (a)] are the *identity arrows.* Suppose further that the Frobenioid $\mathcal{C}$ is of *base-trivial* type, and that the category $\mathcal{C}$ is a *skeleton.* Then every morphism ϕ of $\mathcal{C}$ admits a *factorization*

$$\phi = \alpha \circ \beta \circ \gamma$$

where α is *$\mathcal{P}$-distinguished*; β is a *base-identity pre-step endomorphism*; γ is *$\mathcal{F}$-distinguished.* Moreover, this factorization is *unique* [in the *strict* sense — i.e., not up to isomorphisms, etc.]. [Indeed, the existence and uniqueness of the *factorization* in question follow immediately from Definition 1.3, (iv), (a); the definition of $\mathcal{P}$-, $\mathcal{F}$-distinguished; our assumptions concerning $\mathcal{C}$; the *total epimorphicity* of $\mathcal{C}$; the isomorphism of functors appearing in the definition of a "pull-back morphism" in Definition 1.2, (ii).]

Definition 2.8.

(i) If, for every $A \in \mathrm{Ob}(\mathcal{C})$, it holds that $\mathcal{O}^{\times}(A)$ admits a [uniquely determined] *profinite topology* such that $\mathcal{O}^{\times}(A)$, equipped with this topology, is a topologically finitely generated profinite [abelian] group, then we shall say that $\mathcal{C}$ is *of unit-profinite type.*

(ii) Suppose that M is a topologically finitely generated profinite abelian group. Thus, M decomposes as a direct product of pro-l groups $M[l]$, where l varies over the elements of $\mathfrak{Primes}$ [cf. §0]. We shall refer to the factor $M[l]$ as the *pro-l portion* of M.

(iii) Let M be as in (ii); assume that the group law of M is written *multiplicatively.* If $\zeta : \mathfrak{Primes} \to \mathbb{N}_{\geq 1}$ is a set-theoretic function, then we shall refer to as the *map given by raising to the ζ-th power on M* the map $M \to M$

$$(M \ni)\ a \mapsto a^{\zeta}\ (\in M)$$

given by raising to the $\zeta(l)$-th power on $M[l]$, for $l \in \mathfrak{Primes}$. We shall refer to a set-theoretic function $\zeta : \mathfrak{Primes} \to \mathbb{N}_{\geq 1}$ as being *of co-prime type* if ζ maps each element $l \in \mathfrak{Primes}$ to an element of $\mathbb{N}_{\geq 1}$ that is prime to l. [Thus, if ζ is of co-prime type, then the map given by raising to the ζ-th power will always be *bijective*.]

Proposition 2.9. **(Unit-wise Frobenius Functors II)** *Suppose that the Frobenioid $\mathcal{C}$ is of* **Frobenius-normalized**, **base-trivial**, **isotropic**, *and* **Aut-ample** *[cf. Remark 2.9.2 below] type. Then:*

(i) If the base category $\mathcal{D}$ admits a **terminal object** *[cf. §0], then $\mathcal{C}$ is* **of pre-model type**.

(ii) Let τ be a **characteristic splitting** *on $\mathcal{C}$; $\zeta : \mathfrak{Primes} \to \mathbb{N}_{\geq 1}$ a set-theoretic* **function**. *Suppose that $\mathcal{C}$ is of* **pre-model** *and* **unit-profinite** *type. Then there exists a* **functor**

$$\Psi : \mathcal{C} \to \mathcal{C}$$

— which we shall refer to as the **unit-wise Frobenius functor** *[associated to τ, ζ] — which satisfies the following properties:*

- *(a)* Ψ *is* 1-**compatible**, *relative to the functor $\mathcal{C} \to \mathbb{F}_\Phi$, with the* **identity functor** *on $\mathbb{F}_\Phi$.*
- *(b)* Ψ *maps an object (respectively, morphism of Frobenius type; pre-step; pull-back morphism) of $\mathcal{C}$ to an* **isomorphic** *object (respectively,* **abstractly equivalent** *morphism;* **abstractly equivalent** *morphism;* **abstractly equivalent** *morphism) of $\mathcal{C}$.*
- *(c) If $A \in \mathrm{Ob}(\mathcal{C})$, then there exists an isomorphism $\Psi(A) \cong A$ such that the endomorphism of $\mathcal{O}^\times(A)$ induced by Ψ followed by conjugation by this isomorphism is given by* **raising to the ζ-th power**.
- *(d) If ζ is* **of co-prime type** *[cf. Definition 2.8, (iii)], then Ψ is an* **equivalence of categories**. *If $\mathcal{C}$ is* **of unit-trivial type**, *then Ψ is isomorphic to the* **identity functor**.

Proof. By well-known general nonsense in category theory, we may assume, without loss of generality, for the remainder of the proof of Proposition 2.9, that the category $\mathcal{C}$ is a *skeleton*. Thus, $\mathcal{C}^{\text{pl-bk}}$ is also a *skeleton*. Now we consider assertion (i). Observe [cf. Definition 1.3, (i), (c); the fact that $\mathcal{C}$ is of *base-trivial* type] that if $A \in \mathrm{Ob}(\mathcal{C})$ projects to a *terminal object* of $\mathcal{D}$, then A is *pseudo-terminal* [cf. §0]. Note that by Definition 1.3, (i), (a), and our assumptions on $\mathcal{D}$, it follows that such an object A always exists; let us *fix* one such object A. Thus, the natural projection functor determines an *equivalence of categories*

$$\mathcal{C}_A^{\text{pl-bk}} \xrightarrow{\sim} \mathcal{D}$$

[cf. Definition 1.3, (i), (c)]. Note that it follows immediately from the existence of this equivalence of categories [together with the fact that A maps to a *terminal* *object* of $\mathcal{D}$] that the natural functor $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C}^{\text{pl-bk}}$ is *injective* on isomorphism classes of objects. In particular, if $\mathcal{Q} \subseteq \mathcal{C}_A^{\text{pl-bk}}$ is a *skeletal subcategory* [cf. §0], then [relative to some sufficiently large universe with respect to which, say, the category $\mathcal{C}$ is *small*] the natural map

$$\text{Ob}(\mathcal{Q}) \to \text{Ob}(\mathcal{C}^{\text{pl-bk}}) = \text{Ob}(\mathcal{C})$$

is *bijective* [cf. the fact that $\mathcal{C}$ is of *base-trivial* type; the equivalence $\mathcal{C}_A^{\text{pl-bk}} \xrightarrow{\sim} \mathcal{D}$]. Thus, the subcategory $\mathcal{P} \subseteq \mathcal{C}$ determined by the image of the objects and arrows of $\mathcal{Q}$ in $\mathcal{C}$ is a skeleton which satisfies the conditions of Definition 2.7, (i) [cf. the fact that $\mathcal{C}$ is of *base-trivial* and *isotropic* type; Definition 1.3, (i), (a)] — that is to say, $\mathcal{P}$ is a *base-section.*

Next, let us observe [cf. the fact that $\mathcal{C}$ is of *base-trivial* and *isotropic* type; Definition 1.3, (i), (a)] that A is *Frobenius-trivial*, hence that there exists a morphism of monoids

$$\mathcal{F}_A : \mathbb{N}_{\geq 1} \to \text{End}_{\mathcal{C}}(A)$$

whose composite with the morphism of monoids $\deg_{\text{Fr}}(-) : \text{End}_{\mathcal{C}}(A) \to \mathbb{N}_{\geq 1}$ is the identity morphism on $\mathbb{N}_{\geq 1}$, and whose image consists of base-identity endomorphisms of Frobenius type of A. Thus, by Proposition 1.11, (iii), we conclude that $\mathcal{F}_A$ extends to a *$\mathcal{P}$-Frobenius-section*

$$\mathcal{F} : \mathbb{N}_{\geq 1} \to \text{End}(\mathcal{P} \to \mathcal{C})$$

— hence that $\mathcal{C}$ is *of pre-model type*, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). Observe [cf. the fact that $\mathcal{C}$ is of *base-trivial* and *isotropic* type; the fact that $\mathcal{C}$ is a *skeleton*] we may apply Remark 2.7.2 to conclude that every morphism $\phi : C \to D$ of $\mathcal{C}$ admits a *unique factorization*

$$\phi = \alpha \circ \beta \circ \gamma$$

in $\mathcal{C}$, where α is *$\mathcal{P}$-distinguished*; β is a *base-identity pre-step endomorphism*; γ is *$\mathcal{F}$-distinguished.* Now [cf. the proof of Proposition 2.5, (iii)] by applying the *characteristic splitting* [cf. Definition 2.3, (a)] $\mathcal{O}^\times(C) \times \tau(C) \xrightarrow{\sim} \mathcal{O}^\rhd(C)$, we may write

$$\beta = \beta_0 \cdot \beta_1 \in \mathcal{O}^\rhd(C)$$

[where $\beta_0 \in \mathcal{O}^\times(C)$, $\beta_1 \in \tau(C)$]. Set

$$\Psi(\beta) \stackrel{\text{def}}{=} \beta_0^\zeta \cdot \beta_1; \quad \Psi(\phi) \stackrel{\text{def}}{=} \alpha \circ \Psi(\beta) \circ \gamma$$

[where "$(-)^\zeta$" is as defined in Definition 2.8, (iii)]. Since β is *completely determined* by ϕ, it follows that Ψ is *well-defined* [as a "map on arrows"]. Moreover, it follows from the definition of $\mathcal{P}$- and $\mathcal{F}$-distinguished morphisms [together with the fact that raising to the ζ-th power defines an *endomorphism* of the functor in monoids "$\mathcal{O}^\times(-)$" on $\mathcal{C}^{\text{lin}}$ which commutes with raising to the d-th power, for $d \in \mathbb{N}_{\geq 1}$ — cf.

our assumption that $\mathcal{C}$ is of *Frobenius-normalized* type] that Ψ is, in fact, a *functor*, and that Ψ satisfies properties (a), (b), (c), (d) in the statement of Proposition 2.9, (ii). This completes the proof of Proposition 2.9. ○

Remark 2.9.1. By *"base-changing"* the Frobenioid $\mathcal{C}$ via various functors $\mathcal{D}' \to \mathcal{D}$ as in Proposition 1.6, it follows that one may obtain *"unit-wise Frobenius functors"* as in Proposition 2.9, (ii), for many Frobenioids whose base categories *do not necessarily admit terminal objects* [as is required in the hypotheses of Proposition 2.9, (i)].

Remark 2.9.2. We shall see later [cf. Theorem 5.1, (iii)] that in fact, the Aut-ampleness hypothesis in the statement of Proposition 2.9 is *superfluous*.

Section 3: Category-theoreticity of the Base and Frobenius Degree

In the present §3, we show various results in the "opposite direction" to the direction represented by the various *Frobenius functors* of §2. Namely, we show that various natural structures — such as *Frobenius degrees* and the natural projection functor to the *base category* — are *preserved* by equivalences of categories between Frobenioids.

In the following discussion, we maintain the notation of §1, §2. Also, we assume that we have been given a *divisorial monoid* Φ on a connected, totally epimorphic category $\mathcal{D}$ and a *Frobenioid* $\mathcal{C} \to \mathbb{F}_\Phi$.

Definition 3.1.

(i) We shall say that $\mathcal{C}$ is *of quasi-isotropic type* if it holds that $A \in \mathrm{Ob}(\mathcal{C})$ is non-isotropic if and only if it is an *iso-subanchor* [cf. §0]. [Thus, if $\mathcal{C}$ is of isotropic type, then $\mathcal{C}$ is of quasi-isotropic type — cf. Remark 3.1.1 below.] We shall say that $\mathcal{C}$ is *of standard type* if the following conditions are satisfied: (a) $\mathcal{C}$ is of *quasi-isotropic* and *Frobenius-isotropic* type; (b) if $\mathcal{C}$ is *of group-like type*, then $\mathcal{C}^{\mathrm{istr}}$ admits a *Frobenius-compact* object; (c) $\mathcal{C}$ is *of Frobenius-normalized type*; (d) $\mathcal{D}$ is *of FSMFF-type*; (e) Φ is *non-dilating*. We shall say that a category $\mathcal{E}$ is *Frobenius-slim* if every homomorphism of monoids

$$\mathbb{F} \to \mathrm{Aut}(\mathcal{E}_A \to \mathcal{E})$$

[where $A \in \mathrm{Ob}(\mathcal{E})$] factors through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$. [Thus, every *slim* category is *Frobenius-slim*.]

(ii) Write $\mathcal{C}^{\text{Fr-tp}} \subseteq \mathcal{C}$ for the subcategory of $\mathcal{C}$ determined by the *morphisms of Frobenius type*; $\mathcal{C}^{\text{bi-Fr}} \subseteq \mathcal{C}^{\text{Fr-tp}} \times \mathcal{C}^{\text{Fr-tp}}$ for the subcategory of the *product category* $\mathcal{C}^{\text{Fr-tp}} \times \mathcal{C}^{\text{Fr-tp}}$ determined by pairs of morphisms of Frobenius type of the *same* Frobenius degree. For $A, B \in \mathrm{Ob}(\mathcal{C})$, we shall write

$$\mathrm{Hom}^{\mathrm{pf}}_{\mathcal{C}}(A, B) \stackrel{\mathrm{def}}{=} \varinjlim_{(A \to A', B \to B') \in \mathrm{Ob}({}_{(A,B)}\mathcal{C}^{\text{bi-Fr}})} \mathrm{Hom}_{\mathcal{C}}(A', B')$$

where the inductive limit is parametrized by [say, some *small skeletal subcategory* of] ${}_{(A,B)}\mathcal{C}^{\text{bi-Fr}}$; the map

$$\mathrm{Hom}_{\mathcal{C}}(A', B') \to \mathrm{Hom}_{\mathcal{C}}(A'', B'')$$

induced by a morphism $(A' \to A'', B' \to B'')$ in ${}_{(A,B)}\mathcal{C}^{\text{bi-Fr}}$ from an object $(A \to A', B \to B')$ of ${}_{(A,B)}\mathcal{C}^{\text{bi-Fr}}$ to an object $(A \to A'', B \to B'')$ of ${}_{(A,B)}\mathcal{C}^{\text{bi-Fr}}$ is the map determined by the assignment "$\phi \mapsto \phi'$" of Proposition 1.10, (i). We shall refer to an element of $\mathrm{Hom}^{\mathrm{pf}}_{\mathcal{C}}(A, B)$ as a *perfected morphism* $A \to B$.

(iii) Suppose that the Frobenioid $\mathcal{C}$ is of *Frobenius-isotropic* type. Then we shall write

$$\mathcal{C}^{\mathrm{pf}}$$

for the category — which we shall refer to as the *perfection* of $\mathcal{C}$ — defined as follows: The *objects* of $\mathcal{C}^{\mathrm{pf}}$ are pairs (A, n), where $A \in \mathrm{Ob}(\mathcal{C})$, $n \in \mathbb{N}_{\geq 1}$. The *morphisms* of $\mathcal{C}^{\mathrm{pf}}$ are given by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{pf}}}((A, n), (B, m)) \stackrel{\mathrm{def}}{=} \mathrm{Hom}^{\mathrm{pf}}_{\mathcal{C}}(A', B')$$

where (A, n) and (B, m) are objects of $\mathcal{C}^{\mathrm{pf}}$; $A \to A'$ is a morphism of Frobenius type in $\mathcal{C}$ of Frobenius degree m; $B \to B'$ is a morphism of Frobenius type in $\mathcal{C}$ of Frobenius degree n; one verifies immediately [cf. Definition 1.3, (ii)] that this set of morphisms of $\mathcal{C}^{\mathrm{pf}}$ from (A, n) to (B, m) is *independent* [up to uniquely determined natural bijections] of the choice of morphisms of Frobenius type $A \to A'$, $B \to B'$; composition of morphisms of $\mathcal{C}^{\mathrm{pf}}$ is defined in the evident fashion. [Thus, in words, the pair (A, n) is to be thought of as an *"n-th root of A"*.] Also, we obtain a *natural functor* $\mathcal{C} \to \mathcal{C}^{\mathrm{pf}}$ [by mapping "$A \mapsto (A, 1)$"].

(iv) Two co-objective [cf. §0] morphisms $\alpha_1, \alpha_2 : A \to B$ of $\mathcal{C}^{\mathrm{istr}}$ will be called *unit-equivalent* if there exist morphisms $\gamma : A \to C$, $\beta : C \to B$ [in $\mathcal{C}^{\mathrm{istr}}$] and an automorphism $\delta \in \mathcal{O}^\times(C)$ such that $\alpha_1 = \beta \circ \gamma$, $\alpha_2 = \beta \circ \delta \circ \gamma$. In this situation, we shall write $\alpha_1 \stackrel{\mathcal{O}^\times}{\approx} \alpha_2$. [Thus, if $\mathcal{C}$ is *of unit-trivial type*, then two co-objective morphisms of $\mathcal{C}^{\mathrm{istr}}$ are *unit-equivalent* if and only if they are *equal*.] By Proposition 3.3, (ii), below, it follows that "$\stackrel{\mathcal{O}^\times}{\approx}$" determines an *equivalence relation* on the set of morphisms $A \to B$ in $\mathcal{C}^{\mathrm{istr}}$ which is, moreover, *closed under composition of morphisms*; we shall write

$$\mathrm{Hom}^{\text{un-tr}}_{\mathcal{C}^{\mathrm{istr}}}(A, B)$$

for the set of *unit-equivalence classes* of morphisms $A \to B$. Also, we shall write

$$\mathcal{C}^{\text{un-tr}}$$

for the category whose *objects* are the objects of $\mathcal{C}^{\mathrm{istr}}$, and whose *morphisms* are given by "$\mathrm{Hom}^{\text{un-tr}}_{\mathcal{C}^{\mathrm{istr}}}(-, -)$", and refer to $\mathcal{C}^{\text{un-tr}}$ as the *unit-trivialization of* $\mathcal{C}$.

Remark 3.1.1. Observe that:

> An iso-subanchor of the Frobenioid $\mathcal{C}$ is *never* isotropic. [In particular, if $\mathcal{C}$ is of *isotropic* type, then $\mathcal{C}$ is of *quasi-isotropic* type.]

Indeed, by Proposition 1.10, (iv), an *anchor* is never isotropic. Thus, by Definition 1.3, (vii), (b), a *subanchor* is never isotropic. Now if $B \to A$ is a *mono-minimal categorical quotient* [cf. §0] in $\mathcal{C}$ of B by a group $G \subseteq \mathrm{Aut}_{\mathcal{C}}(B)$ such that B is a subanchor and A is isotropic, then applying the *isotropification functor* of Proposition 1.9, (v), yields a factorization $B \to B' \to A$, where $B \to B'$ is an isotropic hull [hence a *monomorphism* — cf. Definition 1.3, (v), (a)], such that G acts compatibly [relative to the arrow $B \to B'$] on B'; thus, by the definition of the term "mono-minimal" it follows that the arrow $B \to B'$ is an *isomorphism*, i.e., that B is *isotropic* — a contradiction. This completes the proof of the "observation".

Remark 3.1.2. Observe that for any *residually finite group* G [i.e., a group G such that the intersection of the normal subgroups of finite index of G is trivial]:

> Any homomorphism of monoids $\mathbb{F} \to G$ factors through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$.

[Indeed, it suffices to show this when G is *finite*. When G is finite, it follows immediately from the definition of $\mathbb{F}$ [cf. Definition 1.1, (iii)] that the image of $1 \in \mathbb{Z}_{\geq 0}$ in G is an element $\gamma \in G$ such that for every $d \in \mathbb{N}_{\geq 1}$, there exists an element $\delta_d \in G$ such that $\delta_d \cdot \gamma \cdot \delta_d^{-1} = \gamma^d$. Thus, by taking d to be the order of γ, we conclude that γ is the identity, hence that the homomorphism of monoids $\mathbb{F} \to G$ factors through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$, as desired.] In particular, it follows that if $\mathcal{E}$ is a category such that for every $A \in \text{Ob}(\mathcal{E})$, the group $\text{Aut}(\mathcal{E}_A \to \mathcal{E})$ is *residually finite*, then $\mathcal{E}$ is *Frobenius-slim*.

Remark 3.1.3. The phenomenon discussed in Remark 3.1.2 may be regarded as an example of the following *fundamental dichotomy* [which is, in a certain sense, a *central theme* of the theory of the present paper] between the structure of the *base category* $\mathcal{D}$ and the *"Frobenius structure"* constituted by $\mathbb{N}_{\geq 1}$:

base category	*Frobenius*
"indifferent to order"	"order-conscious"
groups	non-group-like monoids

This sort of phenomenon may be observed in "classical scheme theory" for instance in the *invariance of the étale site* of a scheme in positive characteristic under the *Frobenius morphism*. Here, it is useful to recall that a typical example of a "base category" is constituted by the subcategory of connected objects of a *Galois category* [which is easily verified to be of *FSM-*, hence also of *FSMFF-type*]. By contrast, categories such as $\text{Order}(\mathbb{Z}_{\geq 0})$, $\text{Order}(\mathbb{N}_{\geq 1})$ or [the one-object categories determined by] $\mathbb{Z}_{\geq 0}$, $\mathbb{N}_{\geq 1}$ are *not* of *FSMFF-type*. In this context, it is interesting to note that categories such as $\text{Order}(-)$ of a *finite subset of* $\mathbb{Z}_{\geq 0}$ of cardinality ≥ 2 [with the induced ordering] constitute a sort of *"borderline case"*, which is of *FSMFF-*, but *not of FSM-*, type.

Proposition 3.2. (Perfections of Frobenioids) *Suppose that the Frobenioid $\mathcal{C}$ is of* **Frobenius-isotropic** *type. Then:*

(i) There is a **natural 1-commutative diagram of functors**

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^{\text{pf}} \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}_{\Phi} & \longrightarrow & \mathbb{F}_{\Phi^{\text{pf}}} \end{array}$$

— where the vertical arrow on the left is the functor that defines the Frobenioid structure on $\mathcal{C}$; the vertical arrow on the right is induced by the vertical arrow on

the left; the lower horizontal arrow is induced by the natural morphism of monoids $\Phi \to \Phi^{\mathrm{pf}}$*; the upper horizontal arrow is the natural functor* $\mathcal{C} \to \mathcal{C}^{\mathrm{pf}}$ *of Definition 3.1, (iii). In particular, the functor* $\mathcal{C}^{\mathrm{pf}} \to \mathbb{F}_{\Phi^{\mathrm{pf}}}$ *determines a* **pre-Frobenioid** *structure on* $\mathcal{C}^{\mathrm{pf}}$.

(ii) An arrow of $\mathcal{C}^{\mathrm{pf}}$ *is a(n)* **morphism of Frobenius type** *(respectively,* **pre-step***;* **base-isomorphism***;* **base-identity endomorphism***;* **isomorphism***;* **pull-back morphism***;* **isometry***;* **co-angular morphism***;* **LB-invertible morphism***; morphism of a given* **Frobenius degree***) if and only if a cofinal collection of the system of arrows of* $\mathcal{C}$ *that determine this arrow of* $\mathcal{C}^{\mathrm{pf}}$ *[cf. Definition 3.1, (ii)] is so.*

(iii) The category $\mathcal{C}^{\mathrm{pf}}$*, equipped with the functor* $\mathcal{C}^{\mathrm{pf}} \to \mathbb{F}_{\Phi^{\mathrm{pf}}}$ *of the diagram of (i), is a* **Frobenioid** *of* **perfect** *and* **isotropic** *type. Moreover, there is a natural* **equivalence of categories** $\mathcal{C}^{\mathrm{pf}} \xrightarrow{\sim} (\mathcal{C}^{\mathrm{pf}})^{\mathrm{pf}}$.

Proof. In light of our assumption that the Frobenioid $\mathcal{C}$ is of *Frobenius-isotropic* type, assertions (i), (ii), (iii) follow immediately from the definitions; Proposition 1.10, (i) [cf. also Proposition 2.1, (iii)]. ○

Proposition 3.3. (Base-identity Pre-steps and Units)

(i) Write

$$\mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}} \subseteq \mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})$$

[where $\mathcal{C}^{\text{pl-bk}}$ *is as in Definition 1.3, (i), (c);* $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C}$ *is the natural functor] for the submonoid consisting of those natural transformations such that the various endomorphisms of objects of* $\mathcal{C}$ *that occur in the natural transformation are all* **base-isomorphisms***. Then if* $\mathcal{D}$ *is* **Frobenius-slim***, then the image of* $1 \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$ *under any homomorphism of monoids*

$$\mathbb{F} \to \mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$$

determines an element of $\mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$ *with the property that the various endomorphisms of objects of* $\mathcal{C}$ *that occur in the natural transformation determined by this element are all* **base-identity pre-steps** *[i.e., lie in "*$\mathcal{O}^{\triangleright}(-)$*"]. Conversely, if* $\mathcal{C}$ *is of* **Frobenius-normalized** *type, and A is* **Frobenius-trivial***, then every base-identity pre-step endomorphism of A arises as the endomorphism of A induced by the image of* $1 \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$ *via a homomorphism of monoids* $\mathbb{F} \to \mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$.

(ii) Two co-objective morphisms $\alpha_1, \alpha_2 : A \to B$ *of* $\mathcal{C}^{\mathrm{istr}}$ *are* **unit-equivalent** *if and only if they map to the* **same morphism of** $\mathbb{F}_{\Phi}$*, i.e., if and only if the following three conditions are satisfied: (a)* $\deg_{\mathrm{Fr}}(\alpha_1) = \deg_{\mathrm{Fr}}(\alpha_2)$*; (b)* $\mathrm{Div}(\alpha_1) = \mathrm{Div}(\alpha_2)$*; (c)* $\mathrm{Base}(\alpha_1) = \mathrm{Base}(\alpha_2)$.

(iii) There is a natural functor

$$\mathcal{C}^{\mathrm{istr}} \to \mathcal{C}^{\text{un-tr}}$$

which is **full** *and* **essentially surjective***; moreover, this functor is an* **equivalence of categories** *if and only if* $\mathcal{C}^{\text{istr}}$ *is* **of unit-trivial type***.*

(iv) The functor $\mathcal{C}^{\text{istr}} \to \mathbb{F}_\Phi$ *factors naturally through* $\mathcal{C}^{\text{un-tr}}$*, hence determines a functor*

$$\mathcal{C}^{\text{un-tr}} \to \mathbb{F}_\Phi$$

which is **faithful** *and* **essentially surjective***; moreover, this functor determines a natural structure of* **Frobenioid** *on* $\mathcal{C}^{\text{un-tr}}$*, with respect to which* $\mathcal{C}^{\text{un-tr}}$ *is of* **isotropic** *and* **unit-trivial** *type. Finally, an arrow of* $\mathcal{C}^{\text{un-tr}}$ *is a(n)* **morphism of Frobenius type** *(respectively,* **pre-step***;* **base-isomorphism***;* **isomorphism***;* **pull-back morphism***;* **isometry***;* **co-angular morphism***;* **LB-invertible morphism***; morphism of a given* **Frobenius degree***) if and only if it arises from such an arrow of* $\mathcal{C}^{\text{istr}}$*.*

(v) The functor

$$\mathcal{C} \to \mathbb{F}_\Phi$$

is an **equivalence of categories** *if and only if the Frobenioid* $\mathcal{C}$ *is of* Aut**-ample**, **unit-trivial***, and* **base-trivial** *type.*

Proof. First, we consider assertion (i). Note that since the composite of the functor $\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C}$ with the natural projection functor $\mathcal{C} \to \mathcal{D}$ factors as the composite of the equivalence of categories [involving *pull-back morphisms*] of Definition 1.3, (i), (c), $\mathcal{C}_A^{\text{pl-bk}} \xrightarrow{\sim} \mathcal{D}_{A_\mathcal{D}}$ [where $A_\mathcal{D} \stackrel{\text{def}}{=} \text{Base}(A)$] with the natural functor $\mathcal{D}_{A_\mathcal{D}} \to \mathcal{D}$, it follows that any homomorphism of monoids $\mathbb{F} \to \text{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$ determines a *homomorphism of monoids*

$$\mathbb{F} \to \text{Aut}(\mathcal{D}_{A_\mathcal{D}} \to \mathcal{D})$$

— which, if $\mathcal{D}$ is *Frobenius-slim* [cf. Definition 3.1, (i)], necessarily *factors* through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$ — together with a *homomorphism of monoids*

$$\mathbb{F} \to \mathbb{N}_{\geq 1}$$

obtained by considering the *Frobenius degree* of the induced endomorphism of A — which [in light of the fact that the monoid $\mathbb{N}_{\geq 1}$ is *commutative*, together with the structure of $\mathbb{F}$ — cf. Definition 1.1, (iii)] also necessarily *factors* through the natural surjection $\mathbb{F} \twoheadrightarrow \mathbb{N}_{\geq 1}$. Thus, we conclude that if $\mathcal{D}$ is *Frobenius-slim*, then the image of $1 \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$ under the given homomorphism of monoids $\mathbb{F} \to \text{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$ determines an element of $\text{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$ with the property that the various endomorphisms of objects of $\mathcal{C}$ that occur in the natural transformation determined by this element are all *base-identity pre-steps*, as desired.

The "converse assertion" [when $\mathcal{C}$ is of *Frobenius-normalized* type, and A is *Frobenius-trivial*] may be verified by choosing a homomorphism of monoids

$$\mathbb{N}_{\geq 1} \to \text{End}_\mathcal{C}(A)$$

as in the definition of the term "Frobenius-trivial" [cf. the homomorphism "ζ" of Definition 1.2, (iv)], which, together with the homomorphism of monoids

$$\mathbb{Z}_{\geq 0} \to \mathrm{End}_{\mathcal{C}}(A)$$

that maps $1 \in \mathbb{Z}_{\geq 0}$ to a given base-identity pre-step endomorphism of A, yields [cf. our assumption that $\mathcal{C}$ is of *Frobenius-normalized* type!] a *homomorphism of monoids*

$$\mathbb{F} \to \mathrm{End}_{\mathcal{C}}(A)$$

— which, by applying Proposition 1.11, (iii), *lifts* to a *homomorphism of monoids* $\mathbb{F} \to \mathrm{End}(\mathcal{C}_A^{\text{pl-bk}} \to \mathcal{C})^{\text{bs-iso}}$, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). Since assertion (ii) clearly only concerns the Frobenioid $\mathcal{C}^{\text{istr}}$ [cf. Proposition 1.9, (v)], we may replace $\mathcal{C}$ by $\mathcal{C}^{\text{istr}}$ and assume for the remainder of the proof of assertion (ii) that $\mathcal{C}$ is *of isotropic type*. Now the *necessity* of the three conditions (a), (b), (c) follows immediately [cf. Remark 1.1.1] from the fact that endomorphisms of "$\mathcal{O}^{\times}$" are *LB-invertible base-identity linear endomorphisms*. To show the *sufficiency* of these three conditions, we apply the *factorization* of Definition 1.3, (iv), (a) [cf. also Proposition 1.4, (i)], the *essential uniqueness of morphisms of Frobenius type of a given Frobenius degree* [cf. Definition 1.3, (ii)], and the equivalence of categories [involving *pull-back morphisms*] of Definition 1.3, (i), (c), to α_1, α_2. Then conditions (a), (c) imply that there exist morphisms $\gamma : A \to C$; $\beta_1, \beta_2 : C \to D$; $\delta : D \to B$, where γ is a morphism of Frobenius type, β_1 and β_2 are base-equivalent co-angular pre-steps, and $\delta : D \to B$ is a pull-back morphism such that $\alpha_1 = \delta \circ \beta_1 \circ \gamma$, $\alpha_2 = \delta \circ \beta_2 \circ \gamma$. Since δ, γ are *LB-invertible*, it thus follows from condition (b) [cf. also Remark 1.1.1] that $\mathrm{Div}(\beta_1) = \mathrm{Div}(\beta_2)$, hence [by Definition 1.3, (vi)] that $\beta_2 = \zeta \circ \beta_1$, for some $\zeta \in \mathcal{O}^{\times}(D)$. Since $\alpha_1 = \delta \circ (\beta_1 \circ \gamma)$, $\alpha_2 = \delta \circ \zeta \circ (\beta_1 \circ \gamma)$, we thus conclude that $\alpha_1 \overset{\mathcal{O}^{\times}}{\approx} \alpha_2$, as desired. This completes the proof of assertion (ii). Now assertion (iii) is immediate from the definitions.

In light of assertions (ii), (iii), assertion (iv) is immediate from the definitions. As for assertion (v), the *necessity* of the condition in the statement of assertion (v) follows immediately from Proposition 1.5, (i), (ii). To verify the *sufficiency* of this condition, let us first observe that if $\mathcal{C}$ is of *unit-trivial* and *base-trivial* type, then [by the existence of isotropic hulls in $\mathcal{C}$ — cf. Definition 1.3, (vii), (a)] it follows that $\mathcal{C}$ is also of *isotropic* type, hence that we have a natural equivalence of categories $\mathcal{C} \overset{\sim}{\to} \mathcal{C}^{\text{un-tr}}$ [cf. assertion (iii)]. Thus, by assertion (iv), it follows that the natural functor $\mathcal{C} \to \mathbb{F}_{\Phi}$ is *faithful* and *essentially surjective*. Since $\mathcal{C}$ is of *base-trivial* and Aut-*ample* type, it follows from the *factorization* of Definition 1.3, (iv), (a) [cf. also the *existence* and *uniqueness* of morphisms of Frobenius type of a given Frobenius degree asserted in Definition 1.3, (ii); the equivalence of categories involving pull-back morphisms of Definition 1.3, (i), (c)], that to show that $\mathcal{C} \to \mathbb{F}_{\Phi}$ is *full*, it suffices to show that it is surjective on base-identity pre-step endomorphisms [i.e., on "$\mathcal{O}^{\rhd}(-)$"]; but, by our assumption that $\mathcal{C}$ is of *base-trivial* and Aut-*ample* type, this follows immediately from the *first equivalence of categories* of Definition 1.3, (iii), (d). This completes the proof of assertion (v). ◯

Theorem 3.4. (Category-theoreticity of the Base and Frobenius Degree) *For $i = 1, 2$, let Φ_i be a* **divisorial monoid** *on a connected, totally epimorphic category $\mathcal{D}_i$; $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ a* **Frobenioid***;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. Then:*

(i) Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$ are **of quasi-isotropic type***. Then Ψ preserves the* **isotropic objects**, **isotropic hulls**, *and* **isometric pre-steps***. Moreover, there exists a* 1**-unique** *functor $\Psi^{\mathrm{istr}} : \mathcal{C}_1^{\mathrm{istr}} \to \mathcal{C}_2^{\mathrm{istr}}$ that fits into a 1-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\mathrm{istr}} & \xrightarrow{\Psi^{\mathrm{istr}}} & \mathcal{C}_2^{\mathrm{istr}} \end{array}$$

[where the vertical arrows are the natural "isotropification functors" of Proposition 1.9, (v); the horizontal arrows are equivalences of categories]. Finally, if $\mathcal{D}_1$, $\mathcal{D}_2$ are **slim**, *and $\mathcal{C}_1$, $\mathcal{C}_2$ are* **of Frobenius-normalized type**, *then each of the composite functors of this diagram is* **rigid**.

(ii) Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$ are **of quasi-isotropic type**, *and that $\mathcal{D}_1$, $\mathcal{D}_2$ are* **of FSMFF-type**. *Then Ψ preserves* **pre-steps**, **co-angular pre-steps**, *and* **group-like objects**.

(iii) Suppose that: (a) $\mathcal{C}_1$, $\mathcal{C}_2$ are **of standard type**; *(b) if $\mathcal{C}_1$, $\mathcal{C}_2$ are* **of group-like type**, *then both Ψ and some quasi-inverse to Ψ preserve base-isomorphisms. Then Ψ preserves* **morphisms of Frobenius type**, **linear morphisms**, **base-isomorphisms**, **co-angular morphisms**, **pull-back morphisms**, **isometries**, *and* **LB-invertible morphisms**. *Moreover, there exists an automorphism of monoids*

$$\Psi^{\mathbb{N}_{\geq 1}} : \mathbb{N}_{\geq 1} \xrightarrow{\sim} \mathbb{N}_{\geq 1}$$

such that Ψ maps morphisms of **Frobenius degree** *d to morphisms of Frobenius degree $\Psi^{\mathbb{N}_{\geq 1}}(d)$; if $\mathcal{C}_1$, $\mathcal{C}_2$ admit a* **non-group-like** *object, then $\Psi^{\mathbb{N}}_{\geq 1}$ is the* **identity** *automorphism. Also, there exists a* 1**-unique** *functor $\Psi^{\mathrm{pf}} : \mathcal{C}_1^{\mathrm{pf}} \to \mathcal{C}_2^{\mathrm{pf}}$ that fits into a 1-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\mathrm{pf}} & \xrightarrow{\Psi^{\mathrm{pf}}} & \mathcal{C}_2^{\mathrm{pf}} \end{array}$$

[where the vertical arrows are the natural functors of Proposition 3.2, (i); the horizontal arrows are equivalences of categories]. Finally, if $\mathcal{D}_1$, $\mathcal{D}_2$ are **slim**, *then each of the composite functors of this diagram is* **rigid**.

(iv) Suppose that: (a) $\mathcal{C}_1$, $\mathcal{C}_2$ are **of standard type**; *(b) if $\mathcal{C}_1$, $\mathcal{C}_2$ are* **of group-like type**, *then both Ψ and some quasi-inverse to Ψ preserve base-isomorphisms;*

(c) $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are* **Frobenius-slim**. *Then* Ψ *preserves the* **submonoids** *"*$\mathcal{O}^{\triangleright}(-)$*", "*$\mathcal{O}^{\times}(-)$*";* $\Psi^{\mathbb{N}_{\geq 1}}$ *is the* **identity** *automorphism. Moreover, there exists a* **1-unique** *functor* $\Psi^{\text{un-tr}} : \mathcal{C}_1^{\text{un-tr}} \to \mathcal{C}_2^{\text{un-tr}}$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1^{\text{istr}} & \stackrel{\Psi^{\text{istr}}}{\longrightarrow} & \mathcal{C}_2^{\text{istr}} \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\text{un-tr}} & \stackrel{\Psi^{\text{un-tr}}}{\longrightarrow} & \mathcal{C}_2^{\text{un-tr}} \end{array}$$

[where the vertical arrows are the natural functors of Proposition 3.3, (iii); the horizontal arrows are equivalences of categories]. Finally, if $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are* **slim***, then each of the composite functors of this diagram is* **rigid**.

(v) Suppose that: (a) $\mathcal{C}_1$*,* $\mathcal{C}_2$ *are* **of standard type***; (b) if* $\mathcal{C}_1$*,* $\mathcal{C}_2$ *are* **of group-like type***, then both* Ψ *and some quasi-inverse to* Ψ *preserve base-isomorphisms; (c)* $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are* **slim**. *Then* Ψ *preserves the* **base-identity endomorphisms** *and* **base-equivalent pairs** *of co-objective morphisms. Moreover, there exists a* **1-unique** *functor* $\Psi^{\text{Base}} : \mathcal{D}_1 \to \mathcal{D}_2$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \stackrel{\Psi}{\longrightarrow} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \stackrel{\Psi^{\text{Base}}}{\longrightarrow} & \mathcal{D}_2 \end{array}$$

[where the vertical arrows are the natural projection functors; the horizontal arrows are equivalences of categories]. Finally, each of the composite functors of this diagram is **rigid**.

Proof. First, we consider assertion (i). Since *iso-subanchors* are manifestly preserved by any equivalence of categories, it follows from our assumption that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of quasi-isotropic type* that Ψ preserves *isotropic objects*. Now, with the exception of the final statement concerning the rigidity of the composite functors, the remainder of assertion (i) follows formally from [the definitions and] Proposition 1.9, (v), (vi), (vii). The final statement concerning the rigidity of the composite functors may be verified as follows: By Proposition 1.13, (ii), it suffices to show, for each $A \in \text{Ob}(\mathcal{C}^{\text{istr}})$ that the automorphism $\alpha \in \mathcal{O}^{\times}(A)$ induced by an automorphism $\in \text{Aut}(\mathcal{C}_1 \to \mathcal{C}_1^{\text{istr}})$ is *trivial*. But, by Definition 1.3, (i), (a), (b); (iii), (c), it suffices to show this when A is *Frobenius-trivial*, in which case the triviality of α follows from the *functoriality* of α with respect to base-identity endomorphisms of A of arbitrary of Frobenius degree [which implies, since $\mathcal{C}_1$, $\mathcal{C}_2$ are *of Frobenius-normalized type*, that $\alpha^d = \alpha$, for all $d \in \mathbb{N}_{\geq 1}$, hence that α is trivial, as desired]. Next, we consider assertion (ii). By assertion (i) [cf. also Proposition 1.9, (v)], and the characterization of *co-angular pre-steps* given in Proposition 1.7, (iv), we reduce immediately to the case where $\mathcal{C}_1$, $\mathcal{C}_2$ are *of isotropic type*. Then [since any equivalence of categories manifestly preserves FSM-morphisms and irreducible morphisms] the fact that Ψ preserves *pre-steps* follows formally from Proposition 1.14, (ii), (iii). Since Ψ preserves pre-steps, it thus follows from Proposition 1.8, (iii) [cf.

also Proposition 1.4, (i)], that Ψ preserves *group-like objects*. This completes the proof of assertions (i), (ii).

Next, we consider assertion (iii). First, I *claim* that to verify assertion (iii), it suffices to prove that, for each prime $p_1 \in \mathfrak{Primes}$, there exists a prime $p_2 \in \mathfrak{Primes}$, which is equal to p_1 if $\mathcal{C}_1$, $\mathcal{C}_2$ are *not of group-like type*, such that Ψ^{istr} maps p_1-*Frobenius morphisms* to p_2-Frobenius morphisms. Indeed, the assignment $p_1 \mapsto p_2$ determines a *homomorphism of monoids*

$$\Psi^{\mathbb{N}_{\geq 1}} : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$$

which [by considering a quasi-inverse to Ψ] is easily seen to be an *automorphism*. Moreover, by Proposition 1.10, (v), the condition of the claim implies that Ψ^{istr} preserves *morphisms of Frobenius type*, hence also *linear morphisms* [by Proposition 1.7, (iii)], and maps morphisms of *Frobenius degree* d to morphisms of Frobenius degree $\Psi^{\mathbb{N}_{\geq 1}}(d)$ [i.e., since arbitrary morphisms may be written as composites of prime-Frobenius morphisms and linear morphisms — cf. Remark 1.1.1; Definition 1.3, (iv), (a); Proposition 1.10, (v)]. Since the *isotropification functor* preserves Frobenius degrees, this implies that Ψ maps morphisms of *Frobenius degree* d to morphisms of Frobenius degree $\Psi^{\mathbb{N}_{\geq 1}}(d)$, hence that Ψ preserves *linear morphisms* and *morphisms of Frobenius type* [by Proposition 1.7, (iii)]. Moreover, by assertions (i), (ii), Ψ preserves *isometric pre-steps* and *pre-steps*, hence *base-isomorphisms* [i.e., composites of pre-steps and morphisms of Frobenius type — cf. Proposition 1.7, (ii)], *pull-back morphisms* [cf. Proposition 1.7, (ii)], *isometries* [i.e., morphisms that map via the isotropification functor to composites of a morphism of Frobenius type and a pull-back morphism — cf. Propositions 1.4, (i), (v); 1.9, (v)], *co-angular morphisms* [cf. Definition 1.2, (iii); assertion (i) for isometric pre-steps], and *LB-invertible morphisms*. Now it follows immediately from the definition of $\mathcal{C}^{\mathrm{pf}}$ [cf. Definition 3.1, (iii)] that we obtain a 1-unique 1-commutative diagram as in the statement of assertion (iii). Finally, to verify the asserted *rigidity* of composite functors, it suffices [cf. the argument applied in the proof of assertion (i)] to apply Proposition 1.13, (ii), and to consider the *functoriality* of the automorphisms in question with respect to base-identity endomorphisms of Frobenius-trivial objects of arbitrary Frobenius degree. This completes the proof of the *claim*.

Thus, to complete the proof of assertion (iii), we may assume [for the remainder of the proof of assertion (iii)] that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of isotropic type* [cf. assertion (i)]. Then it suffices to prove that, for each prime $p_1 \in \mathfrak{Primes}$, there exists a prime $p_2 \in \mathfrak{Primes}$, which is equal to p_1 if $\mathcal{C}_1, \mathcal{C}_2$ are *not of group-like type*, such that Ψ maps p_1-*Frobenius morphisms* to p_2-Frobenius morphisms. Let us call an object $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$ (p_1, p_2)-*admissible* if Ψ maps every p_1-Frobenius morphism with domain A_1 to a p_2-Frobenius morphism of $\mathcal{C}_2$. Now let us consider the following assertions:

(F1) For each prime $p_1 \in \mathfrak{Primes}$, there exist a prime $p_2 \in \mathfrak{Primes}$ and a (p_1, p_2)-admissible object of $\mathcal{C}_1$.

(F2) For every pair of primes $p_1, p_2 \in \mathfrak{Primes}$ and every morphism $\zeta_1 : A_1 \to B_1$ in $\mathcal{C}_1$, A_1 is (p_1, p_2)-*admissible* if and only if B_1 is.

(F3) If $\mathcal{C}_1$, $\mathcal{C}_2$ are *not of group-like type*, then for each prime $p \in \mathfrak{Primes}$, there exist a (p,p)-admissible object of $\mathcal{C}_1$.

Observe, moreover, that since $\mathcal{C}_1$ is *connected*, to complete the proof of assertion (iii), it suffices to prove (F1), (F2), (F3).

First, we consider assertion (F1). Let us first consider the case where $\mathcal{C}_1$, $\mathcal{C}_2$ are *of group-like type*. Then all pre-steps of $\mathcal{C}_1$, $\mathcal{C}_2$ are *isomorphisms*; Ψ preserves *base-isomorphisms*. Thus, for any $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$, the prime-Frobenius morphisms with domain A_1 are precisely the irreducible base-isomorphisms with domain A_1 [cf. Proposition 1.14, (i)]. In particular, Ψ preserves the prime-Frobenius morphisms; hence, we conclude that assertion (F1) holds. Next, let us consider the case where $\mathcal{C}_1$, $\mathcal{C}_2$ are *not of group-like type*. Then if A_1 is *non-group-like*, then [cf. Definition 1.3, (i), (a); Proposition 1.8, (iii)], there exists a base-isomorphic [i.e., to A_1], hence non-group-like, *Frobenius-trivial* object of $\mathcal{C}_1$. Thus, we may assume without loss of generality that A_1 is *Frobenius-trivial*. Then for any $p_1 \in \mathfrak{Primes}$, there exists a *base-identity* [hence Div-identity] p_1*-Frobenius endomorphism* ϕ_1 of A_1. Since [by assertion (ii)] Ψ preserves *pre-steps*, it thus follows formally from the characterization of "Div-identity prime-Frobenius endomorphisms" given in Proposition 1.14, (v), that Ψ maps ϕ_1 to a prime-Frobenius endomorphism of $A_2 \stackrel{\mathrm{def}}{=} \Psi(A_1)$. This completes the proof of assertion (F1).

Next, we consider assertion (F2). First, observe that if the morphism $\zeta_1 : A_1 \to B_1$ is a *pre-step*, then [since, by assertion (ii), Ψ preserves *pre-steps*] it follows by applying Proposition 1.14, (iv), to commutative diagrams such as the one given in Proposition 1.10, (i), that assertion (F2) holds. Thus, by Definition 1.3, (i), (a), (b), (c), we may assume without loss of generality that B_1 is *Frobenius-trivial*, and that ζ_1 is a *pull-back morphism*. Now, by applying Proposition 1.11, (iii), it follows that for every $p_1 \in \mathfrak{Primes}$, there exist *base-identity* p_1*-Frobenius endomorphisms* $\phi_1 \in \mathrm{End}_{\mathcal{C}}(A_1)$, $\psi_1 \in \mathrm{End}_{\mathcal{C}}(B_1)$ such that $\psi_1 \circ \zeta_1 = \zeta_1 \circ \phi_1$. In particular, if we write $\phi_2 \stackrel{\mathrm{def}}{=} \Psi(\phi_1)$, $\psi_2 \stackrel{\mathrm{def}}{=} \Psi(\psi_1)$, $\zeta_2 \stackrel{\mathrm{def}}{=} \Psi(\zeta_1)$, then $\psi_2 \circ \zeta_2 = \zeta_2 \circ \phi_2$, and ϕ_2, ψ_2 are *irreducible*. Thus, by Proposition 1.14, (iv), we obtain that ϕ_2 is a p_2-Frobenius morphism if and only if ψ_2 is. This completes the proof of assertion (F2).

Finally, we consider assertion (F3). Let $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$ be a *non-group-like*, *Frobenius-trivial* object [cf. the proof of assertion (F1)]. By assertions (F1), (F2), it follows already that Ψ preserves *prime-Frobenius morphisms*. Thus, to complete the proof of assertion (F3), [since the Frobenius degree of a prime-Frobenius morphism is always a *prime number*] it suffices to show that if $\zeta_1, \theta_1 \in \mathrm{End}_{\mathcal{C}_1}(A_1)$ are prime-Frobenius base-identity endomorphisms such that $\deg_{\mathrm{Fr}}(\zeta_1) < \deg_{\mathrm{Fr}}(\theta_1)$, then $\deg_{\mathrm{Fr}}(\zeta_2) < \deg_{\mathrm{Fr}}(\theta_2)$ [where $\zeta_2 \stackrel{\mathrm{def}}{=} \Psi(\zeta_1)$, $\theta_2 \stackrel{\mathrm{def}}{=} \Psi(\theta_1)$]. But, by the first equivalence of categories of Definition 1.3, (iii), (d) [cf. also Proposition 1.10, (i)], the condition "$\deg_{\mathrm{Fr}}(\zeta_1) < \deg_{\mathrm{Fr}}(\theta_1)$" is equivalent to the following condition:

> If we write β_{ζ_1} (respectively, β_{θ_1}) for the step "$\beta \circ \alpha$" of Proposition 1.14, (v), obtained when one takes "ϕ" of *loc. cit.* to be ζ_1 (respectively, θ_1) [and "α" of *loc. cit.* to be some fixed step], then $\beta_{\theta_1} = \gamma_1 \circ \beta_{\zeta_1}$, for some step γ_1.

Thus, if we write β_{ζ_2} (respectively, β_{θ_2}) for the step "$\beta \circ \alpha$" of Proposition 1.14, (v), obtained when one takes "ϕ" of *loc. cit.* to be ζ_2 (respectively, θ_2) [and "α" of *loc. cit.* to be some fixed step], then $\beta_{\theta_2} = \gamma_2 \circ \beta_{\zeta_2}$, for some step γ_2 [since, by assertion (ii), we already know that Ψ preserves pre-steps], which [again by the first equivalence of categories of Definition 1.3, (iii), (d); Proposition 1.10, (i)] implies that $\deg_{\mathrm{Fr}}(\zeta_2) < \deg_{\mathrm{Fr}}(\theta_2)$, as desired. This completes the proof of assertion (F3), hence also the proof of assertion (iii).

Next, let us observe that by assertion (i) [cf. also Proposition 1.9, (v)], it suffices to verify assertions (iv), (v), under the further assumption that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of isotropic type*; thus, we assume for the remainder of the proof of Theorem 3.4 that $\mathcal{C}_1$, $\mathcal{C}_2$ are of isotropic type. Also, to simplify notation [for the remainder of the proof of Theorem 3.4], let us write

$$\mathcal{P}_i \stackrel{\mathrm{def}}{=} \mathcal{C}_i^{\text{pl-bk}}$$

[cf. Definition 1.3, (i), (c)], for $i = 1, 2$.

Next, let us consider assertion (iv). Now, for $i = 1, 2$, it follows formally [in light of our assumption that $\mathcal{D}_i$ is *Frobenius-slim*] from Proposition 3.3, (i) [cf. also Definition 1.3, (i), (a), (b); (iii), (c)], that if $C \in \mathrm{Ob}(\mathcal{C}_i)$, then the endomorphisms of $\mathcal{O}^{\triangleright}(C)$ are precisely the endomorphisms $\gamma \in \mathrm{End}_{\mathcal{C}_i}(C)$ such that the following condition is satisfied:

> There exist pre-steps $\phi : A \to B$, $\psi : A \to C$ and endomorphisms $\alpha \in \mathrm{End}_{\mathcal{C}_i}(A)$, $\beta \in \mathrm{End}_{\mathcal{C}_i}(B)$ such that $\beta \circ \phi = \phi \circ \alpha$, $\gamma \circ \psi = \psi \circ \alpha$, and, moreover, α arises as the endomorphism of A induced by the image of $1 \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$ via a homomorphism of monoids $\mathbb{F} \to \mathrm{End}((\mathcal{P}_i)_A \to \mathcal{C}_i)^{\text{bs-iso}}$.

By assertions (ii), (iii), it follows that Ψ preserves *pre-steps*, *base-isomorphisms*, and *pull-back morphisms*, hence that Ψ preserves endomorphisms satisfying the above condition. Thus, we conclude that Ψ preserves the submonoids "$\mathcal{O}^{\triangleright}(-)$", "$\mathcal{O}^{\times}(-)$", as desired. The existence of a a 1-unique functor $\Psi^{\text{un-tr}} : \mathcal{C}_1^{\text{un-tr}} \to \mathcal{C}_2^{\text{un-tr}}$ that fits into a 1-commutative diagram as in the statement of assertion (iv) then follows formally from the definition of "$\mathcal{C}_1^{\text{un-tr}}$", "$\mathcal{C}_2^{\text{un-tr}}$"; since "$\mathcal{C}_1^{\text{un-tr}}$", "$\mathcal{C}_2^{\text{un-tr}}$" are of *unit-trivial* type, the asserted rigidity follows formally from Proposition 1.13, (ii).

Thus, to complete the proof of assertion (iv), it suffices to show that $\Psi^{\mathbb{N}_{\geq 1}}$ is the *identity* automorphism. If $\mathcal{C}_1$, $\mathcal{C}_2$ are *not of group-like type*, then this already follows formally from assertion (iii). Thus, let us assume for the remainder of the proof of assertion (iv) that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of group-like type*. Observe that there exists an object $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$ such that $A_2 \stackrel{\mathrm{def}}{=} \Psi(A_1)$ is *Frobenius-compact* [cf. Definition 3.1; the fact that Ψ is an *equivalence of categories*]. By Proposition 1.10, (vi), A_1, A_2 are *Frobenius-trivial.* Let $\phi_1 \in \mathrm{End}_{\mathcal{C}_1}(A_1)$ be a *base-identity prime-Frobenius endomorphism.* By assertion (iii), $\phi_2 \stackrel{\mathrm{def}}{=} \Psi(\phi_1)$ is also a prime-Frobenius morphism. Write $\phi_2 = \alpha \circ \psi_2$, where ψ_2 is a base-identity prime-Frobenius endomorphism of

A_2, and $\alpha \in \mathrm{Aut}_{\mathcal{C}_2}(A_2)$ [cf. Definition 1.3, (ii)]. Now since $\mathcal{C}_1$, $\mathcal{C}_2$ are *of Frobenius-normalized type* [cf. Definition 3.1, (i), (c)], it follows that for every $u_1 \in \mathcal{O}^\times(A_1)$, $u_1^{p_1} \circ \phi_1 = \phi_1 \circ u_1$ [where $p_1 \stackrel{\mathrm{def}}{=} \deg_{\mathrm{Fr}}(\phi_1)$]. Thus, for $u_2 \in \mathcal{O}^\times(A_2)$, we obtain

$$\begin{aligned} u_2^{p_1} \circ \phi_2 &= \phi_2 \circ u_2 = \alpha \circ \psi_2 \circ u_2 = \alpha \circ u_2^{p_2} \circ \psi_2 \\ &= \alpha \circ u_2^{p_2} \circ \alpha^{-1} \circ \alpha \circ \psi_2 = \alpha \circ u_2^{p_2} \circ \alpha^{-1} \circ \phi_2 \end{aligned}$$

[where $p_2 \stackrel{\mathrm{def}}{=} \deg_{\mathrm{Fr}}(\phi_2)$], hence [by the *total epimorphicity* of $\mathcal{C}_2$]

$$u_2^{p_1} = \alpha \circ u_2^{p_2} \circ \alpha^{-1}$$

— i.e., α acts on $\mathcal{O}^\times(A_2)^{\mathrm{pf}}$ by multiplication by p_1/p_2. Since A_2 is *Frobenius-compact*, we thus conclude that $p_1 = p_2$. This completes the proof of assertion (iv).

Finally, we consider assertion (v). Now, for $i = 1, 2$, if $A \in \mathrm{Ob}(\mathcal{C}_i) = \mathrm{Ob}(\mathcal{P}_i)$, $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D}_i)$, then the natural projection functor $\mathcal{C}_i \to \mathcal{D}_i$ determines a natural *equivalence of categories*

$$(\mathcal{P}_i)_A \xrightarrow{\sim} (\mathcal{D}_i)_{A_{\mathcal{D}}}$$

[cf. Definition 1.3, (i), (c)]. Moreover, if $A' \in \mathrm{Ob}(\mathcal{C}_i) = \mathrm{Ob}(\mathcal{P}_i)$, $A'_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A') \in \mathrm{Ob}(\mathcal{D}_i)$, then any arrow $A' \to A$ determines a functor

$$(\mathcal{P}_i)_{A'} \to (\mathcal{P}_i)_A$$

by sending an object $\phi : C' \to A'$ of $(\mathcal{P}_i)_{A'}$ to the object $C \to A$ of $(\mathcal{P}_i)_A$ which is the pull-back morphism of $\mathcal{C}_i$ that appears in the *factorization of the composite* $C' \to A' \to A$ given in Definition 1.3, (iv), (a). Moreover, one verifies immediately that this functor fits into a *natural* 1*-commutative diagram*

$$\begin{array}{ccc} (\mathcal{P}_i)_{A'} & \longrightarrow & (\mathcal{P}_i)_A \\ \downarrow & & \downarrow \\ (\mathcal{D}_i)_{A'_{\mathcal{D}}} & \longrightarrow & (\mathcal{D}_i)_{A_{\mathcal{D}}} \end{array}$$

[where the upper horizontal arrow is the functor just defined; the vertical arrows are the *equivalences* that arise from the natural projection functor $\mathcal{C}_i \to \mathcal{D}_i$; the lower horizontal arrow is the natural functor [cf. §0] determined by the arrow $A'_{\mathcal{D}} \to A_{\mathcal{D}}$ obtained by projecting the given arrow $A' \to A$ to $\mathcal{D}_i$].

Next, observe that since the category $\mathcal{D}_i$, hence also the categories $(\mathcal{D}_i)_{A_{\mathcal{D}}}$, $(\mathcal{P}_i)_A$, are *slim*, it follows that the collection of categories "$(\mathcal{P}_i)_A$" [where i is fixed; A ranges over the objects of $\mathcal{C}_i$] and functors "$(\mathcal{P}_i)_{A'} \to (\mathcal{P}_i)_A$" [arising from arrows $A' \to A$ of $\mathcal{C}_i$] determine a 2*-slim* [cf. Definition A.1, (i)] 2*-category of* 1*-categories*, whose *"coarsification"* [cf. Definition A.1, (ii)] we denote by $\mathcal{Q}_i$, together with a *natural functor*

$$\mathcal{C}_i \to \mathcal{Q}_i$$

[i.e., which maps $A \mapsto (\mathcal{P}_i)_A$, $(A' \to A) \mapsto \{(\mathcal{P}_i)_{A'} \to (\mathcal{P}_i)_A\}$]. Similarly, the collection of categories "$(\mathcal{D}_i)_{A_\mathcal{D}}$" [where i is fixed; $A_\mathcal{D}$ ranges over the objects of $\mathcal{D}_i$] and functors "$(\mathcal{D}_i)_{A'_\mathcal{D}} \to (\mathcal{D}_i)_{A_\mathcal{D}}$" [arising from arrows $A'_\mathcal{D} \to A_\mathcal{D}$ of $\mathcal{D}_i$] determine a 2-*category of* 1-*categories*, whose coarsification we denote by $\mathcal{E}_i$, together with a *natural functor*

$$\mathcal{D}_i \to \mathcal{E}_i$$

— which, in fact, may be identified with the *"slim exponentiation functor"* of Proposition A.2, hence, in particular, is an *equivalence of categories*. Thus, since the natural projection functor $\mathcal{C}_i \to \mathcal{D}_i$ is *essentially surjective* [cf. Definition 1.3, (i), (a)], it follows that the natural projection functor $\mathcal{C}_i \to \mathcal{D}_i$ induces a *faithful, essentially surjective functor*

$$\mathcal{Q}_i \to \mathcal{E}_i$$

which may be composed with a quasi-inverse to the natural equivalence $\mathcal{D}_i \xrightarrow{\sim} \mathcal{E}_i$ just discussed to obtain a *faithful, essentially surjective functor*

$$\mathcal{Q}_i \to \mathcal{D}_i$$

[which is well-defined up to isomorphism].

Next, let us observe that if $A, A' \in \mathrm{Ob}(\mathcal{C}_i)$, $A_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$, $A'_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A')$, then *any* morphism $\phi_\mathcal{D} : A_\mathcal{D} \to A'_\mathcal{D}$ may be written in the form

$$\phi_\mathcal{D} = \mathrm{Base}(\psi) \circ \mathrm{Base}(\gamma) \circ \mathrm{Base}(\alpha)^{-1}$$

— where $\alpha : B \to A$, $\gamma : B \to C$, are *pre-steps*; $\psi : C \to A'$ is a *pull-back morphism* [cf. Definition 1.3, (i), (b), (c)]. Since [by the above discussion] any *base-isomorphism* $\zeta : D \to E$ of $\mathcal{C}_i$ induces an *equivalence of categories* $(\mathcal{P}_i)_{D_\mathcal{D}} \xrightarrow{\sim} (\mathcal{P}_i)_{E_\mathcal{D}}$ [where $D, E \in \mathrm{Ob}(\mathcal{C}_i)$, $D_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(D)$, $E_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(E)$], it thus follows that any collection of morphisms α, γ, ψ as just described determine a *"new functor"*

$$(\mathcal{P}_i)_{A_\mathcal{D}} \to (\mathcal{P}_i)_{A'_\mathcal{D}}$$

[i.e., by inverting the equivalence of categories induced by α and then composing with the functors induced by γ, ψ]. Thus, by enlarging the 2-slim 2-category of 1-categories considered above [i.e., whose coarsification we called $\mathcal{Q}_i$] by considering these "new functors", we obtain a [slightly larger] 2-slim 2-category of 1-categories, whose coarsification we denote by $\mathcal{R}_i$. In particular, we obtain a [faithful] embedding $\mathcal{Q}_i \hookrightarrow \mathcal{R}_i$ with the property that the functor $\mathcal{Q}_i \to \mathcal{D}_i$ considered above admits a *natural extension* to a functor

$$\mathcal{R}_i \to \mathcal{D}_i$$

which [by the above discussion] is clearly an *equivalence of categories*.

On the other hand, since [by assertions (ii), (iii)] Ψ preserves *pre-steps*, *pull-back morphisms*, and *factorizations* as in Definition 1.3, (iv), (a), it follows that Ψ

induces a 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{Q}_1 & \xrightarrow{\Psi^{\mathcal{Q}}} & \mathcal{Q}_2 \\ \downarrow & & \downarrow \\ \mathcal{R}_1 & \xrightarrow{\Psi^{\mathcal{R}}} & \mathcal{R}_2 \end{array}$$

— where the vertical functors are the natural functors of the above discussion, and the the horizontal functors are *equivalences of categories* induced by Ψ. Thus, by composing with the natural equivalences of categories $\mathcal{R}_i \xrightarrow{\sim} \mathcal{D}_i$ of the above discussion, we obtain a 1-commutative diagram as in the statement of assertion (v), which is clearly 1-*unique* [cf. Definition 1.3, (i), (a), (b), (c)]. Finally, the asserted rigidity follows formally from Proposition 1.13, (i). This completes the proof of assertion (v). ◯

Remark 3.4.1. With regard to assumption (b) of Theorem 3.4, (iii), (iv), (v), we observe the following: Suppose, in the situation of Theorem 3.4, that $\mathcal{C}_1$, $\mathcal{C}_2$ are of *group-like* and *quasi-isotropic* type. Then if Ψ and some quasi-inverse to Ψ preserve *Frobenius degrees*, then they also preserve *base-isomorphisms*. Indeed, by Theorem 3.4, (i), we may assume, without loss of generality, that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of isotropic type*. Then if Ψ and some quasi-inverse to Ψ preserve Frobenius degrees, then they preserve *linear morphisms*, hence *morphisms of Frobenius type* [cf. Proposition 1.7, (iii)] and *base-isomorphisms* [i.e., morphisms of Frobenius type, since $\mathcal{C}_1$, $\mathcal{C}_2$ are of *group-like* and *isotropic* type — cf. Propositions 1.4, (i); 1.7, (ii); 1.8, (iii)].

One way to understand the *meaning of the conditions imposed* in the various portions of Theorem 3.4 is by considering examples in which some of the conditions hold, but others do not.

Example 3.5. Base Categories with FSMI-endomorphisms. Let $\mathcal{D}$ be a one-object category whose unique object has endomorphism monoid $\mathbb{F}$; $\mathcal{C}$ a one-object category whose unique object has endomorphism monoid $\mathbb{F} \times \mathbb{F}$. Thus, the projection $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ to the first factor determines a functor $\mathcal{C} \to \mathcal{D}$; $\mathcal{C}$ may be identified with the *elementary Frobenioid* determined by the [manifestly *non-dilating*] monoid on $\mathcal{D}$ that assigns to the unique object of $\mathcal{D}$ the monoid $\mathbb{Z}_{\geq 0}$ and to every morphism of $\mathcal{D}$ the identity automorphism of $\mathbb{Z}_{\geq 0}$. In particular, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized and isotropic type*, which is *not of group-like type* [cf. Proposition 1.5, (i), (ii)]. On the other hand, one verifies immediately that every morphism of $\mathcal{D}$ is an *FSM-morphism*, and that the endomorphism $1 \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{F}$ of the unique object of $\mathcal{D}$ is *irreducible*. Thus, $\mathcal{D}$ admits an *FSMI-endomorphism*, which implies [cf. §0] that $\mathcal{D}$ *fails* to be of FSMFF-type. Moreover, the self-equivalence of $\mathcal{C}$ determined by the automorphism of monoids

$$\mathbb{F} \times \mathbb{F} \xrightarrow{\sim} \mathbb{F} \times \mathbb{F}$$

given by *switching the two factors* clearly *fails to preserve pre-steps* [cf. Theorem 3.4, (ii)].

Example 3.6. Frobenioids of Standard and Group-like Type. Let

$$G \stackrel{\text{def}}{=} \mathbb{Z} \oplus \Big(\bigoplus_{p \in \mathfrak{Primes}} \mathbb{Z}/p\mathbb{Z} \Big)$$

[regarded as an abelian group]; $\mathcal{D}$ a one-object category whose unique object has endomorphism monoid $\mathbb{F}_G$; $\mathcal{C}$ a one-object category whose unique object has endomorphism monoid $\mathbb{F}_G \times \mathbb{F}_G$. Thus, if $A \in \mathrm{Ob}(\mathcal{D})$, then each automorphism of an object of $\mathcal{D}$ arising from an element $\mathrm{Aut}(\mathcal{D}_A \to \mathcal{D})$ is contained in the subgroup of *infinitely divisible* elements of G [cf. the proof of Proposition 1.13, (iii)], hence is *trivial* — that is to say, $\mathcal{D}$ is *slim*. Moreover, the projection $\mathbb{F}_G \times \mathbb{F}_G \to \mathbb{F}_G$ to the first factor determines a functor $\mathcal{C} \to \mathcal{D}$; $\mathcal{C}$ may be identified with the *elementary Frobenioid* determined by the [manifestly *non-dilating*] monoid on $\mathcal{D}$ that assigns to the unique object of $\mathcal{D}$ the monoid G and to every morphism of $\mathcal{D}$ the identity automorphism of G. In particular, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized, isotropic, and group-like type* [cf. Proposition 1.5, (i), (iii)]. One verifies immediately that every morphism of $\mathcal{D}$ is either an isomorphism or a non-monomorphism [cf. the existence of the *torsion subgroup* $\bigoplus_{p \in \mathfrak{Primes}} \mathbb{Z}/p\mathbb{Z} \subseteq G$], and that the irreducible morphisms of $\mathcal{D}$ are precisely the morphisms that project via the natural surjection $\mathbb{F}_G \to \mathbb{N}_{\geq 1}$ to primes of $\mathbb{N}_{\geq 1}$. Thus, it follows immediately that $\mathcal{D}$ is *of FSM-*, hence also *of FSMFF-type*. Moreover, since $G^{\mathrm{pf}} \cong \mathbb{Q} \neq 0$, and the first factor of $\mathbb{F}_G$ in the product $\mathbb{F}_G \times \mathbb{F}_G$ *commutes* with the G [i.e., "$\mathcal{O}^\times(-)$"] of the second factor of $\mathbb{F}_G$, it follows that the unique object of $\mathcal{C}$ is *Frobenius-compact*. Thus, $\mathcal{C}$ is *of standard type*. On the other hand, the self-equivalence of $\mathcal{C}$ determined by the automorphism of monoids

$$\mathbb{F}_G \times \mathbb{F}_G \stackrel{\sim}{\to} \mathbb{F}_G \times \mathbb{F}_G$$

given by *switching the two factors* clearly *fails to preserve base-isomorphisms* [cf. Theorem 3.4, (iii)].

Example 3.7. Dilating Monoids. Let G, $\mathcal{D}$ be as in Example 3.6; Φ the monoid on $\mathcal{D}$ that associates to the unique object of $\mathcal{D}$ the monoid $G \times \mathbb{Z}_{\geq 0}$ and to a morphism $f \in \mathbb{F}_G$ of $\mathcal{D}$ that projects to an element $d_f \in \mathbb{N}_{\geq 1}$ the endomorphism of $G \times \mathbb{Z}_{\geq 0}$ that acts trivially on G and by multiplication by d_f on $\mathbb{Z}_{\geq 0}$. Thus, [as observed in Example 3.6] $\mathcal{D}$ is *of FSMFF-type*, but Φ clearly *fails* to be *non-dilating*. Write $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{F}_\Phi$. Thus, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized and isotropic type*, which is *not of group-like type* [cf. Proposition 1.5, (i), (ii)]. Moreover, $\mathcal{C}$ is a one-object category whose unique object has endomorphism monoid M given by the product set

$$\mathbb{Z}_{\geq 0} \times (\mathbb{F}_G \times \mathbb{F}_G)$$

equipped with the following monoid structure: If $a_1, a_2 \in \mathbb{Z}_{\geq 0}$; $b_1, b_2 \in \mathbb{F}_G \times \mathbb{F}_G$, where b_1 projects to an element $(n, m) \in \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1}$, then

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 + n \cdot m \cdot a_2, b_1 \cdot b_2)$$

[cf. the description of elementary Frobenioids in Definition 1.1, (iii)]. Thus, by *switching the two factors of* $\mathbb{F}_G$, and keeping the unique factor of $\mathbb{Z}_{\geq 0}$ fixed, we obtain an automorphism of the monoid M, hence a self-equivalence of $\mathcal{C}$, that preserves pre-steps [cf. Theorem 3.4, (ii)], but *fails to preserve base-isomorphisms* [cf. Theorem 3.4, (iii)].

Example 3.8. Permutation of Primes. Let $\alpha : \mathbb{N}_{\geq 1} \xrightarrow{\sim} \mathbb{N}_{\geq 1}$ be an automorphism of monoids of *order* 2; $N \stackrel{\text{def}}{=} (\mathbb{N}_{\geq 1})^{\text{gp}}$ [so α acts on N]; $U \stackrel{\text{def}}{=} \mathbb{Q}$; $V \stackrel{\text{def}}{=} \mathbb{Q}$; $W \stackrel{\text{def}}{=} \mathbb{Q}$; $G \stackrel{\text{def}}{=} U \rtimes N$, where we let $n \in N$ $(\subseteq \mathbb{Q})$ act on U by n^{-1}; $\mathcal{D}$ the one-object category whose unique object has endomorphism monoid G; Φ the [manifestly *non-dilating*] monoid on $\mathcal{D}$ that associates to the unique object of $\mathcal{D}$ the monoid $V \times W$ and to a morphism $g \in G$ that projects to an element $n \in N$ the automorphism of $V \times W$ given by $(\alpha(n), \alpha(n) \cdot n^{-1})$ [i.e., the automorphism that acts on V by $\alpha(n)$ and on W by $\alpha(n) \cdot n^{-1}$]; $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{F}_\Phi$. Thus, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized, isotropic, and group-like type* [cf. Proposition 1.5, (i), (iii)]; the category $\mathcal{D}$ is manifestly of FSM-, hence also *of FSMFF-type* [cf. §0]. Since the unique object of $\mathcal{C}$ has "$\mathcal{O}^\times(-)$" equal to $V \times W$, it follows from our definition of Φ that this object is *Frobenius-compact*. Thus, $\mathcal{C}$ is *of standard type*. On the other hand, if $A \in \text{Ob}(\mathcal{D})$, then $\text{Aut}(\mathcal{D}_A \to \mathcal{D}) \cong G$; since there exist injections of monoids $\mathbb{F} \hookrightarrow G$, it thus follows that $\mathcal{D}$ *fails to be Frobenius-slim*. The monoid M of endomorphisms of the unique object of $\mathcal{C}$ may be described as the product set

$$U \times V \times W \times N \times \mathbb{N}_{\geq 1}$$

equipped with the following monoid structure: if $u_1, u_2 \in U$; $v_1, v_2 \in V$; $w_1, w_2 \in W$; $n_1, n_2 \in N$; $m_1, m_2 \in \mathbb{N}_{\geq 1}$, then

$$\begin{aligned}(u_1, &v_1, w_1, n_1, m_1) \cdot (u_2, v_2, w_2, n_2, m_2) = \\ &(u_1 + n_1^{-1} \cdot u_2, v_1 + m_1 \cdot \alpha(n_1) \cdot v_2, w_1 + m_1 \cdot \alpha(n_1) \cdot n_1^{-1} \cdot w_2, n_1 \cdot n_2, m_1 \cdot m_2)\end{aligned}$$

[cf. the description of elementary Frobenioids in Definition 1.1, (iii)]. In particular, a routine verification reveals that the assignment

$$(u, v, w, n, m) \mapsto (v, u, w, \alpha(n)^{-1} \cdot m^{-1}, \alpha(m))$$

[where $u \in U$, $v \in V$, $w \in W$, $n \in N$, $m \in \mathbb{N}_{\geq 1}$] determines an *automorphism of the monoid* M, hence a self-equivalence of $\mathcal{C}$, which clearly *preserves base-isomorphisms*, but *fails to preserve "$\mathcal{O}^\times(-)$"* [i.e., the subspace $\{0\} \times V \times W \subseteq U \times V \times W$] or *Frobenius degrees* [when α is not equal to the identity] — cf. Theorem 3.4, (iii), (iv).

Example 3.9. Non-preservation of Units. Let $N \stackrel{\text{def}}{=} (\mathbb{N}_{\geq 1})^{\text{gp}}$; $U \stackrel{\text{def}}{=} \mathbb{Q}$; $V \stackrel{\text{def}}{=} \mathbb{Q}$; $W \stackrel{\text{def}}{=} \mathbb{Z}_{\geq 0}$; $G \stackrel{\text{def}}{=} U \rtimes N$, where we let $n \in N$ $(\subseteq \mathbb{Q})$ act on U by n^{-1}; $\mathcal{D}$ the one-object category whose unique object has endomorphism monoid G; Φ the [manifestly *non-dilating*] monoid on $\mathcal{D}$ that associates to the unique object of

$\mathcal{D}$ the monoid $V \times W$ and to a morphism $g \in G$ that projects to an element $n \in N$ the automorphism of $V \times W$ given by $(n, 1)$ [i.e., the automorphism that acts on V by n and on W by 1]; $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{F}_\Phi$. Thus, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized and isotropic type*, which is *not of group-like type* [cf. Proposition 1.5, (i), (ii)]; $\mathcal{D}$ is manifestly of FSM-, hence also *of FSMFF-type* [cf. §0]. Thus, $\mathcal{C}$ is *of standard type.* On the other hand, [cf. Example 3.8] $\mathcal{D}$ *fails to be Frobenius-slim.* The monoid M of endomorphisms of the unique object of $\mathcal{C}$ may be described as the product set

$$U \times V \times W \times N \times \mathbb{N}_{\geq 1}$$

equipped with the following monoid structure: if $u_1, u_2 \in U$; $v_1, v_2 \in V$; $w_1, w_2 \in W$; $n_1, n_2 \in N$; $m_1, m_2 \in \mathbb{N}_{\geq 1}$, then

$$(u_1, v_1, w_1, n_1, m_1) \cdot (u_2, v_2, w_2, n_2, m_2) = \\ (u_1 + n_1^{-1} \cdot u_2, v_1 + m_1 \cdot n_1 \cdot v_2, w_1 + m_1 \cdot w_2, n_1 \cdot n_2, m_1 \cdot m_2)$$

[cf. the description of elementary Frobenioids in Definition 1.1, (iii)]. In particular, a routine verification reveals that the assignment

$$(u, v, w, n, m) \mapsto (v, u, w, n^{-1} \cdot m^{-1}, m)$$

[where $u \in U$, $v \in V$, $w \in W$, $n \in N$, $m \in \mathbb{N}_{\geq 1}$] determines an *automorphism of the monoid* M, hence a self-equivalence of $\mathcal{C}$, which clearly *fails to preserve "$\mathcal{O}^\times(-)$", "$\mathcal{O}^\rhd(-)$"* [i.e., the subspaces $\{0\} \times V \times \{0\}, \{0\} \times V \times W \subseteq U \times V \times W$] — cf. Theorem 3.4, (iv).

Example 3.10. Non-slim Base Categories. Let G be a group, whose center we denote by $Z(G)$; $\mathcal{D}$ a one-object category whose unique object has endomorphism monoid G; $\mathcal{C}$ a one-object category whose unique object has endomorphism monoid $G \times \mathbb{F}$. Thus, the projection $G \times \mathbb{F} \to G$ determines a functor $\mathcal{C} \to \mathcal{D}$; $\mathcal{C}$ may be identified with the *elementary Frobenioid* determined by the [manifestly *non-dilating*] monoid on $\mathcal{D}$ that assigns to the unique object of $\mathcal{D}$ the monoid $\mathbb{Z}_{\geq 0}$ and to every morphism of $\mathcal{D}$ the identity automorphism of $\mathbb{Z}_{\geq 0}$. In particular, $\mathcal{C}$ is a *Frobenioid of Frobenius-normalized and isotropic type*, which is *not of group-like type* [cf. Proposition 1.5, (i), (ii)]; $\mathcal{D}$ is manifestly of FSM-, hence also *of FSMFF-type* [cf. §0]. Thus, $\mathcal{C}$ is *of standard type.* On the other hand, if $\alpha : \mathbb{F} \to Z(G)$ is any *nontrivial* homomorphism of monoids that factors as the composite of the natural surjection $\mathbb{F} \to \mathbb{N}_{\geq 1}$ with a homomorphism of monoids $\mathbb{N}_{\geq 1} \to Z(G)$, then the automorphism of monoids

$$G \times \mathbb{F} \xrightarrow{\sim} G \times \mathbb{F}$$

$$(g, f) \mapsto (g \cdot \alpha(f), f)$$

[where $g \in G$, $f \in \mathbb{F}$] determines a self-equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{C}$ which clearly *fails to preserve base-identity endomorphisms of Frobenius type* [cf. Theorem 3.4, (v)].

Finally, before proceeding, we consider the case of *Frobenioids of group-like type* in a bit more detail.

Proposition 3.11. (Frobenioids of Isotropic, Unit-trivial, and Group-like Type) *For $i = 1, 2$, let Φ_i be the* **zero monoid** *[more precisely: any functor $\mathcal{D}_i \to \mathfrak{Mon}$ all of whose values are monoids of cardinality one] on a connected, totally epimorphic category $\mathcal{D}_i$* **of FSMFF-type***; $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ a* **Frobenioid** *of* **isotropic***,* **unit-trivial***, and* **group-like type***;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. Then:*

(i) The functor $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ is an **equivalence of categories***.*

(ii) Ψ preserves **base-isomorphisms***,* **pull-back morphisms***,* **linear morphisms***, and* **morphisms of Frobenius type***.*

(iii) Suppose that both Ψ and some quasi-inverse to Ψ preserve **base-identity endomorphisms***. Then there exists a 1-***unique** *functor $\Psi^{\text{Base}} : \mathcal{D}_1 \to \mathcal{D}_2$ that fits into a 1-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\text{Base}}} & \mathcal{D}_2 \end{array}$$

[where the vertical arrows are the natural projection functors; the horizontal arrows are equivalences of categories]. Finally, if $\mathcal{D}_1$, $\mathcal{D}_2$ are **slim***, then each of the composite functors of this diagram is* **rigid***.*

Proof. First, we consider assertion (i). By Proposition 3.3, (iii), (iv), the functor $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ is *essentially surjective* and *faithful*. Since the Frobenioid $\mathcal{C}_i$ is of *group-like* and *isotropic type*, it follows that every pre-step of $\mathcal{C}_i$ is an *isomorphism* [cf. Propositions 1.4, (i); 1.8, (iii)], hence that the Frobenioid $\mathcal{C}_i$ is of Aut-*ample* and *base-trivial* [cf. Definition 1.3, (i), (b)], as well as *unit-trivial*, type. Thus, it follows from Proposition 3.3, (v), that the functor $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ is an *equivalence of categories.* This completes the proof of assertion (i).

Next, we consider assertion (ii). Observe that since $\mathcal{D}_i$ is *of FSMFF-type*, it follows that $\mathcal{D}_i$ has no *FSMI-endomorphisms* [cf. §0], hence that a morphism of $\mathcal{C}_i$ is an FSMI-endomorphism if and only if it is a *prime-Frobenius endomorphism* [cf. Propositions 1.11, (vi); 1.14, (i); the evident structure of $\mathbb{F}_{\Phi_i}$]. Thus, Ψ preserves the prime-Frobenius endomorphisms, hence also prime-Frobenius morphisms [since every prime-Frobenius morphism is abstractly equivalent to a prime-Frobenius endomorphism]. But this implies that Ψ preserves the *morphisms of Frobenius type* [cf. Proposition 1.10, (v)], hence also the *linear morphisms* [cf. Proposition 1.7, (iii)]. Since the [co-angular] pre-steps of $\mathcal{C}_i$ are isomorphisms [cf. Proposition 1.8, (iii)], it thus follows that Ψ preserves the *pull-back morphisms* [cf. Proposition 1.7, (iii)], as well as the *base-isomorphisms* [cf. Proposition 1.7, (ii)]. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). Write $\mathcal{N}$ for the one-object category whose unique object has endomorphism monoid equal to $\mathbb{N}_{\geq 1}$. Then we have *equivalences of categories*

$$\mathcal{C}_i \xrightarrow{\sim} \mathbb{F}_{\Phi_i} \xrightarrow{\sim} \mathcal{D}_i \times \mathcal{N}$$

[cf. assertion (i)]. Moreover, one verifies immediately that the *base-identity endomorphisms* of $\mathcal{C}_i$ are precisely the endomorphisms of $\mathcal{C}_i \xrightarrow{\sim} \mathcal{D}_i \times \mathcal{N}$ that arise from elements of $\mathbb{N}_{\geq 1}$; let us refer to such endomorphisms as "$\mathbb{N}_{\geq 1}$-*endomorphisms*". Thus, it follows from our assumption concerning the preservation of base-identity endomorphisms that the $\mathbb{N}_{\geq 1}$-endomorphisms are *preserved* by Ψ. Note, moreover, that $\mathcal{D}_i$ may be reconstructed from $\mathcal{C}_i$ by considering *equivalence classes of morphisms of* $\mathcal{C}_i$, where two morphisms of $\mathcal{C}_i$ are regarded as equivalent if they admit composites with an $\mathbb{N}_{\geq 1}$-endomorphism which are equal. Thus, we obtain a 1-*commutative diagram* as in the statement of assertion (ii). Finally, the *rigidity assertion* in the statement of assertion (ii) follows immediately from Proposition 1.13, (i). ○

Section 4: Category-theoreticity of the Divisor Monoid

In the present §4, we show that the *monoid on the base category* that appears in the definition of a Frobenioid [cf. Definition 1.3] may, under suitable conditions, be reconstructed entirely *category-theoretically.* Together with the results of §3, this allows us to conclude, under suitable conditions, that the *functor to an elementary Frobenioid* that appears in the definition of a Frobenioid [cf. Definition 1.3] may be recovered entirely from the structure of a Frobenioid as an *abstract category* [cf. Corollary 4.11].

In the following discussion, we maintain the notation of §1, §2, §3. Also, we assume that we have been given a *divisorial monoid* Φ on a connected, totally epimorphic category $\mathcal{D}$ and a *Frobenioid* $\mathcal{C} \to \mathbb{F}_\Phi$.

Proposition 4.1. **(Primary Steps)** *Suppose further that* $\mathcal{C}$ *is of* **perfect** *and* **isotropic** *type, and that* Φ *is* **perf-factorial**. *Let* $A \in \mathrm{Ob}(\mathcal{C})$ *be* Div-*Frobenius-trivial;*

$$\phi : B \to A, \quad \psi : A \to C, \quad \delta : D \to E, \quad \epsilon : E \to F, \quad \iota : I \to F$$

steps *of* $\mathcal{C}$. *For* $n \in \mathbb{N}_{\geq 1}$, *let* $\alpha_n \in \mathrm{End}_{\mathcal{C}}(A)$ *be a* Div**-identity endomorphism of Frobenius type** *such that* $\deg_{\mathrm{Fr}}(\alpha_n) = n$. *Then:*

(i) ϕ *is* **primary** *if and only if, for every factorization* $\phi = \phi_A \circ \phi_B$, *where* $\phi_B : B \to B'$, $\phi_A : B' \to A$ *are steps, there exists a commutative diagram*

$$\begin{array}{ccccc} B & \xrightarrow{\phi_B} & B' & \xrightarrow{\phi_A} & A \\ & & \big\downarrow \beta' & & \big\downarrow \alpha_n \\ & & B'' & \xrightarrow{\zeta} & A \end{array}$$

where $n \in \mathbb{N}_{\geq 1}$; β' *is a morphism of Frobenius type; and* $\zeta = \phi \circ \zeta'$; *and* $\zeta' : B'' \to B$ *is a pre-step.*

(ii) Suppose that ϕ *is primary. Then the composite* $\psi \circ \phi : B \to C$, *hence also* ψ, *is* **primary** *if and only if, for every factorization* $\psi \circ \phi = \psi' \circ \phi'$, *where* $\phi' : B \to A'$, $\psi' : A' \to C$ *are steps, there exist factorizations* $\phi = \zeta \circ \phi''$, $\phi' = \zeta' \circ \phi''$, *where* $\phi'' : B \to A''$ *is a step, and* $\zeta : A'' \to A$, $\zeta' : A'' \to A'$ *are pre-steps.*

(iii) $\epsilon_*(\mathrm{Div}(\epsilon)), \iota_*(\mathrm{Div}(\iota)) \in \Phi(F)$ *[where we write* ϵ_*, ι_* *for the respective bijections induced by the functor* Φ*] have* **disjoint supports** *[cf. Definition 2.4, (i), (d)] if and only if every pre-step* $\zeta : Z \to F$ *such that there exist pre-steps* ϵ', ι' *satisfying* $\epsilon = \zeta \circ \epsilon'$, $\iota = \zeta \circ \iota'$ *is, in fact, an isomorphism. In this case, we shall say that* ϵ, ι *are* **co-primary**. *If* ϵ, ι *are co-primary, then there exists a* **cartesian diagram** *in the category of pre-steps*

$$\begin{array}{ccc} U & \xrightarrow{\epsilon'} & E \\ \big\downarrow \iota' & & \big\downarrow \epsilon \\ I & \xrightarrow{\iota} & F \end{array}$$

such that $\epsilon_*(\epsilon'_*(\mathrm{Div}(\epsilon'))) = \iota_*(\mathrm{Div}(\iota))$, $\iota_*(\iota'_*(\mathrm{Div}(\iota'))) = \epsilon_*(\mathrm{Div}(\epsilon))$*; if* ϵ, ι *are* **primary***, then so are* ϵ', ι'.

(iv) δ *is* **primary** *if and only if there exists a* $\mathfrak{p} \in \mathrm{Prime}(\Phi(F))$ *such that the following condition holds: For every primary* $\epsilon' : E' \to F$ *such that* $\epsilon'_*(\mathrm{Div}(\epsilon')) \notin \mathfrak{p}$ *[where we write* ϵ'_* *for the bijection induced by the functor* Φ*], there exists a factorization* $\epsilon = \epsilon' \circ \zeta$*, where* ζ *is a pre-step, if and only if there exists a factorization* $\epsilon \circ \delta = \epsilon' \circ \theta$*, where* θ *is a pre-step.*

(v) ϵ *is* **primary** *if and only if there exists a* $\mathfrak{p} \in \mathrm{Prime}(\Phi(D))$ *such that the following condition holds: For every primary* $\delta' : D \to E'$ *such that* $\mathrm{Div}(\delta') \notin \mathfrak{p}$*, there exists a factorization* $\delta = \zeta \circ \delta'$*, where* ζ *is a pre-step, if and only if there exists a factorization* $\epsilon \circ \delta = \theta \circ \delta'$*, where* θ *is a pre-step.*

Proof. First, we consider assertion (i). By applying the *second equivalence of categories* of Definition 1.3, (iii), (d), to the various pre-steps over A, it follows that, if we write $x_\phi \stackrel{\mathrm{def}}{=} \phi_*(\mathrm{Div}(\phi)) \in \Phi(A)$ [where we write ϕ_* for the bijection induced by the functor Φ], then the condition of assertion (i) may be *translated* into the *language of monoids* as follows:

For every equation $x_\phi = x_A + x_B$ in $\Phi(A)$, where $x_A, x_B \neq 0$, we have $x_\phi \preccurlyeq x_A$.

Now the *equivalence* of this condition with the condition that x_ϕ is *primary* follows immediately from the definition of the term "primary" [cf. §0], together with the fact that $\Phi(A)$ is *perfect* [cf. Proposition 1.10, (iii)]. This completes the proof of assertion (i).

Next, we consider assertion (ii). Again, we apply Definition 1.3, (iii), (d), to the various pre-steps over C, to obtain the following *translation* of the condition of assertion (ii) into the *language of monoids* [where we set $x_\phi \stackrel{\mathrm{def}}{=} \psi_*(\phi_*(\mathrm{Div}(\phi)))$, $x_\psi \stackrel{\mathrm{def}}{=} \psi_*(\mathrm{Div}(\psi)) \in \Phi(C)$]:

For every equation $x_\phi + x_\psi = x_{\phi'} + x_{\psi'}$ in $\Phi(A)$, where $x_{\phi'}, x_{\psi'} \neq 0$, there exists a $0 \neq x_{\phi''} \in \Phi(A)$ such that $x_{\phi''} \leq x_\phi$, $x_{\phi''} \leq x_{\phi'}$.

Now the *necessity* of this condition follows immediately from the structure of the $\Phi(A)_{\mathfrak{p}}$, where $\mathfrak{p} \in \mathrm{Prime}(\Phi(A))$ [cf. Definition 2.4, (i), (b)], whereas the *sufficiency* of this condition follows by taking $x_{\phi'} \leq x_\psi$ [cf. Definition 2.4, (i), (c), (d); the fact that $\Phi(A)$ is *perfect*]. This completes the proof of assertion (ii).

Next, we consider assertion (iii). By applying the *second equivalence of categories* of Definition 1.3, (iii), (d), to the various pre-steps over F, we obtain the following *translation* of the condition of assertion (iii) into the *language of monoids* [where we set $x_\epsilon \stackrel{\mathrm{def}}{=} \epsilon_*(\mathrm{Div}(\epsilon))$, $x_\iota \stackrel{\mathrm{def}}{=} \iota_*(\mathrm{Div}(\iota)) \in \Phi(F)$]:

Every $x_\zeta \in \Phi(F)$ such that $x_\zeta \leq x_\epsilon$, $x_\zeta \leq x_\iota$ is, in fact, equal to 0.

The *necessity* and *sufficiency* of this condition then follow immediately by considering the *"primary factorizations"* of x_ϵ, x_ι [cf. Definition 2.4, (i), (c), (d); the fact that $\Phi(A)$ is *perfect*]. The cartesian diagram [with the desired properties] then follows from the fact that "for $x_U \in \Phi(F)$, $x_\epsilon + x_\iota \leq x_U$ if and only if $x_\epsilon \leq x_U$, $x_\iota \leq x_U$" [cf. Definition 2.4, (i), (c), (d); the fact that $\Phi(A)$ is *perfect*]. This completes the proof of assertion (iii).

Next, we consider assertion (iv). This time, we apply the *second equivalence of categories* of Definition 1.3, (iii), (d), to the various pre-steps over F, to obtain the following *translation* of the condition of assertion (iv) concerning $\mathfrak{p} \in \text{Prime}(\Phi(F))$ into the *language of monoids* [where we set $x_\delta \stackrel{\text{def}}{=} \epsilon_*(\delta_*((\text{Div}(\delta))), x_\epsilon \stackrel{\text{def}}{=} \epsilon_*(\text{Div}(\epsilon)) \in \Phi(F)$]:

For every primary element $x_{\epsilon'} \notin \mathfrak{p}$, $x_{\epsilon'} \leq x_\epsilon$ if and only if $x_{\epsilon'} \leq x_\delta + x_\epsilon$.

The *necessity* and *sufficiency* of this condition then follow immediately by comparing the *"primary factorizations"* of x_ϵ, $x_\delta + x_\epsilon$ [cf. Definition 2.4, (i), (c), (d); the fact that $\Phi(A)$ is *perfect*]. Also, we observe that assertion (v) follows by an entirely similar argument obtained by "reversing the direction of the arrows". This completes the proof of assertions (iv), (v). ◯

Theorem 4.2. **(Category-theoreticity of Primary Steps)** *For $i = 1, 2$, let Φ_i be a* **perf-factorial divisorial monoid** *on a connected, totally epimorphic category $\mathcal{D}_i$; $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ a* **Frobenioid** *of* **standard** *and* **isotropic** *type, which is* **not** *of group-like type;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. Then:*

(i) Ψ preserves **primary steps***,* Div-**identity endomorphisms***,* Div-**Frobenius-trivial objects***, and* **universally** Div-**Frobenius-trivial objects***.*

(ii) There exists a **unique isomorphism** *Ψ^{Prime} between the functors*

$$\text{Ob}(\mathcal{C}_i^{\text{bs-iso}}) \ni A_i \mapsto \text{Prime}(\Phi_i(A_i))$$

[where $i = 1, 2$] on $\mathcal{C}_i^{\text{bs-iso}}$ which satisfies the following property: Suppose that $A_2 = \Psi(A_1)$; $\mathfrak{p}_1 \in \text{Prime}(\Phi_1(A_1))$, $\mathfrak{p}_2 \in \text{Prime}(\Phi_2(A_2))$ correspond under Ψ^{Prime}. For $i = 1, 2$, write

$$\{_{A_i}(\mathcal{C}_i^{\text{coa-pre}})\}_{\mathfrak{p}_i} \xrightarrow{\sim} \text{Order}(\Phi_i(A_i)_{\mathfrak{p}_i}); \quad \{(\mathcal{C}_i^{\text{coa-pre}})_{A_i}\}_{\mathfrak{p}_i} \xrightarrow{\sim} \text{Order}(\Phi_i(A_i)_{\mathfrak{p}_i})^{\text{opp}}$$

for the respective full subcategories and restricted equivalences of categories determined by the full subcategory

$$\text{Order}(\Phi_i(A_i)_{\mathfrak{p}_i}) \subseteq \text{Order}(\Phi_i(A_i))$$

arising from the **submonoid** $\Phi_i(A_i)_{\mathfrak{p}_i} \subseteq \Phi_i(A_i)$. *Then the map induced by* Ψ *on pre-steps [cf. (i); Theorem 3.4, (ii)] induces* **equivalences of categories**

$$\{_{A_1}(\mathcal{C}_1^{\text{coa-pre}})\}_{\mathfrak{p}_1} \xrightarrow{\sim} \{_{A_2}(\mathcal{C}_2^{\text{coa-pre}})\}_{\mathfrak{p}_2}; \quad \{(\mathcal{C}_1^{\text{coa-pre}})_{A_1}\}_{\mathfrak{p}_1} \xrightarrow{\sim} \{(\mathcal{C}_2^{\text{coa-pre}})_{A_2}\}_{\mathfrak{p}_2}$$

hence **equivalences of categories** *as follows:*

$$\text{Order}(\Phi_1(A_1)_{\mathfrak{p}_1}) \xrightarrow{\sim} \text{Order}(\Phi_2(A_2)_{\mathfrak{p}_2}); \quad \text{Order}(\Phi_1(A_1)_{\mathfrak{p}_1})^{\text{opp}} \xrightarrow{\sim} \text{Order}(\Phi_2(A_2)_{\mathfrak{p}_2})^{\text{opp}}$$

(iii) If, moreover, in the situation of (ii), the A_i *are* Div**-Frobenius-trivial**, *then the last two equivalences of categories of (ii) arise from* **isomorphisms of monoids**

$$\Phi_1(A_1)_{\mathfrak{p}_1} \xrightarrow{\sim} \Phi_2(A_2)_{\mathfrak{p}_2}; \quad \Phi_1(A_1)_{\mathfrak{p}_1} \xrightarrow{\sim} \Phi_2(A_2)_{\mathfrak{p}_2}$$

which we shall refer to, respectively, as the **right-hand** *and* **left-hand** *isomorphisms induced by* Ψ *[cf. Example 4.3 below].*

Proof. First, we observe that by Proposition 1.10, (vi), every group-like object is *Frobenius-trivial*, hence, in particular, Div-*Frobenius-trivial*; moreover, [by the definition of a "group-like object"] every endomorphism of a group-like object is a Div-identity endomorphism, and every pre-step to or from a group-like object is an isomorphism [cf. Propositions 1.4, (i), (iii); 1.8, (iii)]. Thus, since Ψ preserves *non-group-like objects* [cf. Theorem 3.4, (ii)] and *pull-back morphisms* [cf. Theorem 3.4, (iii)], we may assume for the remainder of the proof of Theorem 4.2, without loss of generality, that the objects under consideration are *non-group-like*. Now by Proposition 1.14, (v) [cf. also Theorem 3.4, (ii)], it follows immediately that Ψ preserves *non-group-like* Div-*Frobenius-trivial objects*, as well as Div-*identity prime-Frobenius endomorphisms* of such objects. Since Ψ preserves *morphisms of Frobenius type* and *Frobenius degrees* [cf. Theorem 3.4, (iii)], we thus conclude that to complete the proof of assertion (i), it suffices to prove that Ψ preserves *primary steps* and Div-*identity endomorphisms*. Moreover, to prove the remainder of assertion (i) [i.e., that Ψ preserves *primary steps* and Div-*identity endomorphisms*] and assertions (ii), (iii), clearly it suffices to do so *after* passing to the *perfections* of the $\mathcal{C}_i$ [cf. Theorem 3.4, (iii)]; thus, for the remainder of the proof of Theorem 4.2, we may assume, without loss of generality, that the $\mathcal{C}_i$ are *of perfect type* [cf. also Proposition 5.5, (iii), below].

Now let $A_1 \in \text{Ob}(\mathcal{C}_1)$ be a non-group-like Div-Frobenius-trivial object; $A_2 \stackrel{\text{def}}{=} \Psi(A_1)$. Then it follows formally from Proposition 4.1, (i), (ii) [cf. also Theorem 3.4, (ii), (iii)] that Ψ maps *primary steps* to or from A_1 to primary steps to or from A_2 in such a way that primary steps $B_1 \to A_1$, $A_1 \to C_1$ with *primary composite* $B_1 \to C_1$ are mapped to primary steps $B_2 \to A_2$, $A_2 \to C_2$ with *primary composite* $B_2 \to C_2$. Next, let

$$A_1 \to F_1$$

be a *primary step*. Then it follows immediately from Proposition 4.1, (iii), together with what we have already shown concerning primary steps to or from A_1, that Ψ maps *primary steps* to or from F_1 to primary steps to or from $F_2 \stackrel{\text{def}}{=} \Psi(F_1)$ in such a way that primary steps $F_1' \to F_1$, $F_1 \to F_1''$ with *primary composite* $F_1' \to F_1''$ are mapped to primary steps $F_2' \to F_2$, $F_2 \to F_2''$ with *primary composite* $F_2' \to F_2''$. [Indeed, to see this, it suffices to consider the following two situations [depending on whether the primary steps $A_1 \to F_1$, $F_1' \to F_1$ are *co-primary* or not]: (a) primary steps $B_i \to A_i$, $A_i \to C_i$ with primary composite such that the primary steps to or from F_i under consideration are *subordinate* to the primary composite $B_i \to C_i$; (b) commutative diagrams

$$\begin{array}{ccc} A_i' & \longrightarrow & F_i' \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & F_i \\ \downarrow & & \downarrow \\ A_i'' & \longrightarrow & F_i'' \end{array}$$

[where $i = 1, 2$] in which both the upper and lower squares are cartesian diagrams as in Proposition 4.1, (iii), and all the arrows originating from A_i, as well as the vertical composite $A_i' \to A_i \to A_i''$, are primary steps.]

Next, observe that for a suitable choice of *non-group-like* Div-*Frobenius-trivial* A_1 [e.g., a *Frobenius-trivial* A_1 — cf. Definition 1.3, (i), (a), (b)], it follows that for any object $C_1 \in \mathrm{Ob}(\mathcal{C}_1)$ that is base-isomorphic to A_1, there exist pre-steps $B_1 \to C_1$, $B_1 \to A_1$. Moreover, observe that [by applying the equivalences of categories of Definition 1.3, (iii), (d)] any *primary step to or from* B_1, as well as any *primary composite* of a primary step to B_1 with a primary step from B_1, may always be written in the form

$$D_1 \to E_1$$

where the composite $D_1 \to E_1 \to F_1$ of the above arrow $D_1 \to E_1$ with some arrow $E_1 \to F_1$ *factors* as a composite $D_1 \to B_1 \to A_1 \to F_1$ in which $D_1 \to B_1$ is a pre-step, $B_1 \to A_1$ is the pre-step introduced above, and $A_1 \to F_1$ is a primary step [so in the case of a primary step from B_1, $D_1 = B_1$; in the case of a primary step to B_1, $E_1 = B_1$]. Thus, by applying Proposition 4.1, (iv) [to the arrows $D_1 \to E_1 \to F_1$], together with what we have already shown concerning primary steps to or from F_1, we conclude that Ψ maps *primary steps* to or from B_1 to primary steps to or from $B_2 \stackrel{\mathrm{def}}{=} \Psi(B_1)$ in such a way that primary steps $B_1' \to B_1$, $B_1 \to B_1''$ with *primary composite* $B_1' \to B_1''$ are mapped to primary steps $B_2' \to B_2$, $B_2 \to B_2''$ with *primary composite* $B_2' \to B_2''$.

In a similar vein, we observe that [by applying the equivalences of categories of Definition 1.3, (iii), (d)] a *primary step to or from* C_1, as well as any *primary composite* of a primary step to C_1 with a primary step from C_1, may always be written in the form

$$E_1 \to F_1$$

where the composite $D_1 \to E_1 \to F_1$ of some arrow $D_1 \to E_1$ with the above arrow $E_1 \to F_1$ *factors* as a composite $D_1 \to B_1 \to C_1 \to F_1$ in which $D_1 \to B_1$ is a primary pre-step, $B_1 \to C_1$ is the pre-step introduced above, and $C_1 \to F_1$ is a pre-step [so in the case of a primary step from C_1, $E_1 = C_1$; in the case of a primary step to C_1, $F_1 = C_1$]. Thus, by applying Proposition 4.1, (v) [to the arrows $D_1 \to E_1 \to F_1$], together with what we have already shown concerning primary steps to or from D_1 [i.e., where we regard "D_1" as a "sort of B_1", which is possible in light of the existence of the composite pre-step $D_1 \to B_1 \to A_1$], we conclude that Ψ maps *primary steps* to or from C_1 to primary steps to or from $C_2 \stackrel{\text{def}}{=} \Psi(C_1)$ in such a way that primary steps $C_1' \to C_1$, $C_1 \to C_1''$ with *primary composite* $C_1' \to C_1''$ are mapped to primary steps $C_2' \to C_2$, $C_2 \to C_2''$ with *primary composite* $C_2' \to C_2''$.

Since C_1 was, in effect, allowed to be an *arbitrary non-group-like object* of $\mathcal{C}_1$, we thus conclude that Ψ *preserves primary steps*. Moreover, by thinking, for $A_i \in \text{Ob}(\mathcal{C}_i)$ [where $i = 1, 2$] of an element of $\text{Prime}(\Phi_i(A_i))$ as an *equivalence class* of primary steps to or from A_i [where the correspondence between elements of $\text{Prime}(\Phi_i(A_i))$ and equivalence classes of primary steps is defined by "$\text{Div}(-)$" — cf. the equivalences of categories of Definition 1.3, (iii), (d)], we thus obtain that Ψ induces a *bijection*

$$\Psi^{\text{Prime}}(A_1) : \text{Prime}(\Phi_1(A_1)) \xrightarrow{\sim} \text{Prime}(\Phi_2(A_2))$$

[where $A_2 \stackrel{\text{def}}{=} \Psi(A_1)$] as well as corresponding *equivalences of categories*

$$\{_{A_1}(\mathcal{C}_1^{\text{coa-pre}})\}_{\mathfrak{p}_1} \xrightarrow{\sim} \{_{A_2}(\mathcal{C}_2^{\text{coa-pre}})\}_{\mathfrak{p}_2}; \quad \{(\mathcal{C}_1^{\text{coa-pre}})_{A_1}\}_{\mathfrak{p}_1} \xrightarrow{\sim} \{(\mathcal{C}_2^{\text{coa-pre}})_{A_2}\}_{\mathfrak{p}_2}$$

[where $\mathfrak{p}_1 \in \text{Prime}(\Phi_1(A_1))$, $\mathfrak{p}_2 \in \text{Prime}(\Phi_2(A_2))$ correspond via $\Psi^{\text{Prime}}(A_1)$].

To check the *functoriality* of $\Psi^{\text{Prime}}(-)$ with respect to arbitrary base-isomorphisms, it suffices to check it with respect to *morphisms of Frobenius type* and *pre-steps* [cf. Proposition 1.7, (ii)]. In the case of a *morphism of Frobenius type* $B_i \to A_i$ [where $i = 1, 2$], the desired functoriality follows by considering commutative diagrams

$$\begin{array}{ccc} B_i' & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ A_i' & \longrightarrow & A_i \end{array}$$

[cf. Proposition 1.10, (i)] — where the vertical morphisms are morphisms of Frobenius type, and the horizontal morphisms are *primary steps*. In the case of a *pre-step* $B_i \to A_i$ [where $i = 1, 2$], the desired functoriality follows by considering a commutative diagram

$$\begin{array}{ccc} B_i' & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i' & \longrightarrow & C_i \\ \downarrow & & \downarrow \\ A_i' & \longrightarrow & A_i \end{array}$$

— where all of the morphisms are pre-steps; all of the horizontal morphisms, as well as the vertical morphisms and composite morphisms of the upper square, are either isomorphisms or *primary steps*; either the vertical morphisms of the lower square are isomorphisms, or the lower square is a *cartesian diagram* as in Proposition 4.1, (iii). This completes the proof of the *functoriality of* $\Psi^{\text{Prime}}(-)$, hence of assertion (ii).

Next, we observe that Ψ preserves Div-*identity endomorphisms*. Indeed, since the Φ_i are *non-dilating*, it follows that if $A \in \text{Ob}(\mathcal{C}_i)$ [where $i = 1, 2$], then $\alpha \in \text{End}_{\mathcal{C}_i}(A)$ is a Div-*identity endomorphism* if and only if α admits a factorization $\alpha = \beta \circ \gamma$, where $\beta : B \to A$ is a pull-back morphism, and $\gamma : A \to B$ is a base-isomorphism, such that for *every primary step* $A' \to A$, there exists a commutative diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} \\ B' & \longrightarrow & B \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ A'' & \longrightarrow & A \end{array}$$

in which the horizontal morphisms are *primary steps*; the upper horizontal morphism is the given primary step; the equivalence classes of the primary steps $A' \to A$, $B' \to B$ correspond via the bijection $\text{Prime}(\Phi_i(\gamma)) : \text{Prime}(\Phi_i(B)) \overset{\sim}{\to} \text{Prime}(\Phi_i(A))$ [cf. the *functoriality* of $\Psi^{\text{Prime}}(-)$]; β' is a pull-back morphism [cf. Proposition 1.11, (v)]; the primary steps $A' \to A$, $A'' \to A$ determine the same element of $\text{Prime}(\Phi_i(A))$. This completes the proof of assertion (i).

Finally, we consider assertion (iii). Thus, we assume that the A_i are Div-*Frobenius-trivial*. By considering commutative diagrams of the form

$$\begin{array}{ccc} B_i & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ B'_i & \longrightarrow & A_i \end{array} \qquad \begin{array}{ccc} A_i & \longrightarrow & C_i \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & C'_i \end{array}$$

— where the vertical morphisms are morphisms of Frobenius type [cf. Proposition 1.10, (i)], the morphisms $A_i \to A_i$ are Div-identity endomorphisms, and the horizontal morphisms are *primary steps* — it follows that the equivalences of categories in question arise from *bijections of sets*

$$\Phi_1(A_1)_{\mathfrak{p}_1} \overset{\sim}{\to} \Phi_2(A_2)_{\mathfrak{p}_2}; \quad \Phi_1(A_1)_{\mathfrak{p}_1} \overset{\sim}{\to} \Phi_2(A_2)_{\mathfrak{p}_2}$$

that are *compatible* both with "$\leq$" and with multiplication by elements of $\mathbb{N}_{\geq 1}$. In light of the well-known structure of the monoids $\mathbb{Q}_{\geq 0}$, $\mathbb{R}_{\geq 0}$ [cf. Definition 2.4, (i), (b)], this is enough to conclude that these bijections of sets are, in fact, *isomorphisms of monoids*, as desired. This completes the proof of assertion (iii). ○

Example 4.3. Independence of Right-hand and Left-hand Isomorphisms. As the following example shows, the *right-hand* and *left-hand* isomorphisms of Theorem 4.2, (iii), do *not necessarily coincide* [cf. Remark 4.9.1 below]:

Let $\mathcal{D}$ be a *one-morphism category*; Φ the monoid on $\mathcal{D}$ whose value on the unique object of $\mathcal{D}$ is the monoid $\mathbb{Q}_{\geq 0}$. Now we define a category $\mathcal{C}$ as follows: The *objects* of $\mathcal{C}$ are the elements of $\mathbb{Q}$. The *morphisms* $a \to b$ of $\mathcal{C}$ from an object $a \in \mathbb{Q}$ to an object $b \in \mathbb{Q}$ are the elements $d \in \mathbb{N}_{\geq 1}$ such that $d \cdot a \leq b$; composition of morphisms is defined by multiplication of elements of $\mathbb{N}_{\geq 1}$. We shall refer to the element $d \in \mathbb{N}_{\geq 1}$ determined by a morphism of $\mathcal{C}$ as the *Frobenius degree* of the morphism. Thus, we obtain a natural functor

$$\mathcal{C} \to \mathbb{F}_{\Phi}$$

by assigning to a morphism $\phi : a \to b$ [where $a, b \in \mathbb{Q}$] the *zero divisor* $b - \deg_{\mathrm{Fr}}(\phi) \cdot a \in \mathbb{Q}_{\geq 0}$ and *Frobenius degree* $\deg_{\mathrm{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$. Since $\mathcal{C}$ is clearly connected and totally epimorphic, this functor determines a *pre-Frobenioid structure* on $\mathcal{C}$. Moreover, the object $0 \in \mathbb{Q}$ is *Frobenius-trivial*; $\phi : a \to b$ is a *morphism of Frobenius type* if and only if $b = \deg_{\mathrm{Fr}}(\phi) \cdot a$; $\phi : a \to b$ is a pre-step if and only if $\deg_{\mathrm{Fr}}(\phi) = 1$; all morphisms of $\mathcal{C}$ are base-isomorphisms; all pull-back morphisms of $\mathcal{C}$ are isomorphisms; all "$\mathcal{O}^{\triangleright}(-)$" of $\mathcal{C}$ are trivial; no object of $\mathcal{C}$ is group-like. Thus, one verifies immediately that $\mathcal{C}$ is a *Frobenioid of isotropic type*. Since $\mathcal{D}$ is clearly of FSMFF-type, and Φ is non-dilating, it follows that $\mathcal{C}$ is also *of standard type*, over a *slim* base category $\mathcal{D}$. Now one verifies immediately that if $\lambda \in \mathbb{Q}_{>0}$, then the assignment, for $a \in \mathbb{Q}_{\geq 0}$,

$$a \mapsto a; \quad -a \mapsto -\lambda \cdot a$$

determines a *self-equivalence of categories*

$$\Psi_{\lambda} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

that preserves Frobenius degrees [cf. Theorem 3.4, (iii)]. On the other hand, it follows immediately from the construction of Ψ_{λ} that the *right-hand* isomorphism of Theorem 4.2, (iii), is the *identity* on $\mathbb{Q}_{\geq 0}$, while the *left-hand* isomorphism of Theorem 4.2, (iii), is given by *multiplication by* λ on $\mathbb{Q}_{\geq 0}$.

In order to proceed further toward the goal of *"reconstructing Φ category-theoretically from $\mathcal{C}$"*, it is necessary to find natural conditions on the Frobenioid $\mathcal{C}$ that will allow us to rule out *"pathologies"* of the sort discussed in Example 4.3. One approach to doing this is the introduction of the *birationalization of a Frobenioid*, as follows.

Proposition 4.4. (Birationalization of a Frobenioid I) *For $A, B \in \mathrm{Ob}(\mathcal{C})$, write:*

$$\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A, B) \stackrel{\mathrm{def}}{=} \varinjlim_{(A' \to A) \in \mathcal{C}_A^{\text{coa-pre}}} \mathrm{Hom}_{\mathcal{C}}(A', B)$$

where the inductive limit is parametrized by [say, some **small skeletal subcategory** *of] $\mathcal{C}_A^{\text{coa-pre}}$, and the transition morphism induced by a co-angular pre-step $A'' \to A'$ [regarded as a morphism in $\mathcal{C}_A^{\text{coa-pre}}$] is the natural morphism* $\mathrm{Hom}_{\mathcal{C}}(A', B) \to \mathrm{Hom}_{\mathcal{C}}(A'', B)$. *Then:*

(i) Composition of morphisms in $\mathcal{C}$ determines a natural composition map

$$\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A,C)$$

[where $A, B, C \in \mathrm{Ob}(\mathcal{C})$], hence a **category** $\mathcal{C}^{\mathrm{birat}}$, *whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by "$\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}$". Moreover, there exists a natural 1-commutative diagram of functors*

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathbb{F}_{\Phi} & & \\ \downarrow & & \downarrow & & \\ \mathcal{C}^{\mathrm{birat}} & \longrightarrow & \mathbb{F}_{\Phi^{\mathrm{gp}}} & \longrightarrow & \mathbb{F}_{0_{\mathcal{D}}} \end{array}$$

where the functors between elementary Frobenioids are those induced by the natural morphisms of monoids $\Phi \to \Phi^{\mathrm{gp}} \to 0_{\mathcal{D}}$; $0_{\mathcal{D}}$ is the monoid on $\mathcal{D}$ all of whose values on objects of $\mathcal{D}$ are equal to the monoid with one element [so $\mathbb{F}_{0_{\mathcal{D}}}$ is the product category of $\mathcal{D}$ with the one-object category determined by the monoid $\mathbb{N}_{\geq 1}$].

(ii) The functor $\mathcal{C}^{\mathrm{birat}} \to \mathbb{F}_{0_{\mathcal{D}}}$ of (i) determines a structure of **Frobenioid of group-like type** *on $\mathcal{C}^{\mathrm{birat}}$. Moreover, the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{birat}}$ is* **faithful**. *In particular, for every $A \in \mathrm{Ob}(\mathcal{C})$ with image A^{birat} in $\mathcal{C}^{\mathrm{birat}}$, the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{birat}}$ determines an* **injection** *of groups $\mathcal{O}^{\rhd}(A)^{\mathrm{gp}} \hookrightarrow \mathcal{O}^{\times}(A^{\mathrm{birat}})$. We shall refer to the functor "$\mathcal{O}^{\times}(-)$" on $\mathcal{D}$ associated to the Frobenioid $\mathcal{C}^{\mathrm{birat}}$ [cf. Proposition 2.2, (ii), (iii)] as the* **rational function monoid** *of the Frobenioid $\mathcal{C}$.*

(iii) There exists a unique **subfunctor of groups** $\Phi^{\mathrm{birat}} \subseteq \Phi^{\mathrm{gp}}$ *such that the functor $\mathcal{C}^{\mathrm{birat}} \to \mathbb{F}_{\Phi^{\mathrm{gp}}}$ of (i) factors through the subcategory $\mathbb{F}_{\Phi^{\mathrm{birat}}} \subseteq \mathbb{F}_{\Phi^{\mathrm{gp}}}$ determined by Φ^{birat}, and, moreover, the resulting functor*

$$\mathcal{C}^{\mathrm{birat}} \to \mathbb{F}_{\Phi^{\mathrm{birat}}}$$

induces, for each $A^{\mathrm{birat}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{birat}})$, a **surjection** $\mathcal{O}^{\times}(A^{\mathrm{birat}}) \twoheadrightarrow \Phi^{\mathrm{birat}}(A^{\mathrm{birat}})$, *whose kernel is the image, via the injection $\mathcal{O}^{\rhd}(A)^{\mathrm{gp}} \hookrightarrow \mathcal{O}^{\times}(A^{\mathrm{birat}})$ of (ii), of $\mathcal{O}^{\times}(A) \subseteq \mathcal{O}^{\rhd}(A)^{\mathrm{gp}}$.*

(iv) A morphism of $\mathcal{C}$ maps to a(n) **co-angular morphism** *(respectively,* **isomorphism**; **morphism of Frobenius type**; **pull-back morphism**; **morphism of a given Frobenius degree**; **isometry**; **pre-step**; **base-isomorphism***) of $\mathcal{C}^{\mathrm{birat}}$ if and only if it is a(n) co-angular morphism (respectively, co-angular pre-step; co-angular base-isomorphism; co-angular linear morphism; morphism of a given Frobenius degree; arbitrary morphism; pre-step; base-isomorphism) of $\mathcal{C}$. A morphism of $\mathcal{C}^{\mathrm{birat}}$ is a* **base-identity endomorphism** *if and only if arises from a pair $(\alpha : A' \to A; \phi : A' \to A)$, where α is a co-angular pre-step in the indexing category of the inductive limit defining $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A,A)$, and α and ϕ are base-equivalent. An object of $\mathcal{C}$ maps to an* **isotropic** *object of $\mathcal{C}^{\mathrm{birat}}$ if and only if it is an isotropic object of $\mathcal{C}$.*

Proof. First, we consider assertion (i). Given morphisms $\phi' : A' \to B$, $\psi' : B' \to C$ [in $\mathcal{C}$] and co-angular pre-steps $\alpha : A' \to A$, $\beta : B' \to B$ [in $\mathcal{C}$], it follows from Proposition 1.11, (vii), that there exists a commutative diagram

$$\begin{array}{ccccccc} & & A'' & \xrightarrow{\phi''} & B' & \xrightarrow{\psi'} & C \\ & & \downarrow \alpha' & & \downarrow \beta & & \\ A & \xleftarrow{\alpha} & A' & \xrightarrow{\phi'} & B & & \end{array}$$

where α' [hence also $\alpha \circ \alpha'$] is a co-angular pre-step. Then we take the composite of the image of ϕ' in $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A, B)$ with the image of ψ' in $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(B, C)$ to be the image of $\psi' \circ \phi''$ in $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A, C)$. To show that this assignment is *independent* of the choice of α', ϕ'', it suffices to consider commutative diagrams

$$\begin{array}{ccccc} A^* & \xrightarrow{\alpha^*} & A''' & \xrightarrow{\phi'''} & B' \\ \downarrow \alpha'' & & \downarrow \alpha''' & & \downarrow \beta \\ A'' & \xrightarrow{\alpha'} & A' & \xrightarrow{\phi'} & B \end{array}$$

[where α'', α''', α^* are co-angular pre-steps] and to observe that since β is a *monomorphism* [cf. Definition 1.3, (v), (a)], the fact that

$$\beta \circ \phi''' \circ \alpha^* = \phi' \circ \alpha''' \circ \alpha^* = \phi' \circ \alpha' \circ \alpha'' = \beta \circ \phi'' \circ \alpha''$$

implies that $\phi''' \circ \alpha^* = \phi'' \circ \alpha''$, i.e., that $\psi' \circ \phi''$, $\psi' \circ \phi'''$ determine the *same element* of $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(A, C)$. Also, one verifies immediately that composite of morphisms of $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(-, -)$ is *associative*. This completes the definition of the category $\mathcal{C}^{\mathrm{birat}}$. Then by assigning to the pair $(\alpha : A' \to A, \phi' : A' \to B)$ the element

$$\Phi(\alpha)^{-1}\{\mathrm{Div}(\phi') - \deg_{\mathrm{Fr}}(\phi') \cdot \mathrm{Div}(\alpha)\} \in \Phi(A)^{\mathrm{gp}}$$

[cf. Remark 1.1.1] one verifies immediately that the functor $\mathcal{C} \to \mathbb{F}_{\Phi}$ induces a functor $\mathcal{C}^{\mathrm{birat}} \to \mathbb{F}_{\Phi^{\mathrm{gp}}}$, as well as a 1-commutative diagram as in the statement of assertion (i). This completes the proof of assertion (i).

Next, we observe that it follows formally from the definition of $\mathcal{C}^{\mathrm{birat}}$ that $\mathcal{C}^{\mathrm{birat}}$ is *connected*; moreover, [cf. the discussion of the composition of arrows of $\mathcal{C}^{\mathrm{birat}}$ in the proof of assertion (i)] the *total epimorphicity* of $\mathcal{C}^{\mathrm{birat}}$ follows immediately from that of $\mathcal{C}$. Thus, the functor $\mathcal{C}^{\mathrm{birat}} \to \mathbb{F}_{0_{\mathcal{D}}}$ determines a structure of *pre-Frobenioid* on $\mathcal{C}^{\mathrm{birat}}$. Now the portion of assertion (iv) concerning *morphisms of a given Frobenius degree*, *isometries* [cf. the monoid structure of the monoid $0_{\mathcal{D}}!$], *pre-steps*, *base-isomorphisms*, and *base-identity endomorphisms* of $\mathcal{C}^{\mathrm{birat}}$ follows immediately from the definitions. The portion of assertion (iv) concerning *co-angular pre-steps* of $\mathcal{C}$ follows immediately from the definition of "$\mathrm{Hom}_{\mathcal{C}}^{\mathrm{birat}}(-, -)$"; Proposition 1.7, (v) [for co-angular pre-steps].

To verify the portion of assertion (iv) concerning *co-angular morphisms*, we reason as follows: Given a morphism $A^{\mathrm{birat}} \to B^{\mathrm{birat}}$ in $\mathcal{C}^{\mathrm{birat}}$, any factorization

$A^{\text{birat}} \to C^{\text{birat}} \to D^{\text{birat}} \to B^{\text{birat}}$ in $\mathcal{C}^{\text{birat}}$, where either $A^{\text{birat}} \to C^{\text{birat}}$ or $D^{\text{birat}} \to B^{\text{birat}}$ is a base-isomorphism, $C^{\text{birat}} \to D^{\text{birat}}$ is a(n) [isometric] pre-step, and $D^{\text{birat}} \to B^{\text{birat}}$ is linear, arises [cf. the proof of assertion (i)] from a factorization $A' \to C' \to D' \to B$ in $\mathcal{C}$, where either $A' \to C'$ or $D' \to B$ is a base-isomorphism, $C' \to D'$ is a pre-step, and $D' \to B$ is linear. Thus, if $A' \to B$ is co-angular, then [by applying the factorization of Definition 1.3, (v), (b), to $C' \to D'$, we conclude that] $C' \to D'$ is a co-angular pre-step, so $C^{\text{birat}} \to D^{\text{birat}}$ is an isomorphism; in particular, it follows that $A^{\text{birat}} \to B^{\text{birat}}$ is *co-angular*. On the other hand, if $A^{\text{birat}} \to B^{\text{birat}}$ is co-angular, then $C^{\text{birat}} \to D^{\text{birat}}$ is an isomorphism, which [by the portion of assertion (iv) concerning isomorphisms of $\mathcal{C}^{\text{birat}}$] implies that $C' \to D'$ is a *co-angular pre-step*, hence an *isomorphism* whenever it is an isometry [cf. Proposition 1.4, (iii)]; thus, $A' \to B$, hence also any morphism $A \to B$ appearing in a factorization $A' \to A \to B$ [where $A' \to A$ is a co-angular pre-step], is *co-angular*.

The portion of assertion (iv) concerning *morphisms of Frobenius type* now follows formally from the portion of assertion (iv) concerning co-angular morphisms, isometries, and base-isomorphisms. Next, let us observe that it is immediate from the definition of a *pull-back morphism* [cf. Definition 1.2, (ii)] that any pull-back morphism of $\mathcal{C}$ maps to a pull-back morphism of $\mathcal{C}^{\text{birat}}$. Since, moreover, a morphism of $\mathcal{C}$ is a co-angular linear morphism if and only if it is a composite of a co-angular pre-step and a pull-back morphism [cf. Propositions 1.4, (iv); 1.7, (iii)], it thus follows [cf. the portion of assertion (iv) concerning co-angular pre-steps of $\mathcal{C}$] that every co-angular linear morphism of $\mathcal{C}$ maps to a pull-back morphism of $\mathcal{C}^{\text{birat}}$. On the other hand, if $\phi : A \to B$ is a morphism of $\mathcal{C}$ that maps to a pull-back morphism $\phi^{\text{birat}} : A^{\text{birat}} \to B^{\text{birat}}$ of $\mathcal{C}^{\text{birat}}$, then it follows that ϕ is *linear*, hence that it factors as a composite $\gamma \circ \alpha$, where $\alpha : A \to C$ is a pre-step, and $\gamma : C \to B$ is a pull-back morphism [cf. Proposition 1.7, (iii)]. Thus, we obtain an equation $\phi^{\text{birat}} = \gamma^{\text{birat}} \circ \alpha^{\text{birat}}$ in $\mathcal{C}^{\text{birat}}$, where ϕ^{birat}, γ^{birat} are pull-back morphisms, and α^{birat} is a base-isomorphism; but [by the isomorphism of functors appearing in the definition of a "pull-back morphism" in Definition 1.2, (ii)] this implies formally that α^{birat} is an *isomorphism*, hence [by the portion of assertion (iv) concerning co-angular pre-steps of $\mathcal{C}$] that α is a co-angular pre-step, as desired. Finally, the portion of assertion (iv) concerning *isotropic objects* follows immediately from the portion of assertion (iv) concerning pre-steps and co-angular pre-steps; Proposition 1.4, (i), (iii); Proposition 1.9, (iv). This completes the proof of assertion (iv).

In light of the *"dictionary"* provided by assertion (iv) [cf. also Proposition 1.4, (iv); the equivalence of categories of Proposition 1.9, (ii)], it is now a routine exercise to check that $\mathcal{C}^{\text{birat}}$ is, in fact, a *Frobenioid of group-like type*. Moreover, it is immediate from the definitions [and the *total epimorphicity* of $\mathcal{C}$] that the functor $\mathcal{C} \to \mathcal{C}^{\text{birat}}$ is *faithful* and determines an injection $\mathcal{O}^{\rhd}(A)^{\text{gp}} \hookrightarrow \mathcal{O}^{\times}(A^{\text{birat}})$, for $A \in \text{Ob}(\mathcal{C})$. This completes the proof of assertion (ii). Now assertion (iii) follows immediately from the existence of the functor $\mathcal{C}^{\text{birat}} \to \mathbb{F}_{\Phi^{\text{gp}}}$ of assertion (i) [cf. also Proposition 1.5, (ii)]; here, we note that the computation of the *kernel* of the surjection of assertion (iii) follows from Definition 1.3, (vi). ○

Definition 4.5.

(i) We shall say that an object of $\mathcal{C}$ is *birationally Frobenius-normalized* if its image in $\mathcal{C}^{\text{birat}}$ is Frobenius-normalized. [Thus, any birationally Frobenius-normalized object of $\mathcal{C}$ is Frobenius-normalized — cf. Proposition 4.4, (ii), (iv).] If every object of $\mathcal{C}$ is birationally Frobenius-normalized, then we shall say that $\mathcal{C}$ is *of birationally Frobenius-normalized type*. If $\mathcal{C}$ is of pre-model and birationally Frobenius-normalized type, then we shall say that $\mathcal{C}$ is *of model type*.

(ii) Suppose that Φ is *perf-factorial*; $A \in \text{Ob}(\mathcal{C})$. Then we shall say that A is *strictly rational* if, for every prime $\mathfrak{p} \in \text{Prime}(\Phi(A))$, there exists an element $a - b \in \Phi^{\text{birat}}(A)$, where $a, b \in \Phi(A)$ such that $\mathfrak{p} \in \text{Supp}(a)$, $\mathfrak{p} \notin \text{Supp}(b)$ [cf. Definition 2.4, (i), (d)]. We shall say that A is *rational* if there exists a pull-back morphism $B \to A$ in $\mathcal{C}$, where B is strictly rational. If [Φ is perf-factorial, and] every object of $\mathcal{C}$ is rational (respectively, strictly rational), then we shall say that $\mathcal{C}$ is *of rational* (respectively, *strictly rational*) *type*.

(iii) We shall say that $\mathcal{C}$ is *of rationally standard type* if the following conditions are satisfied: (a) $\mathcal{C}$ is of birationally Frobenius-normalized, rational, and standard type; (b) $(\mathcal{C}^{\text{un-tr}})^{\text{birat}}$ admits a Frobenius-compact object.

(iv) We shall say that $\mathcal{D}$ is Div-*slim* [relative to Φ] if, for every $A \in \text{Ob}(\mathcal{D})$, the homomorphism

$$\text{Aut}(\mathcal{D}_A \to \mathcal{D}) \to \text{Aut}(\mathcal{D}_A \to \mathfrak{Mon})$$

[induced by composition with the functor $\Phi : \mathcal{D} \to \mathfrak{Mon}$] is *injective*. [Thus, if $\mathcal{D}$ is *slim*, then it is Div-*slim*.]

Remark 4.5.1. We observe in passing that it is immediate from the definitions that if $\mathcal{C}$ is *of rationally standard type* (respectively, *of standard type*), then so is $\mathcal{C}^{\text{istr}}$.

Example 4.6. Frobenius-normalized vs. Birationally Frobenius-normalized. As the following example shows, it is not necessarily the case that a Frobenioid of Frobenius-normalized type is of birationally Frobenius-normalized type: Let G be an abelian group, written additively. For each $p \in \mathfrak{Primes}$, let $\xi_p \in G$. Then if we write $M \stackrel{\text{def}}{=} G \times \mathbb{Z} \times \mathbb{Z}$, then the assignment

$$M \ni (g, a, b) \mapsto (p \cdot g + a \cdot \xi_p, p \cdot a, p \cdot b) \in M$$

determines an endomorphism $\alpha_p \in \text{End}(M)$ of the module M such that α_p commutes with all $\alpha_{p'}$, for $p' \in \mathfrak{Primes}$. Thus, we obtain a homomorphism $\mathbb{N}_{\geq 1} \to \text{End}(M)$, i.e., an action of $\mathbb{N}_{\geq 1}$ on M; write α_n for the image in $\text{End}(M)$ of $n \in \mathbb{N}_{\geq 1}$. Write N for the monoid whose underlying set is equal to the direct product

$$M \times \mathbb{N}_{\geq 1}$$

and whose monoid structure is given as follows: If $\lambda, \mu \in M$; $l, m \in \mathbb{N}_{\geq 1}$, then $(\lambda, l) \cdot (\mu, m) = (\lambda + \alpha_l(\mu), l \cdot m)$. Now let $\mathcal{D}$ be a one-morphism category; Φ the monoid on $\mathcal{D}$ whose unique value is given by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Let $\mathcal{C}$ be the category whose *objects* A_n are indexed by elements $n \in \mathbb{Z}$, and whose *morphisms* $A_{n_1} \to A_{n_2}$ [where $n_1, n_2 \in \mathbb{Z}$] consist of elements $(g, a, b, d) \in N$ such that $a \geq 0$, $b \geq 0$, $n_2 - d \cdot n_1 = a + b$; composition of morphisms is determined by the product structure of N. The assignment $(g, a, b, d) \mapsto (a, b, d)$ then determines a functor

$$\mathcal{C} \to \mathbb{F}_\Phi$$

[which lies over $\mathcal{D}$]. Moreover, one checks immediately that, relative to this last functor, $\mathcal{C}$ is a *Frobenioid of isotropic and standard type* which is *not* of group-like type. Also, we observe that the object $A_0 \in \mathrm{Ob}(\mathcal{C})$ is *Frobenius-trivial*, and that for every $A \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{O}^\times(A) = \mathcal{O}^\triangleright(A) = G$. On the other hand, one computes easily that for $A^{\mathrm{birat}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{birat}})$, $\mathcal{O}^\times(A^{\mathrm{birat}}) = M_0$, where we write $M_0 \subseteq M$ for the subgroup of $(g, a, b) \in M$ such that $a + b = 0$. Moreover, the morphisms $(0, 0, 0, d) \in N$ determine a homomorphism $\mathbb{N}_{\geq 1} \to \mathrm{End}_{\mathcal{C}}(A_0) \to \mathrm{End}_{\mathcal{C}^{\mathrm{birat}}}(A_0^{\mathrm{birat}})$ [where we write use the superscript "birat" to denote the image of objects of $\mathcal{C}$ in $\mathcal{C}^{\mathrm{birat}}$], hence an action of $\mathbb{N}_{\geq 1}$ on $\mathcal{O}^\times(A^{\mathrm{birat}}) = M_0$, which is easily verified to coincide with the restriction to M_0 of the original action of $\mathbb{N}_{\geq 1}$ on M. Now observe that $\mathcal{C}$ is *of [strictly] rational type* [cf. Definition 4.5, (ii)], and, moreover, every object of $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$ is *Frobenius-compact*. On the other hand, if the $\xi_p \neq 0$ [so α_p does not act on M_0 by multiplication by p], then $\mathcal{C}$ *fails to be of birationally Frobenius-normalized type*. [In a similar vein, we note that although $\mathcal{C}^{\mathrm{birat}}$ is "very similar" to an elementary Frobenioid, the presence of the "ξ_p's" means that it is not, in general, an elementary Frobenioid.]

Example 4.7. Frobenius-slim vs. Div**-slim.**

(i) Suppose that the functor $\Phi : \mathcal{D} \to \mathfrak{Mon}$ maps every automorphism of $\mathcal{D}$ to an *identity* automorphism of $\mathfrak{Mon}$. Then it follows formally that $\mathcal{D}$ is Div-slim if and only if $\mathcal{D}$ is slim. In particular, if, for instance, $\mathcal{D}$ is a *one-object category*, $A \in \mathrm{Ob}(\mathcal{D})$, and $\mathrm{End}_{\mathcal{D}}(A)$ is a *nontrivial residually finite group* G, then

$$\mathrm{Aut}(\mathcal{D}_A \to \mathcal{D}) = \mathrm{Ker}(\mathrm{Aut}(\mathcal{D}_A \to \mathcal{D}) \to \mathrm{Aut}(\mathcal{D}_A \to \mathfrak{Mon})) = G$$

— so [cf. Remark 3.1.2] $\mathcal{D}$ is *Frobenius-slim*, but not Div-*slim*.

(ii) Let $V \stackrel{\mathrm{def}}{=} \mathbb{Q}$; $N \stackrel{\mathrm{def}}{=} (\mathbb{N}_{\geq 1})^{\mathrm{gp}}$; $G \stackrel{\mathrm{def}}{=} V \rtimes N$, where N ($\subseteq \mathbb{Q}$) acts on V multiplicatively. Let $\mathcal{D}$ be a *one-object category*, $A \in \mathrm{Ob}(\mathcal{D})$; suppose that $\mathrm{End}_{\mathcal{D}}(A) = G$ [so $\mathrm{Aut}(\mathcal{D}_A \to \mathcal{D}) = G$]. Then clearly there exists an injection $\mathbb{F} \hookrightarrow G$, so $\mathcal{D}$ *fails to be Frobenius-slim*. On the other hand, if $\Phi : \mathcal{D} \to \mathfrak{Mon}$ is the functor determined by the monoid

$$\bigoplus_{g \in G} \mathbb{Z}_{\geq 0}$$

[i.e., the copies of $\mathbb{Z}_{\geq 0}$ are indexed by the elements of G] equipped with the G-action obtained by letting G act by left multiplication on the indices of the copies of $\mathbb{Z}_{\geq 0}$, then the natural map

$$\mathrm{Aut}(\mathcal{D}_A \to \mathcal{D}) = G \to \mathrm{Aut}(\mathcal{D}_A \to \mathfrak{Mon})$$

is clearly *injective*, so $\mathcal{D}$ is Div-*slim* [relative to Φ].

Proposition 4.8. (Birationalization of a Frobenioid II)

(i) If $\mathcal{C}$ *is of* **isotropic** *type, then so is* $\mathcal{C}^{\text{birat}}$.

(ii) If $\mathcal{C}$ *is of* **perfect** *and* **isotropic** *type, then so is* $\mathcal{C}^{\text{birat}}$.

(iii) If $\mathcal{C}$ *is of* **rationally standard** *type, then* $(\mathcal{C}^{\text{istr}})^{\text{birat}}$ *is* **of standard type**.

(iv) If $\mathcal{C}$ *is of* **isotropic** *and* **pre-model** *type, then so is* $\mathcal{C}^{\text{birat}}$.

Proof. Assertion (i) follows formally from Proposition 4.4, (iv). To prove assertion (ii), observe that the *naive Frobenius functor* [cf. Proposition 2.1] determines a natural 1-commutative diagram [cf. Proposition 4.4, (ii), (iv)]

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Psi} & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\text{birat}} & \xrightarrow{\Psi^{\text{birat}}} & \mathcal{C}^{\text{birat}} \end{array}$$

in which the vertical arrows are the natural functor $\mathcal{C} \to \mathcal{C}^{\text{birat}}$ of Proposition 4.4, (i); the horizontal arrows are the "naive Frobenius functor" of Proposition 2.1; the upper horizontal arrow is an *equivalence of categories* [by our assumption that $\mathcal{C}$ is of perfect type; Proposition 2.1, (iii)]. Since, moreover, Ψ and any quasi-inverse to Ψ preserve [necessarily co-angular, since $\mathcal{C}$ is of isotropic type] *pre-steps*, it thus follows immediately [cf. the definition of "$\mathcal{C}^{\text{birat}}$"] that Ψ^{birat} is also an *equivalence of categories*. But this implies [cf. Proposition 2.1, (iii)] that $\mathcal{C}^{\text{birat}}$ is *of perfect type*, as desired. In light of assertion (i), this completes the proof of assertion (ii). Finally, assertion (iii) follows formally from the definitions [cf. also assertion (i)]; assertion (iv) follows formally Proposition 4.4, (iv) [cf. also assertion (i)]. ○

We are now ready to *"reconstruct* Φ *category-theoretically from* $\mathcal{C}$*"*.

Theorem 4.9. (Category-theoreticity of Divisor Monoids) *For* $i = 1, 2$, *let* Φ_i *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}_i$*;* $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *a* **Frobenioid** *of* **rationally standard** *type;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories**. *Then there exists an* **isomorphism of functors**

$$\Psi^{\Phi} : \Phi_1 \xrightarrow{\sim} \Phi_2$$

[where we regard, for $i = 1, 2$, *the functor* $\Phi_i : \mathcal{D}_i \to \mathfrak{Mon}$ *as a functor on* $\mathcal{C}_i$, *by restriction via the natural projection functor* $\mathcal{C}_i \to \mathcal{D}_i$*] lying over* Ψ, *which is*

compatible *[when the $\mathcal{C}_i$ are of* **isotropic***, but* **not** *of group-like type] with the isomorphism Ψ^{Prime} of Theorem 4.2, (ii).*

Proof. First, we observe [cf. Theorem 3.4, (i), (ii)] that we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are of *isotropic* type [cf. Remark 4.5.1], but *not* of group-like type [since Theorem 4.9 is vacuous if $\mathcal{C}_1$, $\mathcal{C}_2$ are of group-like type].

Next, I *claim* that to complete the proof of Theorem 4.9, it suffices to show that the *right-hand* and *left-hand* isomorphisms of Theorem 4.2, (iii), *coincide* [cf. Remark 4.9.1 below], for all *universally* Div-*Frobenius-trivial objects* [e.g., Frobenius-trivial objects — cf. Remark 1.11.1]. Indeed, if the right-hand and left-hand isomorphisms of Theorem 4.2, (iii), coincide for all universally Div-Frobenius-trivial objects, then it follows immediately from the construction of the *isomorphism of functors* Ψ^{Prime} in the proof of Theorem 4.2, (ii), that Ψ^{Prime} extends, for $A_i \in \text{Ob}(\mathcal{C}_i^{\text{bs-iso}})$, $\mathfrak{p}_i \in \text{Prime}(\Phi_i(A_i))$ [where $i = 1, 2$] such that $A_2 = \Psi(A_1)$, $\mathfrak{p}_2 = \Psi^{\text{Prime}}(\mathfrak{p}_1)$, to an *isomorphism of monoids*

$$\Phi_1(A_1)^{\text{pf}}_{\mathfrak{p}_1} \xrightarrow{\sim} \Phi_2(A_2)^{\text{pf}}_{\mathfrak{p}_2}$$

which is *functorial in* A_1 [regarded as an object of $\mathcal{C}_1^{\text{bs-iso}}$]. Thus, by allowing the $\mathfrak{p}_i$ to vary, we obtain, for $A_i \in \text{Ob}(\mathcal{C}_i^{\text{bs-iso}})$ [where $i = 1, 2$] such that $A_2 = \Psi(A_1)$, an *isomorphism of monoids*

$$\Phi_1(A_1)^{\text{pf}}_{\text{factor}} \xrightarrow{\sim} \Phi_2(A_2)^{\text{pf}}_{\text{factor}}$$

[cf. Definition 2.4, (i), (c)] which is *functorial in* A_1 [regarded as an object of $\mathcal{C}_1^{\text{bs-iso}}$]. Moreover, by applying, say, the first equivalence of categories of Definition 1.3, (iii), (d), to obtain pre-steps $\phi : A \to B$ of $\mathcal{C}_i$ with arbitrary prescribed zero divisor and considering primary steps $\psi : A \to C$ such that $\phi = \zeta \circ \psi$ for some pre-step ζ, one concludes immediately that this subset maps the subset $\Phi_1(A_1) \subseteq \Phi_1(A_1)^{\text{pf}}_{\text{factor}}$ [cf. Definition 2.4, (i), (c)] onto the subset $\Phi_2(A_2) \subseteq \Phi_2(A_2)^{\text{pf}}_{\text{factor}}$, hence determines an *isomorphism of monoids*

$$\Phi_1(A_1) \xrightarrow{\sim} \Phi_2(A_2)$$

which is *functorial in* A_1 [regarded as an object of $\mathcal{C}_1^{\text{bs-iso}}$]. Finally, the *functoriality* of this isomorphism of monoids with respect to *pull-back morphisms* follows immediately by *"pulling back pre-steps"*, as in Proposition 1.11, (v). This completes the proof of the *claim*.

To prove that the *right-hand* and *left-hand* isomorphisms of Theorem 4.2, (iii), *coincide* for all universally Div-Frobenius-trivial objects, we reason as follows. First of all, by passing to *perfections* [cf. Theorem 3.4, (iii)], we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are *of perfect type* [cf. also Proposition 5.5, (iii), below]. Let A be a *universally* Div-*Frobenius-trivial object* of $\mathcal{C}_i$ [where $i = 1, 2$]. Since the *right-hand* and *left-hand* isomorphisms of Theorem 4.2, (iii), are clearly *compatible with pull-back morphisms* [cf. Proposition 1.11, (v); the proof of Theorem 4.2, (iii)], and Ψ *preserves pull-back morphisms* [cf. Theorem 3.4, (iii)], it follows that we may assume without loss of generality that A is *strictly rational.* Let us refer to pairs of primary steps $\beta : A \to B$, $\gamma : C \to A$ such that $\text{Div}(\beta) = (\Phi_i(\gamma))^{-1}(\text{Div}(\gamma))$ as *twin-primary steps*. Then, it suffices to show, for each $\mathfrak{p} \in \text{Prime}(\Phi_1(A))$, the existence of twin-primary steps with zero divisor in $\mathfrak{p}$ that are mapped by Ψ to twin-primary steps of $\mathcal{C}_2$.

On the other hand, since A is *strictly rational*, it follows [cf. Definition 4.5, (ii)] that there exist, for each $\mathfrak{p} \in \text{Prime}(\Phi_i(A))$, *cartesian commutative diagrams* of pre-steps as in Proposition 4.1, (iii),

$$\begin{array}{ccc} C & \xrightarrow{\gamma'} & D \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ B & \xrightarrow{\beta} & A \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\gamma'} & D \\ \downarrow{\scriptstyle\gamma''} & & \downarrow{\scriptstyle\delta'} \\ A & \xrightarrow{\alpha} & F \end{array}$$

in which α, β are *twin-primary* with zero divisor in $\mathfrak{p}$; the pre-steps $\zeta \stackrel{\text{def}}{=} \beta \circ \gamma : C \to A$, $\gamma'' : C \to A$ are Div-*equivalent* [e.g., base-equivalent]. [Indeed, Definition 4.5, (ii) [cf. also the equivalences of categories of Definition 1.3, (iii), (d)], guarantees the existence of base-equivalent pre-steps ζ, γ'' — which may, moreover, be taken to be *co-primary* [cf. Proposition 4.1, (iii); Definition 2.4, (i), (c), (d)], by our assumption that $\mathcal{C}_i$ is *of perfect type* — such that ζ admits a factorization $\beta \circ \gamma$, where β is primary with zero divisor that maps via $\Phi(\beta)^{-1}$ to an element of $\mathfrak{p}$, and [again by our assumption that $\mathcal{C}_i$ is *of perfect type*] $\mathfrak{p}$ is not contained in the support of $(\Phi(\zeta))^{-1}(\text{Div}(\gamma))$.] Conversely, given any pair of cartesian diagrams of pre-steps as in Proposition 4.1, (iii),

$$\begin{array}{ccc} C & \xrightarrow{\gamma'} & D \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ B & \xrightarrow{\beta} & A \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\gamma'} & D \\ \downarrow{\scriptstyle\gamma''} & & \downarrow{\scriptstyle\delta'} \\ A & \xrightarrow{\alpha} & F \end{array}$$

in which α, β are *primary* with zero divisor in $\mathfrak{p}$; the pre-steps $\zeta \stackrel{\text{def}}{=} \beta \circ \gamma : C \to A$, $\gamma'' : C \to A$ are Div-*equivalent* [e.g., base-equivalent], it follows immediately that α, β are *twin-primary*. On the other hand, since Ψ preserves *pre-steps* [cf. Theorem 3.4, (ii)], *primary steps* [cf. Theorem 4.2, (i)], Div-*equivalent pairs of base-isomorphisms* [cf. Theorem 4.2, (ii); the fact that Φ_i is *non-dilating*], and *cartesian diagrams* as in Proposition 4.1, (iii) [cf. Proposition 4.1, (iii), or, alternatively, Theorem 4.2, (ii)], we thus conclude that for each $\mathfrak{p} \in \text{Prime}(\Phi_1(A))$, there exist twin-primary steps with zero divisor in $\mathfrak{p}$ that are mapped by Ψ to twin-primary steps of $\mathcal{C}_2$. This completes the proof of Theorem 4.9. ◯

Remark 4.9.1. One verifies immediately that the Frobenioid of Example 4.3 is *not* of rational type.

Corollary 4.10. **(Category-theoreticity of the Birationalization)** *For* $i = 1, 2$, *let* Φ_i *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}_i$ **of FSMFF-type**; $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *a* **Frobenioid of quasi-isotropic type**;

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. Then there exists a* 1**-unique** *functor* Ψ^{birat} : $\mathcal{C}_1^{\text{birat}} \to \mathcal{C}_2^{\text{birat}}$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\text{birat}} & \xrightarrow{\Psi^{\text{birat}}} & \mathcal{C}_2^{\text{birat}} \end{array}$$

[where the vertical arrows are the natural functors of Proposition 4.4, (i); the horizontal arrows are equivalences of categories]. Finally, if $\mathcal{D}_1$*,* $\mathcal{D}_2$ *are* **slim***, and* $\mathcal{C}_1$*,* $\mathcal{C}_2$ *are of* **birationally Frobenius-normalized** *type, then each of the composite functors of this diagram is* **rigid***.*

Proof. The existence and 1-uniqueness of a 1-*commutative diagram* as in the statement of Corollary 4.10 follows immediately from the definition of "$\mathcal{C}_i^{\text{birat}}$" [cf. Proposition 4.4, (i)], and the fact that Ψ preserves *co-angular pre-steps* [cf. Theorem 3.4, (ii)]. The *rigidity assertion* then follows immediately from Proposition 1.13, (i), by considering base-identity endomorphisms of Frobenius type of Frobenius-trivial objects of $\mathcal{C}_i$, under the hypothesis that the $\mathcal{C}_i$ are *birationally Frobenius-normalized* [cf., e.g., the proof of the rigidity assertion of Theorem 3.4, (i)]. ◯

Corollary 4.11. (Category-theoreticity of the Functor to an Elementary Frobenioid I) *For* $i = 1, 2$*, let* Φ_i *be a* **perf-factorial divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}_i$ *which is* Div**-slim** *[with respect to* Φ_i*];* $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *a* **Frobenioid of standard type***;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. If* $\mathcal{C}_1$*,* $\mathcal{C}_2$ *are* **of group-like type***, then we also assume that both* Ψ *and some quasi-inverse to* Ψ *preserve base-isomorphisms. Then:*

(i) There exists a 1**-unique** *functor* $\Psi^{\text{un-tr}} : \mathcal{C}_1^{\text{un-tr}} \to \mathcal{C}_2^{\text{un-tr}}$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1^{\text{istr}} & \xrightarrow{\Psi^{\text{istr}}} & \mathcal{C}_2^{\text{istr}} \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\text{un-tr}} & \xrightarrow{\Psi^{\text{un-tr}}} & \mathcal{C}_2^{\text{un-tr}} \end{array}$$

[where the vertical arrows are the natural projection functors; the horizontal arrows are equivalences of categories; Ψ^{istr} *is the restriction of* Ψ *to* $\mathcal{C}_1^{\text{istr}}$ *— cf. Theorem 3.4, (i)]. Moreover, each of the composite functors of this diagram is* **rigid***.*

(ii) There exists a 1**-unique** *functor* $\Psi^{\text{Base}} : \mathcal{D}_1 \to \mathcal{D}_2$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\text{Base}}} & \mathcal{D}_2 \end{array}$$

[where the vertical arrows are the natural projection functors; the horizontal arrows are equivalences of categories]. Moreover, if $\mathcal{D}_1$, $\mathcal{D}_2$ are **slim***, then each of the composite functors of this diagram is* **rigid***.*

(iii) Suppose further that $\mathcal{C}_1$, $\mathcal{C}_2$ are **of rationally standard type***. Then there exists an* **isomorphism of functors**

$$\Psi^{\Phi} : \Phi_1 \xrightarrow{\sim} \Phi_2$$

[where we regard, for $i = 1, 2$, the functor $\Phi_i : \mathcal{D}_i \to \mathfrak{Mon}$ as a functor on $\mathcal{D}_i$] lying over the equivalence of categories Ψ^{Base} of (i), which is **compatible** *[when the $\mathcal{C}_i$ are of* **isotropic***, but* **not** *of group-like type] with the isomorphism Ψ^{Prime} of Theorem 4.2, (ii). In particular, Ψ^{Base}, Ψ^{Φ} induce an* **equivalence of categories** *$\Psi^{\mathbb{F}} : \mathbb{F}_{\Phi_1} \xrightarrow{\sim} \mathbb{F}_{\Phi_2}$.*

(iv) Suppose further that $\mathcal{C}_1$, $\mathcal{C}_2$ are **of rationally standard type***. Then there exists a* **1-commutative diagram**

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathbb{F}_{\Phi_1} & \xrightarrow{\Psi^{\mathbb{F}}} & \mathbb{F}_{\Phi_2} \end{array}$$

[where the vertical arrows are the functors that define the Frobenioid structures on $\mathcal{C}_1$, $\mathcal{C}_2$; the horizontal arrows are equivalences of categories]. Moreover, each of the composite functors of this diagram is **rigid***.*

Proof. First, we observe [cf. Theorem 3.4, (i)] that we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are of *isotropic* type [cf. Remark 4.5.1]. Also, if $\mathcal{C}_1$, $\mathcal{C}_2$ are of group-like type [cf. Theorem 3.4, (ii)], then "Div-slimness" amounts to "slimness", so assertions (i), (ii) follow from Theorem 3.4, (iv), (v); assertion (iii) is vacuous; assertion (iv) follows from the fact that Ψ preserves Frobenius degrees [cf. Theorem 3.4, (iii), (iv)]. Thus, we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are *not* of group-like type.

Now we consider assertion (i). To show the existence and 1-uniqueness of a 1-commutative diagram as in the statement of assertion (i), it suffices to show that Ψ preserves "$\mathcal{O}^{\times}(-)$" [cf. the proof of Theorem 3.4, (iv)]. But observe that, for $A \in \text{Ob}(\mathcal{C}_i)$, an element $f \in \mathcal{O}^{\times}(A)$ determines an automorphism [cf. the proof of Proposition 3.3, (i)]

$$\phi_f \in \text{Aut}((\mathcal{C}_i^{\text{pl-bk}})_A \to \mathcal{C}_i)$$

that maps to the identity in $\text{Aut}((\mathcal{D}_i)_{A_{\mathcal{D}}} \to \mathcal{D}_i)$ [where $A_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(A)$ — cf. the equivalence of categories $(\mathcal{C}_i^{\text{pl-bk}})_A \xrightarrow{\sim} (\mathcal{D}_i)_{A_{\mathcal{D}}}$ of Definition 1.3, (i), (c)], hence also to the identity in $\text{Aut}((\mathcal{D}_i)_{A_{\mathcal{D}}} \to \mathfrak{Mon})$ [i.e., via composition with Φ_i]. Since $\mathcal{D}_i$ is Div-*slim*, it thus follows that the elements of $\mathcal{O}^{\times}(A) \subseteq \text{Aut}_{\mathcal{C}_i}(A)$ may be characterized as the automorphisms of A that arise from automorphisms

$$\phi \in \text{Aut}((\mathcal{C}_i^{\text{pl-bk}})_A \to \mathcal{C}_i)$$

such that every automorphism [of an object of $\mathcal{C}_i$] induced by ϕ is a Div-*identity automorphism*. Thus, since Ψ preserves *pull-back morphisms* [cf. Theorem 3.4, (iii)] and Div-*identity automorphisms* [cf. Theorem 4.2, (i); our assumption that the Φ_i are *perf-factorial*], we thus conclude that Ψ *preserves* "$\mathcal{O}^\times(-)$", as desired. This completes the proof of the existence and 1-uniqueness of a 1-commutative diagram as in the statement of assertion (i).

The *rigidity assertion* in the statement of assertion (i) follows by observing that if $\alpha \in \mathrm{Aut}(\mathcal{C}_i \to \mathcal{C}_i^{\text{un-tr}})$, then every automorphism [of an object of $\mathcal{C}_i^{\text{un-tr}}$] induced by α is a Div-*identity automorphism*. [Indeed, this follows by applying the *functoriality* of α to [co-angular] *pre-steps*, in light of the *second equivalence of categories* of Definition 1.3, (iii), (d).] In particular, it follows that if $A \in \mathrm{Ob}(\mathcal{C}_i)$, $A_{\mathcal{D}} \stackrel{\text{def}}{=} \mathrm{Base}(A)$, then the element

$$\alpha_A \in \mathrm{Aut}((\mathcal{C}_i^{\text{pl-bk}})_A \to \mathcal{D}_i) \xrightarrow{\sim} \mathrm{Aut}((\mathcal{D}_i)_{A_{\mathcal{D}}} \to \mathcal{D}_i)$$

determined by α maps [under composition with $\Phi_i : \mathcal{D}_i \to \mathfrak{Mon}$] to the identity element of $\mathrm{Aut}((\mathcal{D}_i)_{A_{\mathcal{D}}} \to \mathfrak{Mon})$. Thus, since $\mathcal{D}_i$ is Div-*slim*, it follows that every automorphism [of an object of $\mathcal{C}_i^{\text{un-tr}}$] induced by α is a *base-identity automorphism*, hence *trivial* [since $\mathcal{C}_i^{\text{un-tr}}$ is of *unit-trivial* type]. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us observe that we obtain a 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\text{birat}} & \xrightarrow{\Psi^{\text{birat}}} & \mathcal{C}_2^{\text{birat}} \end{array}$$

[cf. Corollary 4.10]. Since, moreover, the *base-isomorphisms* of $\mathcal{C}_i^{\text{birat}}$ are precisely the morphisms of $\mathcal{C}_i^{\text{birat}}$ which are abstractly equivalent to morphisms that arise from base-isomorphisms of $\mathcal{C}_i$ [cf. Proposition 4.4, (iv)], it follows that Ψ^{birat} preserves *base-isomorphisms*, hence also *pull-back morphisms* [cf. Proposition 1.7, (ii)]. Thus, since $\mathcal{D}_i$ is Div-*slim*, the *base-identity endomorphisms* of $A \in \mathrm{Ob}(\mathcal{C}_i^{\text{birat}})$ may be characterized as the endomorphisms of A that arise from endomorphisms

$$\phi \in \mathrm{End}((\mathcal{C}_i^{\text{birat}})_A^{\text{pl-bk}} \to \mathcal{C}_i^{\text{birat}})$$

such that every endomorphism [of an object of $\mathcal{C}_i^{\text{birat}}$] induced by ϕ projects to an *automorphism* of $\mathcal{D}_i$ that is mapped by Φ_i to an *identity automorphism*. Since, by Theorem 4.2, (ii) [cf. also the fact that the Φ_i are *perf-factorial* and *non-dilating*], it follows immediately from the definition of $\mathcal{C}_i^{\text{birat}}$ that Ψ^{birat} *preserves* those endomorphisms [of an object of $\mathcal{C}_i^{\text{birat}}$] that project to an *automorphism* of $\mathcal{D}_i$ that is mapped by Φ_i to an *identity automorphism*, we thus conclude that Ψ^{birat} *preserves the base-identity endomorphisms* [hence, in particular, that Ψ^{birat} preserves "$\mathcal{O}^\times(-)$"]. Thus, we obtain a 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1^{\text{birat}} & \xrightarrow{\Psi^{\text{birat}}} & \mathcal{C}_2^{\text{birat}} \\ \downarrow & & \downarrow \\ (\mathcal{C}_1^{\text{birat}})^{\text{un-tr}} & \xrightarrow{(\Psi^{\text{birat}})^{\text{un-tr}}} & (\mathcal{C}_2^{\text{birat}})^{\text{un-tr}} \end{array}$$

[where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories]. Since, moreover, the Frobenioids $(\mathcal{C}_i^{\text{birat}})^{\text{un-tr}}$ are of *isotropic*, *unit-trivial*, and *group-like* type, we thus conclude that we obtain a 1-commutative diagram

$$\begin{array}{ccc} (\mathcal{C}_1^{\text{birat}})^{\text{un-tr}} & \xrightarrow{(\Psi^{\text{birat}})^{\text{un-tr}}} & (\mathcal{C}_2^{\text{birat}})^{\text{un-tr}} \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\text{Base}}} & \mathcal{D}_2 \end{array}$$

[cf. Proposition 3.11, (iii)]. Thus, by composing diagrams, we obtain a 1-commutative diagram as in the statement of assertion (ii), which is easily verified to be 1-unique. Finally, the *rigidity assertion* in the statement of assertion (ii) follows from Proposition 1.13, (i). This completes the proof of assertion (ii).

Next, we observe that assertion (iii) follows formally from assertion (ii); Theorem 4.9 [cf. also Definition 1.3, (i), (a), (b); the technique of using the equivalence of categories "$\mathcal{D}^* \xrightarrow{\sim} \mathcal{D}$" applied in Proposition 2.2, (ii)]. Finally, we consider assertion (iv). In light of the structure of an *elementary Frobenioid* [cf. Definition 1.1, (iii)], the existence of a 1-commutative diagram as in the statement of assertion (iv) now follows simply by *concatenating* assertions (ii), (iii), with the fact that Ψ *preserves Frobenius degrees* [cf. Theorem 3.4, (iii)]. Finally, the *rigidity assertion* follows via the same argument as was applied to prove the rigidity assertion that appears in the statement of assertion (i). This completes the proof of assertion (iv). ○

Remark 4.11.1. Note that since "slim always implies Div-slim", it follows that, at least when the divisorial monoids involved are *perf-factorial*, Corollary 4.11, (ii), *constitutes a substantial strengthening* of Theorem 3.4, (v).

Remark 4.11.2. Observe that in Example 3.9, since the subgroup U of $G = \text{Aut}(\mathcal{D}_A \to \mathcal{D})$ [where $A \in \text{Ob}(\mathcal{D})$] acts trivially on $V \times W$, it follows that $\mathcal{D}$ *fails to be* Div-*slim*, so the *non-preservation of units* that occurs in this example does *not contradict* Corollary 4.11, (i), (ii). In a similar vein, in Example 3.10, since $G = \text{Aut}(\mathcal{D}_A \to \mathcal{D})$ [where $A \in \text{Ob}(\mathcal{D})$] acts trivially on $\mathbb{Z}_{\geq 0}$, it follows that $\mathcal{D}$ *fails to be* Div-*slim*, so the *non-preservation of base-identity endomorphisms* that occurs in this example does *not contradict* Corollary 4.11, (ii).

Corollary 4.12. (Category-theoreticity of the Functor to an Elementary Frobenioid II) *For* $i = 1, 2$, *let* Φ_i *be a* **divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}_i$ *which is* **Frobenius-slim***;* $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *a* **Frobenioid of rationally standard type***;* $0_{\mathcal{D}_i}$ *the monoid on* $\mathcal{D}_i$ *that assigns to every object of* $\mathcal{D}_i$ *the monoid with one element;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories**. *If* $\mathcal{C}_1$, $\mathcal{C}_2$ *are* **of group-like type***, then we also assume that both* Ψ *and some quasi-inverse to* Ψ *preserve base-isomorphisms. Then* *there exists a* **1-unique** *functor* $\Psi^0 : \mathbb{F}_{0_{\mathcal{D}_1}} \to \mathbb{F}_{0_{\mathcal{D}_2}}$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathbb{F}_{0_{\mathcal{D}_1}} & \xrightarrow{\Psi^0} & \mathbb{F}_{0_{\mathcal{D}_2}} \end{array}$$

[where the vertical arrows are the natural projection functors, determined by the Frobenius degree and the projection to $\mathcal{D}_i$ *[cf. Proposition 4.4, (i)]; the horizontal arrows are equivalences of categories]. Moreover, if* $\mathcal{D}_1$, $\mathcal{D}_2$ *are* **slim**, *then each of the composite functors of this diagram is* **rigid**.

Proof. First, we observe [cf. Theorem 3.4, (i)] that we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are of *isotropic* type [cf. Remark 4.5.1]. Now the natural projection functors $\mathcal{C}_i \to \mathbb{F}_{0_{\mathcal{D}_i}}$ may be identified with the natural functors $\mathcal{C}_i \to \mathcal{C}_i^{\text{birat}} \to (\mathcal{C}_i^{\text{birat}})^{\text{un-tr}}$ [cf. Proposition 3.11, (i)]. In particular, if $\mathcal{C}_1$, $\mathcal{C}_2$ are of group-like type [cf. Theorem 3.4, (ii)], then [since $\mathcal{C}_i \xrightarrow{\sim} \mathcal{C}_i^{\text{birat}}$] Corollary 4.12 follows from Theorem 3.4, (iv). Thus, we may assume without loss of generality that $\mathcal{C}_1$, $\mathcal{C}_2$ are *not* of group-like type. Then Corollary 4.12 follows by applying Corollary 4.10 to pass from $\mathcal{C}_i$ to $\mathcal{C}_i^{\text{birat}}$ [where we note that, by Proposition 4.4, (iv), and Theorem 3.4, (iii), it follows that the resulting equivalence of categories Ψ^{birat} preserves *base-isomorphisms*], followed by Theorem 3.4, (iv) [where we note that, by Proposition 4.8, (iii), $\mathcal{C}_i^{\text{birat}}$ is *of standard type*], which allows us to pass from $\mathcal{C}_i^{\text{birat}}$ to $(\mathcal{C}_i^{\text{birat}})^{\text{un-tr}}$, as desired. Finally, the *rigidity assertion* follows from Proposition 1.13, (i). This completes the proof of Corollary 4.12. ○

Remark 4.12.1. One verifies immediately that if one takes the group G of Example 3.10 to be *residually finite*, then the Frobenioid of Example 3.10 is of *rationally standard* and *unit-trivial* type [but *not* of group-like type] over a *Frobenius-slim* base category [which is *not* Div-slim — cf. Remark 4.11.2]. In particular, one may apply Corollary 4.12 to the self-equivalence of categories of Example 3.10. On the other hand, since this self-equivalence *fails to preserve base-identity endomorphisms of Frobenius type*, it follows that it is not possible to replace the "$\mathbb{F}_{0_{\mathcal{D}_i}}$" in the diagram of Corollary 4.12 by "$\mathcal{D}_i$".

Section 5: Model Frobenioids

In the present §5, we study the extent to which an arbitrary Frobenioid of isotropic type may be *constructed explicitly* as a *"model Frobenioid"*. This study of "model Frobenioids" will be of use in the consideration of the concrete examples of Frobenioids that we discuss in §6 below.

In the following discussion, we maintain the notation of §1, §2, §3, §4. In particular, we assume that we have been given a *divisorial monoid* Φ on a connected, totally epimorphic category $\mathcal{D}$ and a *Frobenioid* $\mathcal{C} \to \mathbb{F}_\Phi$.

Theorem 5.1. **(Divisorial Descriptions)** *Suppose that the Frobenioid $\mathcal{C}$ is of* **isotropic** *type. Let $A, A' \in \mathrm{Ob}(\mathcal{C})$ be* **Frobenius-trivial***; $A_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$; $A'_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A') \in \mathrm{Ob}(\mathcal{D})$; $\mathcal{D}^{\mathrm{isom}} \subseteq \mathcal{D}$ the subcategory determined by the* **isomorphisms** *of $\mathcal{D}$; $\mathcal{D}^{\mathrm{isom}}_D \stackrel{\mathrm{def}}{=} (\mathcal{D}^{\mathrm{isom}})_D$ [for $D \in \mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{D}^{\mathrm{isom}})$]. Write*

$$\mathrm{Pic}_\Phi(A) \stackrel{\mathrm{def}}{=} \Phi^{\mathrm{gp}}(A)/\Phi^{\mathrm{birat}}(A)$$

[cf. Proposition 4.4, (iii)] and $\mathrm{Pic}_\mathcal{C}(A)$ for the set of isomorphism classes of $\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}}$ [where the fiber product category is taken with respect to the natural functors $\mathcal{C} \to \mathcal{D}$, $\mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}} \to \mathcal{D}$ — cf. §0]. Then:

(i) The assignment that maps a **pair of pre-steps**

$$(\phi : B \to A, \psi : B \to C)$$

to the object

$$(C, \mathrm{Base}(\phi) \circ \mathrm{Base}(\psi)^{-1}) \in \mathrm{Ob}(\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}})$$

on the one hand and to the element

$$\Phi(\phi)^{-1}(\mathrm{Div}(\psi) - \mathrm{Div}(\phi)) \in \Phi^{\mathrm{gp}}(A)$$

on the other hand determines a **bijection** *$\mathrm{Pic}_\Phi(A) \xrightarrow{\sim} \mathrm{Pic}_\mathcal{C}(A)$. Moreover, if $(C, \zeta : C_\mathcal{D} \xrightarrow{\sim} A_\mathcal{D}) \in \mathrm{Ob}(\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}})$ [where $C \in \mathrm{Ob}(\mathcal{C})$, $C_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(C)$] corresponds, via this bijection, to an element $\gamma \in \mathrm{Pic}_\Phi(A)$, and $\kappa : C \to C'$ is a* **morphism of Frobenius type***, then $(C', \zeta \circ \mathrm{Base}(\kappa)^{-1}) \in \mathrm{Ob}(\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}})$ corresponds to the element $\deg_{\mathrm{Fr}}(\kappa) \cdot \gamma \in \mathrm{Pic}_\Phi(A)$.*

(ii) If

$$(B, \lambda : B_\mathcal{D} \xrightarrow{\sim} A_\mathcal{D}) \in \mathrm{Ob}(\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A_\mathcal{D}}); \quad (B', \lambda' : B'_\mathcal{D} \xrightarrow{\sim} A'_\mathcal{D}) \in \mathrm{Ob}(\mathcal{C} \times_\mathcal{D} \mathcal{D}^{\mathrm{isom}}_{A'_\mathcal{D}})$$

[where $B_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(B)$; $B'_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(B')$], then there **exists** *a morphism*

$$\phi : B \to B'$$

in $\mathcal{C}$ *of Frobenius degree* d *such that* $\text{Base}(\phi) = (\lambda')^{-1} \circ \theta \circ \lambda$, *where* $\theta : A_{\mathcal{D}} \to A'_{\mathcal{D}}$ *is a morphism of* $\mathcal{D}$, *and* $\text{Div}(\phi) = z \in \Phi(B)$ *if and only if the classes* $\beta \in \text{Pic}_\Phi(A)$, $\beta' \in \text{Pic}_\Phi(A')$ *determined by* B, B', *respectively, via the bijection of (i) satisfy the following relation:*

$$d \cdot \beta + z|_{A_{\mathcal{D}}} = (\Phi(\theta))(\beta') \in \text{Pic}_\Phi(A)$$

[where, by abuse of notation, we denote by $z|_{A_{\mathcal{D}}}$ *the image of* $\Phi(\lambda)^{-1}(z) \in \Phi(A)$ *in* $\text{Pic}_\Phi(A)$*]. Moreover, if such a morphism exists, then its* **unit-equivalence class** *[i.e., its image in* $\mathcal{C}^{\text{un-tr}}$, *or, equivalently,* $\mathbb{F}_\Phi$ *— cf. Proposition 3.3, (iv)] is* **unique**.

(iii) The subcategory

$$\mathcal{C}^{\text{Fr-tr}} \subseteq \mathcal{C}$$

determined by the **Frobenius-trivial objects** *and* **isometric morphisms** *is a* **Frobenioid** *of* **isotropic**, **group-like**, **base-trivial**, *and* Aut-**ample** *type. In particular, the isomorphism class of a Frobenius-trivial object of* $\mathcal{C}$ *is completely determined by the isomorphism class of its projection to* $\mathcal{D}$*; all Frobenius-trivial objects of* $\mathcal{C}$ *are* Aut*-ample.*

(iv) Suppose that $\mathcal{C}$ *is of* **unit-trivial** *type. Then* **any** *skeletal subcategory* $\mathcal{P} \subseteq (\mathcal{C}^{\text{Fr-tr}})^{\text{pl-bk}}$ *determines a* **base-section** *of* $\mathcal{C}$*; any base-section of* $\mathcal{C}$ *admits an associated* **Frobenius-section** $\mathcal{F}$. *Moreover,* $\mathcal{C}$ *is of* **model** *type.*

Proof. First, we consider assertion (i). Let us refer to a(n) [ordered] pair of pre-steps as an *A-pair* if the first pre-step has codomain A, and the second pre-step has the same domain as the first; let us say that two A-pairs $(\phi : B \to A, \psi : B \to C)$; $(\phi' : B' \to A, \psi : B' \to C')$ are *isomorphic* if there exist isomorphisms $\iota_B : B \xrightarrow{\sim} B'$, $\iota_C : C \xrightarrow{\sim} C'$ such that $\phi' \circ \iota_B = \phi$, $\psi' \circ \iota_B = \iota_C \circ \psi$. Then observe that by the equivalences of categories of Definition 1.3, (iii), (d), it follows that the assignment

$$(\phi : B \to A, \psi : B \to C) \mapsto (\Phi(\phi)^{-1}(\text{Div}(\phi)), \Phi(\phi)^{-1}(\text{Div}(\psi))) \in \Phi(A) \times \Phi(A)$$

determines a *bijection* from the set of isomorphism classes of A-pairs onto $\Phi(A) \times \Phi(A)$; in particular, we obtain a map $\Phi(A) \times \Phi(A) \to \text{Pic}_{\mathcal{C}}(A)$. Moreover, relative to this bijection, replacing an element $(x, y) \in \Phi(A) \times \Phi(A)$ by an element $(x + z, y + z) \in \Phi(A) \times \Phi(A)$ [where $z \in \Phi(A)$] corresponds to replacing $(\phi : B \to A, \psi : B \to C)$ by $(\phi \circ \delta, \psi \circ \delta)$, for some pre-step δ; in particular, such replacements do not affect the element of $\text{Pic}_{\mathcal{C}}(A)$ determined by the A-pair.

Now I *claim* that the map $\Phi(A) \times \Phi(A) \to \text{Pic}_{\mathcal{C}}(A)$ of the above discussion *factors* through $\text{Pic}_\Phi(A)$. Indeed, suppose that $(x, y) \in \Phi(A) \times \Phi(A)$, $(x', y') \in \Phi(A) \times \Phi(A)$ map to the same element of $\text{Pic}_\Phi(A)$. Then, by the definition of "$\Phi^{\text{birat}}(A)$" [cf. the statements and proofs of Proposition 4.4, (i), (iii)], it follows that there exists a pair of base-equivalent pre-steps $\delta_1, \delta_2 : D \to A$ such that

$$\Phi(\delta_1)^{-1}(\text{Div}(\delta_1)) + x' + y + z = \Phi(\delta_2)^{-1}(\text{Div}(\delta_2)) + x + y' + z$$

for some $z \in \Phi(A)$ [cf. also the definition of "gp" in §0]; thus, by replacing δ_1, δ_2 by the composite of δ_1, δ_2 with an appropriate pre-step [cf. Definition 1.3, (iii), (d)], we may assume that

$$\Phi(\delta_1)^{-1}(\text{Div}(\delta_1)) = x + y' + z'; \quad \Phi(\delta_2)^{-1}(\text{Div}(\delta_2)) = x' + y + z'$$

for some $z' \in \Phi(A)$ [for instance, one natural choice for z' is $\Phi(\delta_1)^{-1}(\text{Div}(\delta_1)) + x' + y + z = \Phi(\delta_2)^{-1}(\text{Div}(\delta_2)) + x + y' + z$]; by replacing (x, y) by $(x + z', y + z')$ [cf. discussion of the the preceding paragraph], it follows that we may assume, without loss of generality, that $z' = 0$. Next, by applying the *first equivalence of categories* of Definition 1.3, (iii), (d), we observe that there exists a pre-step $\delta^\dagger : D \to D^\dagger$ such that $\text{Div}(\delta^\dagger) = \Phi(\delta_i)(x + x' + y + y')$, where $i = 1, 2$ [and we note that $\Phi(\delta_i)$ is *independent* of i, since δ_1, δ_2 are *base-equivalent*]. Thus, [again by Definition 1.3, (iii), (d)] we conclude that there exist *base-equivalent pre-steps* $\delta^A_1, \delta^A_2 : A \to D^\dagger$ such that $\delta^\dagger = \delta^A_2 \circ \delta_1 = \delta^A_1 \circ \delta_2$. In particular, we have $\text{Div}(\delta^A_1) = x + y'$, $\text{Div}(\delta^A_2) = x' + y$.

Let $\epsilon : E \to A$ be a pre-step with $\Phi(\epsilon)^{-1}(\text{Div}(\epsilon)) = x + x'$ [cf. Definition 1.3, (iii), (d)]; $(\phi : B \to A, \psi : B \to C)$ an A-pair that corresponds to (x, y); $(\phi' : B' \to A, \psi' : B' \to C')$ an A-pair that corresponds to (x', y'). Then since $x, x' \leq x + x'$, it follows [cf. Definition 1.3, (iii), (d)] that there exist factorizations $\epsilon = \phi \circ \eta$, $\epsilon = \phi' \circ \eta'$, where $\eta : E \to B$, $\eta' : E \to B'$ are pre-steps. Moreover, by applying the the *second equivalence of categories* of Definition 1.3, (iii), (d), to $D^\dagger$, we conclude from the existence of the *composites* of $\epsilon : E \to A$ with $\delta^A_1, \delta^A_2 : A \to D^\dagger$ that there exists a pre-step $\epsilon^F : F \to D^\dagger$ and a pair of *base-equivalent pre-steps* $\delta^E_1, \delta^E_2 : E \to F$ such that the following relations hold:

$$\epsilon^F \circ \delta^E_1 = \delta^A_1 \circ \epsilon; \quad \epsilon^F \circ \delta^E_2 = \delta^A_2 \circ \epsilon$$

$$\text{Div}(\delta^E_1) = (\Phi(\epsilon))(x + y'); \quad \text{Div}(\delta^E_2) = (\Phi(\epsilon))(x' + y)$$

[so $\Phi(\epsilon^F)^{-1}(\text{Div}(\epsilon^F)) = \Phi(\delta^A_i)^{-1}(x + x')$, for $i = 1, 2$]. On the other hand, since $\text{Div}(\psi \circ \eta) = (\Phi(\epsilon))(x' + y) = \text{Div}(\delta^E_2)$, $\text{Div}(\psi' \circ \eta') = (\Phi(\epsilon))(x + y') = \text{Div}(\delta^E_1)$, we thus conclude [cf. Definition 1.3, (iii), (d), applied to the pairs of pre-steps $(\psi \circ \eta : E \to C, \delta^E_2 : E \to F)$ and $(\psi' \circ \eta' : E \to C', \delta^E_1 : E \to F)$ emanating from E] that there exists an isomorphism $\iota : C \xrightarrow{\sim} C'$ such that $\text{Base}(\psi' \circ \eta') = \text{Base}(\iota \circ \psi \circ \eta)$, $\text{Base}(\iota \circ \psi) \circ \text{Base}(\phi)^{-1} = \text{Base}(\psi') \circ \text{Base}(\phi')^{-1}$. That is to say, we have a [not necessarily commutative!] *diagram of pre-steps*

$$\begin{array}{ccccc} & & E & \stackrel{\eta}{\longrightarrow} & B & \stackrel{\phi}{\longrightarrow} & A \\ & & \downarrow \eta' & & \downarrow \iota\circ\psi & & \\ A & \stackrel{\phi'}{\longleftarrow} & B' & \stackrel{\psi'}{\longrightarrow} & C' & & \end{array}$$

whose projection to $\mathcal{D}$ is a *commutative diagram of isomorphisms* that is compatible with identification of the two copies of $A_{\mathcal{D}}$. In particular, we conclude that $(C, \text{Base}(\phi) \circ \text{Base}(\psi)^{-1})$, $(C', \text{Base}(\phi') \circ \text{Base}(\psi')^{-1})$ determine the same element of $\text{Pic}_{\mathcal{C}}(A)$. This completes the proof of the *claim*.

Thus, we obtain a map $\text{Pic}_\Phi(A) \to \text{Pic}_{\mathcal{C}}(A)$. It follows immediately from Definition 1.3, (i), (b), that this map is a *surjection*. To show that this map is *injective*, it suffices to consider $(x, y) \in \Phi(A) \times \Phi(A)$, $(x', y') \in \Phi(A) \times \Phi(A)$ that map to the same element of $\text{Pic}_{\mathcal{C}}(A)$. Let $(\phi : B \to A, \psi : B \to C)$ be an A-pair that corresponds to (x, y); $(\phi' : B' \to A, \psi : B' \to C)$ an A-pair that corresponds to (x', y'). By our assumption that (x, y) and (x', y') map to the same element of $\mathrm{Pic}_{\mathcal{C}}(A)$, it follows that we may assume that $\mathrm{Base}(\phi') \circ \mathrm{Base}(\psi')^{-1} = \mathrm{Base}(\phi) \circ \mathrm{Base}(\psi)^{-1}$. Thus, by applying Definition 1.3, (iii), (d), we obtain a [not necessarily commutative!] *diagram of pre-steps*

$$\begin{array}{ccccccc} & & E & \xrightarrow{\eta} & B & \xrightarrow{\phi} & A \\ & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \psi} & & \\ A & \xleftarrow{\phi'} & B' & \xrightarrow{\psi'} & C & & \end{array}$$

such that $\phi \circ \eta = \phi' \circ \eta'$, and whose projection to $\mathcal{D}$ is a *commutative diagram of isomorphisms* that is compatible with identification of the two copies of $A_{\mathcal{D}}$. Thus, it follows that $\psi \circ \eta, \psi' \circ \eta' : E \to C$ are *base-equivalent*, hence determine an element of $\Phi^{\mathrm{birat}}(C)$, which may be transported via ψ, ϕ [or, equivalently, ψ', ϕ'] to an element of $\Phi^{\mathrm{birat}}(A) \subseteq \Phi^{\mathrm{gp}}(A)$ which [cf. the discussion of the preceding paragraph] is easily verified to be $x + y' - x' - y \in \Phi^{\mathrm{gp}}(A)$. This completes the proof of the injectivity, hence also of the *bijectivity* of the map $\mathrm{Pic}_{\Phi}(A) \to \mathrm{Pic}_{\mathcal{C}}(A)$. Also, the portion of assertion (i) concerning *morphisms of Frobenius type* follows easily by considering commutative diagrams as in Proposition 1.10, (i). This completes the proof of assertion (i). Now assertion (ii) follows formally from assertion (i) [cf. also Remark 1.1.1; the *factorization* of Definition 1.3, (iv), (a); the *faithfulness* portion of Proposition 3.3, (iv)].

Next, we consider assertion (iii). First, let us observe that by assertion (i), any isomorphism $\alpha_{\mathcal{D}} : A'_{\mathcal{D}} \xrightarrow{\sim} A_{\mathcal{D}}$ determines an object $(A', \alpha) \in \mathrm{Ob}(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}^{\mathrm{isom}}_{A_{\mathcal{D}}})$ which [in light of the fact that A' is *Frobenius-trivial*, hence admits base-identity endomorphisms of Frobenius type of arbitrary prescribed Frobenius degree] corresponds [via the bijection of assertion (i)] to an element $\xi \in \mathrm{Pic}_{\Phi}(A)$ such that $d \cdot \xi = \xi$, for all $d \in \mathbb{N}_{\geq 1}$. Thus, taking $d = 2$ implies that $\xi = 0$, i.e., [cf. the definition of $\mathrm{Pic}_{\mathcal{C}}(A)$] that there exists an isomorphism $\alpha : A' \xrightarrow{\sim} A$ such that $\alpha_{\mathcal{D}} = \mathrm{Base}(\alpha)$. In particular, we conclude that *base-isomorphic Frobenius-trivial objects* of $\mathcal{C}$ are, in fact, *isomorphic*, and that all Frobenius-trivial objects of $\mathcal{C}$ are Aut-*ample*. In light of these observations, it follows immediately that $\mathcal{C}^{\text{Fr-tr}}$ satisfies the conditions of Definition 1.3, i.e., that $\mathcal{C}^{\text{Fr-tr}}$ is a *Frobenioid* [of *isotropic*, *group-like*, *base-trivial*, and Aut-*ample* type]. This completes the proof of assertion (iii).

Finally, we consider assertion (iv). First, we observe that since $\mathcal{C}$ is of *unit-trivial* type, it follows immediately [cf., e.g., Proposition 3.3, (iii), (iv)] that given any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a *pull-back morphism* $A \to B$ (respectively, *base-identity endomorphism of Frobenius type* of A) is *uniquely determined* by its projection to $\mathcal{D}$ (respectively, by its Frobenius degree). Moreover, by assertion (iii), it follows immediately that if $A, B \in \mathrm{Ob}(\mathcal{C}^{\text{Fr-tr}})$, then any morphism $\mathrm{Base}(A) \to \mathrm{Base}(B)$ [in $\mathcal{D}$] lifts to a pull-back morphism of $\mathcal{C}^{\text{Fr-tr}}$. Thus, we conclude that the *natural projection functor*

$$(\mathcal{C}^{\text{Fr-tr}})^{\text{pl-bk}} \to \mathcal{D}$$

is an *equivalence of categories*, hence that *any skeletal subcategory* $\mathcal{P} \subseteq (\mathcal{C}^{\text{Fr-tr}})^{\text{pl-bk}}$ determines a *base-section* of $\mathcal{C}$, and that any base-section of $\mathcal{C}$ admits an associated

Frobenius-section. Moreover, since $\mathcal{C}$ is of *unit-trivial* type, it follows immediately from the structure of an *elementary Frobenioid* [cf. the description of the kernel in Proposition 4.4, (iii)] that $\mathcal{C}$ is of *birationally Frobenius-normalized* type, hence also of *model* type, as desired. This completes the proof of assertion (iv). ◯

The *explicit descriptions* of Theorem 5.1, (i), (ii), motivate the following construction/result.

Theorem 5.2. **(Model Frobenioids)** *Let* $\underline{\Phi} : \mathcal{D} \to \mathfrak{Mon}$ *be a* **divisorial monoid** *on* $\mathcal{D}$*;* $\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$ *a* **group-like monoid** *on* $\mathcal{D}$*;* $\text{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\text{gp}}$ *a* **homomorphism** *of monoids on* $\mathcal{D}$*. Denote the group-like monoid determined by the image of* $\text{Div}_{\mathbb{B}}$ *by* $\underline{\Phi}^{\text{birat}} \subseteq \underline{\Phi}^{\text{gp}}$*. Then:*

(i) A well-defined **category** $\underline{\mathcal{C}}$ *may be constructed in the following fashion. The* **objects** *of* $\underline{\mathcal{C}}$ *are pairs of the form*

$$(A_{\mathcal{D}}, \alpha)$$

where $A_{\mathcal{D}} \in \text{Ob}(\mathcal{D})$*,* $\alpha \in \underline{\Phi}(A_{\mathcal{D}})^{\text{gp}}$*; set* $\text{Base}(A) \stackrel{\text{def}}{=} A_{\mathcal{D}}$*,* $\underline{\Phi}(A) \stackrel{\text{def}}{=} \underline{\Phi}(A_{\mathcal{D}})$*,* $\mathbb{B}(A) \stackrel{\text{def}}{=} \mathbb{B}(A_{\mathcal{D}})$*. A* **morphism**

$$\phi : A \stackrel{\text{def}}{=} (A_{\mathcal{D}}, \alpha) \to B \stackrel{\text{def}}{=} (B_{\mathcal{D}}, \beta)$$

[where $A_{\mathcal{D}}, B_{\mathcal{D}} \in \text{Ob}(\mathcal{D})$*,* $\alpha \in \underline{\Phi}(A)^{\text{gp}}$*,* $\beta \in \underline{\Phi}(B)^{\text{gp}}$*] is defined to be a collection of data as follows: (a) an element* $\deg_{\text{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$*, which we shall refer to as the* **Frobenius degree** *of* ϕ*; (b) a morphism* $\text{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$*, which we shall refer to as the* **projection** *to* $\mathcal{D}$ *to* ϕ*; (c) an element* $\text{Div}(\phi) \in \underline{\Phi}(A)$*, which we shall refer to as the* **zero divisor** *of* ϕ*; (d) an element* $u_\phi \in \mathbb{B}(A)$ *whose image* $\text{Div}_{\mathbb{B}}(u_\phi) \in \underline{\Phi}(A)^{\text{gp}}$ *satisfies the relation*

$$\deg_{\text{Fr}}(\phi) \cdot \alpha + \text{Div}(\phi) = (\underline{\Phi}(\text{Base}(\phi)))(\beta) + \text{Div}_{\mathbb{B}}(u_\phi)$$

in $\underline{\Phi}(A)^{\text{gp}}$*. The composite* $\psi \circ \phi$ *of two morphisms*

$$\phi = (\deg_{\text{Fr}}(\phi), \text{Base}(\phi), \text{Div}(\phi), u_\phi) : A \to B$$

$$\psi = (\deg_{\text{Fr}}(\psi), \text{Base}(\psi), \text{Div}(\psi), u_\psi) : B \to C$$

is defined as follows:

$$\psi \circ \phi \stackrel{\text{def}}{=} \Big(\deg_{\text{Fr}}(\psi) \cdot \deg_{\text{Fr}}(\phi), \text{Base}(\psi) \circ \text{Base}(\phi),$$
$$(\underline{\Phi}(\text{Base}(\phi)))(\text{Div}(\psi)) + \deg_{\text{Fr}}(\psi) \cdot \text{Div}(\phi), (\mathbb{B}(\text{Base}(\phi)))(u_\psi) + \deg_{\text{Fr}}(\psi) \cdot u_\phi\Big)$$

[cf. Remark 1.1.1]. Moreover, the Frobenius degree, projection to $\mathcal{D}$*, and zero divisor determine a* **functor** $\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$*.*

(ii) The category $\underline{\mathcal{C}}$ *is a* **Frobenioid** *[with respect to the functor* $\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$*] of* **isotropic** *and* **model** *— hence, in particular,* **birationally Frobenius-normalized** *— type. We shall refer to* $\underline{\mathcal{C}}$ *as the* **model Frobenioid** *defined by the* **divisor monoid** $\underline{\Phi}$ *and the* **rational function monoid** $\mathbb{B}$ *[which we regard as equipped with the homomorphism* $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\mathrm{gp}}$*]. Moreover, there is a natural isomorphism of functors between the functor "*$\mathcal{O}^{\times}(-)$*" on* $\mathcal{D}$ *associated to the Frobenioid* $\underline{\mathcal{C}}^{\mathrm{birat}}$ *[cf. Propositions 2.2, (ii), (iii); 4.4, (ii)] and the functor* $\mathbb{B}$*; this isomorphism is compatible with the homomorphisms* $\mathcal{O}^{\times}(-) \to \underline{\Phi}^{\mathrm{gp}}$ *[cf. Proposition 4.4, (iii)],* $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\mathrm{gp}}$.

(iii) $\underline{\mathcal{C}}$ *is of* **standard** *type if and only if the following conditions are satisfied: (a) if* $\underline{\Phi}$ *is the zero monoid, then* $\underline{\mathcal{C}}$ *admits a* **Frobenius-compact** *object; (b)* $\mathcal{D}$ *is* **of FSMFF-type***; (c)* $\underline{\Phi}$ *is* **non-dilating**. $\underline{\mathcal{C}}$ *is of* **rationally standard** *type if and only if the following conditions are satisfied: (a)* $\underline{\mathcal{C}}$ *is of rational and standard type; (b)* $(\underline{\mathcal{C}}^{\text{un-tr}})^{\mathrm{birat}}$ *admits a Frobenius-compact object.*

(iv) Suppose that $\Phi = \underline{\Phi}$*;* $\mathbb{B}$ *is the* **rational function monoid** *on* $\mathcal{D}$ *associated to the Frobenioid* $\mathcal{C}$ *[cf. Proposition 4.4, (ii)];* $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\mathrm{gp}}$ *is the natural homomorphism* $\mathcal{O}^{\times}(-) \to \Phi^{\mathrm{gp}} = \underline{\Phi}^{\mathrm{gp}}$ *[cf. Proposition 4.4, (iii)];* $\mathcal{C}$ *is of* **model** *type. Then there exists an* **equivalence of categories**

$$\mathcal{C} \xrightarrow{\sim} \underline{\mathcal{C}}$$

that is **1-compatible** *with the functors* $\mathcal{C} \to \mathbb{F}_{\Phi}$, $\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$.

Proof. Assertions (i), (ii) follow via a routine verification [which, in the case of assertion (ii), is reminiscent of the verification that "elementary Frobenioids are Frobenioids" in Proposition 1.5, (i)]; in light of assertion (ii), assertion (iii) follows formally from the definitions [cf. Definitions 3.1, (i); 4.5, (iii)]. Here, we observe that the objects $A = (A_{\mathcal{D}}, \alpha)$ such that $\alpha = 0$ are *Frobenius-trivial*, and that these objects, together with the morphisms $\phi = (\deg_{\mathrm{Fr}}(\phi), \mathrm{Base}(\phi), \mathrm{Div}(\phi), u_{\phi}) : A \to B$ such that $\mathrm{Div}(\phi) = 0$, $u_{\phi} = 1$ [i.e., u_{ϕ} is the identity element of $\mathbb{B}(A)$], determine a *base-Frobenius pair* of $\underline{\mathcal{C}}$.

Finally, we consider assertion (iv). We may assume without loss of generality that $\mathcal{C}$, hence also $\mathcal{C}^{\text{Fr-tr}}$, is a *skeleton*. Let $(\mathcal{P}, \mathcal{F})$ be a *base-Frobenius pair* of $\mathcal{C}$ [cf. our assumption that $\mathcal{C}$ is of *model* type]. Thus, $\mathcal{P}$ may be regarded as a subcategory of $\mathcal{C}^{\text{Fr-tr}}$. If $C \in \mathrm{Ob}(\mathcal{C})$, then let us refer to a(n)[ordered] pair of pre-steps in $\mathcal{C}$

$$(B \to A, A \to C)$$

such that $A \in \mathrm{Ob}(\mathcal{P})$ as an $\mathcal{FP}$*-path for* C. Write

$$\mathcal{C}'$$

for the *category* $\mathcal{C}'$ whose *objects* are objects of $\mathcal{C}$ equipped with an $\mathcal{FP}$-path, and whose *morphisms* are the morphisms between the objects regarded as objects of $\mathcal{C}$. Thus, we have a natural functor $\mathcal{C}' \to \mathcal{C}$ [obtained by forgetting the $\mathcal{FP}$-paths], which is manifestly an *equivalence of categories*. Thus, it suffices to construct an equivalence of categories $\mathcal{C}' \xrightarrow{\sim} \underline{\mathcal{C}}$ that is *compatible* with the functors $\mathcal{C}' \to \mathcal{C} \to \mathbb{F}_\Phi$, $\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$.

Next, observe that we may apply Remark 2.7.2 to $\mathcal{C}^{\text{Fr-tr}}$ [which is of *base-trivial* type, by Theorem 5.1, (iii)] to conclude that every morphism ϕ of $\mathcal{C}^{\text{Fr-tr}}$ admits a *unique factorization*

$$\phi = \phi_{\mathcal{P}} \circ \phi_{\mathcal{O}^\times} \circ \phi_{\mathcal{F}}$$

in $\mathcal{C}^{\text{Fr-tr}}$, where $\phi_{\mathcal{P}}$ is *$\mathcal{P}$-distinguished*; $\phi_{\mathcal{O}^\times}$ is a *base-identity automorphism*; $\phi_{\mathcal{F}}$ is *$\mathcal{F}$-distinguished*. Let us write

$$\mathcal{E} \subseteq \mathcal{C}^{\text{birat}}$$

for the full subcategory determined by the image of the objects in $\mathcal{P}$. Then observe that the category $\mathcal{E}$ is also a *skeleton*; that the Frobenioid $\mathcal{E} \xrightarrow{\sim} \mathcal{C}^{\text{birat}}$ is also of *isotropic* and *base-trivial* type [cf. Proposition 4.8, (i); Theorem 5.1, (iii)]; and that $(\mathcal{P}, \mathcal{F})$ determine a *base-Frobenius pair* of $\mathcal{E}$. Thus, we may apply Remark 2.7.2 to $\mathcal{E}$ to conclude that every morphism ψ of $\mathcal{E}$ admits a *unique factorization*

$$\psi = \psi_{\mathcal{P}} \circ \psi_{\mathcal{O}^\times} \circ \psi_{\mathcal{F}}$$

in $\mathcal{E}$, where $\psi_{\mathcal{P}}$ is *$\mathcal{P}$-distinguished*; $\psi_{\mathcal{O}^\times}$ is a *base-identity automorphism*; $\psi_{\mathcal{F}}$ is *$\mathcal{F}$-distinguished*.

Now observe that to every object $C \in \text{Ob}(\mathcal{C})$ equipped with an $\mathcal{FP}$-path $(\zeta_A : B \to A, \zeta_C : B \to C)$, we may associate an object

$$(\text{Base}(A), \Phi(\zeta_A)^{-1}(\text{Div}(\zeta_C) - \text{Div}(\zeta_A)) \in \Phi^{\text{gp}}(A))$$

of $\underline{\mathcal{C}}$ [cf. Theorem 5.1, (i)]. If $C' \in \text{Ob}(\mathcal{C})$ is equipped with an $\mathcal{FP}$-path $(\zeta_{A'} : B' \to A', \zeta_{C'} : B' \to C')$, then we may associate to any morphism $\phi : C \to C'$ a morphism

$$\Big(\deg_{\text{Fr}}(\phi), \quad \text{Base}(\zeta_{A'}) \circ \text{Base}(\zeta_{C'})^{-1} \circ \text{Base}(\phi) \circ \text{Base}(\zeta_C) \circ \text{Base}(\zeta_A)^{-1} : A \to A',$$
$$(\Phi(\zeta_A)^{-1} \circ \Phi(\zeta_C))(\text{Div}(\phi)) \in \Phi(A),$$
$$\{\zeta_{A'}^{\text{birat}} \circ (\zeta_{C'}^{\text{birat}})^{-1} \circ \phi^{\text{birat}} \circ \zeta_C^{\text{birat}} \circ (\zeta_A^{\text{birat}})^{-1}\}_{\mathcal{O}^\times} \in \mathcal{O}^\times(A^{\text{birat}})\Big)$$

[where the superscript "birat's" denote the images of the respective objects and morphisms of $\mathcal{C}$ in $\mathcal{C}^{\text{birat}}$] of $\underline{\mathcal{C}}$. Now in light of the fact that $\mathcal{C}$ is of model — hence, in particular, *birationally Frobenius-normalized* — type, it is a routine exercise to verify that these assignments determine a *functor*

$$\mathcal{C}' \to \underline{\mathcal{C}}$$

that is *compatible* with the functors $\mathcal{C}' \to \mathcal{C} \to \mathbb{F}_\Phi$, $\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$. Indeed, this is immediate for the first three entries of the data that define a morphism of $\underline{\mathcal{C}}$; for the final entry, it follows from the existence of the *unique factorizations* of morphisms of $\mathcal{E}$ discussed above. Note that these factorizations also imply that this functor

$\mathcal{C}' \to \underline{\mathcal{C}}$ is *faithful*. Moreover, this functor $\mathcal{C}' \to \underline{\mathcal{C}}$ is manifestly *essentially surjective* [cf. Theorem 5.1, (i)] and *full* [cf. Theorem 5.1, (ii)], hence an *equivalence of categories*, as desired. This completes the proof of assertion (iv). ◯

Remark 5.2.1. It follows formally from Theorem 5.2, (ii), (iv), that the Frobenioid "$\mathcal{C}$" of Example 4.6 constitutes an example of a Frobenioid of *isotropic*, *standard*, and *[strictly] rational* type, which is *not* of *group-like* or *model* type.

Proposition 5.3. (Realifications of Frobenioids) *Suppose that* Φ *is* **perf-factorial**. *Then we shall refer to as the* **realification**

$$\mathcal{C}^{\mathrm{rlf}}$$

of the Frobenioid $\mathcal{C}$ *the* **model Frobenioid** *[cf. Theorem 5.2, (ii)] associated to the* **divisor monoid**

$$\Phi^{\mathrm{rlf}}$$

[i.e., the **"realification"** *of Definition 2.4, (i)] and the* **rational function monoid** $\mathbb{R} \cdot \Phi^{\mathrm{birat}} \subseteq (\Phi^{\mathrm{rlf}})^{\mathrm{gp}}$ *[i.e., for* $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, $(\mathbb{R} \cdot \Phi^{\mathrm{birat}})(A_{\mathcal{D}})$ *is the* $\mathbb{R}$*-vector subspace of* $(\Phi^{\mathrm{rlf}})^{\mathrm{gp}}(A_{\mathcal{D}})$ *generated by* $\Phi^{\mathrm{birat}}(A_{\mathcal{D}})$*]. Moreover, the Frobenioid* $\mathcal{C}^{\text{un-tr}}$ *(respectively,* $(\mathcal{C}^{\text{un-tr}})^{\mathrm{pf}}$*) is of* **model** *type and may be obtained as the model Frobenioid associated to the divisor monoid* Φ *(respectively,* Φ^{pf}*) and the rational function monoid* Φ^{birat} *(respectively,* $\mathbb{Q} \cdot \Phi^{\mathrm{birat}} = \Phi^{\mathrm{birat}} \otimes_{\mathbb{Z}} \mathbb{Q} = (\Phi^{\mathrm{birat}})^{\mathrm{pf}}$*). In particular, if* $\mathcal{C}$ *is of* **Frobenius-isotropic** *type, then there is a* **natural** **1-commutative diagram** *of functors*

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{istr}} & \longrightarrow & \mathcal{C}^{\mathrm{pf}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{C}^{\text{un-tr}} & \longrightarrow & (\mathcal{C}^{\text{un-tr}})^{\mathrm{pf}} & \longrightarrow & \mathcal{C}^{\mathrm{rlf}} \end{array}$$

[where the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{istr}}$ *is the isotropification functor of Proposition 1.9, (v); the remaining functors are the functors that arise naturally from the construction of the "unit-trivialization", "perfection", and "realification"].*

Proof. Since Frobenioids of unit-trivial type are always of *model* type [cf. Theorem 5.1, (iv)], the various assertions in the statement of Proposition 5.3 follow immediately from the definitions and Theorem 5.2, (ii), (iv). ◯

Corollary 5.4. (Category-theoreticity of the Realification) *For* $i = 1, 2$, *let* Φ_i *be a* **perf-factorial divisorial monoid** *on a connected, totally epimorphic category* $\mathcal{D}_i$ *which is* Div**-slim** *[with respect to* Φ_i*];* $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ *a* **Frobenioid of rationally standard type***;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories**. *If* $\mathcal{C}_1$, $\mathcal{C}_2$ *are* **of group-like type***, then we also assume that both* Ψ *and some quasi-inverse to* Ψ *preserve base-isomorphisms. Then* *there exists a* **1-unique** *functor* $\Psi^{\mathrm{rlf}} : \mathcal{C}_1^{\mathrm{rlf}} \to \mathcal{C}_2^{\mathrm{rlf}}$ *that fits into a* 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\mathrm{rlf}} & \xrightarrow{\Psi^{\mathrm{rlf}}} & \mathcal{C}_2^{\mathrm{rlf}} \end{array}$$

[where the vertical arrows are the natural functors of Proposition 5.3; the horizontal arrows are equivalences of categories]. Moreover, each of the composite functors of this diagram is **rigid**. *Finally, the formation of* Ψ^{rlf} *from* Ψ *is* **1-compatible** *with the* 1*-commutative diagram of Proposition 5.3 [involving perfections, unit-trivializations, etc.].*

Proof. In light of the definition of the *realification* [cf. Proposition 5.3], Corollary 5.4 follows immediately from Corollaries 4.10; 4.11, (iii), (iv). [Here, we note that the *rigidity* assertion of Corollary 5.4 follows by a similar argument applied to prove the rigidity assertion in Corollary 4.11, (i), (iv).] ◯

Before continuing, we note the following [portions of which were in fact applied in the proofs of Theorems 4.2, 4.9].

Proposition 5.5. (Perfection, Unit-trivialization and Realification of Types) *Suppose that* $\mathcal{C}$ *is of* **Frobenius-isotropic** *and* **Frobenius-normalized** *type. Then:*

(i) If $A \in \mathrm{Ob}(\mathcal{C}^{\mathrm{istr}})$ *maps to an object* $A^{\mathrm{pf}} \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pf}})$*, then the natural functor* $\mathcal{C} \to \mathcal{C}^{\mathrm{pf}}$ *determines a* **natural isomorphism** $\mathcal{O}^{\rhd}(A)^{\mathrm{pf}} \xrightarrow{\sim} \mathcal{O}^{\rhd}(A^{\mathrm{pf}})$.

(ii) There is a natural **equivalence of categories** *[compatible with the functors to the respective elementary Frobenioids] between* $(\mathcal{C}^{\mathrm{pf}})^{\text{un-tr}}$ *and* $(\mathcal{C}^{\text{un-tr}})^{\mathrm{pf}}$ *and between* $(\mathcal{C}^{\mathrm{pf}})^{\mathrm{birat}}$ *and* $(\mathcal{C}^{\mathrm{birat}})^{\mathrm{pf}}$.

(iii) If $\mathcal{C}$ *is of* **standard** *(respectively,* **rationally standard***;* **model***) type, then so is* $\mathcal{C}^{\mathrm{pf}}$. *Moreover,* $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$ *are always of* **model** *type. Finally, suppose further that* $\mathcal{C}$ *is* **not** *of group-like type. Then if* $\mathcal{C}$ *is of* **standard** *(respectively,* **rationally standard***) type, then so are* $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$.

(iv) If $\mathcal{C}$ *is the* **model Frobenioid** *associated to data* Φ, $\mathbb{B}$, $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \Phi^{\mathrm{gp}}$ *[cf. Theorem 5.2, (ii)], then there is a natural* **equivalence of categories** *[compatible with the functors to the respective elementary Frobenioids] between* $\mathcal{C}^{\mathrm{pf}}$ *(respectively,* $\mathcal{C}^{\text{un-tr}}$*;* $\mathcal{C}^{\mathrm{rlf}}$*) and the model Frobenioid associated to the data* Φ^{pf}, $\mathbb{B}^{\mathrm{pf}}$, $\mathbb{B}^{\mathrm{pf}} \to (\Phi^{\mathrm{gp}})^{\mathrm{pf}}$ *(respectively,* Φ, Φ^{birat}, $\Phi^{\mathrm{birat}} \hookrightarrow \Phi^{\mathrm{gp}}$*;* Φ^{rlf}, $\mathbb{R} \cdot \Phi^{\mathrm{birat}}$, $\mathbb{R} \cdot \Phi^{\mathrm{birat}} \hookrightarrow (\Phi^{\mathrm{rlf}})^{\mathrm{gp}}$*).*

Proof. Assertion (i) follows immediately for *Frobenius-trivial* A by considering *base-identity endomorphisms of Frobenius type* of A and applying the hypothesis that $\mathcal{C}$ is of *Frobenius-normalized type*; the case of *arbitrary* A then follows by considering *"pairs of pre-steps"* as in Theorem 5.1, (i) [cf. also Definition 1.3, (iii), (c)]. Next, we consider assertion (ii). One checks immediately that [in light of our hypothesis that $\mathcal{C}$ is of *Frobenius-isotropic* type] we may assume without loss of generality that $\mathcal{C}$ is of *isotropic* type. Then it follows immediately from the definition of the *perfection* [cf. Definition 3.1, (iii)] that it suffices to obtain natural bijections between the respective sets of morphisms between the images of two given objects of $\mathcal{C}$ in $(\mathcal{C}^{\mathrm{pf}})^{\text{un-tr}}$, $(\mathcal{C}^{\text{un-tr}})^{\mathrm{pf}}$ (respectively, $(\mathcal{C}^{\mathrm{pf}})^{\mathrm{birat}}$), $(\mathcal{C}^{\mathrm{birat}})^{\mathrm{pf}}$). But this follows immediately from the definitions, together with Proposition 3.2, (ii), applied to "pre-steps" and "units" [i.e., base-identity automorphisms]. Next, we consider assertion (iii). First, we observe that $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$ are of *model* type [cf. Theorem 5.1, (iv); Proposition 5.3; Theorem 5.2, (ii)], hence of *isotropic* and *birationally Frobenius-normalized* type [cf. Definitions 2.7, (iii); 4.5, (i)]. Next, let us observe that by assertion (ii), we have natural equivalences $((\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}})^{\mathrm{pf}} \xrightarrow{\sim} ((\mathcal{C}^{\mathrm{pf}})^{\text{un-tr}})^{\mathrm{birat}}$, $(\mathcal{C}^{\mathrm{birat}})^{\mathrm{pf}} \xrightarrow{\sim} (\mathcal{C}^{\mathrm{pf}})^{\mathrm{birat}}$; moreover, since $\mathcal{C}^{\text{un-tr}}$ is of *birationally Frobenius-normalized* type, it follows that $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$ is of *Frobenius-normalized* type, so assertion (i) may be applied to $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$. In light of these observations, assertion (iii) for $\mathcal{C}^{\mathrm{pf}}$ follows immediately from the definitions [cf. also Proposition 3.2, (ii), (iii)] by observing that $\mathcal{C}^{\mathrm{pf}}$ is of *isotropic* type, and that by assertion (i), if $\mathcal{C}^*$ is $\mathcal{C}$ or $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$ [or $\mathcal{C}^{\mathrm{birat}}$, when $\mathcal{C}$ is of birationally Frobenius-normalized type], and $A \in \mathrm{Ob}((\mathcal{C}^*)^{\mathrm{istr}})$, then $\mathcal{O}^{\triangleright}(-)$ of the image of A in $(\mathcal{C}^*)^{\mathrm{pf}}$ is the *perfection* of $\mathcal{O}^{\triangleright}(A)$. Now suppose that $\mathcal{C}$, hence also $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$, are *not* of group-like type. Since $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$ admits a *Frobenius-compact* object, the same is true for $(\mathcal{C}^{\mathrm{rlf}})^{\mathrm{birat}}$. Also, we observe that the pull-back morphisms of $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$ are precisely the *linear isometries* [cf. Proposition 1.4, (ii)]. In light of these observations, it follows immediately from the definitions that if $\mathcal{C}$ is of *standard* (respectively, *rationally standard*) type, then so are $\mathcal{C}^{\text{un-tr}}$, $\mathcal{C}^{\mathrm{rlf}}$. Finally, assertion (iv) is immediate from the definitions [cf. also assertions (i), (ii); Proposition 5.3]. ○

Finally, we conclude the theory of the present §5 by discussing a certain issue which is closely related to the issue of being *of model type*. Namely, instead of working at the level of the entire category $\mathcal{C}$, or $\mathcal{C}^{\text{Fr-tr}}$, we consider the issue of being "of model type" at the level of a *single Frobenius-trivial object*:

Proposition 5.6. (Base-Sections of Frobenius-Trivial Objects) *Suppose that $\mathcal{C}$ is of* **model** *[hence, in particular,* **isotropic** *— cf. Definition 2.7, (iii)] and* **unit-profinite** *type. Let $(\mathcal{P}, \mathcal{F})$ be a* **base-Frobenius pair** *of $\mathcal{C}$; $A \in \mathrm{Ob}(\mathcal{P})$ a* **Frobenius-trivial** *object; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$. Then the* **pair**

$$\Big(\sigma : \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}}) \hookrightarrow \mathrm{Aut}_{\mathcal{C}}(A), \quad \phi : \mathbb{N}_{\geq 1} \to \mathrm{End}_{\mathcal{C}}(A)\Big)$$

— where σ is a group homomorphism whose composite with the **natural surjection** *$\mathrm{Aut}_{\mathcal{C}}(A) \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}})$ [cf. Theorem 5.1, (iii)] is the* **identity***, and ϕ is a homomorphism of monoids — determined by "restricting" $\mathcal{P}$, $\mathcal{F}$ to A, in fact,* **depends only** *on the data $(\mathcal{C}, A)$, and, in particular, is* **independent** *of the data $(\mathcal{F}, \mathcal{P})$ — up to* **conjugation** *[as a pair!] by an element of $\mathcal{O}^{\times}(A)$. We shall refer to such a pair (σ, ϕ) as a* **base-Frobenius pair** *of A; when $\mathcal{F}$ is regarded as being*

known only up to composition with automorphisms of the monoid $\mathbb{N}_{\geq 1}$*, we shall refer to such a pair as a* **quasi-base-Frobenius pair** *of* A.

Proof. Let

$$\left(\sigma' : \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}}) \hookrightarrow \mathrm{Aut}_{\mathcal{C}}(A), \quad \phi' : \mathbb{N}_{\geq 1} \to \mathrm{End}_{\mathcal{C}}(A)\right)$$

be another such pair, that arises from a *base-Frobenius pair* $(\mathcal{P}', \mathcal{F}')$ of $\mathcal{C}$. Write $\mathbb{E} \subseteq \mathrm{End}_{\mathcal{C}}(A)$ for the *submonoid of base-isomorphisms*; $\phi_n \stackrel{\mathrm{def}}{=} \phi(n) \in \mathbb{E}$, $\phi'_n \stackrel{\mathrm{def}}{=} \phi'(n) \in \mathbb{E}$, for $n \in \mathbb{N}_{\geq 1}$. Then I *claim* that it suffices to show the existence of a $u \in \mathcal{O}^\times(A) \subseteq \mathbb{E}$ such that

$$u \cdot \phi_p \cdot u^{-1} = \phi'_p$$

for all $p \in \mathfrak{Primes}$. Indeed, if, for $\alpha \in \mathrm{Aut}_{\mathcal{D}}(A_{\mathcal{D}})$, we write $\sigma_\alpha \stackrel{\mathrm{def}}{=} \sigma(\alpha)$, $\sigma'_\alpha \stackrel{\mathrm{def}}{=} \sigma'(\alpha)$ — so $\sigma'_\alpha = v_\alpha \cdot u \cdot \sigma_\alpha \cdot u^{-1}$, for some $v_\alpha \in \mathcal{O}^\times(A) \subseteq \mathbb{E}$ — then it follows from the *functoriality* of $\mathcal{F}$, $\mathcal{F}'$ that, for $p \in \mathfrak{Primes}$,

$$\sigma_\alpha \cdot \phi_p = \phi_p \cdot \sigma_\alpha; \quad \sigma'_\alpha \cdot \phi'_p = \phi'_p \cdot \sigma'_\alpha$$

— hence [since $\mathcal{C}$, being of *model type*, is also of *[birationally] Frobenius-normalized* type — cf. Definition 4.5, (i)] that

$$\begin{aligned} u \cdot v_\alpha \cdot \phi_p \cdot \sigma_\alpha \cdot u^{-1} &= v_\alpha \cdot u \cdot \sigma_\alpha \cdot \phi_p \cdot u^{-1} = v_\alpha \cdot (u \cdot \sigma_\alpha \cdot \phi_p \cdot u^{-1}) \\ &= v_\alpha \cdot (u \cdot \sigma_\alpha \cdot u^{-1}) \cdot (u \cdot \phi_p \cdot u^{-1}) = \sigma'_\alpha \cdot \phi'_p = \phi'_p \cdot \sigma'_\alpha \\ &= (u \cdot \phi_p \cdot u^{-1}) \cdot v_\alpha \cdot (u \cdot \sigma_\alpha \cdot u^{-1}) = (u \cdot v_\alpha^p \cdot \phi_p \cdot u^{-1}) \cdot (u \cdot \sigma_\alpha \cdot u^{-1}) \\ &= u \cdot v_\alpha^p \cdot \phi_p \cdot \sigma_\alpha \cdot u^{-1} \end{aligned}$$

— which [by the *total epimorphicity* of $\mathcal{C}$] implies that $v_\alpha = v_\alpha^p$, for all $p \in \mathfrak{Primes}$. Thus, by taking $p = 2$, we obtain that $v_\alpha = 1$. Since ϕ, ϕ' are *homomorphisms*, and $\mathbb{N}_{\geq 1}$ is generated by $\mathfrak{Primes}$, this completes the proof of the *claim*.

To verify the existence of a $u \in \mathcal{O}^\times(A)$ as in the above *claim*, let us first observe that if $M \subseteq \mathcal{O}^\times(A) \subseteq \mathbb{E}$ is any subgroup such that for any $m \in M$, $f \in \mathbb{E}$, there exists an $m' \in M$ such that $f \cdot m = m' \cdot f$, then there is a *natural monoid structure* on the set of cosets $\mathbb{E}_M \stackrel{\mathrm{def}}{=} M \backslash \mathbb{E} = \{M \cdot f\}_{f \in \mathbb{E}}$, together with a natural surjection of monoids $\mathbb{E} \twoheadrightarrow \mathbb{E}_M$. For $p \in \mathfrak{Primes}$, let us write

$$M_p \subseteq \mathcal{O}^\times(A)$$

for the closed subgroup topologically generated by the *pro-l portions* $(\mathcal{O}^\times(A))[l]$ [cf. Definition 2.8, (ii)] of $\mathcal{O}^\times(A)$, as l ranges over the primes $\neq p$. Note that since the Frobenioid $\mathcal{C}^{\text{Fr-tr}}$ is of Aut-*ample* type [cf. Theorem 5.1, (iii)], it follows that any $f \in \mathbb{E}$ admits a factorization $f = f_0 \cdot f_1$, where f_0 is an automorphism, and f_1 is a *base-identity* endomorphism. Thus, [by applying, again, the fact that $\mathcal{C}$, being of *model type*, is also of *[birationally] Frobenius-normalized* type — cf. Definition 4.5, (i)] it follows that "for any $m \in M_p$, there exists an $m' \in M_p$ such that $f \cdot m = m' \cdot f$". In particular, it makes sense to speak of the monoid $\mathbb{E}_{M_p}$. Let us use the symbol " $\overset{p}{\approx}$ " to denote the equality of the images in $\mathbb{E}_p$ of elements of $\mathbb{E}$. Now since we have a natural isomorphism

$$\prod_{p \in \mathfrak{Primes}} \mathcal{O}^\times(A)[p] \xrightarrow{\sim} \mathcal{O}^\times(A)$$

[cf. Definition 2.8, (ii)], it thus follows that to prove the existence of a $u \in \mathcal{O}^\times(A)$ as desired, it suffices to show, for each $p \in \mathfrak{Primes}$, the existence of a $u_p \in \mathcal{O}^\times(A)[p]$ such that $u_p \cdot \phi_l \cdot u_p^{-1} \overset{p}{\approx} \phi'_l$, for all $l \in \mathfrak{Primes}$ [i.e., we then take u to be the "infinite product" of the u_p].

Now observe that for each $l \in \mathfrak{Primes}$, $\phi'_l \overset{p}{\approx} v_l \cdot \phi_l$, for some $v_l \in \mathcal{O}^\times(A)[p]$. Since, for $w \in \mathcal{O}^\times(A)[p]$, we have, for $l \in \mathfrak{Primes}$, $w \cdot \phi_l \cdot w^{-1} \overset{p}{\approx} w^{1-l} \cdot \phi_l$ [where we recall again that $\mathcal{C}$, being of *model type*, is also of *[birationally] Frobenius-normalized* type — cf. Definition 4.5, (i)], and $\mathcal{O}^\times(A)[p]$ is a [topologically finitely generated] *pro-p group*, it follows that there exists a $u_p \in \mathcal{O}^\times(A)[p]$ such that $u_p \cdot \phi_p \cdot u_p^{-1} \overset{p}{\approx} \phi'_p$, as well as a $w_l \in \mathcal{O}^\times(A)[p]$ such that $w_l \cdot u_p \cdot \phi_l \cdot u_p^{-1} \overset{p}{\approx} \phi'_l$, for each $l \in \mathfrak{Primes}$. On the other hand, since ϕ, ϕ' are *homomorphisms*, it follows that

$$\phi_{l_1} \cdot \phi_{l_2} \overset{p}{\approx} \phi_{l_2} \cdot \phi_{l_1}; \quad \phi'_{l_1} \cdot \phi'_{l_2} \overset{p}{\approx} \phi'_{l_2} \cdot \phi'_{l_1}$$

[for $l_1, l_2 \in \mathfrak{Primes}$]. Thus, for $l \in \mathfrak{Primes}$, we have

$$\begin{aligned} w_l \cdot u_p \cdot \phi_p \cdot \phi_l \cdot u_p^{-1} &\overset{p}{\approx} w_l \cdot u_p \cdot \phi_l \cdot \phi_p \cdot u_p^{-1} \overset{p}{\approx} w_l \cdot u_p \cdot \phi_l \cdot u_p^{-1} \cdot u_p \cdot \phi_p \cdot u_p^{-1} \\ &\overset{p}{\approx} \phi'_l \cdot \phi'_p \overset{p}{\approx} \phi'_p \cdot \phi'_l \overset{p}{\approx} u_p \cdot \phi_p \cdot u_p^{-1} \cdot w_l \cdot u_p \cdot \phi_l \cdot u_p^{-1} \\ &\overset{p}{\approx} u_p \cdot w_l^p \cdot \phi_p \cdot u_p^{-1} \cdot u_p \cdot \phi_l \cdot u_p^{-1} \overset{p}{\approx} w_l^p \cdot u_p \cdot \phi_p \cdot \phi_l \cdot u_p^{-1} \end{aligned}$$

— which [by the *total epimorphicity* of $\mathcal{C}$] implies that $w_l \overset{p}{\approx} w_l^p$ [for all $l \in \mathfrak{Primes}$]. Since $\mathcal{O}^\times(A)[p]$ is a [topologically finitely generated] *pro-p group*, we thus conclude that $w_l \overset{p}{\approx} 1$. This completes the proof of the existence of a $u \in \mathcal{O}^\times(A)$ as desired, and hence of Proposition 5.6. ◯

Remark 5.6.1. The notion of a *"base-section of a Frobenius-trivial object"* [i.e., in the notation of Proposition 5.6, a section "σ"] is intended to be an *abstract category-theoretic translation* of the notion of a *"tautological section of a trivial line bundle"* [cf. Remark 2.7.1; the Frobenioids of Examples 6.1, 6.3 below].

Corollary 5.7. **(Category-theoreticity of Base-Sections)** *For $i = 1, 2$, let Φ_i be a* **perf-factorial divisorial monoid** *on a connected, totally epimorphic category $\mathcal{D}_i$ which is* Div-**slim** *[with respect to Φ_i]; $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ a* **Frobenioid of standard type***;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories***. If $\mathcal{C}_1$, $\mathcal{C}_2$ are* **of group-like type***, then we also assume that both* Ψ *and some quasi-inverse to* Ψ *preserve* **base-isomorphisms***. Then:*

(i) Ψ *maps* **base-sections** *(respectively,* **quasi-base-Frobenius pairs***) of* $\mathcal{C}_1$ *to base-sections (respectively, quasi-base-Frobenius pairs) of* $\mathcal{C}_2$*. In particular,* $\mathcal{C}_1$ *is of* **model** *type if and only if* $\mathcal{C}_2$ *is.*

(ii) $\mathcal{C}_1$ *is of* **unit-profinite** *type if and only if* $\mathcal{C}_2$ *is.*

(iii) Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$ *are of* **model** *and* **unit-profinite** *type. Then* Ψ *maps every* **quasi-base-Frobenius pair** *of a Frobenius-trivial object* $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$ *to a quasi-base-Frobenius pair of a Frobenius-trivial object* $A_2 \in \mathrm{Ob}(\mathcal{C}_2)$*.*

(iv) Suppose, moreover, when $\mathcal{C}_1$, $\mathcal{C}_2$ *are* **of group-like type***, that both* Ψ *and some quasi-inverse to* Ψ *preserve* **Frobenius degrees***. Then the prefix "quasi-" may be removed from the statements of (i), (iii).*

Proof. Indeed, sorting through the definitions, to verify assertions (i), (ii), (iii), (iv) it suffices to show that Ψ preserves isotropic objects, prime-Frobenius morphisms, pull-back morphisms, birationalizations, the natural projection functor $\mathcal{C}_i \to \mathcal{D}_i$ [hence, in particular, the units "$\mathcal{O}^\times(-)$"], and [in the case of the final portion of assertion (iv), when $\mathcal{C}_1$, $\mathcal{C}_2$ are *not* of group-like type] Frobenius degrees. But this follows from Theorem 3.4, (i), (iii); Corollary 4.10; Corollary 4.11, (ii) [cf. also Remark 3.4.1]. This completes the proof of Corollary 5.7. ◯

Section 6: Some Motivating Examples

In the present §6, we discuss some of the principal *motivating examples* from *arithmetic geometry* of the theory of Frobenioids. In particular, in the case of number fields, one of these examples provides an interesting *"category-theoretic interpretation"* of some results of classical number theory, such as the *Dirichlet unit theorem* and *Tchebotarev's density theorem*, as well as a result in transcendence theory due to *Lang* [cf. Theorem 6.4, (i), (iii), (iv)].

Example 6.1. A Frobenioid of Geometric Origin. Let V be a *proper normal* [geometrically integral] *variety* over a field k; K the *function field* of V; $\widetilde{K}/K$ a [possibly infinite] *Galois extension*; $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}/K)$; $\mathbb{D}_K$ a set of $\mathbb{Q}$-*Cartier prime divisors* on V. The connected objects of the Galois category $\mathcal{B}(G)$ [cf. §0] may be thought of as schemes $\mathrm{Spec}(L)$, where $L \subseteq \widetilde{K}$ is a finite [necessarily separable] extension of K. If we write $V[L]$ for the *normalization* of V in L [so $V[L]$ is also a proper normal variety], then let us write $\mathbb{D}_L$ for the set of prime divisors of $V[L]$ that map into [possibly subvarieties of codimension ≥ 1 of] prime divisors of $\mathbb{D}_K$. If for every $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{B}(G)^0)$ [cf. §0], every prime divisor of $\mathbb{D}_L$ is $\mathbb{Q}$-Cartier, then we shall say that $\mathbb{D}_K$ is $\widetilde{K}$-$\mathbb{Q}$-*Cartier*. In the following, we shall assume that $\mathbb{D}_K$ is $\widetilde{K}$-$\mathbb{Q}$-Cartier. If $L \subseteq \widetilde{K}$ is a finite extension, then let us write

$$\Phi(L) \subseteq \mathbb{Z}_{\geq 0}[\mathbb{D}_L] \subseteq \mathbb{Z}[\mathbb{D}_L]$$

for the *monoid of Cartier effective divisors* D on $V[L]$ with support in $\mathbb{D}_L$ [i.e., D such that every prime divisor in the support of D belongs to $\mathbb{D}_L$] and

$$\mathbb{B}(L) \subseteq L^{\times}$$

for the *group of rational functions* f on $V[L]$ such that every prime divisor at which f has a zero or a pole belongs to $\mathbb{D}_L$. Observe that $\Phi(L)^{\mathrm{gp}} \subseteq \mathbb{Z}[\mathbb{D}_L]$ may be identified with the *group of Cartier divisors* on $V[L]$, and that

$$\Phi(L)^{\mathrm{pf}} = \mathbb{Q}_{\geq 0}[\mathbb{D}_L] \subseteq \mathbb{Q}[\mathbb{D}_L] = (\Phi(L)^{\mathrm{pf}})^{\mathrm{gp}}$$

[since $\mathbb{D}_K$ is $\widetilde{K}$-$\mathbb{Q}$-Cartier]; moreover, one verifies immediately that $\Phi(L)$ is *perf-factorial*, that there is a *natural bijection* $\mathrm{Prime}(\Phi(L)) \xrightarrow{\sim} \mathbb{D}_L$, and that the supports of elements of $\Phi(L)$ are precisely the *finite subsets* of $\mathbb{D}_L$. By assigning to a rational function f the divisor obtained by subtracting the divisor of poles of f from the divisor of zeroes of f, we obtain a natural homomorphism

$$\mathbb{B}(L) \to \Phi(L)^{\mathrm{gp}}$$

which is *functorial* in L. In particular, the assignments $L \mapsto \Phi(L)$, $L \mapsto \mathbb{B}(L)$ determine, respectively, a *perf-factorial divisorial monoid* Φ on $\mathcal{D} \stackrel{\mathrm{def}}{=} \mathcal{B}(G)^0$ and a *group-like monoid* $\mathbb{B}$ on $\mathcal{D}$, equipped with a homomorphism [of monoids on $\mathcal{D}$] $\mathbb{B} \to \Phi^{\mathrm{gp}}$. Thus, by Theorem 5.2, (ii), this data determines a [model] *Frobenioid*

$$\mathcal{C}_{V,\widetilde{K},\mathbb{D}_K}$$

of *isotropic* and *birationally Frobenius-normalized* type. Note that an *object* of $\mathcal{C}_{V,\widetilde{K},\mathbb{D}_K}$ that projects to $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{B}(G)^0)$ may be thought of as a *line bundle* $\mathcal{L}$ on $V[L]$ that is representable by a Cartier divisor D with support in $\mathbb{D}_L$. If $\mathcal{L}$ is such a line bundle on $V[L]$, and $\mathcal{M}$ is such a line bundle on $V[M]$ [where $M \subseteq \widetilde{K}$ is a finite extension of K], then one verifies immediately that a *morphism* $\mathcal{L} \to \mathcal{M}$ in $\mathcal{C}_{V,\widetilde{K},\mathbb{D}_K}$ may be thought of as consisting of the following data: (a) a morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(M)$ over $\mathrm{Spec}(K)$ [which thus induces a morphism $V[L] \to V[M]$ over V]; (b) an element $d \in \mathbb{N}_{\geq 1}$; (c) a morphism of line bundles $\mathcal{L}^{\otimes d} \to \mathcal{M}|_{V[L]}$ on $V[L]$ whose zero locus is a Cartier divisor supported in $\mathbb{D}_L$. Also, we observe that [since $V[L]$ is a *proper normal variety*] for $A \in \mathrm{Ob}(\mathcal{C}_{V,\widetilde{K},\mathbb{D}_K})$ that projects to $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{B}(G)^0)$, we have

$$\mathcal{O}^\times(A) = \mathcal{O}^\triangleright(A) = k_L^\times$$

where k_L denotes the algebraic closure of k in L [so k_L is a finite separable extension of k].

Theorem 6.2. **(Geometric Frobenioids)** *For* $i = 1, 2$*, let* V_i *be a* **proper normal** *[geometrically integral]* **variety** *over a field* k_i*;* K_i *the* **function field** *of* V_i*;* $\widetilde{K}_i/K_i$ *a [possibly infinite]* **Galois extension***;* $G_i \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_i/K_i)$*;* $\mathcal{D}_i \stackrel{\mathrm{def}}{=} \mathcal{B}(G_i)^0$*;* $\mathbb{D}_{K_i} \neq \emptyset$ *a* $\widetilde{K}_i$**-$\mathbb{Q}$-Cartier** *set of prime divisors on* V_i*. For* $\mathrm{Spec}(L_i) \in \mathrm{Ob}(\mathcal{D}_i)$*, write* $V_i[L_i]$ *for the* **normalization** *of* V_i *in* L_i*;* $\mathbb{D}_{L_i}$ *for the* **set of prime divisors** *of* $V_i[L_i]$ *that map into [possibly subvarieties of codimension* ≥ 1 *of] prime divisors of* $\mathbb{D}_{K_i}$*;*

$$\Phi_i(L_i) \subseteq \mathbb{Z}_{\geq 0}[\mathbb{D}_{L_i}] \subseteq \mathbb{Z}[\mathbb{D}_{L_i}]$$

for the **monoid of Cartier effective divisors** *on* $V_i[L_i]$ *with support in* $\mathbb{D}_{L_i}$*;*

$$\mathbb{B}_i(L_i) \subseteq L_i^\times$$

for the **group of rational functions** *on* $V_i[L_i]$ *whose zeroes and poles are supported on* $\mathbb{D}_{L_i}$*;* $\mathbb{B}_i(L_i) \to \Phi_i(L_i)^{\mathrm{gp}}$ *for the natural map;*

$$\mathcal{C}_i$$

for the associated **model Frobenioid** *of Theorem 5.2, (ii). Then:*

(i) Let

$$\psi : V_2 \to V_1$$

be a **dominant morphism** *of schemes such that the following conditions are satisfied: (a)* $\mathbb{D}_{K_2}$ *is equal to the set of prime divisors of* V_2 *that map into a prime divisor of* $\mathbb{D}_{K_1}$*; (b) the resulting inclusion of function fields* $K_1 \hookrightarrow K_2$ *satisfies the condition that the composite inclusion* $K_1 \hookrightarrow K_2 \hookrightarrow \widetilde{K}_2$ *factors through* $\widetilde{K}_1$*; (c)* K_1 *is* **separably closed** *in* K_2*. Then* ψ *induces a* **functor**

$$\Psi : \mathcal{C}_1 \to \mathcal{C}_2$$

[well-defined up to isomorphism] that is **compatible** *with Frobenius degrees, the functor* $\mathcal{D}_1 \to \mathcal{D}_2$ *induced by the inclusion of fields* $K_1 \hookrightarrow K_2$*, and the natural transformations* $\Phi_1 \to \Phi_2|_{\mathcal{D}_1}$, $\mathbb{B}_1 \to \mathbb{B}_2|_{\mathcal{D}_1}$ *induced by pulling back divisors and rational functions, respectively, via* ψ*.*

(ii) Assume that the data labeled by the index "1" is **equal** *to the data labeled by the index "2" [so in the following, we shall omit these indices]. Also, let us suppose that* k *is of* **positive characteristic** $\boldsymbol{p}$. *Then the* **Frobenius morphism** $\psi : V \to V$ *satisfies the conditions of (i), hence determines a functor*

$$\Psi : \mathcal{C} \to \mathcal{C}$$

which is isomorphic to the **naive Frobenius functor** *[of degree* p *on* $\mathcal{C}$*] of Proposition 2.1.*

(iii) We maintain the assumption of (ii) concerning indices. Then the Frobenioid $\mathcal{C}$ *is of* **isotropic**, **standard**, *and* **birationally Frobenius-normalized** *type, but* **not** *of group-like type. If, moreover, for every finite extension* $L \subseteq \widetilde{K}$ *of* K*, and every* $D \in \mathbb{D}_L$*, it holds that* D *lies in the support of the image in* $\Phi(L)^{\mathrm{gp}}$ *of an element of* $\mathbb{B}(L)$*, then* $\mathcal{C}$ *is of* **rationally standard** *type.*

(iv) We maintain the assumption of (ii) concerning indices. Then $\mathcal{D}$ *is* **Frobenius-slim**. *Let* $Z \subseteq G$ *be the subgroup of elements that commute with some open subgroup of* G*. Then* $\mathcal{D}$ *is* **slim** *if and only if* $Z = \{1\}$*;* $\mathcal{D}$ *is* Div**-slim** *[relative to* Φ*] if and only if, for every* $1 \neq z \in Z$*, there exists a finite Galois extension* $L \subseteq \widetilde{K}$ *of* K *such that* z *acts nontrivially on* $\Phi(L)$*.*

Proof. First, we consider assertion (i). Note that by assumptions (b), (c) [in the statement of assertion (i)], it follows that any finite extension $L_1 \subseteq \widetilde{K}_1$ of K_1 determines a finite extension $L_2 \stackrel{\mathrm{def}}{=} L_1 \cdot K_2 \subseteq \widetilde{K}_2$ of K_2 such that $[L_2 : K_2] = [L_1 : K_1]$. Thus, ψ determines a functor $\mathcal{D}_1 \to \mathcal{D}_2$. Moreover, by assumption (a) [in the statement of assertion (i)], it follows that by pulling back [Cartier] divisors and rational functions via ψ, we obtain *compatible natural transformations* $\Phi_1 \to \Phi_2|_{\mathcal{D}_1}$, $\mathbb{B}_1 \to \mathbb{B}_2|_{\mathcal{D}_1}$. Thus, it follows formally from the definition of the category underlying a *model Frobenioid* in Theorem 5.2, (i), that we obtain a functor $\Psi : \mathcal{C}_1 \to \mathcal{C}_2$ satisfying the properties stated in assertion (i). From this definition of the functor Ψ, it then follows immediately from the definition of the "Frobenius morphism in characteristic p", together with the definition of the "naive Frobenius functor" of Proposition 2.1 — i.e., in a word, that both functors are obtained by *"raising to the p-th power"* — that these two functors are isomorphic. This completes the proof of assertions (i), (ii).

Next, we consider assertion (iii). The fact that $\mathcal{C}$ is of *isotropic* and *birationally Frobenius-normalized* type follows formally from Theorem 5.2, (ii). The fact that $\mathcal{C}$ is *not* of group-like type is immediate from our assumption that $\mathbb{D}_K \neq \emptyset$ [and the definition of Φ]. It is immediate that every monomorphism of $\mathcal{D}$ is an isomorphism, hence that $\mathcal{D}$ is of *FSM-type* [hence also of *FSMFF-type* — cf. §0]. If a K-linear automorphism α of a finite extension $L \subseteq \widetilde{K}$ of K induces an automorphism of $\Phi(L)$

which preserves the primes of L, then it is immediate from the fact that α induces an automorphism of the scheme $V[L]$ that α maps every prime divisor $D \in \Phi(L)$ to D [i.e., not to some $n \cdot D$, where $n \geq 2$]; thus, we conclude that Φ is *non-dilating*, hence that $\mathcal{C}$ is of *standard* type. Now suppose that for every finite extension $L \subseteq \widetilde{K}$ of K, and every $D \in \mathbb{D}_L$, it holds that D lies in the support of the image in $\Phi(L)^{\mathrm{gp}}$ of an element of $\mathbb{B}(L)$. Then it follows formally [cf. Definition 4.5, (ii)] that $\mathcal{C}$ is of *[strictly] rational* type [since Φ has already been observed to be *perf-factorial* — cf. Example 6.1]. Thus, $\mathcal{C}$ satisfies condition (a) of Definition 4.5, (iii). Now I *claim* that every object of $(\mathcal{C}^{\text{un-tr}})^{\text{birat}}$ is *Frobenius-compact*. Indeed, if α is a K-linear automorphism of a finite extension $L \subseteq \widetilde{K}$ of K that acts by multiplication by $\lambda \in \mathbb{Q}_{>0}$ on $\Phi^{\text{birat}}(L)^{\text{pf}}$ ($\neq 0$), then since α induces an automorphism of the variety $V[L]$, it follows that the *order* $\in \mathbb{Q}_{>0}$ of the zero [or pole] of highest order of an element $f \in \Phi^{\text{birat}}(L)^{\text{pf}}$ is *preserved* by α, hence that $\lambda = 1$. This completes the proof of the *claim*, and hence of the fact that $\mathcal{C}$ is of *rationally standard* type.

Finally, we consider assertion (iv). First, we observe that if $L \subseteq \widetilde{K}$ is a finite extension of K that corresponds to an open subgroup $H \subseteq G$, then there is a *natural isomorphism*

$$(Z \supseteq)\ Z_G(H) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}_{\mathrm{Spec}(L)} \to \mathcal{D})$$

[cf. [Mzk7], Corollary 1.1.6]. Since G is *profinite*, hence, in particular, *residually finite*, it follows formally that Z, $Z_G(H)$ are also residually finite, hence that $\mathcal{D}$ is *Frobenius-slim*, by Remark 3.1.2. Also, since Z is the union of subgroups of G of the form "$Z_G(H)$", it follows formally that $\mathcal{D}$ is *slim* if and only if $Z = \{1\}$, and that $\mathcal{D}$ is Div-*slim* [relative to Φ] if and only if, for every $1 \neq z \in Z$, there exists a finite Galois extension $L' \subseteq \widetilde{K}$ of K such that z acts nontrivially on $\Phi(L')$. This completes the proof of assertion (iv). ◯

Remark 6.2.1. Theorem 6.2, (ii), constitutes the principal justification for the name *"Frobenius functor"* that was applied to various functors in §2. From this point of view, the decomposition of the naive Frobenius functor of Proposition 2.1 into *"unit-linear"* and *"unit-wise"* Frobenius functors [cf. the proof of Corollary 2.6] may be thought of as corresponding to the decomposition of the Frobenius morphism in positive characteristic algebraic geometry over a fixed base into the composite of a "relative Frobenius morphism", which is *linear* over the fixed base, with the Frobenius morphism of the fixed base.

Example 6.3. A Frobenioid of Arithmetic Origin. Let F be a *number field* [cf. §0]. Write $\mathbb{V}(F)$ for the set of *valuations* on F [where we identify complex archimedean valuations with their complex conjugates]; $\mathcal{O}_F$ for the *ring of integers* of F. If $v \in \mathbb{V}(F)$, then we shall write F_v for the *completion* of F at v; $\mathcal{O}_v^\times \subseteq F_v^\times$ for the *group of units* [i.e., elements of valuation 1 of $F_v^\times$]; $\mathcal{O}_v^\triangleright \subseteq F_v^\times$ for the *multiplicative monoid* of elements of valuation ≤ 1; $\boldsymbol{\mu}(F) \subseteq \mathcal{O}_F^\times$ for the *group of roots of unity* in F; $\mathrm{ord}(F_v) \stackrel{\mathrm{def}}{=} F_v^\times/\mathcal{O}_v^\times$; $\mathrm{ord}(\mathcal{O}_v^\triangleright) \stackrel{\mathrm{def}}{=} \mathcal{O}_v^\triangleright/\mathcal{O}_v^\times \subseteq \mathrm{ord}(F_v)$. Thus, $\mathrm{ord}(F_v) = \mathrm{ord}(\mathcal{O}_v^\triangleright)^{\mathrm{gp}}$; $\mathrm{ord}(F_v) \cong \mathbb{Z}$, $\mathrm{ord}(\mathcal{O}_v^\triangleright) \cong \mathbb{Z}_{\geq 0}$ if v is *nonarchimedean*;

$\mathrm{ord}(F_v) \cong \mathbb{R}$, $\mathrm{ord}(\mathcal{O}_v^{\rhd}) \cong \mathbb{R}_{\geq 0}$ if v is *archimedean*. We shall refer to an element of the monoid

$$\Phi(F) \stackrel{\mathrm{def}}{=} \bigoplus_{v \in \mathbb{V}(F)} \mathrm{ord}(\mathcal{O}_v^{\rhd})$$

as an *effective arithmetic divisor* on F, and to an element of the group

$$\Phi(F)^{\mathrm{gp}} = \bigoplus_{v \in \mathbb{V}(F)} \mathrm{ord}(F_v)$$

as an *arithmetic divisor* on F. Thus, there is a natural homomorphism of groups

$$\mathbb{B}(F) \stackrel{\mathrm{def}}{=} F^{\times} \to \Phi(F)^{\mathrm{gp}}$$

[given by mapping an element $f \in F^{\times}$ to the images of f in the various factors $F_v^{\times}/\mathcal{O}_v^{\times} = \mathrm{ord}(F_v)$, all but a finite number of which are zero]. Note, moreover, that Φ, $\mathbb{B}$, as well as the homomorphism $\mathbb{B} \to \Phi^{\mathrm{gp}}$ are *functorial* in the number field F. Thus, if $\widetilde{F}$ is a [not necessarily finite] *Galois extension* of F, $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F)$, $\mathcal{D} \stackrel{\mathrm{def}}{=} \mathcal{B}(G)^0$, then Φ, $\mathbb{B}$ determine *monoids* on $\mathcal{D}$, and we have a natural *homomorphism* $\mathbb{B} \to \Phi^{\mathrm{gp}}$. Moreover, for each finite extension $L \subseteq \widetilde{F}$ of F, one verifies immediately that $\Phi(L) \neq 0$ is *perf-factorial*, that there is a *natural bijection* $\mathrm{Prime}(\Phi(L)) \stackrel{\sim}{\to} \mathbb{V}(L)$, and that the supports of elements of $\Phi(L)$ are precisely the finite subsets of $\mathbb{V}(L)$. Thus, by Theorem 5.2, (ii), this data determines a [model] *Frobenioid*

$$\mathcal{C}_{\widetilde{F}/F}$$

of *isotropic* and *birationally Frobenius-normalized* type. Note that an object of $\mathcal{C}_{\widetilde{F}/F}$ that projects to $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{B}(G)^0)$ may be thought of as an *arithmetic line bundle* $\mathcal{L}$ on L [i.e., a line bundle on $\mathrm{Spec}(\mathcal{O}_L)$, equipped with Hermitian metrics at the archimedean primes — cf. [Szp], pp. 13-14]. If $\mathcal{L}$ is an arithmetic line bundle on L, and $\mathcal{M}$ is an arithmetic line bundle on M [where $M \subseteq \widetilde{F}$ is a finite extension of F], then one verifies immediately that a *morphism* $\mathcal{L} \to \mathcal{M}$ in $\mathcal{C}_{\widetilde{F}/F}$ may be thought of as consisting of the following data: (a) a morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(M)$ over $\mathrm{Spec}(F)$; (b) an element $d \in \mathbb{N}_{\geq 1}$; (c) a nonzero morphism of arithmetic line bundles $\mathcal{L}^{\otimes d} \to \mathcal{M}|_L$ on L. Also, we observe that for $A \in \mathrm{Ob}(\mathcal{C}_{\widetilde{F}/F})$ that projects to $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{B}(G)^0)$, we have

$$\mathcal{O}^{\times}(A) = \mathcal{O}^{\rhd}(A) = \boldsymbol{\mu}(L)$$

[cf., for instance, [Szp], p. 15]. Also, observe that we have a natural *arithmetic degree* homomorphism

$$\deg_L^{\mathrm{arith}} : \Phi(L)^{\mathrm{gp}} \to \mathbb{R}$$

obtained as follows: If v is *archimedean*, so we have a natural embedding of topological fields $\mathbb{R} \hookrightarrow F_v$, then the restriction of $\deg_L^{\mathrm{arith}}$ to the factor $\mathrm{ord}(F_v)$ maps the image of $\lambda \in \mathbb{R}_{>0}$ to $-[F_v : \mathbb{R}] \cdot \log(\lambda)$. If v is *nonarchimedean*, then the restriction of $\deg_L^{\mathrm{arith}}$ to the factor $\mathrm{ord}(F_v)$ maps the image of an element $\lambda \in \mathcal{O}_v^{\rhd}$ to the natural logarithm of the cardinality of the finite set $\mathcal{O}_v/(\lambda)$ [where $\mathcal{O}_v$ is the ring of integers of F_v]. Thus, one verifies immediately that $\deg_L^{\text{arith}}$ *vanishes* on the image of $\mathbb{B}(L)$.

Remark 6.3.1. In light of Examples 6.1, 6.3, many readers might expect that the next natural step is to attempt to apply the theory of Frobenioids to study arithmetic line bundles on *higher-dimensional arithmetic varieties*. This leads, however, to numerous complications which are beyond the scope of the present paper. Moreover, it is not even clear to the author at the time of writing that this constitutes a natural direction in which to further develop the theory of Frobenioids.

Theorem 6.4. (Arithmetic Frobenioids) *For $i = 1, 2$, let F_i be a* **number field***; $\widetilde{F}_i/F_i$ a [possibly infinite]* **Galois extension***; $G_i \stackrel{\text{def}}{=} \text{Gal}(\widetilde{F}_i/F_i)$; $\mathcal{D}_i \stackrel{\text{def}}{=} \mathcal{B}(G_i)^0$; Φ_i the monoid on $\mathcal{D}_i$ given by the* **effective arithmetic divisors***; $\mathbb{B}_i$ the group-like monoid on $\mathcal{D}_i$ given by the* **multiplicative group** *of the field extension of F_i in question; $\mathbb{B}_i \to \Phi_i^{\text{gp}}$ the natural map;*

$$\mathcal{C}_i$$

the associated **model Frobenioid** *of Theorem 5.2, (ii). Then:*

(i) Assume that the data labeled by the index "1" is **equal** *to the data labeled by the index "2" [so in the remainder of the present assertion (i), we shall omit these indices]. Then the Frobenioids $\mathcal{C}$, $\mathcal{C}^{\text{pf}}$, $\mathcal{C}^{\text{rlf}}$, $\mathcal{C}^{\text{un-tr}}$, $(\mathcal{C}^{\text{pf}})^{\text{un-tr}}$ are of* **isotropic** *and* **rationally standard** *type, but* **not** *of group-like type; $\mathcal{D}$ is* **Frobenius-slim** *and* Div**-slim** *[with respect to Φ, Φ^{pf}, Φ^{rlf}]. Moreover, $\mathcal{D}$ is* **slim** *if and only if the subgroup of elements of G that commute with some open subgroup of G is trivial. Finally, if $A \in \text{Ob}(\mathcal{C}^{\text{rlf}})$ is a* **Frobenius-trivial** *object that projects to the object of $\mathcal{D}$ determined by a finite extension $L \subseteq \widetilde{F}$ of F, then $\deg_L^{\text{arith}}$ determines an* **isomorphism of groups**

$$\delta_A : \text{Pic}_\Phi(A) \xrightarrow{\sim} \mathbb{R}$$

[cf. Theorem 5.1, (i)].

(ii) Let

$$\Psi^{\text{rlf}} : \mathcal{C}_1^{\text{rlf}} \xrightarrow{\sim} \mathcal{C}_2^{\text{rlf}}$$

be an **equivalence of categories** *between the* **realifications** *[cf. Proposition 5.3] of $\mathcal{C}_1$, $\mathcal{C}_2$. Then there exists an element $\deg(\Psi^{\text{rlf}}) \in \mathbb{R}_{>0}$ such that for all* **Frobenius-trivial** *$A_1 \in \text{Ob}(\mathcal{C}_1)$, $A_2 \in \text{Ob}(\mathcal{C}_2)$ such that $A_2 = \Psi^{\text{rlf}}(A_1)$ [where we recall that Ψ^{rlf} preserves Frobenius-trivial objects — cf. (i); Corollary 4.11, (iv)], the composite of δ_{A_2} with the isomorphism $\text{Pic}_\Phi(A_1) \xrightarrow{\sim} \text{Pic}_\Phi(A_2)$ determined by Ψ^{rlf} [cf. (i) above; Corollary 4.10; Corollary 4.11, (iii)] is equal to $\deg(\Psi^{\text{rlf}}) \cdot \delta_{A_1}$.*

(iii) If the equivalence of categories Ψ^{rlf} of (ii) arises from an **equivalence of categories**

$$(\Psi^{\text{pf}})^{\text{un-tr}} : (\mathcal{C}_1^{\text{pf}})^{\text{un-tr}} \xrightarrow{\sim} (\mathcal{C}_2^{\text{pf}})^{\text{un-tr}}$$

between the **unit-trivialized perfections** *of* $\mathcal{C}_1$, $\mathcal{C}_2$ *[cf. (i); Corollary 5.4], then* $\deg(\Psi^{\mathrm{rlf}}) \in \mathbb{Q}_{>0}$. *In particular, if* $A_1 \in \mathrm{Ob}((\mathcal{C}_1^{\mathrm{pf}})^{\text{un-tr}})$ *[whose projection to* $\mathcal{D}_1$ *we denote by* $\mathrm{Spec}(L_1)$*],* $A_2 \in \mathrm{Ob}((\mathcal{C}_2^{\mathrm{pf}})^{\text{un-tr}})$ *[whose projection to* $\mathcal{D}_2$ *we denote by* $\mathrm{Spec}(L_2)$*],* $A_2 = (\Psi^{\mathrm{pf}})^{\text{un-tr}}(A_1)$, *then the bijection*

$$\mathbb{V}(L_1) \xrightarrow{\sim} \mathrm{Prime}(\Phi_1(L_1)) \xrightarrow{\sim} \mathrm{Prime}(\Phi_2(L_2)) \xrightarrow{\sim} \mathbb{V}(L_2)$$

induced by $(\Psi^{\mathrm{pf}})^{\text{un-tr}}$ *[cf. (i); Corollary 4.11, (iii)] maps a valuation* $v_1 \in \mathbb{V}(L_1)$ *lying over a valuation* v_0 *of* $\mathbb{Q}$ *to a valuation* $v_2 \in \mathbb{V}(L_2)$ *lying over the valuation* v_0 *of* $\mathbb{Q}$.

(iv) If the equivalence of categories Ψ^{rlf} *of (ii) arises from an* **equivalence of categories**

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

between $\mathcal{C}_1$, $\mathcal{C}_2$ *[cf. (i); (iii); Theorem 3.4, (iii), (iv)], then* $\deg(\Psi^{\mathrm{rlf}}) = 1$. *If, moreover, there exists a finite extension* $L_1 \subseteq \widetilde{F}_1$ *of* F_1 *which is* **Galois** *over* $\mathbb{Q}$, *then the corresponding [i.e., via the equivalence* $\mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$ *induced by* Ψ *— cf. (i); Corollary 4.11, (ii)] finite extension* $L_2 \subseteq \widetilde{F}_2$ *of* F_2 *is isomorphic to* L_1 *in a fashion that is compatible with an isomorphism* $F_1 \cong F_2$.

Proof. First, we consider assertion (i). We have already seen in Example 6.3 that the Frobenioid $\mathcal{C}$ is of *isotropic* and *birationally Frobenius-normalized* type, and that Φ is *nonzero* [so $\mathcal{C}$ is *not* of group-like type] and *perf-factorial*. As was observed in the proof of Theorem 6.2, (iii), (iv), $\mathcal{D}$ is *Frobenius-slim* and of *FSM-type*, hence also of *FSMFF-type*. Moreover, since any automorphism of a number field that fixes all of the valuations of the number field is clearly equal to the identity automorphism, it follows immediately that Φ is *non-dilating*, and that $\mathcal{D}$ is Div-*slim* [relative to Φ, hence also relative to Φ^{pf}, Φ^{rlf}]. Also, it is immediate from the definition of $\mathbb{B}$ that $\mathcal{C}$ is of *[strictly] rational* type, and that every object of $(\mathcal{C}^{\text{un-tr}})^{\mathrm{birat}}$ is *Frobenius-compact*. Thus, we conclude that $\mathcal{C}$ [hence also $\mathcal{C}^{\mathrm{pf}}$, $\mathcal{C}^{\mathrm{rlf}}$, $\mathcal{C}^{\text{un-tr}}$, $(\mathcal{C}^{\mathrm{pf}})^{\text{un-tr}}$ — cf. Proposition 5.5, (iii)] is of *rationally standard* type. The proof of the criterion for $\mathcal{D}$ to be *slim* is entirely similar to the proof given for Theorem 6.2, (iv). Finally, to show that the surjection

$$\delta_A : \mathrm{Pic}_\Phi(A) \twoheadrightarrow \mathbb{R}$$

is, in fact, an *isomorphism*, it suffices to verify that the image of $\Phi^{\mathrm{birat}}(L) \otimes_{\mathbb{Z}} \mathbb{R} = (L^\times) \otimes_{\mathbb{Z}} \mathbb{R}$ in $(\Phi^{\mathrm{rlf}}_{\mathrm{factor}})^{\mathrm{gp}}(L)$ is equal to the set of elements of $(\Phi^{\mathrm{rlf}}_{\mathrm{factor}})^{\mathrm{gp}}(L)$ with finite support whose image under $\deg^{\mathrm{arith}}_L$ is 0. But this is an immediate consequence of the well-known *Dirichlet unit theorem* of classical number theory [cf., e.g., [Lang2], p. 104]. This completes the proof of assertion (i).

Now assertion (ii) follows by observing that the isomorphism of groups

$$\mathrm{Pic}_\Phi(A_1) \xrightarrow{\sim} \mathrm{Pic}_\Phi(A_2)$$

determined by Ψ^{rlf} [cf. assertion (i); Corollary 4.10; Corollary 4.11, (iii)] is *compatible* with the "order structure" induced on both sides [via δ_{A_1}, δ_{A_2}] by the "order structure" of $\mathbb{R}$. [Indeed, this compatibility follows from the fact that the isomorphism in question arises from an isomorphism of monoids $\Phi_1^{\mathrm{rlf}}(A_1) \xrightarrow{\sim} \Phi_2^{\mathrm{rlf}}(A_2)$.] This completes the proof of assertion (ii).

Next, we observe that assertion (iii) follows formally from assertion (ii), by applying Lemma 6.5, (ii), below in the following fashion: If $\deg(\Psi^{\mathrm{rlf}}) \notin \mathbb{Q}_{>0}$, then one verifies immediately that there exist three nonarchimedean valuations $w_1, w_3, w_5 \in \mathbb{V}(L_1)$ lying over primes $p_1, p_3, p_5 \in \mathfrak{Primes}$, respectively, with the property that $w_1 \mapsto w_2 \in \mathbb{V}(L_2)$, $w_3 \mapsto w_4 \in \mathbb{V}(L_2)$, $w_5 \mapsto w_6 \in \mathbb{V}(L_2)$, where w_2, w_4, w_6 lie over primes $p_2, p_4, p_6 \in \mathfrak{Primes}$, respectively, such that $p_1, p_2, p_3, p_4, p_5, p_6$ are *distinct*. But this implies that

$$\log(p_1)/\log(p_2), \log(p_3)/\log(p_4), \log(p_5)/\log(p_6) \in (\deg(\Psi^{\mathrm{rlf}}))^{-1} \cdot \mathbb{Q}_{>0}$$

in contradiction to Lemma 6.5, (ii). Thus, $\deg(\Psi^{\mathrm{rlf}}) \in \mathbb{Q}_{>0}$. The final portion of assertion (iii) concerning valuations of $\mathbb{Q}$ now follows from Lemma 6.5, (i). This completes the proof of assertion (iii).

Finally, we consider assertion (iv). Suppose that $v_1 \in \mathbb{V}(L_1)$ maps to $v_2 \in \mathbb{V}(L_2)$ [cf. the notation of the statement of assertion (iii)]. For $i = 1, 2$, write

$$\deg(L_i, v_i)$$

for the number of valuations $\in \mathbb{V}(L_i)$, including v_i, that lie over the same valuation of $\mathbb{Q}$ as v_i. Then by *Tchebotarev's density theorem* [cf., e.g., [Lang2], Chapter VIII, §4, Theorem 10], it follows that $[L_i : \mathbb{Q}]$ is equal to the *maximum* of the $\deg(L_i, v_i)$, as v_i ranges over the elements of $\mathbb{V}(L_i)$. Moreover, if v_i is nonarchimedean and lies over a prime $p_i \in \mathbb{V}(L_i)$, then p_i *splits completely* in L_i if and only if $\deg(L_i, v_i) = [L_i : \mathbb{Q}]$. Thus, it follows that if v_1, v_2 lie over a prime $p \in \mathfrak{Primes}$ [cf. assertion (iii)], then [again by assertion (iii)] p *splits completely* in L_1 if and only if p splits completely in L_2. If this is the case, then it follows that $\deg_{L_i}^{\mathrm{arith}}$ maps a generator of the monoid $\Phi_i(L_i)_{v_i}$ ($\cong \mathbb{Z}_{\geq 0}$) to $\log(p)$. Thus, we conclude that $\deg(\Phi^{\mathrm{rlf}}) = 1$, as desired. Note that this implies that v_1 is of degree 1 [i.e., $\deg_{L_1}^{\mathrm{arith}}$ maps a generator of the monoid $\Phi_1(L_1)_{v_1}$ ($\cong \mathbb{Z}_{\geq 0}$) to $\log(p)$] if and only if v_2 is of degree 1. Thus, if L_1 is *Galois* over $\mathbb{Q}$, then whenever v_2 is of degree 1, it follows that v_1 is of degree 1, hence that p splits completely in L_1 [since L_1 is Galois over $\mathbb{Q}$]. But this implies [again by Tchebotarev's density theorem — cf., e.g., [NSW], Theorem 12.2.5] that $L_1 \subseteq L_2$, hence that $L_1 = L_2$ [since we have already seen that $[L_1 : \mathbb{Q}] = [L_2 : \mathbb{Q}]$]. This completes the proof of assertion (iv). ○

Lemma 6.5. (Transcendental Properties of Logarithms of Prime Numbers)

(i) The real numbers $\log(p) \in \mathbb{R}$, *where* p *ranges over the prime numbers, are* **linearly independent** *over* $\mathbb{Q}$.

(ii) Let $p_1, p_2, \ldots, p_6$ *be distinct prime numbers. Then there do not exist* $\lambda_1, \lambda_2 \in \mathbb{Q}_{>0}$ *such that:* $\log(p_1)/\log(p_2) = \lambda_1 \cdot \log(p_3)/\log(p_4) = \lambda_2 \cdot \log(p_5)/\log(p_6)$.

Proof. Assertion (i) is a formal consequence of the fact that $\mathbb{Z}$ is a *unique factorization domain*. Assertion (ii) is a consequence of a *theorem of Lang* [cf. [Lang1]; [Baker], p. 119]: Indeed, since the $\log(p_i)$ are *linearly independent* over $\mathbb{Q}$ [by assertion (i)], it follows that each of the following two sets of numbers is also *linearly independent* over $\mathbb{Q}$:

$$\{\log(p_2), \log(p_4), \log(p_6)\}; \quad \{1, \log(p_3)/\log(p_4)\}$$

Moreover, all *six* products of one element from the first set and one element from the second set are of the form $\mu \cdot \log(p_i)$, where $\mu \in \mathbb{Q}_{>0}$. Thus, the exponential of each of these products is *algebraic*, in contradiction to Lang's theorem. $\bigcirc$

Appendix: Slim Exponentiation

In the present Appendix, we discuss some elementary general nonsense concerning *slim categories.*

Definition A.1.

(i) A 2-category of 1-categories will be called 2-*slim* [cf. [Mzk7], Definition 1.2.4, (iii)] if every 1-morphism [i.e., functor] in the 2-category has no nontrivial automorphisms.

(ii) If $\mathcal{D}$ is a 2-category of 1-categories, then we shall write

$$|\mathcal{D}|$$

for the *associated* 1-*category* whose *objects* are objects of $\mathcal{D}$ and whose *morphisms* are *isomorphism classes* of morphisms of $\mathcal{D}$ [cf. [Mzk7], Definition 1.2.4, (iv)]. We shall also refer to $|\mathcal{D}|$ as the *coarsification* of $\mathcal{C}$.

Remark A.1.1. The name *"coarsification"* is motivated by the theory of "coarse moduli spaces" associated to (say) "fine moduli stacks" [cf. [Mzk7], Remark 1.2.4.1].

The following result may be regarded as a generalization of [Mzk7], Proposition 1.2.5, (ii) [a result concerning anabelioids], to the case of *arbitrary slim categories.*

Proposition A.2. (Slim Exponentiation) *Let* $\mathcal{C}$ *be a* **slim category** *[cf. §0]. Let* $\mathcal{D}$ *be the* **2-category of 1-categories** *defined as follows: The* **objects** *of* $\mathcal{D}$ *are the categories* $\mathcal{C}_A$ *[cf. §0], where* $A \in \mathrm{Ob}(A)$*. The* **1-morphisms** *of* $\mathcal{D}$ *are the functors*

$$f_! : \mathcal{C}_A \to \mathcal{C}_B$$

[cf. §0] induced by **morphisms** $f : A \to B$ *of* $\mathcal{C}$*. The* **2-morphisms** *of* $\mathcal{D}$ *are isomorphisms between these functors [cf. §0]. Then* $\mathcal{D}$ *is* **2-slim***. Moreover, the functor*

$$\mathfrak{E} : \mathcal{C} \to |\mathcal{D}|$$

$$A \mapsto \mathcal{C}_A; \quad f \mapsto f_!$$

determines an **equivalence of categories** $\mathcal{C} \xrightarrow{\sim} |\mathcal{D}|$*. We shall refer to the functor* $\mathfrak{E}$ *as the* **slim exponentiation functor***.*

Proof. The fact that $\mathcal{D}$ is 2-*slim* follows immediately from the assumption that $\mathcal{C}$ is *slim*. Now it is immediate from the definitions that $\mathfrak{E}$ is *full* and *essentially surjective*. To verify that $\mathfrak{E}$ is *faithful*, let us first observe that given any two morphisms $f, g : A \to B$ of $\mathcal{C}$, an isomorphism between the functors $f_!, g_! : \mathcal{C}_A \to \mathcal{C}_B$ determines an isomorphism between the composites of the functors $f_!$, $g_!$ with the natural functor $\mathcal{C}_B \to \mathcal{C}$. On the other hand, these two composite functors $\mathcal{C}_A \to \mathcal{C}$ both *coincide* with the natural functor $\mathcal{C}_A \to \mathcal{C}$ [i.e., that maps an object $C \to A$ of $\mathcal{C}_A$ to the object C of $\mathcal{C}$]. Thus, any isomorphism $f_! \xrightarrow{\sim} g_!$ determines an *automorphism* of the natural functor $\mathcal{C}_A \to \mathcal{C}$, which [by the *slimness* of $\mathcal{C}$!] is the *identity automorphism*. But this implies [by applying the isomorphism $f_! \xrightarrow{\sim} g_!$ to the object $A \xrightarrow{\mathrm{id}_A} A$ of $\mathcal{C}_A$] that $f = g$, as desired. ○

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Chart of Types of Morphisms in a Frobenioid

type of morphism	*projection to base*	*zero divisor*	*Frobenius degree*
linear	?	?	1
isometry	?	0	?
base-isomorphism	isomorphism	?	?
base-FSM-morphism	FSM-morphism	?	?
pull-back morphism	?	0	1
pre-step	isomorphism	?	1
step	isomorphism	$\neq 0$	1
primary pre-step	isomorphism	primary	1
isometric pre-step	isomorphism	0	1
LB-invertible	?	0	?
morphism of Frobenius type	isomorphism	0	?
prime-Frobenius morphism	isomorphism	0	prime

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