THE GEOMETRY OF FROBENIOIDS II: POLY-FROBENIOIDS

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Abstract. We develop the theory of Frobenioids associated to *nonarchimedean* [mixed-characteristic] and *archimedean local fields*. In particular, we show that the resulting Frobenioids satisfy the properties necessary to apply the *main results of the general theory of Frobenioids*. Moreover, we show that the *reciprocity map* in the nonarchimedean case, as well as a certain *archimedean analogue* of this reciprocity map, admit natural *Frobenioid-theoretic translations*, which are, moreover, *purely category-theoretic*, to a substantial extent [i.e., except for the extent to which this category-theoreticity is obstructed by certain *"Frobenius endomorphisms"* of the relevant Frobenioids]. Finally, we show that certain Frobenioids which naturally encode the *global arithmetic of a number field* may be *"grafted"* [i.e., glued] onto the Frobenioids associated to nonarchimedean and archimedean primes of the number field to obtain *"poly-Frobenioids"*. These poly-Frobenioids encode, in a purely category-theoretic fashion, most of the important aspects of the classical framework of the arithmetic geometry of number fields.

Introduction

In the present paper, we continue to develop the theory of *Frobenioids*, along the lines of [Mzk5]. By comparison with the theory of [Mzk5], however, here we introduce the following *two new themes*:

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(A) the *localization* of the global arithmetic Frobenioids associated to number fields in [Mzk5], Example 6.3;

(B) the *intertwining* of the "étale-like" and "Frobenius-like" portions of the structure of the local Frobenioids introduced in the course of (A).

Both of these themes may be regarded as "special cases" of the more general theme, discussed in [Mzk5], §I2, of *translating — via Frobenioids — into category-theoretic language* the familiar scheme-theoretic framework of *arithmetic geometry over number fields.*

The first theme (A) is also motivated by the ultimate goal of the author, discussed in [Mzk5], §I3, of developing a sort of "arithmetic Teichmüller theory for number fields". That is to say, if one thinks of Teichmüller theory as a sort of *"surgery operation"* on the ["global"] objects under consideration, then it is natural to require that these "global objects" be *"dissectible into local objects"*, so that one is free to perform such "local dissections" in the course of a "surgery operation". [Indeed, here it may be helpful to recall that the classical theory of *deformations of proper curves* proceeds precisely by "dissecting" a proper curve into "*local* affine opens" whose deformation theory is *"trivial"* and then studying how these local affine opens may be *glued* together in a fashion that *deforms* the original tautological gluing. Alternatively, in classical complex Teichmüller theory, Teichmüller deformations of a Riemann surface are obtained by *locally* integrating the square root of a given [global] square differential so as to obtain *local* holomorphic coordinates, which one then proceeds to deform in a linear, real analytic fashion.]

With regard to the second theme (B), it is interesting to observe that this theme runs, in some sense, in the *opposite direction* to the theme discussed in [Mzk5], §I4, of *distinguishing* or *separating out* from one another the "étale-like" and "Frobenius-like" portions of the structure of a Frobenioid. That is to say, the content of this theme (B) is that, for certain special types of Frobenioids [i.e., that arise in the localization theory of (A)], although the "étale-like" and "Frobenius-like" portions of the Frobenioid are distinguishable or separable from one another, they are, nonetheless, *intertwined* in a fairly essential way. In the case of the *nonarchimedean* local Frobenioids that arise in (A), this intertwining may be observed by considering Frobenioid-theoretic versions of the *Kummer* and *reciprocity* maps that occur in the classical theory of nonarchimedean [mixed-characteristic] local fields. In the case of the *archimedean* local Frobenioids that arise in (A), this intertwining may be observed by considering certain *"circles"* — i.e., copies of $\mathbb{S}^1$ — that occur [under suitable conditions] in the *base category* and *units* of the Frobenioid.

In §1, we observe that by applying the theory of [Mzk5], it is an "easy exercise" to define Frobenioids associated to p-adic local fields — which we refer to as *p-adic Frobenioids* [cf. Example 1.1] — and to show that these Frobenioids satisfy the properties necessary to apply the *main results of the theory of [Mzk5]* [cf. Theorem 1.2, (i); Example 1.4, (iii)]. We also observe that the theory of *temperoids* in [Mzk2] furnishes [cf. Example 1.3] an example of a natural *base category* for these Frobenioids — especially, for instance, if one is interested in the *arithmetic of hyperbolic curves* [cf. Remark 5.6.2] — and discuss how base categories of Frobenioids at localizations of a number field may be modified slightly so as to reflect the *geometry* *of such localizations* [cf. Example 1.4, Proposition 1.5]. One important aspect of the arithmetic of nonarchimedean [mixed-characteristic] local fields is the existence of natural *Kummer* and *reciprocity* maps [cf. the discussion of theme (B) above]. In §2 [cf. especially Theorem 2.4], we observe that the Frobenioid-theoretic versions of these maps are *"category-theoretic"* to a substantial extent, but *not completely* [cf. Remark 2.4.2] — a situation that reflects the existence of certain *"Frobenius [self-]equivalences"* in the case of Frobenioids.

In §3, we take up the study of Frobenioids associated to archimedean local fields — which we refer to as *archimedean Frobenioids* [cf. Example 3.3]. Again, by applying the general framework of [Mzk5], it is a relatively easy matter to define such Frobenioids. Unlike the nonarchimedean case, however, the archimedean case exhibits an additional "layer of complexity" — namely, the existence of *non-isotropic objects* and *non-co-angular morphisms*, which reflect the *"category-theoretic aspects" of the geometry of the circle* $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ [cf. Lemma 3.2; the Appendix]. Indeed, it is precisely the example of archimedean Frobenioids that motivated both the *terminology* [e.g., "isotropic", "co-angular"] and the *results* of this portion of the general theory of Frobenioids in [Mzk5]. After showing that the archimedean Frobenioids just defined satisfy the properties necessary to apply the *main results of the theory of [Mzk5]* [cf. Theorem 3.6, (i), (ii); Remark 4.2.1], we proceed in §4 to discuss the base categories — which we refer to as *angloids* [cf. Example 3.3] — obtained by "appending an extra copy of the circle to a given base category" [cf. Proposition 4.1; Corollary 4.2]. This theory of angloids — in particular, the natural *homeomorphisms* that one obtains between the "circle in the base angloid" and the "circle in the group of units of the Frobenioid" — may, in some sense, be regarded as an *archimedean analogue* of the theory of the reciprocity map in the nonarchimedean case [cf. the discussion of theme (B) above; Remark 4.2.1]. Finally, we discuss [the "non-circular portion" of] typical base categories that occur in the archimedean context — namely, temperoids [cf. Example 4.4] and *categories naturally associated to hyperbolic Riemann surfaces* [cf. Example 4.3].

In §5, we take up the issue of *"grafting"* [i.e., gluing] the local Frobenioids introduced in §1, §3 onto the global Frobenioids of [Mzk5], Example 6.3. We begin by discussing this sort of "grafting operation" for categories in substantial generality, and show, in particular, that, under suitable conditions, one may *dissect* such a grafted category into its *local and global components* [cf. Proposition 5.2, (iv)]. Next, we consider categories — which we refer to as *poly-Frobenioids* — obtained by grafting Frobenioids. By combining various generalities on "grafting" with the extensive theory of [Mzk5], we obtain the result that "most natural aspects" of the theory of poly-Frobenioids are *category-theoretic* [i.e., preserved by arbitrary equivalences of categories between poly-Frobenioids satisfying certain conditions — cf. Theorem 5.5]. Finally, we discuss in some detail the *poly-Frobenioids* obtained by considering the localizations of number fields at nonarchimedean and archimedean primes [cf. Example 5.6]. This example is, in some sense, representative of the *entire theory constituted by [Mzk5] and the present paper.* In addition, it shows [cf. Remark 5.6.1] that the theory of poly-Frobenioids is *strictly more general* than the theory of Frobenioids [i.e., that "poly-Frobenioids are not just Frobenioids of a certain special type in disguise"].

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Section 0: Notations and Conventions

In addition to the *"Notations and Conventions"* of [Mzk5], §0, we shall employ the following "Notations and Conventions" in the present paper:

Categories:

If $\mathcal{C}$ is a *category*, and $\{\phi_i : A_i \to A\}_{i \in I}$ is a collection of arrows in $\mathcal{C}$ all of which have codomain equal to A and nonempty [i.e., non-initial] domains, then we shall say that this collection *strongly* (respectively, *weakly*) *dissects* A if, for every pair of *distinct* elements $i, j \in I$, $\mathcal{C}$ fails to contain a pair of arrows $\psi_i : B \to A_i$, $\psi_j : B \to A_j$ (respectively, pair of arrows $\psi_i : B \to A_i$, $\psi_j : B \to A_j$ such that $\phi_i \circ \psi_i = \phi_j \circ \psi_j$), where B is a nonempty object of $\mathcal{C}$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is *strongly dissectible* (respectively, *weakly dissectible*) if it admits a strongly (respectively, weakly) dissecting pair of arrows. If A is not weakly (respectively, not strongly) dissectible, then we shall say that it is *strongly indissectible* (respectively, *weakly indissectible*). Thus, if A is strongly dissectible (respectively, indissectible), then it is weakly dissectible (respectively, indissectible). If every object of $\mathcal{C}$ is strongly dissectible (respectively, weakly dissectible; strongly indissectible; weakly indissectible), then we shall say that $\mathcal{C}$ is *of strongly dissectible* (respectively, *weakly dissectible*; *strongly indissectible*; *weakly indissectible*) *type*.

Observe that by considering appropriate *pre-steps* and *pull-back morphisms* as in [Mzk5], Definition 1.3, (i), (b), (c), it follows immediately that [the underlying category of] any Frobenioid over a base category of weakly indissectible (respectively, strongly dissectible; weakly dissectible) type is itself of *weakly indissectible* (respectively, *strongly dissectible*; *weakly dissectible*) *type*.

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a *functor* between categories $\mathcal{C}$, $\mathcal{D}$. Then we shall say that Φ is *arrow-wise essentially surjective* if it is *surjective* on *abstract equivalence classes* [cf. [Mzk5], §0] of arrows. If, for every object $B \in \mathrm{Ob}(\mathcal{D})$, there exists an object $A \in \mathrm{Ob}(\mathcal{C})$, together with a morphism $\Phi(A) \to B$ in $\mathcal{D}$, then we shall say that Φ is *relatively initial*. If, for every object $A \in \mathrm{Ob}(\mathcal{C})$, the object $\Phi(A)$ is *non-initial*, then we shall say that Φ is *totally non-initial*.

Let $\mathcal{C}$ be a *category*. If $\phi : A \to B$ is a morphism of $\mathcal{C}$, write

$$\mathcal{C}_\phi$$

for the category whose *objects* are *factorizations*

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B$$

of ϕ, and whose *morphisms*

$$\{A \xrightarrow{\alpha_1} C_1 \xrightarrow{\beta_1} B\} \longrightarrow \{A \xrightarrow{\alpha_2} C_2 \xrightarrow{\beta_2} B\}$$

are morphisms $\psi : C_1 \to C_2$ of $\mathcal{C}$ such that $\psi \circ \alpha_1 = \alpha_2$, $\beta_2 \circ \psi = \beta_1$. Also, we shall write

$$\mathcal{C}^{\rightarrowtail} \subseteq \mathcal{C}$$

for the subcategory of *monomorphisms* of $\mathcal{C}$. We shall say that a morphism $\phi : A \rightarrowtail B$ of $\mathcal{C}^{\rightarrowtail}$ [i.e., a monomorphism of $\mathcal{C}$] is *totally ordered* if the *category of factorizations* $\mathcal{C}^{\rightarrowtail}_{\phi}$ of ϕ is equivalent to a category of the form $\mathrm{Order}(E)$ [cf. [Mzk5], §0], where E is a *totally ordered set* [so E may be recovered as the set of isomorphism classes of $\mathcal{C}^{\rightarrowtail}_{\phi}$, with the order relation determined by the arrows of $\mathcal{C}^{\rightarrowtail}_{\phi}$]. If, in this situation, E satisfies the property that for any $a, b \in E$ such that $a < b$, there exists a $c \in E$ such that $a < c < b$, then we shall say that ϕ is *continuously ordered.* We shall say that a morphism of $\mathcal{C}^{\rightarrowtail}$ [i.e., a monomorphism of $\mathcal{C}$] is *quasi-totally ordered* (respectively, *quasi-continuously ordered*) if it is an isomorphism or factors as a finite composite of totally ordered (respectively, continuously ordered) morphisms. If every totally ordered morphism (respectively, continuously ordered morphism) of $\mathcal{C}$ is an isomorphism, then we shall say that $\mathcal{C}$ is *of strictly partially ordered* (respectively, *discontinuously ordered*) *type.* If every morphism of $\mathcal{C}$ is totally ordered (respectively, quasi-totally ordered; continuously ordered; quasi-continuously ordered), then we shall say that $\mathcal{C}$ is *of totally ordered* (respectively, *quasi-totally ordered*; *continuously ordered*; *quasi-continuously ordered*) *type.*

Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be *categories*; $\Phi : \mathcal{C} \to \mathcal{E}$, $\Psi : \mathcal{D} \to \mathcal{E}$ *functors.* Then we shall refer to as the *category obtained by grafting* $\mathcal{D}$ *onto* $\mathcal{C}$ *[via* Φ, Ψ*]* the category

$$\mathcal{C} \dashv_{\mathcal{E}} \mathcal{D}$$

whose *objects* are objects of $\mathcal{C}$ or objects of $\mathcal{D}$, and whose *morphisms* are defined as follows: The morphisms $B \to A$, where $A, B \in \mathrm{Ob}(\mathcal{C})$ (respectively, $A, B \in \mathrm{Ob}(\mathcal{D})$), are simply the morphisms $B \to A$ in $\mathcal{C}$ (respectively, $\mathcal{D}$); such morphisms will be referred to as *homogeneous.* The morphisms $B \to A$, where $A \in \mathrm{Ob}(\mathcal{C})$, $B \in \mathrm{Ob}(\mathcal{D})$, are the morphisms $\Psi(B) \to \Phi(A)$ in $\mathcal{E}$; such morphisms will be referred to as *heterogeneous.* If $A \in \mathrm{Ob}(\mathcal{C})$, $B \in \mathrm{Ob}(\mathcal{D})$, then there are no morphisms $A \to B$. Composition of morphisms is defined in the evident fashion. This completes the definition of $\mathcal{C} \dashv_{\mathcal{E}} \mathcal{D}$. Thus, there are natural *full embeddings*

$$\mathcal{C} \hookrightarrow \mathcal{C} \dashv_{\mathcal{E}} \mathcal{D}; \quad \mathcal{D} \hookrightarrow \mathcal{C} \dashv_{\mathcal{E}} \mathcal{D}$$

— so we may regard $\mathcal{C}$, $\mathcal{D}$ as *full subcategories* of $\mathcal{C} \dashv_{\mathcal{E}} \mathcal{D}$.

Let I be a set. For each $i \in I$, let $\mathcal{C}_i$ be a *category.* Then we shall write

$$\coprod_{i \in I} \mathcal{C}_i$$

for the category whose *objects* are objects of one of the $\mathcal{C}_i$, and whose *morphisms* $A \to B$ are the morphisms of $\mathcal{C}_i$ whenever $A, B \in \mathrm{Ob}(\mathcal{C}_i)$ [so there are no morphisms $A \to B$ whenever $A \in \mathrm{Ob}(\mathcal{C}_i)$, $B \in \mathrm{Ob}(\mathcal{C}_j)$, $i \neq j$]. Thus, we have a *natural functor*

$$\coprod_{i \in I} \mathcal{C}_i \to \prod_{i \in I} \mathcal{C}_i^{\top}$$

[where "$\top$" is as in [Mzk5], §0] given by sending an object $A \in \mathrm{Ob}(\mathcal{C}_i)$ to the object of the product category on the right whose component labeled i is A, and whose other components are *initial objects* of the categories $\mathcal{C}_j$, for $j \neq i$.

Section 1: Nonarchimedean Primes

In the present §1, we define and study the basic properties of certain Frobenioids naturally associated to *nonarchimedean [mixed-characteristic] local fields.*

Let $p \in \mathfrak{Primes}$ be a *prime number* [cf. [Mzk5], §0]; write

$$\mathcal{D}_0$$

for the *full subcategory of connected objects of the Galois category* of finite étale coverings of $\mathrm{Spec}(\mathbb{Q}_p)$ [cf. [Mzk5], §0]. Thus, $\mathcal{D}_0$ is a *connected*, *totally epimorphic* category, which is of *FSM-*, hence also of *FSMFF-type* [cf. [Mzk5], §0].

Example 1.1. ***p*-adic Frobenioids.**

(i) If $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$ [i.e., K is a finite extension of $\mathbb{Q}_p$], then write $\mathcal{O}_K \subseteq K$ for the ring of integers of K, $\mathcal{O}_K^\times \subseteq \mathcal{O}_K$ for the group of units, and $\mathcal{O}_K^\triangleright \subseteq \mathcal{O}_K$ for the multiplicative monoid of nonzero elements. Also, we shall use the following notation:

$$\mathrm{ord}(\mathcal{O}_K^\triangleright) \stackrel{\mathrm{def}}{=} \mathcal{O}_K^\triangleright/\mathcal{O}_K^\times \subseteq \mathrm{ord}(K^\times) \stackrel{\mathrm{def}}{=} K^\times/\mathcal{O}_K^\times$$

Thus, the assignment

$$\mathrm{Ob}(\mathcal{D}_0) \ni \mathrm{Spec}(K) \mapsto \mathrm{ord}(\mathcal{O}_K^\triangleright)^{\mathrm{rlf}} \ (\cong \mathbb{R}_{\geq 0}) \in \mathrm{Ob}(\mathfrak{Mon})$$

[where $\mathfrak{Mon}$ is as in [Mzk5], §0; the superscript "rlf" is discussed in [Mzk5], Definition 2.4, (i)] determines a *monoid* Φ_0 on $\mathcal{D}_0$ [cf. [Mzk5], Definition 1.1, (ii)], which is easily verified to be *non-dilating* [cf. [Mzk5], Definition 1.1, (i), (ii)] and *perf-factorial* [cf. [Mzk5], Definition 2.4, (i)]; the assignment

$$\mathrm{Ob}(\mathcal{D}_0) \ni \mathrm{Spec}(K) \mapsto K^\times \in \mathrm{Ob}(\mathfrak{Mon})$$

determines a *group-like* [cf. [Mzk5], Definition 1.1, (i), (ii)] *monoid* $\mathbb{B}_0$ on $\mathcal{D}_0$ together with a natural *homomorphism of monoids*

$$\mathbb{B}_0 \to \Phi_0^{\mathrm{gp}}$$

[i.e., by considering the natural surjection $K^\times \twoheadrightarrow \mathrm{ord}(K^\times)$]. Thus, by [Mzk5], Theorem 5.2, (ii), this data determines a [model] *Frobenioid*

$$\mathcal{C}_0$$

which is easily verified to be of *rationally standard* type [cf. [Mzk5], Theorem 5.2, (iii)] over a *slim* base category $\mathcal{D}_0$ [cf. [Mzk1], Theorem 1.1.1, (ii); [Mzk5], Theorem 6.2, (iv); [Mzk5], Theorem 6.4, (i)]. If Λ is a *monoid type* [cf. [Mzk5], §0], then we define the notation $\mathcal{C}_0^\Lambda$ as follows:

$$\mathcal{C}_0^{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \mathcal{C}_0; \quad \mathcal{C}_0^{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathcal{C}_0^{\mathrm{pf}}; \quad \mathcal{C}_0^{\mathbb{R}} \stackrel{\mathrm{def}}{=} \mathcal{C}_0^{\mathrm{rlf}}$$

[cf. [Mzk5], Propositions 3.2, 5.3]. Thus, $\mathcal{C}_0^{\mathbb{Q}}$ (respectively, $\mathcal{C}_0^{\mathbb{R}}$) is the model Frobenioid associated to the data

$$\Phi_0^{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \Phi_0; \quad \mathbb{B}_0^{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathbb{B}_0^{\mathrm{pf}} \to \Phi_0^{\mathrm{gp}}$$

$$(\text{respectively, } \Phi_0^{\mathbb{R}} \stackrel{\mathrm{def}}{=} \Phi_0; \quad \mathbb{B}_0^{\mathbb{R}} \stackrel{\mathrm{def}}{=} \mathbb{R} \cdot \Phi_0^{\mathrm{birat}} \hookrightarrow \Phi_0^{\mathrm{gp}})$$

[cf. [Mzk5], Proposition 5.5, (iv)]. Note that the monoid Φ_0^{Λ} is isomorphic to the *constant monoid* determined by $\mathbb{R}_{\geq 0} \in \mathrm{Ob}(\mathfrak{Mon})$ on $\mathcal{D}_0$; in particular, the Frobenioid $\mathcal{C}_0^{\mathbb{R}}$ may be naturally identified with the *elementary Frobenioid* determined by this constant monoid. Also, we observe in passing that an *object* of $\mathcal{C}_0$ lying over $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$ may be thought of as a *metrized line bundle* on $\mathrm{Spec}(K)$ [i.e., a $K^{\times}$-torsor V equipped with a *"metric"*, i.e., a choice of an element

$$\mu_V \in \mathrm{ord}(V)^{\mathrm{rlf}}$$

where we write $\mathrm{ord}(V) \stackrel{\mathrm{def}}{=} V/\mathcal{O}_K^{\times}$ ($\cong \mathbb{Z}$), and the superscript "rlf" denotes the result of changing the structure group of the $\mathrm{ord}(K^{\times})$-torsor $\mathrm{ord}(V)$ via the homomorphism $\mathrm{ord}(K^{\times}) \to \mathrm{ord}(K^{\times}) \otimes_{\mathbb{Z}} \mathbb{R}$] — cf. [Mzk5], Theorem 5.2, (i). Then a *morphism* in $\mathcal{C}_0$ from a metrized line bundle (V, μ_V) lying over $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$ to a metrized line bundle (W, μ_W) lying over $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{D}_0)$ consists of the following data: (a) a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(L)$ in $\mathcal{D}_0$; (b) an element $d \in \mathbb{N}_{\geq 1}$; (c) an isomorphism of $K^{\times}$-torsors $V^{\otimes d} \xrightarrow{\sim} W|_K$ [where the "d-th tensor power" and "restriction from L to K" are defined in the evident way] that is *"integral"* with respect to the metrics $\mu_{V^{\otimes d}}$, $\mu_{W|_K}$ induced on $V^{\otimes d}$, $W|_K$ by μ_V, μ_W, respectively [i.e., in the sense that $\mu_{V^{\otimes d}} \in \mathrm{ord}(V^{\otimes d})^{\mathrm{rlf}}$ maps to an element in the $\mathrm{ord}(\mathcal{O}_K^{\triangleright})^{\mathrm{rlf}}$-orbit of $\mu_{W|_K} \in \mathrm{ord}(W|_K)^{\mathrm{rlf}}$].

(ii) Let $\mathcal{D}$ be a *connected*, *totally epimorphic* category, $\mathcal{D} \to \mathcal{D}_0$ a *functor*. Let

$$\Phi \subseteq \Phi_0^{\Lambda}|_{\mathcal{D}} \ (= \Phi_0|_{\mathcal{D}})$$

be a *monoprime* [cf. [Mzk5], §0; the convention of [Mzk5], Definition 1.1, (ii)] *subfunctor in monoids* such that the image of the resulting homomorphism of group-like monoids on $\mathcal{D}$

$$\mathbb{B} \stackrel{\mathrm{def}}{=} \mathbb{B}_0^{\Lambda}|_{\mathcal{D}} \times_{(\Phi_0^{\Lambda})^{\mathrm{gp}}|_{\mathcal{D}}} \Phi^{\mathrm{gp}} \to \Phi^{\mathrm{gp}}$$

determines a subfunctor in *nonzero* monoids of Φ^{gp} [i.e., for every $A \in \mathrm{Ob}(\mathcal{D})$, the homomorphism $\mathbb{B}(A) \to \Phi^{\mathrm{gp}}(A)$ is nonzero]. We shall refer to as a *p-adic Frobenioid* the Frobenioid

$$\mathcal{C}$$

that arises as the *model Frobenioid* associated to this data Φ, $\mathbb{B} \to \Phi^{\mathrm{gp}}$ [cf. [Mzk5], Theorem 5.2, (ii)]. If it holds that

$$\Phi(K) \subseteq \Lambda \cdot \mathrm{ord}(\mathbb{Q}_p^{\times}) \quad (\subseteq \mathrm{ord}(K^{\times}) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathrm{ord}(\mathcal{O}_K^{\triangleright})^{\mathrm{rlf}})^{\mathrm{gp}})$$

$$(\text{respectively,} \quad \mathrm{ord}(K^{\times}) \subseteq \Phi(K) \quad (\subseteq \mathrm{ord}(K^{\times}) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathrm{ord}(\mathcal{O}_K^{\triangleright})^{\mathrm{rlf}})^{\mathrm{gp}}))$$

for every $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$, then we shall say that Φ is *absolutely primitive* (respectively, *fieldwise saturated*).

Theorem 1.2. **(Basic Properties of p-adic Frobenioids)** *In the notation and terminology of Example 1.1:*

(i) If $\Lambda = \mathbb{Z}$ (respectively, $\Lambda = \mathbb{R}$), then $\mathcal{C}$ is of **unit-profinite** *(respectively,* **unit-trivial***) type. For arbitrary Λ, the Frobenioid $\mathcal{C}$ is of* **isotropic***,* **model***,* Aut-**ample***,* $\mathrm{Aut}^{\mathrm{sub}}$**-ample***,* End-**ample***, and* **quasi-Frobenius-trivial** *type, but* **not** *of group-like type. If $\mathcal{D}$ is of* **FSMFF-type***, then $\mathcal{C}$ is of* **rationally standard** *type.*

(ii) Let $A \in \mathrm{Ob}(\mathcal{C})$; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$. Write $A_0 \in \mathrm{Ob}(\mathcal{D}_0)$ for the image of $A_{\mathcal{D}}$ in $\mathcal{D}_0$. Then the natural action of $\mathrm{Aut}_{\mathcal{C}}(A)$ on $\mathcal{O}^{\rhd}(A)$, $\mathcal{O}^{\times}(A)$ **factors** *through $\mathrm{Aut}_{\mathcal{D}_0}(A_0)$. If, moreover, $\Lambda \in \{\mathbb{Z}, \mathbb{Q}\}$, then this factorization determines a* **faithful** *action of the image of $\mathrm{Aut}_{\mathcal{C}}(A)$ in $\mathrm{Aut}_{\mathcal{D}_0}(A_0)$ on $\mathcal{O}^{\rhd}(A)$, $\mathcal{O}^{\times}(A)$.*

(iii) If $\mathcal{D}$ admits a **terminal object***, then $\mathcal{C}$ admits a* **pseudo-terminal** *object.*

(iv) If $\mathcal{D}$ is **slim***, and $\Lambda \in \{\mathbb{Z}, \mathbb{R}\}$, then $\mathcal{C}$ is also* **slim***.*

(v) Suppose that Φ is **absolutely primitive***. Then $\mathcal{C}$ is of* **base-trivial** *type. Moreover, the element $p \in \mathbb{Q}_p^{\times}$ determines a* **characteristic splitting** *[cf. [Mzk5], Definition 2.3] on $\mathcal{C}$.*

Proof. First, we consider assertion (i). In light of the definition of $\mathcal{C}$ as a *model Frobenioid*, it follows from [Mzk5], Theorem 5.2, (ii), that $\mathcal{C}$ is of *isotropic* and *model* type. Since, for $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, the monoid $\mathrm{End}_{\mathcal{D}}(A_{\mathcal{D}})$ acts *trivially* on $\Phi(A_{\mathcal{D}})$, it follows immediately from the construction of a model Frobenioid [cf. [Mzk5], Theorem 5.2, (i)] that $\mathcal{C}$ is of Aut-*ample*, $\mathrm{Aut}^{\mathrm{sub}}$-*ample*, and End-*ample* type. Consideration of the kernel of the homomorphism of group-like monoids on $\mathcal{D}$

$$\mathbb{B} \to \Phi^{\mathrm{gp}}$$

reveals that if, moreover, $\Lambda = \mathbb{Z}$ (respectively, $\Lambda = \mathbb{R}$), then $\mathcal{C}$ is of *unit-profinite* (respectively, *unit-trivial*) type; since the image monoids of this homomorphism are assumed to be *nonzero*, and Φ is *monoprime* [cf. Example 1.1, (ii)], it follows immediately that $\mathcal{C}$ is of *quasi-Frobenius-trivial* and *[strictly] rational* type, but *not* of group-like type. Thus, it follows immediately from [Mzk5], Theorem 5.2, (iii), that if $\mathcal{D}$ is of *FSMFF-type*, then $\mathcal{C}$ is of *rationally standard* type. This completes the proof of assertion (i). Assertions (ii), (iii) follow immediately from the construction of a model Frobenioid [cf. Example 1.1, (i), (ii); [Mzk5], Theorem 5.2, (i)]. Assertion (iv) follows formally from [Mzk5], Proposition 1.13, (iii) [since, by assertion (i) of the present Theorem 1.2, "condition (b)" of *loc. cit.* is always satisfied by objects of $\mathcal{C}$]. Finally, assertion (v) follows by observing that if Φ is *absolutely primitive*, then the homomorphism $\mathbb{B} \to \Phi^{\mathrm{gp}}$ considered above is *surjective*. Thus, it follows formally from the computation of isomorphism classes of $\mathcal{C}$ given in [Mzk5], Theorem 5.1, (i), that $\mathcal{C}$ is of *base-trivial* type; moreover, it is immediate that the image of $p \in \mathbb{Q}_p^\times$ in $K^\times$ [for $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$] determines a *characteristic splitting*. This completes the proof of Theorem 1.2. ◯

Remark 1.2.1. We observe in passing that if Φ is *absolutely primitive*, then by Theorem 1.2, (i), (v), it follows that $\mathcal{C}$ admits *unit-linear Frobenius functors* as in [Mzk5], Proposition 2.5, (iii) [where one takes the "Λ" of *loc. cit.* to be $\mathbb{Z}$], and *unit-wise Frobenius functors* as in [Mzk5], Corollary 2.6. If, moreover, $\Lambda = \mathbb{Z}$, then $\mathcal{C}$ admits *unit-wise Frobenius functors* as in [Mzk5], Proposition 2.9, (ii).

Remark 1.2.2. Consider the functors "$\mathcal{O}^\rhd(-)$", "$\mathcal{O}^\times(-)$" on $\mathcal{D}$ determined by $\mathcal{C}$ [cf. [Mzk5], Proposition 2.2, (ii), (iii)]. Let

$$u_\mathcal{D} : (\mathrm{Ob}(\mathcal{D}) \ni)\ A_\mathcal{D} \mapsto u_{A_\mathcal{D}} \in \mathcal{O}^\times(A_\mathcal{D})$$

be a *"section"* of the functor "$\mathcal{O}^\times(-)$" [i.e., every arrow $\phi : A_\mathcal{D} \to B_\mathcal{D}$ of $\mathcal{D}$ maps $u_{B_\mathcal{D}} \mapsto u_{A_\mathcal{D}}$]. Suppose that Φ is *absolutely primitive*, and that $\Lambda = \mathbb{Z}$ [so $\mathcal{O}^\rhd(A_\mathcal{D})^{\mathrm{char}} = \mathcal{O}^\rhd(A_\mathcal{D})/\mathcal{O}^\times(A_\mathcal{D}) \cong \mathbb{Z}_{\geq 0}$]. Then if τ is a *characteristic splitting* on $\mathcal{C}$, then it follows immediately that $u_\mathcal{D}$ determines a *new characteristic splitting* "$u_\mathcal{D} \cdot \tau$", by taking

$$(u_\mathcal{D} \cdot \tau)(A_\mathcal{D}) \stackrel{\mathrm{def}}{=} (u_{A_\mathcal{D}} \cdot \eta)^{\mathbb{Z}_{\geq 0}} \subseteq \mathcal{O}^\rhd(A_\mathcal{D})$$

for $\eta \in \tau(A_\mathcal{D}) \subseteq \mathcal{O}^\rhd(A_\mathcal{D})$ a generator of $\tau(A_\mathcal{D})$. Moreover, one obtains a *unique automorphism* $\mathcal{U}$ of the data

$$(\Phi, \mathbb{B} \to \Phi^{\mathrm{gp}})$$

which is the *identity* on "$\mathcal{O}^\times(-)$" [regarded as a subfunctor of $\mathbb{B}$] and Φ, but which maps τ [where we regard "$\mathcal{O}^\rhd(-)$" as a subfunctor of $\mathbb{B}$ such that $\mathbb{B} = \mathcal{O}^\rhd(-)^{\mathrm{gp}}$] to $u_\mathcal{D} \cdot \tau$. Thus, $\mathcal{U}$ induces a *self-equivalence*

$$\Psi_\mathcal{U} : \mathcal{C} \stackrel{\sim}{\to} \mathcal{C}$$

of the model Frobenioid $\mathcal{C}$. In particular, $\Psi_\mathcal{U}$ exhibits an example of a situation where $p \in \mathbb{Q}_p^\times$ is mapped to some $p \cdot u \in \mathbb{Q}_p^\times$, where $u \in \mathbb{Z}_p^\times$ — a situation which, of course, *never arises in conventional scheme theory* [cf. the various Frobenius functors discussed in Remark 1.2.1; [Mzk5], Introduction].

Example 1.3. Connected Quasi-temperoids.

(i) If Π is a *topological group*, then we shall write

$$\mathcal{B}^{\mathrm{temp}}(\Pi)$$

for the *category* whose *objects* are *countable* [i.e., of cardinality $\leq$ the cardinality of the set of natural numbers], *discrete* sets equipped with a continuous Π-action and whose *morphisms* are morphisms of Π-sets [cf. [Mzk2], §3]. If Π may be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall say that Π is *tempered* [cf. [Mzk2], Definition 3.1, (i)]. Thus, if Π is *profinite*, then it is tempered, and there is a natural equivalence of categories $\mathcal{B}^{\text{temp}}(\Pi)^0 \xrightarrow{\sim} \mathcal{B}(\Pi)^0$ [where the superscript "0" is as in [Mzk5], §0]. If $Z_\Pi(H) = \{1\}$ for every open subgroup $H \subseteq \Pi$, then we shall say that Π is *temp-slim* [cf. [Mzk2], Definition 3.4, (ii)]. Any category equivalent to a category of the form $\mathcal{B}^{\text{temp}}(\Pi)$ for some tempered topological group Π will be referred to as a *connected temperoid* [cf. [Mzk2], Definition 3.1, (ii)]. We recall that if Π is a tempered topological group, then Π is *temp-slim* if and only if the category $\mathcal{B}^{\text{temp}}(\Pi)$ is *slim* [cf. [Mzk2], Corollary 3.3, Remark 3.4.1)]. In a similar vein, if Π is *residually finite*, then $\mathcal{B}^{\text{temp}}(\Pi)^0$ is *Frobenius-slim* [cf. [Mzk5], Remark 3.1.2]. If $\Pi^\circ \subseteq \Pi$ is an *open subgroup*, then we shall write

$$\mathcal{B}^{\text{temp}}(\Pi, \Pi^\circ) \subseteq \mathcal{B}^{\text{temp}}(\Pi)$$

for the full subcategory of objects that admit a morphism to the object Π/Π° [i.e., the set of cosets Π/Π° equipped with its natural Π-action from the left] of $\mathcal{B}^{\text{temp}}(\Pi)$. Note that there is a *natural equivalence of categories* $\mathcal{B}^{\text{temp}}(\Pi^\circ) \xrightarrow{\sim} \mathcal{B}^{\text{temp}}(\Pi)_{\Pi/\Pi^\circ}$; in particular, we obtain *natural functors* $\mathcal{B}^{\text{temp}}(\Pi^\circ) \to \mathcal{B}^{\text{temp}}(\Pi)$, $\mathcal{B}^{\text{temp}}(\Pi^\circ)^0 \to \mathcal{B}^{\text{temp}}(\Pi)^0$. A *connected quasi-temperoid* is a category that is equivalent to a category of the form $\mathcal{B}^{\text{temp}}(\Pi, \Pi^\circ)$, where Π is a tempered topological group, and $\Pi^\circ \subseteq \Pi$ is an open subgroup [cf. [Mzk2], Definition A.1, (i)]. One verifies immediately that if $\mathcal{E}$ is a connected quasi-temperoid, then the category $\mathcal{E}^0$ ($\subseteq \mathcal{E}$) is *connected*, *totally epimorphic*, of *strongly indissectible* type [cf. §0], and of *FSM-type* [indeed, every *monomorphism* of $\mathcal{E}^0$ is an isomorphism], hence, in particular, of *FSMFF-type* [cf. [Mzk5], §0].

(ii) Let $\phi : \Pi_1 \to \Pi_2$ be an *open homomorphism of tempered topological groups* [i.e., ϕ is a continuous homomorphism, $\phi(\Pi_1)$ is an open subgroup of Π_2, and ϕ induces an isomorphism of topological groups $\Pi_1/\text{Ker}(\phi) \xrightarrow{\sim} \phi(\Pi_1)$]. Then observe that ϕ induces a *natural functor*

$$\phi_* : \mathcal{B}^{\text{temp}}(\Pi_1)^0 \to \mathcal{B}^{\text{temp}}(\Pi_2)^0$$

by composing the functor $\mathcal{B}^{\text{temp}}(\phi(\Pi_1))^0 \to \mathcal{B}^{\text{temp}}(\Pi_2)^0$ of (i) with the functor $\mathcal{B}^{\text{temp}}(\Pi_1)^0 \to \mathcal{B}^{\text{temp}}(\phi(\Pi_1))^0$ obtained by mapping a Π_1-set E to the $\phi(\Pi_1)$-set $E' \stackrel{\text{def}}{=} E/\text{Ker}(\phi)$ given by taking the set of $\text{Ker}(\phi)$-*orbits* of E. When ϕ is *surjective*, one verifies immediately that the functor ϕ_* is *left adjoint* to the *pull-back functor* $\mathcal{B}^{\text{temp}}(\Pi_2)^0 \to \mathcal{B}^{\text{temp}}(\Pi_1)^0$ [i.e., obtained by composing the Π_2-action on a Π_2-set F_2 with ϕ so as to obtain a Π_1-set F_1].

(iii) Let $F = \mathbb{Q}_p$ or $\mathbb{R}$; $\widetilde{F}$ an *algebraic closure* of F. Then by (i), (ii), any *open homomorphism* $\Pi \to Q$ from a *tempered topological group* Π, equipped with an open subgroup $\Pi^\circ \subseteq \Pi$, to a *quotient* $G_F \twoheadrightarrow Q$ of the *absolute Galois group* $G_F \stackrel{\text{def}}{=} \text{Gal}(\widetilde{F}/F)$ of F determines a *functor*

$$\mathcal{B}^{\text{temp}}(\Pi, \Pi^\circ)^0 \hookrightarrow \mathcal{B}^{\text{temp}}(\Pi)^0 \to \mathcal{B}^{\text{temp}}(Q)^0 \hookrightarrow \mathcal{B}^{\text{temp}}(G_F)^0$$

[where the middle arrow "$\to$" is a functor as in (ii)] between *connected*, *totally epimorphic* categories of *FSM-type*. In particular, when $F = \mathbb{Q}_p$, if we set $\mathcal{D} \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(\Pi, \Pi^\circ)^0$, then we obtain a functor $\mathcal{D} \to \mathcal{D}_0 = \mathcal{B}^{\mathrm{temp}}(G_{\mathbb{Q}_p})^0$ [cf. Example 1.1, (ii)] which satisfies the hypotheses of Theorem 1.2, (i). That is to say, in this case, *the main results of the theory of [Mzk5] may be applied to the p-adic Frobenioids* of Example 1.1, (ii).

Example 1.4. Localizations of Number Fields.

(i) Often it is of interest to *relate finite extensions of a local field to finite extensions of number fields*. This may be achieved as follows: Let $\widetilde{F}$ be a *Galois extension* of a *number field* F; v a [not necessarily nonarchimedean!] *valuation* of F. Write F_v for the completion of F at v, $\widetilde{F}_v$ for the Galois extension of F_v determined by $\widetilde{F}/F$, $\mathfrak{D}_v \subseteq \mathrm{Gal}(\widetilde{F}/F)$ for the *decomposition group* of v in $\mathrm{Gal}(\widetilde{F}/F)$ [which is well-defined up to conjugation],

$$\mathcal{Q}_0$$

for the *full subcategory of the Galois category* of finite étale coverings of $\mathrm{Spec}(F)$ determined by objects of the form $\mathrm{Spec}(F') \to \mathrm{Spec}(F)$, where $F' \subseteq \widetilde{F}$ is a finite field extension of F, and

$$\mathcal{P}_0$$

for the *full subcategory of connected objects of the Galois category* of finite étale coverings of $\mathrm{Spec}(F_v)$ determined by objects of the form $\mathrm{Spec}(F') \to \mathrm{Spec}(F_v)$, where $F' \subseteq \widetilde{F}_v$ is a finite field extension of F_v. Thus, restriction to $\mathrm{Spec}(F_v)$ determines a *natural functor*

$$\rho : \mathcal{Q}_0 \to \mathcal{P}_0^{\perp}$$

[where the superscript "$\perp$" is as in [Mzk5], §0; thus, $\mathcal{P}_0^{\perp}$ is naturally equivalent to the full subcategory of the Galois category of finite étale coverings of $\mathrm{Spec}(F_v)$ consisting of objects whose connected components form objects of $\mathcal{P}_0$]. Write

$$\mathcal{E}_0$$

for the category whose *objects* are triples

$$(P, Q, \iota : P \to \rho(Q))$$

— where $P \in \mathrm{Ob}(\mathcal{P}_0)$, $Q \in \mathrm{Ob}(\mathcal{Q}_0)$, and ι is an isomorphism in $\mathcal{P}_0^{\perp}$ of P onto a connected component of $\rho(Q)$ — and whose *morphisms*

$$(P, Q, \iota : P \to \rho(Q)) \to (P', Q', \iota' : P' \to \rho(Q'))$$

are pairs of morphisms $P \to P'$, $Q \to Q'$ in $\mathcal{P}_0$, $\mathcal{Q}_0$, respectively, that are compatible with ι, ι'. Thus, $\mathcal{E}_0$ may be thought of as the category of *"connected finite étale coverings of* $\mathrm{Spec}(F_v)$ *equipped with a localization morphism [that satisfies certain* *properties] to a connected finite étale covering of* $\mathrm{Spec}(F)$ *determined by a finite subextension of the extension* $\widetilde{F}/F$*".*

(ii) One verifies immediately that the category $\mathcal{E}_0$ is *connected* and *totally epimorphic*. Also, we have natural functors $\mathcal{E}_0 \to \mathcal{P}_0$, $\mathcal{E}_0 \to \mathcal{Q}_0$. One verifies immediately that $\mathcal{E}_0 \to \mathcal{P}_0$ is *faithful* and *arrow-wise essentially surjective* [cf. §0], and that if $E_0 \in \mathrm{Ob}(\mathcal{E}_0)$ projects to an object $P_0 \in \mathrm{Ob}(\mathcal{P}_0)$, then the functor $\mathcal{E}_0 \to \mathcal{P}_0$ induces a *bijection* $\mathrm{Aut}_{\mathcal{E}_0}(E_0) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{P}_0}(P_0)$. Note that finite extensions $F_1 \subseteq F_2 \subseteq \widetilde{F}$ of F, together with a valuation v_2 of F_2 lying over a valuation v_1 of F_1 which, in turn, lies over v such that the local extension $(F_2)_{v_2}/(F_1)_{v_1}$ is *trivial* yield examples of *FSM-morphisms of* $\mathcal{E}_0$ which are *not isomorphisms*. Conversely, one verifies immediately that *every* FSM-morphism of $\mathcal{E}_0$ which is not an isomorphism arises in this way. Thus, the category $\mathcal{E}_0$ is *not* [in general] *of FSM-type*. On the other hand, one verifies immediately that $\mathcal{E}_0$ is *of FSMFF-type*.

(iii) Suppose that v is *nonarchimedean* and lies over a rational prime p. Then one has an evident *natural functor* $\mathcal{P}_0 \to \mathcal{D}_0$. Moreover, by applying Proposition 1.5, (ii), (iii), below, given a connected, totally epimorphic category $\mathcal{P}$ and a functor $\mathcal{P} \to \mathcal{P}_0$, one obtains a *connected, totally epimorphic* category $\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{P} \times_{\mathcal{P}_0} \mathcal{E}_0$ equipped with a functor $\mathcal{E} \to \mathcal{P}_0 \to \mathcal{D}_0$ — cf. the category "$\mathcal{D}$", the functor "$\mathcal{D} \to \mathcal{D}_0$" of Example 1.1, (ii). If, moreover, $\mathcal{P}$ is *of FSM-type* [cf., e.g., Example 1.3, (i)], then $\mathcal{E}$ is *of FSMFF-type* [cf. Proposition 1.5, (viii)], hence, in particular, satisfies the hypotheses of Theorem 1.2, (i). That is to say, in this case, *the main results of the theory of [Mzk5] may be applied to the p-adic Frobenioids* of Example 1.1, (ii). Also, we observe that if $\mathcal{P}$ is *Frobenius-slim* or *slim* [cf., e.g., Example 1.3, (i)], then so is $\mathcal{E}$ [cf. Proposition 1.5, (iv)]; if $\mathcal{P}$ is of *strongly indissectible* type [cf., e.g., Example 1.3, (i)], then so is $\mathcal{E}$ [cf. Proposition 1.5, (vii)].

Proposition 1.5. (Localizations of Number Fields) *Let* $\mathcal{P}_0$*,* $\mathcal{E}_0$ *be as in Example 1.4;* $\mathcal{P}$ *a* **category***;* $\mathcal{P} \to \mathcal{P}_0$ *a* **functor***. Denote by*

$$\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{P} \times_{\mathcal{P}_0} \mathcal{E}_0$$

the **categorical fiber product** *[cf. [Mzk5], §0]. [Thus, we have* **1-compatible** *natural projection functors* $\mathcal{E} \to \mathcal{P}$*,* $\mathcal{E} \to \mathcal{E}_0$*,* $\mathcal{E} \to \mathcal{P}_0$*.] We shall refer to a morphism of* $\mathcal{E}$ *that projects to an an isomorphism of* $\mathcal{P}$ *as a* $\mathcal{P}$**-isomorphism***. Then:*

(i) The natural projection functor $\mathcal{E} \to \mathcal{P}$ *is* **faithful** *and* **arrow-wise essentially surjective***. Moreover, if* $E_1, E_2 \in \mathrm{Ob}(\mathcal{E})$ *project to* $P_1, P_2 \in \mathrm{Ob}(\mathcal{P})$*, respectively, then the functor* $\mathcal{E} \to \mathcal{P}$ *induces a* **bijection**

$$\varinjlim_{E_3 \to E_1} \mathrm{Hom}_{\mathcal{E}}(E_3, E_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}}(P_1, P_2)$$

where the inductive limit is over $\mathcal{P}$*-isomorphisms* $E_3 \to E_1$*. That is to say, "*$\mathcal{P}$ *may be* **reconstructed** *from* $\mathcal{E}$ *by* **inverting the** $\mathcal{P}$**-isomorphisms***".*

(ii) $\mathcal{P}$ *is* **connected** *if and only if* $\mathcal{E}$ *is.*

(iii) $\mathcal{P}$ is **totally epimorphic** *if and only if $\mathcal{E}$ is.*

(iv) If $E \in \mathrm{Ob}(\mathcal{E})$ projects to an object $P \in \mathrm{Ob}(\mathcal{P})$, then we have **natural bijections**

$$\mathrm{Aut}(\mathcal{P}_P \to \mathcal{P}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{E}_E \to \mathcal{P}); \quad \mathrm{Aut}(\mathcal{E}_E \to \mathcal{E}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{E}_E \to \mathcal{P})$$

[induced by composition with the natural functors $\mathcal{E} \to \mathcal{P}$, $\mathcal{E}_E \to \mathcal{P}_P$]. In particular, $\mathcal{P}$ is **slim** *if and only if $\mathcal{E}$ is; $\mathcal{P}$ is* **Frobenius-slim** *if and only if $\mathcal{E}$ is.*

(v) Let $\phi_{\mathcal{E}} \in \mathrm{Arr}(\mathcal{E})$; denote by $\phi_{\mathcal{P}} \in \mathrm{Arr}(\mathcal{P})$ the projection of $\phi_{\mathcal{E}}$ to $\mathcal{P}$. Then $\phi_{\mathcal{E}}$ is a **monomorphism** *(respectively,* **fiber-wise surjective***; an* **FSM-morphism***) if and only if $\phi_{\mathcal{P}}$ is.*

(vi) Suppose that $\mathcal{P}$ is of **FSM-type***. Then a morphism of $\mathcal{E}$ is a* **$\mathcal{P}$-isomorphism** *if and only if it is an* **FSM-morphism***.*

(vii) Let $A_{\mathcal{E}} \in \mathrm{Ob}(\mathcal{E})$; denote by $A_{\mathcal{P}} \in \mathrm{Ob}(\mathcal{P})$ the projection of $A_{\mathcal{E}}$ to $\mathcal{P}$. Then $A_{\mathcal{P}}$ is **weakly dissectible** *(respectively,* **weakly indissectible***;* **strongly dissectible***;* **strongly indissectible***) [cf. §0] if and only if $A_{\mathcal{E}}$ is.*

(viii) Suppose that $\mathcal{P}$ is of **FSM-type** *and* **totally epimorphic***. Then $\mathcal{E}$ is of* **FSMFF-type***.*

(ix) Let $\phi_{\mathcal{E}}$, $\phi_{\mathcal{P}}$ be as in (v). Suppose, moreover, that $\mathfrak{D}_v$ is a **finite** *group which is either trivial or of* **prime order** *[which is always the case, if, for instance, v is* **archimedean***]. Then if $\phi_{\mathcal{P}}$ admits a* **factorization** *$\phi_{\mathcal{P}} = \alpha_{\mathcal{P}} \circ \beta_{\mathcal{P}}$ in $\mathcal{P}$, then there exist morphisms $\alpha_{\mathcal{E}}$, $\beta_{\mathcal{E}}$ of $\mathcal{E}$* **lifting** *$\alpha_{\mathcal{P}}$, $\beta_{\mathcal{P}}$, respectively, such that $\phi_{\mathcal{E}} = \alpha_{\mathcal{E}} \circ \beta_{\mathcal{E}}$ in $\mathcal{E}$. In particular, if $\phi_{\mathcal{E}}$ is* **totally ordered** *[cf. §0] (respectively, either* **irreducible** *or an isomorphism), then so is $\phi_{\mathcal{P}}$.*

Proof. First, we consider assertion (i). The *faithfulness* and *arrow-wise essential surjectivity* of $\mathcal{E} \to \mathcal{P}$ follow immediately from the corresponding properties of $\mathcal{E}_0 \to \mathcal{P}_0$ [cf. Example 1.4, (ii)]. The *bijection* of assertion (i) follows immediately from the definitions and the easily verified *observation* that such a bijection exists when $\mathcal{P} \to \mathcal{P}_0$ is the *identity functor* on $\mathcal{P}_0$. Next, we consider assertion (ii). Since $\mathcal{E} \to \mathcal{P}$ is *essentially surjective* [cf. assertion (i)], it is immediate that the connectedness of $\mathcal{E}$ implies that of $\mathcal{P}$. On the other hand, note that when $\mathcal{P}$ is a *one-morphism category*, the connectedness of $\mathcal{E}$ follows immediately from the *bijection* of assertion (i). If $\mathcal{P}$ is an arbitrary connected category [equipped with a functor $\mathcal{P} \to \mathcal{P}_0$], then the connectedness of $\mathcal{E}$ follows immediately from the *arrow-wise essential surjectivity* of $\mathcal{E} \to \mathcal{P}$ [cf. assertion (i)] by considering the "fibers" of the functor $\mathcal{E} \to \mathcal{P}$ [which correspond to the case where $\mathcal{P}$ is a one-morphism category]. Assertion (iii) follows immediately from the definitions, the total epimorphicity of $\mathcal{E}_0$ and $\mathcal{P}_0$, and the *bijection* of assertion (i). Assertion (iv) follows immediately from the *bijection* of assertion (i) and the bijection "$\mathrm{Aut}_{\mathcal{E}_0}(E_0) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{P}_0}(P_0)$" observed in Example 1.4, (ii). Assertions (v), (vii) follow from the *faithfulness* of $\mathcal{E} \to \mathcal{P}$ [cf. assertion (i)], together with the *bijection* of assertion (i), by a routine verification. Assertion (vi) follows formally from assertion (v). This completes the proof of assertions (i), (ii), (iii), (iv), (v), (vi), (vii).

Next, we consider assertion (viii). Since $\mathcal{P}$ is *totally epimorphic*, it follows that if a composite of two morphisms α, β of $\mathcal{P}$ is an isomorphism, then both α and β are isomorphisms [cf. the discussion of [Mzk5], §0]; a similar statement holds for $\mathcal{P}_0$. Thus, by applying the equivalence of assertion (vi) both to the original $\mathcal{P}$ and to the case where $\mathcal{P} \to \mathcal{P}_0$ is the identity functor on $\mathcal{P}_0$, we conclude that a morphism of $\mathcal{E}$ is an FSMI-morphism if and only if it is a $\mathcal{P}$-isomorphism that projects to an FSMI-morphism of $\mathcal{E}_0$. Thus, the fact that $\mathcal{E}$ is of *FSMFF-type* follows formally from the fact that $\mathcal{E}_0$ is of *FSMFF-type* [cf. Example 1.4, (ii)]. This completes the proof of assertion (viii).

Finally, we observe that the portion of assertion (ix) concerning *factorizations* follows immediately from the fact that the assumption on $\mathfrak{D}_v$ implies that if $\alpha_{\mathcal{P}_0} \circ \beta_{\mathcal{P}_0}$ is any composite morphism of $\mathcal{P}_0$, then either $\alpha_{\mathcal{P}_0}$ or $\beta_{\mathcal{P}_0}$ is an isomorphism. The portion of assertion (ix) concerning *irreducibility* then follows formally; the portion of assertion (ix) concerning *totally ordered morphisms* follows from the portion of assertion (ix) concerning factorizations, together with the portion of assertion (v) concerning monomorphisms [cf. §0]. ◯

Section 2: The Kummer and Reciprocity Maps

The purpose of the present §2 is to consider classical *Kummer theory*, as well as the *reciprocity map* of classical local class field theory, in the context of the *"p-adic Frobenioids"* of §1.

We begin with the following general definition.

Definition 2.1. Let $\mathcal{C}$ be a *Frobenioid* [whose Frobenioid structure is given by a functor $\mathcal{C} \to \mathbb{F}_\Phi$, where Φ is a divisorial monoid on a connected, totally epimorphic category $\mathcal{D}$]; $A \in \mathrm{Ob}(\mathcal{C})$; $A_\mathcal{D} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$; $N \in \mathbb{N}_{\geq 1}$.

(i) Write

$$\boldsymbol{\mu}_N(A) \subseteq \mathcal{O}^\times(A)$$

for the subgroup of elements of $\mathcal{O}^\times(A)$ that are annihilated by N. [Thus, if $\mathcal{C}$ is as in Example 1.1, with $\Lambda = \mathbb{Z}$, then $\boldsymbol{\mu}_N(A)$ is isomorphic to a subgroup of $\mathbb{Z}/N\mathbb{Z}$.] We shall refer to $\boldsymbol{\mu}_N(A)$ as the *cyclotomic portion of* $\mathcal{O}^\times(A)$ *[or* $\mathcal{O}^\triangleright(A)$*] of order* N. Write

$$\mathrm{Aut}_{\mathcal{C}/\mathcal{D}}(A) \stackrel{\mathrm{def}}{=} \mathrm{Im}(\mathrm{Aut}_\mathcal{C}(A)) \subseteq \mathrm{Aut}_\mathcal{D}(A_\mathcal{D})$$

for the image of $\mathrm{Aut}_\mathcal{C}(A)$ in $\mathrm{Aut}_\mathcal{D}(A_\mathcal{D})$. [Thus, since $\mathcal{O}^\triangleright(A)$ is *abelian* [cf. [Mzk5], Remark 1.3.1], the natural action of $\mathrm{Aut}_\mathcal{C}(A)$ on $\mathcal{O}^\triangleright(A)$ *factors* through $\mathrm{Aut}_{\mathcal{C}/\mathcal{D}}(A)$.] Then we observe that the submonoid $\boldsymbol{\mu}_N(A) \subseteq \mathcal{O}^\times(A) \subseteq \mathcal{O}^\triangleright(A)$ is *stabilized* by the natural action of $\mathrm{Aut}_{\mathcal{C}/\mathcal{D}}(A)$ on $\mathcal{O}^\triangleright(A)$. We shall say that A is $\boldsymbol{\mu}_N$*-saturated* if the abstract group $\boldsymbol{\mu}_N(A)$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$.

(ii) Suppose that A is $\boldsymbol{\mu}_N$-saturated. Let $H_A \subseteq \mathrm{Aut}_{\mathcal{C}/\mathcal{D}}(A)$ be a subgroup;

$$f \in \mathcal{O}^\triangleright(A)^{H_A}$$

[where the superscript H_A denotes the subset of elements fixed by the action of H_A] an element that admits an N*-th root* $g \in \mathcal{O}^\triangleright(A)$ — i.e., $g^N = f$. Then since the action of H_A on the monoid $\mathcal{O}^\triangleright(A)$ fixes f, it follows that H_A *stabilizes* the subset $\boldsymbol{\mu}_N(A) \cdot g \subseteq \mathcal{O}^\triangleright(A)$ [i.e., the subset of N-th roots of f — cf. [Mzk5], Definition 1.3, (vi)], hence determines a *cohomology class*

$$\kappa_f \in H^1(H_A, \boldsymbol{\mu}_N(A))$$

which is easily verified to be independent of the choice of g, and which we shall refer to as the *Kummer class of* f. Here, we note that H_A acts *trivially* on the cohomology module $H^1(H_A, \boldsymbol{\mu}_N(A))$. The assignment $f \mapsto \kappa_f$ determines a homomorphism

$$(\mathcal{O}^\triangleright(A) \supseteq) \quad \mathcal{O}^\triangleright(A)^{H_A} \bigcap \mathcal{O}^\triangleright(A)^N \to H^1(H_A, \boldsymbol{\mu}_N(A))$$

which we shall refer to as the *Kummer map*.

Now we return to our discussion of the *p-adic Frobenioids* $\mathcal{C}$ of Example 1.1. Let us assume that $\Lambda = \mathbb{Z}$; $\mathcal{D}$ is *any category*

$$\mathcal{B}^{\text{temp}}(\Pi, \Pi^{\circ})^0$$

as in Example 1.3, (iii), where $\Pi \to Q$, $G_{\mathbb{Q}_p} \twoheadrightarrow Q$ [i.e., we take "F" of *loc. cit.* to be $\mathbb{Q}_p$], $\Pi^{\circ} \subseteq \Pi$ are as in *loc. cit.*, and we assume further that the surjection $G_{\mathbb{Q}_p} \twoheadrightarrow Q$ is an *isomorphism*. Write $G \stackrel{\text{def}}{=} \text{Im}(\Pi) \subseteq Q$; $G^{\circ} \stackrel{\text{def}}{=} \text{Im}(\Pi^{\circ}) \subseteq Q$; $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{B}^{\text{temp}}(G, G^{\circ})^0$; K for the *finite extension* of $\mathbb{Q}_p$ determined by the open subgroup $G \subseteq Q$. Thus, we have a *natural functor*

$$\mathcal{D} = \mathcal{B}^{\text{temp}}(\Pi, \Pi^{\circ})^0 \to \mathcal{E} = \mathcal{B}^{\text{temp}}(G, G^{\circ})^0$$

[cf. Example 1.3, (ii), (iii)].

Definition 2.2. Let $H \subseteq G$ be a *normal open subgroup*; $N \in \mathbb{N}_{\geq 1}$; $A \in \text{Ob}(\mathcal{C})$. Write $A_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(A) \in \text{Ob}(\mathcal{D})$; $A_{\mathcal{E}} \in \text{Ob}(\mathcal{E})$ for the projection of A to $\mathcal{E}$.

(i) Note that the natural action by conjugation of $\text{Aut}_{\mathcal{C}}(A)$ on $\mathcal{O}^{\times}(A)$ ($\subseteq \text{Aut}_{\mathcal{C}}(A)$) *factors through the quotient*

$$\text{Aut}_{\mathcal{C}}(A) \twoheadrightarrow G_A \stackrel{\text{def}}{=} \text{Aut}_{\mathcal{C}}(A)/(\text{Ker}(\text{Aut}_{\mathcal{C}}(A) \to \text{Aut}_{\mathcal{E}}(A_{\mathcal{E}})))$$

induced by the functor $\mathcal{C} \to \mathcal{E}$. Moreover, the functor $\mathcal{C} \to \mathcal{E}$ induces a natural inclusion $G_A \hookrightarrow \text{Aut}_{\mathcal{E}}(A_{\mathcal{E}})$, which is an *isomorphism* if, for instance, $A_{\mathcal{D}}$ is *Galois* [where we recall that $\mathcal{C}$ is Aut-*ample* — cf. Theorem 1.2, (i)]. If $A_{\mathcal{D}}$ is Galois, then we have a *natural surjective outer homomorphism*

$$G \twoheadrightarrow \text{Aut}_{\mathcal{E}}(A_{\mathcal{E}}) \xrightarrow{\sim} G_A$$

which thus determines a surjective outer homomorphism

$$H \twoheadrightarrow H_A \stackrel{\text{def}}{=} \text{Im}(H) \quad (\subseteq G_A)$$

— which, in light of the fact that H is *normal* in G, is well-defined, up to *composition with conjugation by an element of* G, despite the fact that the homomorphism $G \twoheadrightarrow G_A$ is only determined up to composition with an inner automorphism.

(ii) We shall say that A is (N, H)-*saturated* if the following conditions are satisfied: (a) A is $\boldsymbol{\mu}_N$-saturated; (b) $A_{\mathcal{D}}$ is Galois; (c) the natural surjective homomorphism $H \twoheadrightarrow H_A$ induces *isomorphisms on first cohomology modules*

$$H^1(H_A, \boldsymbol{\mu}_N(A)) \xrightarrow{\sim} H^1(H, \boldsymbol{\mu}_N(A)); \quad H^1(H_A, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} H^1(H, \mathbb{Z}/N\mathbb{Z})$$

and a *surjection on second cohomology modules* $H^2(H_A, \boldsymbol{\mu}_N(A)) \twoheadrightarrow H^2(H, \boldsymbol{\mu}_N(A))$ [conditions which are unaffected by composition with conjugation by an element of G]. In this situation, we shall write

$$\mathfrak{F}_N(A) \stackrel{\text{def}}{=} H^2(H_A, \boldsymbol{\mu}_N(A))/\text{Ker}(H^2(H_A, \boldsymbol{\mu}_N(A)) \twoheadrightarrow H^2(H, \boldsymbol{\mu}_N(A)))$$

— so $\mathfrak{F}_N(A) \cong H^2(H, \boldsymbol{\mu}_N(A)) \cong \mathbb{Z}/N\mathbb{Z}$ [cf., e.g., [NSW], Chapter 7, Theorem 7.2.6].

(iii) Define

$$\mathcal{O}^{\square}(-) \stackrel{\text{def}}{=} \mathcal{O}^{\triangleright}(-)$$

if Φ is *fieldwise saturated* [cf. Example 1.1, (ii)], and

$$\mathcal{O}^{\square}(-) \stackrel{\text{def}}{=} \mathcal{O}^{\times}(-)$$

if Φ is *not* fieldwise saturated [e.g., if Φ is *absolutely primitive*].

Remark 2.2.1. Thus, it follows immediately from the definitions, together with the *finiteness* of the cohomology modules $H^1(H, \boldsymbol{\mu}_N(A))$, $H^2(H, \boldsymbol{\mu}_N(A))$ [cf., e.g., [NSW], Chapter 7, Theorem 7.2.6], that given an $A' \in \mathrm{Ob}(\mathcal{C})$, it follows that, for any N, H as in Definition 2.2, there exists a *pull-back morphism* $A'' \to A'$ in $\mathcal{C}$ such that A'' is (N, H)*-saturated*.

In the notation of Definition 2.2, suppose that A is (N, H)*-saturated*. Write $A_{\mathcal{E}} = \mathrm{Spec}(L)$ [so L is a finite extension of K]. Note that H_A acts naturally on L, $\mathcal{O}^{\square}(A)$. Moreover, we have a *natural isomorphism*

$$(\mathcal{O}^{\square}(A) \supseteq) \quad \mathcal{O}^{\square}(A)^H \xrightarrow{\sim} \mathcal{O}^{\square}(L^H)$$

[where the superscript "H's" denote the submonoids/subfields of elements on which H_A acts trivially]. Observe that, in light of the definition of the notation "$\mathcal{O}^{\square}(-)$" in Definition 2.2, (iii), the [first cohomology module portion of the] *Galois-theoretic condition* of Definition 2.2, (ii), (c), implies [upon translation into "extension field-theoretic language"] that any element

$$f \in \mathcal{O}^{\square}(A)^H$$

admits an N*-th root* $g \in \mathcal{O}^{\square}(A)$ — i.e., $g^N = f$. Thus, the *Kummer map* [cf. Definition 2.1, (ii)] is defined on *all of* $\mathcal{O}^{\square}(A)^H$.

Next, let us recall that, by the well-known *duality theory of nonarchimedean [mixed-characteristic] local fields* [cf., e.g., [NSW], Chapter 7, Theorem 7.2.6], the *cup product* on group cohomology determines an isomorphism

$$H^1(H, \boldsymbol{\mu}_N(A)) \xrightarrow{\sim} H^{\mathrm{ab}} \otimes H^2(H, \boldsymbol{\mu}_N(A))$$

hence [in light of the condition of Definition 2.2, (ii), (c)] an *isomorphism* [induced by the cup product on group cohomology]

$$H^1(H_A, \boldsymbol{\mu}_N(A)) \xrightarrow{\sim} H_A^{\mathrm{ab}} \otimes \mathfrak{F}_N(A)$$

which is *independent* of the choice of the homomorphism $H \twoheadrightarrow H_A$ among its various G-conjugates and *compatible* with the natural actions of G_A/H_A on its domain and codomain. Write $\eta_f \in H_A^{\mathrm{ab}} \otimes \mathfrak{F}_N(A)$ for the image of κ_f via this last isomorphism.

Definition 2.3. The assignments $f \mapsto \kappa_f$, $f \mapsto \eta_f$ determine homomorphisms

$$\mathcal{O}^{\square}(A)^H \to H^1(H_A, \boldsymbol{\mu}_N(A)); \quad \mathcal{O}^{\square}(A)^H \to H_A^{\mathrm{ab}} \otimes \mathfrak{F}_N(A)$$

which we shall refer to as the *Kummer* and *reciprocity maps* [associated to the (N,H)-saturated object A], respectively.

Theorem 2.4. (Category-theoreticity of the Kummer and Reciprocity Maps) *For $i = 1, 2$, let $p_i \in \mathfrak{Primes}$; $\mathcal{C}_i$ a* **p_i-adic Frobenioid** *[cf. Example 1.1, (ii)] whose* **monoid type** *[i.e., the "Λ" of Example 1.1] is $\mathbb{Z}$, whose defining* **divisor monoid** *[i.e., the "Φ" of Example 1.1] we denote by Φ_i, whose defining* **rational function monoid** *[i.e., the "$\mathbb{B}$" of Example 1.1] we denote by $\mathbb{B}_i$, and whose* **base category** *$\mathcal{D}_i$ satisfies*

$$\mathcal{D}_i = \mathcal{B}^{\mathrm{temp}}(\Pi_i, \Pi_i^{\circ})^0$$

[cf. Example 1.3, (iii)], where $\Pi_i \to Q_i \stackrel{\mathrm{def}}{=} G_{\mathbb{Q}_{p_i}}$ is an open homomorphism of **temp-slim** *tempered topological groups [cf. [Mzk1], Theorem 1.1.1, (ii)], $\Pi_i^{\circ} \subseteq \Pi_i$ is an open subgroup, $G_i \stackrel{\mathrm{def}}{=} \mathrm{Im}(\Pi_i) \subseteq Q_i$; $G_i^{\circ} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\Pi_i^{\circ}) \subseteq Q_i$; $\mathcal{E}_i \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(G_i, G_i^{\circ})^0$. [Thus, we have a natural functor $\mathcal{D}_i \to \mathcal{E}_i$.] Suppose further that we have been given* **open normal subgroups** *$H_1 \subseteq G_1$, $H_2 \subseteq G_2$, together with an* **equivalence of categories**

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

— which [cf. Theorem 1.2, (i); Example 1.3, (i); [Mzk5], Theorem 3.4, (v)] necessarily induces a **1-compatible** *equivalence of categories $\Psi^{\mathrm{Base}} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$, hence an* **outer isomorphism** *of topological groups*

$$\Pi_1 \xrightarrow{\sim} \Pi_2$$

[cf. [Mzk2], Proposition 3.2; [Mzk2], Theorem A.4] that lies over an **outer isomorphism** *of topological groups*

$$G_1 \xrightarrow{\sim} G_2$$

[cf. Theorem 1.2, (ii)]. Assume that this isomorphism $G_1 \xrightarrow{\sim} G_2$ maps H_1 onto H_2. Then [relative to the notation introduced in Definition 2.2]:

(i) Φ_1 is **fieldwise saturated** *if and only if Φ_2 is. Moreover, $p_1 = p_2$; Ψ maps (N, H_1)-***saturated** *objects to (N, H_2)-saturated objects and induces* **isomorphisms** *of monoids/modules*

$$\mathcal{O}^{\square}(A_1)^{H_1} \xrightarrow{\sim} \mathcal{O}^{\square}(A_2)^{H_2}; \quad (H_1)_{A_1} \xrightarrow{\sim} (H_2)_{A_2}; \quad (G_1)_{A_1} \xrightarrow{\sim} (G_2)_{A_2}$$

$$H^1((H_1)_{A_1}, \boldsymbol{\mu}_N(A_1)) \xrightarrow{\sim} H^1((H_2)_{A_2}, \boldsymbol{\mu}_N(A_2)); \quad \mathfrak{F}_N(A_1) \xrightarrow{\sim} \mathfrak{F}_N(A_2)$$

[where $\Psi(A_1) = A_2$; for $i = 1, 2$, $A_i \in \mathrm{Ob}(\mathcal{C}_i)$ is (N, H_i)-saturated] which are **compatible** *with the respective* **Kummer** *and* **reciprocity** *maps*

$$\mathcal{O}^{\square}(A_i)^{H_i} \to H^1((H_i)_{A_i}, \boldsymbol{\mu}_N(A_i)); \quad \mathcal{O}^{\square}(A_i)^{H_i} \to (H_i)_{A_i}^{\mathrm{ab}} \otimes \mathfrak{F}_N(A_i)$$

[where $i = 1, 2$] as well as with the various natural actions of $(G_1)_{A_1}/(H_1)_{A_1}$, $(G_2)_{A_2}/(H_2)_{A_2}$.

(ii) If the Φ_i *are* **fieldwise saturated**, *then the isomorphism* $\mathfrak{F}_N(A_1) \xrightarrow{\sim} \mathfrak{F}_N(A_2)$ *of (i) is* **compatible** *with the natural isomorphisms*

$$\mathfrak{F}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$

[cf., e.g., [NSW], Chapter 7, Theorem 7.2.6].

Proof. First, we consider assertion (i). We begin by observing the [easily verified] fact that Φ_i is *fieldwise saturated* if and only if the following two conditions hold:

(a) The inductive limit monoid

$$\varinjlim \ \Phi_i(B)$$

[where B ranges over the objects of $\mathcal{D}_i$] is *divisible.*

(b) For every pull-back morphism $\phi : B \to C$ of $\mathcal{C}_i$ that projects to a Galois covering $\phi_{\mathcal{D}} : B_{\mathcal{D}} \to C_{\mathcal{D}}$ of $\mathcal{D}_i$, the injection $\mathcal{O}^{\rhd}(C) \hookrightarrow \mathcal{O}^{\rhd}(B)$ induced by ϕ [cf. [Mzk5], Proposition 1.11, (iv)] determines a *bijection*

$$\mathcal{O}^{\rhd}(C) \xrightarrow{\sim} \mathcal{O}^{\rhd}(B)^{\mathrm{Gal}(B/C)}$$

[where the superscript "$\mathrm{Gal}(B/C)$" denotes the submonoid of $\mathrm{Gal}(B/C)$-invariants; $\mathrm{Gal}(B/C) \stackrel{\mathrm{def}}{=} \mathrm{Gal}(B_{\mathcal{D}}/C_{\mathcal{D}})$].

Moreover, since $\mathcal{C}_i$ is a Frobenioid of *rationally standard* type [cf. Theorem 1.2, (i); Example 1.3, (i)] over a *slim* base category [cf. Example 1.3, (i)], it follows from [Mzk5], Corollaries 4.10; 4.11, (ii), (iii), that Ψ induces a 1-*compatible* equivalence of categories $\Psi^{\mathrm{Base}} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$, as well as *compatible* isomorphisms of functors $\Phi_1 \xrightarrow{\sim} \Phi_2$, $\mathbb{B}_1 \xrightarrow{\sim} \mathbb{B}_2$ — where we regard "$\mathcal{O}^{\rhd}(-)$" as a subfunctor of $\mathbb{B}_i$ [such that $\mathbb{B}_i$ is the groupification of "$\mathcal{O}^{\rhd}(-)$"] which is *preserved* by the isomorphism $\mathbb{B}_1 \xrightarrow{\sim} \mathbb{B}_2$ — hence that these conditions (a), (b) are *preserved* by Ψ. Thus, Φ_1 is *fieldwise saturated* if and only if Φ_2 is. The fact that $p_1 = p_2$ follows from the existence of the isomorphism $G_1 \xrightarrow{\sim} G_2$ [cf. [Mzk1], Proposition 1.2.1, (i)]. The remainder of assertion (i) follows immediately from the fact [just observed] that Ψ preserves "$\mathcal{O}^{\rhd}(-)$", together with the *"manifestly group-/category-theoretic" construction* given above of the Kummer and reciprocity maps. This completes the proof of assertion (i).

Next, we consider assertion (ii). For $i = 1, 2$, write K_i for the *finite extension* of $\mathbb{Q}_p$ [where $p \stackrel{\mathrm{def}}{=} p_1 = p_2$] determined by the open subgroup $G_i \subseteq Q_i$, $\overline{K}_i$ for the *algebraic closure* of K_i used to define the Galois group Q_i [so $G_i = \mathrm{Gal}(\overline{K}_i/K_i)$]. Then since Φ_i is *fieldwise saturated*, it follows — by *varying* the objects A_i [that correspond via Ψ] and reconstructing the *multiplicative group* associated to the field determined by the image of A_i in $\mathcal{B}(G_i, G_i^\circ)$ as the groupification of the monoid $\mathcal{O}^\rhd(A_i) = \mathcal{O}^\square(A_i)$ — that Ψ induces a *pair of compatible isomorphisms*

$$G_1 \xrightarrow{\sim} G_2; \quad \overline{K}_1^\times \xrightarrow{\sim} \overline{K}_2^\times$$

— where this pair is well-defined up to composition with *automorphisms of the pair* $(G_2, \overline{K}_2^\times)$ induced by elements of G_2. For $i = 1, 2$, denote by $\boldsymbol{\mu}(\overline{K}_i^\times) \subseteq \overline{K}_i^\times$ the torsion subgroup of $\overline{K}_i^\times$.

Now let us recall [cf. the theory of the Brauer group of a nonarchimedean [mixed-characteristic] local field, as exposed, for instance, in [Serre], §1; [Mzk1], the proof of Proposition 1.2.1, (vii)] that the natural isomorphism

$$H^2(G_i, \boldsymbol{\mu}(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

may be constructed as the composite of the natural isomorphism

$$H^2(G_i, \boldsymbol{\mu}(\overline{K}_i^\times)) \xrightarrow{\sim} H^2(G_i, \overline{K}_i^\times)$$

[induced by the natural inclusion $\boldsymbol{\mu}(\overline{K}_i^\times) \hookrightarrow \overline{K}_i^\times$] with the inverse of the natural isomorphism

$$H^2(G_i^{\mathrm{unr}}, (K_i^{\mathrm{unr}})^\times) \xrightarrow{\sim} H^2(G_i, \overline{K}_i^\times)$$

[induced by the inclusion $(K_i^{\mathrm{unr}})^\times \hookrightarrow \overline{K}_i^\times$ — where $K_i^{\mathrm{unr}} \subseteq \overline{K}_i$ denotes the *maximal unramified extension* of K_i; $G_i^{\mathrm{unr}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K_i^{\mathrm{unr}}/K_i)$], followed by the natural isomorphism

$$H^2(G_i^{\mathrm{unr}}, (K_i^{\mathrm{unr}})^\times) \xrightarrow{\sim} H^2(G_i^{\mathrm{unr}}, \mathbb{Z}) \xrightarrow{\sim} H^2(\widehat{\mathbb{Z}}, \mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

[where the first arrow is induced by the valuation map $(K_i^{\mathrm{unr}})^\times \to \mathbb{Z}$; the isomorphism $G_i^{\mathrm{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}$ is induced by the *Frobenius element*]. Since the isomorphism $G_1 \xrightarrow{\sim} G_2$ is *automatically compatible* with the quotients $G_i \twoheadrightarrow G_i^{\mathrm{unr}}$, as well as with the Frobenius elements $\in G_i^{\mathrm{unr}}$ [cf., e.g., [Mzk1], Proposition 1.2.1, (ii), (iv)], we thus conclude that the natural isomorphisms

$$H^2(G_i, \boldsymbol{\mu}(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

are compatible with the isomorphism

$$H^2(G_1, \boldsymbol{\mu}(\overline{K}_1^\times)) \xrightarrow{\sim} H^2(G_2, \boldsymbol{\mu}(\overline{K}_2^\times))$$

induced by Ψ. Since, moreover, the natural isomorphisms

$$\mathfrak{F}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$

in question are easily verified to be *"isomorphisms induced on subquotients"* by the natural isomorphisms $H^2(G_i, \boldsymbol{\mu}(\overline{K}_i^\times)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ [cf., e.g., [NSW], Chapter 7, Corollary 7.1.4], it thus follows that the natural isomorphisms $\mathfrak{F}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$ are compatible with the isomorphism $\mathfrak{F}_N(A_1) \xrightarrow{\sim} \mathfrak{F}_N(A_2)$ induced by Ψ, as desired. This completes the proof of assertion (ii). ◯

Remark 2.4.1. By allowing $N \in \mathbb{N}_{\geq 1}$ to *vary*, it is a routine exercise to show that:

> One obtains a *profinite version* of Theorem 2.4, (i), so long as one allows for a $(G_i)_{A_i}/(H_i)_{A_i}$*-indeterminacy* [where $i = 1, 2$] in the resulting profinite Kummer and reciprocity maps.

Indeed, when one allows N — hence also the objects A_i, where $i = 1, 2$ — to vary, one must exercise care with respect to the *coefficients* "$\boldsymbol{\mu}_N(A_i)$", as one allows A_i to vary. The fact that the coefficients then vary does not cause a problem — so long as one allows for a $(G_i)_{A_i}/(H_i)_{A_i}$*-indeterminacy* — since [cf. the discussion preceding Definition 2.3] $(H_i)_{A_i}$ acts *trivially* on $H^1((H_i)_{A_i}, \boldsymbol{\mu}_N(A_i))$; $\mathfrak{F}_N(A_i)$. [Here, we note that we implicitly used the [well-known — cf., e.g., [Mzk1], Proposition 1.2.1, (vii)] fact that $(G_i)_{A_i}$ acts *trivially* on $\mathfrak{F}_N(A_i)$ in the proof of Theorem 2.4, (ii).]

Remark 2.4.2. Note [relative to Theorem 2.4, (ii)] that if the Φ_i are *not fieldwise saturated*, then the natural isomorphisms

$$\mathfrak{F}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$$

are *not*, in general, compatible with the isomorphism $\mathfrak{F}_N(A_1) \xrightarrow{\sim} \mathfrak{F}_N(A_2)$ induced by Ψ. Indeed, when the Φ_i are *absolutely primitive* [cf. Remark 1.2.1], an example of such a Ψ is provided by the *unit-wise Frobenius functor* of [Mzk5], Proposition 2.9, (ii), which acts on $\mathfrak{F}_N(A_i)$ [relative to the natural isomorphisms $\mathfrak{F}_N(A_i) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$] by *"raising to the ζ-th power"* [cf. "the compatibility with the reciprocity map" asserted in Theorem 2.4, (i); [Mzk5], Proposition 2.9, (ii), (a), (c)].

Section 3: Archimedean Primes

In the present §3, we define and study the basic properties of certain Frobenioids naturally associated to *archimedean local fields*. In particular, we construct our *first example* [i.e., among the various examples presented thus far in [Mzk5] and the present paper] of a Frobenioid which is *not* of isotropic type. In fact, it is this example that *motivated* the theory [developed in [Mzk5]] of Frobenioids that are not necessarily of isotropic type; in particular, this example is also the source of the terminology *"isotropic"* and *"co-angular"* in the theory of [Mzk5].

Write

$$\mathcal{D}_0$$

for the *subcategory of connected objects of the Galois category* of finite étale coverings of $\mathrm{Spec}(\mathbb{R})$ [cf. [Mzk5], §0]. Thus, $\mathcal{D}_0$ is a *connected*, *totally epimorphic* category, which is of *FSM*-, hence also of *FSMFF-type* [cf. [Mzk5], §0].

We begin by introducing some terminology.

Definition 3.1.

(i) An *archimedean local field* is defined to be a topological field that is isomorphic [as a topological field] to either the [topological] field of real numbers $\mathbb{R}$ — in which case we shall refer to the archimedean local field as being *real* — or the [topological] field of complex numbers $\mathbb{C}$ — in which case we shall refer to the archimedean local field as being *complex*. Recall that the topological field $\mathbb{R}$ has *no nontrivial automorphisms*, while the unique nontrivial automorphism of the topological field $\mathbb{C}$ is given by *complex conjugation*. If K, L are archimedean local fields, then any inclusion of topological rings

$$\iota : K \hookrightarrow L$$

is either an isomorphism or satisfies the property that there exist isomorphisms of topological fields $\alpha_K : K \xrightarrow{\sim} \mathbb{R}$, $\alpha_L : L \xrightarrow{\sim} \mathbb{C}$ such that $\alpha_L \circ \iota \circ \alpha_K^{-1} : \mathbb{R} \hookrightarrow \mathbb{C}$ is the natural inclusion.

(ii) If K is an archimedean local field, then we shall denote by

$$\mathcal{O}_K^\times \subseteq K^\times$$

and refer to as the *group of units* of K the topological subgroup of $K^\times$ of elements of norm 1; let us write

$$\mathbb{S}^1 \stackrel{\mathrm{def}}{=} \mathcal{O}_{\mathbb{C}}^\times \subseteq \mathbb{C}^\times; \quad \mathrm{ord}(K^\times) \stackrel{\mathrm{def}}{=} K^\times/\mathcal{O}_K^\times$$

and observe that the usual absolute value $|-|$ on K determines a *natural isomorphism* $\mathrm{ord}(K^\times) \xrightarrow{\sim} \mathbb{R}_{>0}$. Also, note that we have a *canonical decomposition*

$$\mathcal{O}_K^\times \times \mathrm{ord}(K^\times) \xrightarrow{\sim} K^\times$$

[i.e., given by mapping a pair (z, λ), where $z \in \mathcal{O}_K^\times$, $\lambda \in \mathbb{R}_{>0} \cong \mathrm{ord}(K^\times)$, to $z \cdot \lambda \in K^\times$].

(iii) Let K be an archimedean local field. Then we shall refer to as an *angular region*

$$\mathbb{A} \subseteq K^\times$$

any subset $\mathbb{A}$ of the form $B \times C$ [i.e., relative to the canonical decomposition of (ii)], where $B \subseteq \mathcal{O}_K^\times$ is an open subset whose intersection with each connected component of $\mathcal{O}_K^\times$ is [nonempty and] *connected* [which implies that $B = \mathcal{O}_K^\times$ whenever K is *real*], and $C \subseteq \mathrm{ord}(K^\times) \cong \mathbb{R}_{>0}$ is an interval of the form $(0, \lambda]$, where $\lambda \in \mathbb{R}_{>0}$. In this situation, we shall refer to λ as the *tip* of the angular region $\mathbb{A}$ and to the subset

$$\partial\mathbb{A} \stackrel{\mathrm{def}}{=} \{a \in \mathbb{A} \mid |a| = \lambda\} \subseteq \mathbb{A}$$

as the *boundary* of $\mathbb{A}$ [so $\partial\mathbb{A}$ maps bijectively via the projection $K^\times \to \mathcal{O}_K^\times$ to B]; if $B = \mathcal{O}_K^\times$, then we shall refer to the angular region $\mathbb{A}$ as *isotropic*. If $\mathbb{A}' \subseteq K^\times$ is another angular region, then we shall refer to $\mathbb{A}$, $\mathbb{A}'$ as *co-angular* if the projections of $\mathbb{A}$, $\mathbb{A}'$ to $\mathcal{O}_K^\times$ *coincide*. We shall refer to a subset of K as an *angular region* if it is the union with the subset $\{0\}$ of an angular region of $K^\times$. Note that if V is any *one-dimensional vector space* over K, then it makes sense to speak of angular regions, tips of angular regions ($\in \mathrm{ord}(V) \stackrel{\mathrm{def}}{=} V^\times/\mathcal{O}_K^\times$, where $V^\times \stackrel{\mathrm{def}}{=} V \backslash \{0\}$), boundaries of angular regions, isotropic angular regions, and co-angular angular regions of V, $V^\times$. Also, if, for $i = 1, 2$, $\mathbb{A}_i \subseteq V_i$ is an angular region of a one-dimensional vector space V_i over K, then one verifies immediately that the subset

$$\mathbb{A}_1 \otimes_K \mathbb{A}_2 \stackrel{\mathrm{def}}{=} \{a_1 \otimes_K a_2 \mid a_1 \in \mathbb{A}_1,\ a_2 \in \mathbb{A}_2\} \subseteq V_1 \otimes_K V_2$$

is an angular region of $V_1 \otimes_K V_2$.

(iv) If

$$\iota : K \hookrightarrow L$$

is an inclusion of archimedean local fields, and $\mathbb{A}_K \subseteq V_K$ is an angular region of a one-dimensional vector space V_K over K, then the pair $(V_K, \mathbb{A}_K)$ determines a pair

$$(V_K, \mathbb{A}_K)|_L \stackrel{\mathrm{def}}{=} (V_L, \mathbb{A}_L)$$

where $V_L \stackrel{\mathrm{def}}{=} V_K \otimes_K L$, and $\mathbb{A}_L \subseteq V_L$ is the angular region of V_L defined as follows: If ι is an *isomorphism*, then we take $\mathbb{A}_L \subseteq V_L$ to be the image of $\mathbb{A}_K$ via the natural bijection $V_K \xrightarrow{\sim} V_L$. If ι is *not* an isomorphism, then we take $\mathbb{A}_L \subseteq V_L$ to be the [necessarily isotropic] angular region of V_L given by the $\mathcal{O}_L^\times$-orbit of the image of $\mathbb{A}_K$ via the natural inclusion $V_K \hookrightarrow V_L$.

(v) If $\mathcal{F}$ is a category equipped with a functor $\mathcal{F} \to \mathcal{D}_0$, then we shall write $\mathcal{F}[\mathbb{R}]$ (respectively, $\mathcal{F}[\mathbb{C}]$) for the full subcategory of *real* (respectively, *complex*) objects — i.e., objects that project to an object of $\mathcal{D}_0$ determined by a real (respectively, complex) archimedean local field. If $\mathcal{F} = \mathcal{F}[\mathbb{R}]$ (respectively, $\mathcal{F} = \mathcal{F}[\mathbb{C}]$), then we shall say that $\mathcal{F}$ is *real* (respectively, *complex*). If, for every $A \in \mathrm{Ob}(\mathcal{F}[\mathbb{R}])$, there exists a morphism $B \to A$ in $\mathcal{F}$, where $B \in \mathrm{Ob}(\mathcal{F}[\mathbb{C}])$, together with an automorphism of $\beta \in \mathrm{Aut}_{\mathcal{F}_A}(B)$ that projects to the unique nontrivial automorphism of $\mathcal{D}_0$, then we shall say that $\mathcal{F}$ is *complexifiable*. If $\mathcal{F}$ is connected, and, moreover, the categories $\mathcal{F}[\mathbb{R}]$, $\mathcal{F}[\mathbb{C}]$ are either empty or connected, then we shall say that $\mathcal{F}$ is *RC-connected*. Suppose that $\mathcal{F}$ is *totally epimorphic*. Then we shall say that an object $A \in \mathrm{Ob}(\mathcal{F})$ is an *RC-anchor* if A determines an *anchor* [cf. [Mzk5], §0] of $\mathcal{F}[\mathbb{C}]$. We shall say that an object $A \in \mathrm{Ob}(\mathcal{F})$ is an *RC-subanchor* if there exists a morphism $A \to B$ in $\mathcal{F}$, where B is an RC-anchor. We shall say that an object $A \in \mathrm{Ob}(\mathcal{F})$ is an *RC-iso-subanchor* if there exist an RC-subanchor $B \in \mathrm{Ob}(\mathcal{F})$, a subgroup $G \subseteq \mathrm{Aut}_{\mathcal{F}}(B)$, and a morphism $B \to A$ in $\mathcal{F}$ which is a *mono-minimal categorical quotient* in $\mathcal{F}$ [cf. [Mzk5], §0] of B by G. If every object of [the totally epimorphic category] $\mathcal{F}$ is an RC-iso-subanchor, then we shall say that $\mathcal{F}$ is *of RC-iso-subanchor type*. We shall say that [the totally epimorphic category] $\mathcal{F}$ is *of RC-standard type* if $\mathcal{F}$ is: (a) RC-connected; (b) complexifiable; (c) of FSMFF-type; (d) of RC-iso-subanchor type.

Before proceeding, we note the following elementary result concerning the *geometry of the circle* $\mathbb{S}^1$:

Lemma 3.2. **(Connected Open Subsets of the Circle)** *Let $A, B \subseteq \mathbb{S}^1$ be [nonempty]* **connected open subsets** *such that $A \subseteq B$. We shall refer to a connected open subset $C \subseteq B$ that contains A as an* **(A, B)-subset**. *If $n \in \mathbb{Z}$, write $\phi_n : \mathbb{S}^1 \to \mathbb{S}^1$ for the map given by $\mathbb{S}^1 \ni z \mapsto z^n$.*

(i) $A = B$ if and only if every (A, B)-subset C is, in fact, equal to A. In particular, $A = \mathbb{S}^1$ if and only if every $(A, \mathbb{S}^1)$-subset C is, in fact, equal to A.

(ii) There exists a $w \in \mathbb{S}^1$ such that the translated open subset $w \cdot A$ satisfies $\phi_{-1}(w \cdot A) = w \cdot A$ [i.e., $w \cdot A$ is **invariant** *with respect to* **complex conjugation**]. *For n a positive integer, there exists a [nonempty] connected open subset $A' \subseteq A$ such that $\phi_{-1}(A') = w \cdot A'$, where $w^n = 1$, if and only if $\phi_n(A) \bigcap \{1, -1\} \neq \emptyset$.*

(iii) The **complement** *$\mathbb{S}^1 \backslash A$ is of* **cardinality** *≤ 1 if and only if there does not exist an $(A, \mathbb{S}^1)$-subset A' such that $A' \neq A, \mathbb{S}^1$.*

(iv) Suppose that for some $0 \neq n \in \mathbb{Z}$, $\phi_n(A) \subseteq A$. Then either $A = \mathbb{S}^1$ or $|n| = 1$. Moreover, [in either of these cases] $\phi_n(A) = A$.

(v) There exists a **finite** *subset $E \subseteq \mathbb{Z}$ such that for any $n \in \mathbb{Z} \backslash E$, and any $(A, \mathbb{S}^1)$-subset A', the restriction of ϕ_n to A' is* **surjective**, *but* **not injective**. *In particular, for $n \in \mathbb{Z} \backslash E$, $\phi_n(A) = \phi_n(B) = \mathbb{S}^1$.*

(vi) Suppose that $A \neq B$. Consider the following conditions on A, B:

(a) For any two (A, B)-subsets A_1, A_2, it holds that either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

(b) For any two (A, B)-subsets A_1, A_2 such that $A_1 \subseteq A_2$, $A_1 \neq A_2$, there exists an (A_1, A_2)-subset A_3 such that $A_3 \neq A_1, A_2$.

(c) The complement $B \backslash A$ is **connected**.

(d) If $B = \mathbb{S}^1$, then $B \backslash A$ is of **cardinality** ≤ 1.

(e) $B \neq \mathbb{S}^1$.

Then condition (a) holds if and only if both conditions (c) and (d) hold; both conditions (a) and (b) hold if and only if both conditions (c) and (e) hold. If the pair (A, B) satisfies conditions (a) and (b), then we shall say that the pair (A, B) is **continuously ordered**.

(vii) Suppose that $A \neq B$, $B \neq \mathbb{S}^1$. Then there exists an (A, B)-subset C such that the pairs (A, C); (C, B) are **continuously ordered**.

(viii) Suppose that $B \backslash A$ is of **cardinality** > 1. *Then there exist (A, B)-subsets A_1, A_2 such that $B = A_1 \bigcup A_2$, $A_1 \neq \mathbb{S}^1$, $A_2 \neq \mathbb{S}^1$.*

(ix) Suppose that $A \neq B$, and that there does not exist a connected open subset $D \subseteq B$ such that $A \bigcap D = \emptyset$. Then $B = \mathbb{S}^1$, and $B \backslash A$ is of **cardinality** ≤ 1.

(x) If $A \neq \mathbb{S}^1$, then A is **not** *homeomorphic to $\mathbb{S}^1$.*

(xi) Every **automorphism** *of the* **topological group** $\mathbb{S}^1$ *is equal to either ϕ_1 or ϕ_{-1}.*

(xii) Write

$$\mathrm{Homeo}(\mathbb{S}^1); \quad \mathrm{Trans}(\mathbb{S}^1) \subseteq \mathrm{Homeo}(\mathbb{S}^1); \quad \mathrm{Refl}(\mathbb{S}^1) \subseteq \mathrm{Homeo}(\mathbb{S}^1)$$

for the group of **self-homeomorphisms** *of $\mathbb{S}^1$, the subgroup [naturally isomorphic to $\mathbb{S}^1$] determined by the* **translations** *by elements of $\mathbb{S}^1$, and the subgroup generated by* $\mathrm{Trans}(\mathbb{S}^1)$, ϕ_{-1}, *respectively. Then* $\mathrm{Refl}(\mathbb{S}^1) \cong \mathbb{S}^1 \rtimes (\mathbb{Z}/2\mathbb{Z})$; $\mathrm{Trans}(\mathbb{S}^1) \subseteq \mathrm{Refl}(\mathbb{S}^1)$ *is equal to the subgroup of* **infinitely divisible** *elements; the* **normalizer** *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *of either* $\mathrm{Trans}(\mathbb{S}^1)$ *or* $\mathrm{Refl}(\mathbb{S}^1)$ *is equal to* $\mathrm{Refl}(\mathbb{S}^1)$; *the* **centralizer** *in* $\mathrm{Homeo}(\mathbb{S}^1)$ *of* $\mathrm{Trans}(\mathbb{S}^1)$ *is equal to* $\mathrm{Trans}(\mathbb{S}^1)$.

Proof. Assertions (i), (ii), (iii), (vi), (vii), (viii) are immediate from the well-known structure of $\mathbb{S}^1$ [and, in the case of assertion (vii), the equivalence, given in assertion (vi), of conditions (a) and (b) with conditions (c) and (e)]. To verify assertion (iv), note that, relative to the standard metric on the tangent bundle of $\mathbb{S}^1$ [i.e., the metric induced by the standard euclidean metric on $\mathbb{C}$ via the natural inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{C}$], $|d\phi_n| = |n| \geq 1$ — i.e., "ϕ_n multiplies *lengths* by $|n|$". Suppose that $A \neq \mathbb{S}^1$, $|n| \geq 2 > 1$. Then $|d\phi_n| > 1$, so it follows that the length of $\phi_n(A)$ is *strictly greater* than the length of A, in contradiction to the inclusion $\phi_n(A) \subseteq A$. Thus, either $A = \mathbb{S}^1$ or $|n| = 1$. The fact that $\phi_n(A) = A$ then follows immediately. This completes the proof of assertion (iv). Assertion (v) follows by taking E to be the [manifestly finite] set of n such that $|n|$ times the length of A fails to exceed [i.e., is $\leq$] the length of $\mathbb{S}^1$. To verify assertion (ix), observe that our assumption on (A, B) implies that there does not exist an (A, B)-subset C such that (A, C), (C, B) are *continuously ordered*; thus, by assertion (vii), it follows that $B = \mathbb{S}^1$, and [again by our assumption on (A, B)] that $B \backslash A$ is of *cardinality* ≤ 1. Assertion (x) follows, for instance, by observing that the fundamental group of $\mathbb{S}^1$ is *nontrivial*, whereas the fundamental group of $A \neq \mathbb{S}^1$ is *trivial*. Assertion (xi) follows, for instance, by thinking of $\mathbb{S}^1$ as a *quotient* $\mathbb{R}/\mathbb{Z}$ and applying the easily verified fact that every automorphism of the [additive] topological group $\mathbb{R}$ is given by multiplication by a nonzero element of $\mathbb{R}$. As for assertion (xii), the isomorphism $\mathrm{Refl}(\mathbb{S}^1) \cong \mathbb{S}^1 \rtimes (\mathbb{Z}/2\mathbb{Z})$, as well as the characterization of the subgroup $\mathrm{Trans}(\mathbb{S}^1) \subseteq \mathrm{Refl}(\mathbb{S}^1)$, are immediate from the definitions. In light of this characterization, it follows that the normalizer of $\mathrm{Refl}(\mathbb{S}^1)$ in $\mathrm{Homeo}(\mathbb{S}^1)$ is *contained* in the normalizer of $\mathrm{Trans}(\mathbb{S}^1)$ in $\mathrm{Homeo}(\mathbb{S}^1)$. Since the topology of $\mathrm{Trans}(\mathbb{S}^1)$ ($\cong \mathbb{S}^1$) may be recovered from the action of $\mathrm{Trans}(\mathbb{S}^1)$ on the *topological space* $\mathbb{S}^1$, it follows immediately that any element of the normalizer of $\mathrm{Trans}(\mathbb{S}^1)$ in $\mathrm{Homeo}(\mathbb{S}^1)$ determines, by conjugation, an automorphism of the *topological group* $\mathrm{Trans}(\mathbb{S}^1) \cong \mathbb{S}^1$, hence, by assertion (xi), lies in the subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ generated by $\mathrm{Refl}(\mathbb{S}^1)$ and the *centralizer* of $\mathrm{Trans}(\mathbb{S}^1)$ in $\mathrm{Homeo}(\mathbb{S}^1)$. Thus, to complete the proof of assertion (xii), it suffices to show that this centralizer is equal to $\mathrm{Trans}(\mathbb{S}^1)$, or, equivalently [since $\mathrm{Trans}(\mathbb{S}^1)$ acts *transitively* on $\mathbb{S}^1$], that every element α of this centralizer that fixes $1 \in \mathbb{S}^1$ is equal to the identity. But this follows by considering, for an arbitrary $z \in \mathbb{S}^1$, the corresponding translation $\tau_z \in \mathrm{Trans}(\mathbb{S}^1)$, which yields the desired relation $z = \tau_z(1) = \tau_z(\alpha(1)) = \alpha(\tau_z(1)) = \alpha(z)$. ○

Example 3.3. Archimedean Frobenioids, Angular Frobenioids, and Angloids.

(i) Now we define a category

$$\mathcal{C}_0$$

as follows: The *objects* of $\mathcal{C}_0$ are triples $(\mathrm{Spec}(K), V_K, \mathbb{A}_K)$, where $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$; V_K is a one-dimensional K-vector space; and $\mathbb{A}_K \subseteq V_K$ is an angular region. We shall say that the object $(\mathrm{Spec}(K), V_K, \mathbb{A}_K)$ is *naively isotropic* if $\mathbb{A}_K$ is isotropic. The *morphisms* of $\mathcal{C}_0$

$$\phi : (\mathrm{Spec}(L), V_L, \mathbb{A}_L) \to (\mathrm{Spec}(K), V_K, \mathbb{A}_K)$$

consist of data as follows:

(a) a morphism $\mathrm{Base}(\phi) : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ of $\mathcal{D}_0$;

(b) an element $\deg_{\mathrm{Fr}}(\phi) \stackrel{\mathrm{def}}{=} d \in \mathbb{N}_{\geq 1}$;

(c) an isomorphism of L-vector spaces $V_L^{\otimes d} \xrightarrow{\sim} V_K|_L$ that maps $\mathbb{A}_L^{\otimes d}$ into $\mathbb{A}_K|_L$.

One verifies immediately that this category $\mathcal{C}_0$ is *connected* and *totally epimorphic*. If, whenever L is complex, the image $\mathrm{Im}(\mathbb{A}_L^{\otimes d}) \subseteq V_K|_L$ is co-angular with $\mathbb{A}_K|_L$, then we shall say that ϕ is *naively co-angular*. [Thus, if L is real, then ϕ is always naively co-angular.] Also, we shall write $\mathrm{Div}(\phi) \stackrel{\mathrm{def}}{=} \log(\lambda) \in \mathbb{R}_{\geq 0}$, for the largest $\lambda \in \mathbb{R}_{>0}$ such that $\lambda \cdot \mathrm{Im}(\mathbb{A}_L^{\otimes d}) \subseteq \mathbb{A}_K|_L$. If we denote by

$$\Phi_0 : \mathcal{D}_0 \to \mathfrak{Mon}$$

the functor determined by the assignment

$$(\mathrm{Ob}(\mathcal{D}_0) \ni) \ \mathrm{Spec}(K) \mapsto \mathrm{ord}(K) \cong \mathbb{R}_{>0} \cong \mathbb{R}_{\geq 0}$$

[where the isomorphism $\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ is given by the natural logarithm], then the triple "$(\mathrm{Base}(-), \mathrm{Div}(-), \deg_{\mathrm{Fr}}(-))$" determines a *pre-Frobenioid structure* [cf. [Mzk5], Definition 1.1, (iv)]

$$\mathcal{C}_0 \to \mathbb{F}_{\Phi_0}$$

on $\mathcal{C}_0$. Similarly, if $\mathcal{D}$ is any *connected*, *totally epimorphic* category, and $\mathcal{D} \to \mathcal{D}_0$ is a *functor*, then by setting

$$\mathcal{C} \stackrel{\mathrm{def}}{=} \mathcal{C}_0 \times_{\mathcal{D}_0} \mathcal{D}; \quad \Phi \stackrel{\mathrm{def}}{=} \Phi_0|_{\mathcal{D}}$$

we obtain a *pre-Frobenioid structure*

$$\mathcal{C} \to \mathbb{F}_{\Phi}$$

on $\mathcal{C}$.

(ii) By Lemma 3.2, (i), it follows immediately that an object of $\mathcal{C}$ is *isotropic* [in the sense of [Mzk5], Definition 1.2, (iv)] if and only if it is *naively isotropic*, and that a morphism of $\mathcal{C}$ is *co-angular* [in the sense of [Mzk5], Definition 1.2, (iii)] if and only if it is *naively co-angular*. Also, we note that, by Lemma 3.2, (iv), every *endomorphism* of a *non-isotropic* object of $\mathcal{C}$ is *linear* and *co-angular*; every *endomorphism* of an *isotropic* object of $\mathcal{C}$ is *co-angular*. Moreover, the following conditions on an object of $\mathcal{C}$ are equivalent: (a) the object is *isotropic*; (b) the object is *Frobenius-trivial*; (c) the object is *Frobenius-ample*. Also, we observe that every object of $\mathcal{C}$ is *metrically trivial*. Now a routine verification reveals that $\mathcal{C}$ satisfies the conditions of [Mzk5], Definition 1.3, hence that $\mathcal{C}$ is a *Frobenioid*. Moreover, by Lemma 3.2, (v), it follows that $\mathcal{C}$ is of *Frobenius-isotropic* type, so it makes sense to speak of the *perfection* $\mathcal{C}^{\mathrm{pf}}$ of $\mathcal{C}$ [cf. [Mzk5], Definition 3.1, (iii)]. If Λ is a *monoid type*, then we define $\mathcal{C}^{\Lambda}$ as follows:

$$\mathcal{C}^{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \mathcal{C}; \quad \mathcal{C}^{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\mathrm{pf}}; \quad \mathcal{C}^{\mathbb{R}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\mathrm{rlf}}$$

[cf. [Mzk5], Proposition 5.3]. We shall refer to a Frobenioid $\mathcal{C}^{\Lambda}$ as an *archimedean Frobenioid*.

(iii) Write

$$\mathcal{A}_0 \subseteq \mathcal{C}_0; \quad \mathcal{A} \subseteq \mathcal{C}$$

for the respective subcategories determined by the *isometries*. Thus, we have a natural equivalence of categories $\mathcal{A} \xrightarrow{\sim} \mathcal{A}_0 \times_{\mathcal{D}_0} \mathcal{D}$. Moreover, a routine verification reveals that $\mathcal{A}$ satisfies the conditions of [Mzk5], Definition 1.3, hence that $\mathcal{A}$ is a *Frobenioid* over the base category $\mathcal{D}$, which is, in fact, of *group-like* type. Also, we observe that the isotropic objects of $\mathcal{A}$ are precisely the isotropic objects of $\mathcal{C}$, and that the co-angular morphisms of $\mathcal{A}$ are precisely the co-angular isometries of $\mathcal{C}$. We shall refer to $\mathcal{A}$ as an *angular Frobenioid.* Write

$$\mathcal{N} \stackrel{\text{def}}{=} \mathcal{A}^{\text{lin}} \subseteq \mathcal{A}; \quad \mathcal{N}_0 \stackrel{\text{def}}{=} \mathcal{A}_0^{\text{lin}} \subseteq \mathcal{A}_0$$

[i.e., the respective subcategories determined by the *linear* morphisms — cf. [Mzk5], Definition 1.2, (iv)]. We shall refer to a category $\mathcal{N}$ as a *non-rigidified angloid* [over the base category $\mathcal{D}$]. Note that there is a natural equivalence of categories $\mathcal{N} \xrightarrow{\sim} \mathcal{N}_0 \times_{\mathcal{D}_0} \mathcal{D}$.

(iv) Observe that all *real* objects of $\mathcal{N}_0$ are *isomorphic.* Thus, it makes sense to define

$$\mathcal{R}_0 \stackrel{\text{def}}{=} (\mathcal{N}_0)_A; \quad \mathcal{R} \stackrel{\text{def}}{=} \mathcal{R}_0 \times_{\mathcal{D}_0} \mathcal{D}$$

for $A \in \text{Ob}(\mathcal{N}_0[\mathbb{R}])$. We shall refer to a category $\mathcal{R}$ as a *rigidified angloid* [over the base category $\mathcal{D}$]. If $\mathcal{D}$ is RC-connected, then one verifies immediately that $\mathcal{C}^{\Lambda}$, $\mathcal{A}$, $\mathcal{N}$, $\mathcal{R}$ are *RC-connected* [hence, in particular, connected] and *totally epimorphic.* Although $\mathcal{N}$, $\mathcal{R}$ are not equipped with [pre-]Frobenioid structures, we shall often apply *"Frobenioid-theoretic terminology"* to objects and morphisms of $\mathcal{N}$, $\mathcal{R}$; such terminology is to be interpreted as referring to the images of the objects and morphisms in question in $\mathcal{C}$ via the natural functors $\mathcal{N} \to \mathcal{C}$, $\mathcal{R} \to \mathcal{C}$; also, we shall apply the notation "$\mathcal{O}^{\triangleright}(-)$", "$\mathcal{O}^{\times}(-)$" to objects of $\mathcal{N}$, $\mathcal{R}$ to denote the monoids of *base-identity* endomorphisms [i.e., as objects of $\mathcal{N}$, $\mathcal{R}$], automorphisms [i.e., as objects of $\mathcal{N}$, $\mathcal{R}$], respectively.

(v) Note that if $\mathcal{F}$ is one of the categories $\mathcal{A}$, $\mathcal{N}$, $\mathcal{R}$, then any morphism of $\mathcal{F}$ that is obtained as the *isotropic hull* of an object of $\mathcal{F}$ whose *angular region* is determined by the *complement* in $\mathbb{S}^1$ of a *single element* of $\mathbb{S}^1$ is an *FSMI-morphism* of $\mathcal{F}$ [cf. Lemma 3.2, (iii)]. We shall refer to such a morphism as a *slit morphism* of $\mathcal{F}$. Note that the existence of slit morphisms implies that $\mathcal{F}$ is *not of FSM-type.*

Remark 3.3.1. Note that it follows immediately from the simple, explicit structure of the categories $\mathcal{C}_0$, $\mathcal{A}_0$, $\mathcal{N}_0$, $\mathcal{R}_0$ introduced in Example 3.3 that in any of the categories $\mathcal{C}$, $\mathcal{A}$, $\mathcal{N}$, or $\mathcal{R}$, a *base-isomorphism* between *complex* objects is a *monomorphism* if and only if it induces an *injection* on underlying angular regions.

Proposition 3.4. **(FSM-Morphisms)** *In the notation and terminology of Example 3.3, let $\mathcal{F}$ be one of the following categories: $\mathcal{A}$, $\mathcal{N}$, $\mathcal{R}$. If $\mathcal{F} = \mathcal{A}$ (respectively, $\mathcal{F} = \mathcal{N}$; $\mathcal{F} = \mathcal{R}$), then set $\mathcal{F}_0 = \mathcal{A}_0$ (respectively, $\mathcal{F}_0 = \mathcal{N}_0$; $\mathcal{F}_0 = \mathcal{R}_0$). Then:*

(i) **Fiberwise-surjective morphisms** *of $\mathcal{F}$ project to fiberwise-surjective morphisms of $\mathcal{D}$.*

(ii) Let $\phi : A \to B$ be a **monomorphism** *of $\mathcal{F}$ such that [at least] one of the following two conditions is satisfied [cf. Remark 3.4.1 below]: (a) ϕ projects to an isomorphism of $\mathcal{D}_0$; (b) ϕ admits a factorization $A \to A' \to B$ as a composite of a morphism of Frobenius type $A \to A'$ and a linear morphism $A' \to B$ such that any isotropic hull $A' \to A''$ of A' is either an* **isomorphism** *or a* **slit morphism** *[cf. Example 3.3, (v)]. Then ϕ projects to a monomorphism $\phi_{\mathcal{D}}$ of $\mathcal{D}$.*

(iii) **FSM-morphisms** *of $\mathcal{F}$ project to FSM-morphisms of $\mathcal{D}$.*

(iv) Suppose that $\mathcal{D}$ is **complexifiable***. Then any* **FSM-morphism** *(respectively,* **FSMI-morphism***) of $\mathcal{F}$ that projects to an* **isomorphism** *of $\mathcal{D}$ projects to an FSM-morphism (respectively, FSMI-morphism) of $\mathcal{F}_0$.*

(v) Any morphism of $\mathcal{F}$ that projects to an irreducible morphism (respectively, isomorphism; irreducible morphism) of $\mathcal{D}$, to an isomorphism (respectively, irreducible morphism; irreducible morphism) of $\mathcal{F}_0$, and to a(n) isomorphism (respectively, isomorphism; non-isomorphism) of $\mathcal{D}_0$ is an **irreducible morphism** *of $\mathcal{F}$.*

(vi) If ϕ is a morphism of $\mathcal{F}$ such that $\phi_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\phi)$ admits a **factorization** *$\phi_{\mathcal{D}} = \alpha_{\mathcal{D}} \circ \beta_{\mathcal{D}}$ in $\mathcal{D}$, then there exist morphisms α, β of $\mathcal{F}$ lifting $\alpha_{\mathcal{D}}$, $\beta_{\mathcal{D}}$, respectively, such that $\phi = \alpha \circ \beta$ in $\mathcal{F}$. In particular,* **irreducible** *morphisms of $\mathcal{F}$ project to either isomorphisms or irreducible morphisms of $\mathcal{D}$;* **FSMI-morphisms** *of $\mathcal{F}$ project to either isomorphisms or FSMI-morphisms of $\mathcal{D}$.*

(vii) Let ϕ be a morphism of $\mathcal{F}$ that projects to a **pull-back morphism** *of $\mathcal{F}_0$ and to an* **FSM-morphism** *(respectively,* **FSMI-morphism***) of $\mathcal{D}$. Then ϕ is an* **FSM-morphism** *(respectively,* **FSMI-morphism***) of $\mathcal{F}$.*

(viii) Suppose that $\mathcal{D}$ is complexifiable, RC-connected, and of FSMFF-type. Then $\mathcal{F}$ is **complexifiable**, **RC-connected** *[hence, in particular, connected],* **totally epimorphic***, and* **of FSMFF-type**.

Proof. Assertion (i) follows immediately from the observation that if $A \in \mathrm{Ob}(\mathcal{F})$ projects to an object $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, then any morphism $B_{\mathcal{D}} \to A_{\mathcal{D}}$ in $\mathcal{D}$ lifts to a morphism $B \to A$ of $\mathcal{F}$, for some $B \in \mathrm{Ob}(\mathcal{F})$. To verify assertion (ii), let $\alpha_{\mathcal{D}}, \beta_{\mathcal{D}} : C_{\mathcal{D}} \to A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$ be morphisms of $\mathcal{F}$ such that $\phi_{\mathcal{D}} \circ \alpha_{\mathcal{D}} = \phi_{\mathcal{D}} \circ \beta_{\mathcal{D}}$, where $\phi_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\phi)$. Then to complete the proof of assertion (ii), it suffices to show that $\alpha_{\mathcal{D}} = \beta_{\mathcal{D}}$. Write $\phi_0 : A_0 \to B_0$ for the projection of ϕ to $\mathcal{F}_0$ and $\alpha_{\mathcal{D}_0}, \beta_{\mathcal{D}_0} : C_{\mathcal{D}_0} \to A_{\mathcal{D}_0}$, $\phi_{\mathcal{D}_0} : A_{\mathcal{D}_0} \to B_{\mathcal{D}_0}$ for the respective projections of α, β, ϕ to $\mathcal{D}_0$ [so $\phi_{\mathcal{D}_0} \circ \alpha_{\mathcal{D}_0} = \phi_{\mathcal{D}_0} \circ \beta_{\mathcal{D}_0}$]. Now I *claim* that there exist *linear isometries* $\alpha_0, \beta_0 : C_0 \to A_0$ that lift $\alpha_{\mathcal{D}_0}$, $\beta_{\mathcal{D}_0}$, respectively, and, moreover, satisfy the relation $\phi_0 \circ \alpha_0 = \phi_0 \circ \beta_0$. Indeed, this follows immediately if $\alpha_{\mathcal{D}_0} = \beta_{\mathcal{D}_0}$ [which is the case whenever condition (a) is satisfied, i.e., whenever $\phi_{\mathcal{D}_0}$ is an isomorphism], by taking $\alpha_0 = \beta_0$ to be a pull-back morphism. On the other hand, if $\alpha_{\mathcal{D}_0} \neq \beta_{\mathcal{D}_0}$, but condition (b) is satisfied, then it follows from [the latter portion of] Lemma 3.2, (ii), that there exist α_0, β_0 as desired, where α_0 is related to β_0 by an automorphism of C_0 that lies over the *complex conjugation* automorphism of $C_{\mathcal{D}_0}$. This completes the proof of the *claim*. Thus, by applying the natural equivalence of categories $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_0 \times_{\mathcal{D}_0} \mathcal{D}$, we conclude that there exist *linear isometries* $\alpha, \beta : C \to A$ that simultaneously lift α_0, β_0 and $\alpha_{\mathcal{D}}$, $\beta_{\mathcal{D}}$, respectively, and, moreover, satisfy the relation $\phi \circ \alpha = \phi \circ \beta$. Since ϕ is a *monomorphism* of $\mathcal{F}$, we thus conclude that $\alpha = \beta$, hence that $\alpha_{\mathcal{D}} = \beta_{\mathcal{D}}$, as desired. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii), together with the observation that if ϕ is a *fiberwise-surjective* morphism of $\mathcal{F}$ that does *not* satisfy condition (a) of assertion (ii), then the fiberwise-surjectivity of ϕ implies [cf. Lemma 3.2, (ix)] that ϕ necessarily satisfies condition (b) of assertion (ii).

Next, we consider assertion (iv). Let $\phi : A \to B$ be an *FSM-morphism* of $\mathcal{F}$ that projects to an *isomorphism* of $\mathcal{D}$. Thus, the projection $\phi_0 : A_0 \to B_0$ of ϕ to $\mathcal{F}_0$ projects to an *isomorphism* $\phi_{\mathcal{D}_0}$ of $\mathcal{D}_0$. Now since $\mathcal{D}$ is *complexifiable*, it follows immediately that one can lift arbitrary pairs of morphisms $\alpha_0, \beta_0 : C_0 \to A_0$ of $\mathcal{F}_0$ such that $\phi_0 \circ \alpha_0 = \phi_0 \circ \beta_0$ [which, since $\phi_{\mathcal{D}_0}$ is an *isomorphism*, implies that the respective projections $\alpha_{\mathcal{D}_0}, \beta_{\mathcal{D}_0}$ of α_0, β_0 to $\mathcal{D}_0$ *coincide*] to morphisms $\alpha, \beta : C \to A$ of $\mathcal{F}$ such that $\phi \circ \alpha = \phi \circ \beta$, hence that $\alpha = \beta$ [since ϕ is a *monomorphism*], so $\alpha_0 = \beta_0$, i.e., ϕ_0 is a *monomorphism*. Similarly, [since $\mathcal{D}$ is *complexifiable*] any morphism $\gamma_0 : C_0 \to B_0$ of $\mathcal{F}_0$ lifts to a morphism $\gamma : C \to B$ of $\mathcal{F}$; thus, the *fiberwise-surjectivity* of ϕ implies that of ϕ_0. If ϕ is *irreducible*, then it follows immediately from the fact that ϕ projects to an isomorphism of $\mathcal{D}$ that ϕ_0 is also irreducible. This completes the proof of assertion (iv).

Assertion (v) follows immediately from the fact that since $\mathcal{F}_0$, $\mathcal{D}$ are *total epimorphic*, if a composite morphism $\alpha \circ \beta$ of $\mathcal{F}_0$ or $\mathcal{D}$ is an isomorphism, then so are α, β. Assertion (vi) follows immediately from the fact that if $\alpha_{\mathcal{D}_0} \circ \beta_{\mathcal{D}_0}$ is any composite morphism of $\mathcal{D}_0$, then either $\alpha_{\mathcal{D}_0}$ or $\beta_{\mathcal{D}_0}$ is an isomorphism [cf. also assertion (iii)].

Next, we consider assertion (vii). Since every non-isomorphism of $\mathcal{D}_0$ [hence also every pull-back morphism of $\mathcal{F}_0$ which is not an isomorphism — cf. [Mzk5], Proposition 1.7, (v), for pull-back morphisms] is *irreducible*, it follows from assertion (v) that it suffices to show that ϕ is an *FSM-morphism*. Since $\phi : A \to B$ projects to a *pull-back morphism* $\phi_0 : A_0 \to B_0$ of $\mathcal{F}_0$, it follows that given any morphism $\psi_0 : C_0 \to B_0$, it holds that any factorization $\psi_{\mathcal{D}_0} = \phi_{\mathcal{D}_0} \circ \zeta_{\mathcal{D}_0}$ [for some morphism $\zeta_{\mathcal{D}_0} : C_{\mathcal{D}_0} \to A_{\mathcal{D}_0}$ of $\mathcal{D}_0$], where $\phi_{\mathcal{D}_0} : A_{\mathcal{D}_0} \to B_{\mathcal{D}_0}$, $\psi_{\mathcal{D}_0} : C_{\mathcal{D}_0} \to B_{\mathcal{D}_0}$ are the respective projections of ϕ_0, ψ_0 to $\mathcal{D}_0$, *lifts* to a factorization $\psi_0 = \phi_0 \circ \zeta_0$ for some $\zeta_0 : C_0 \to A_0$ [cf. the definition of a "pull-back morphism" in [Mzk5], Definition 1.2, (ii)]. Thus, [cf. also the observation made in the proof of assertion (i) that if $E \in \mathrm{Ob}(\mathcal{F})$ projects to an object $E_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$, then any morphism $F_{\mathcal{D}} \to E_{\mathcal{D}}$ in $\mathcal{D}$ lifts to a morphism $F \to E$ of $\mathcal{F}$] the *fiberwise-surjectivity* of ϕ follows immediately from that of the projection $\phi_{\mathcal{D}}$ of ϕ to $\mathcal{D}$. To show that ϕ is a *monomorphism*, let $\alpha, \beta : C \to A$ be morphisms of $\mathcal{F}$ such that $\phi \circ \alpha = \phi \circ \beta$. Since $\phi_{\mathcal{D}}$ is a *monomorphism*, it follows that the respective projections $\alpha_{\mathcal{D}}$, $\beta_{\mathcal{D}}$ of α, β to $\mathcal{D}$ *coincide*, hence that the respective projections $\alpha_{\mathcal{D}_0}$, $\beta_{\mathcal{D}_0}$ of α, β to $\mathcal{D}_0$ *coincide*. Thus, again by the definition of a "pull-back morphism" [cf. [Mzk5], Definition 1.2, (ii)], we conclude that the respective projections α_0, β_0 of α, β to $\mathcal{F}_0$ *coincide*, hence that $\alpha = \beta$, as desired. This completes the proof of assertion (vii).

Finally, we consider assertion (viii). It is immediate from the definition of $\mathcal{F}$ that $\mathcal{F}$ is *complexifiable*, *RC-connected* [hence, in particular, connected] and *totally epimorphic.*

Next, I *claim* that $\mathcal{F}_0$ is *of FSMFF-type.* Indeed, let $\phi : A \to B$ be an *FSM-morphism* of $\mathcal{F}_0$. Then by assertion (iii), it follows that ϕ projects to an *isomorphism* of $\mathcal{D}_0$. Suppose that A, B are *real.* Now if $\mathcal{F}_0 = \mathcal{N}_0$ or $\mathcal{F}_0 = \mathcal{R}_0$, then one concludes immediately that ϕ is an LB-invertible pre-step, hence an *isomorphism* [cf. [Mzk5], Proposition 1.4, (iii)]. If $\mathcal{F}_0 = \mathcal{A}_0$, then there exists a complex isotropic object C of $\mathcal{F}_0$ and a linear morphism $C \to A$; thus, the existence of *torsion elements* in $\mathcal{O}^\times(C)$ of *arbitrary order* when $\mathcal{F}_0 = \mathcal{A}_0$ implies [since ϕ is a *monomorphism*] that the LB-invertible base-isomorphism [cf. the definition of $\mathcal{F}_0$!] ϕ is *linear*, hence an *isomorphism.* In particular, it follows that *every FSMI-morphism of* $\mathcal{F}_0$ *has complex domain and codomain.*

Now suppose that A, B are *complex.* Then observe that if the isometric base-isomorphism ϕ is *linear*, then it is an isometric pre-step, hence [cf. the *fiberwise-surjectivity* of ϕ; Lemma 3.2, (ix); Example 3.3, (v)] either an *isomorphism* or a *slit morphism.* Moreover, even if ϕ is *not* linear, by Lemma 3.2, (iv) [which may be interpreted as asserting that *"angular regions never shrink"*], (v), there exists an integer N such that given *any composite*

$$\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_2 \circ \phi_1$$

of [not necessarily FSMI-!] morphisms $\phi_1, \ldots, \phi_n$ such that the domain of ϕ_1 is equal to A, it holds that the *cardinality* of the set of j [where $j = 1, \ldots, n$] such that ϕ_j is an *FSMI-morphism* [which implies that ϕ_j is a *monomorphism* which is a *base-isomorphism* between *complex* objects — cf. Remark 3.3.1] and *non-linear* is $\leq N$. In this situation, any *linear FSMI-morphism* ϕ_j [where $j = 1, \ldots, n$] is necessarily a *slit morphism* [as observed above] — which implies that the codomain of ϕ_j is *isotropic*, so no $\phi_{j'}$, where $j' > j$, can be a slit morphism; we thus conclude that the *cardinality* of the set of $j \in \{1, \ldots, n\}$ such that ϕ_j is an *FSMI-morphism* is $\leq N + 1$.

Next, observe that [by [Mzk5], Proposition 1.7, (ii), applied to the Frobenioid $\mathcal{A}_0$] ϕ admits a *factorization* $\phi = \alpha \circ \beta$, where β is a *morphism of Frobenius type*, and α is an *isometric pre-step* [hence, in particular, a *monomorphism* — cf., e.g., [Mzk5], Definition 1.3, (v), (a)]. Since ϕ is *fiberwise-surjective*, it follows formally that α is an *FSM-morphism*, hence that α is either a *slit morphism* [hence, in particular, an *FSMI-morphism* — cf. Example 3.3, (v)] or an *isomorphism.* On the other hand, β may be written as a *composite* $\beta = \beta_b \circ \ldots \circ \beta_1$ of *prime-Frobenius morphisms* $\beta_1, \ldots, \beta_b$. Since each of the β_j [where $j = 1, \ldots, b$] is *co-angular*, and ϕ [hence also β] is a *monomorphism*, it follows immediately by induction on j that each of the β_j [where $j = 1, \ldots, b$] is a *monomorphism* [i.e., more concretely: induces an *injection* on underlying angular regions — cf. Remark 3.3.1], hence [by a *formal* argument, since ϕ is *fiberwise-surjective*, and α is a *monomorphism*] that each of the β_j is an *FSM-morphism.* Moreover, since β_j is a *prime-Frobenius morphism*, it follows immediately from the fact that β_j induces a *bijection* on underlying angular regions that β_j is *irreducible*, hence an *FSMI-morphism.* Thus, we conclude that ϕ

factors as a *composite of finitely many FSMI-morphisms*. This completes the proof of the *claim* that $\mathcal{F}_0$ is *of FSMFF-type*.

To prove that $\mathcal{F}$ is *of FSMFF-type*, we reason as follows: First, since FSMI-morphisms of $\mathcal{F}$ project to either isomorphisms or FSMI-morphisms of $\mathcal{D}$ [cf. assertion (vi)], and $\mathcal{D}$ is *of FSMFF-type*, it follows that for every $A \in \mathrm{Ob}(\mathcal{F})$, there exists an integer N such that given *any composite*

$$\phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_2 \circ \phi_1$$

of FSMI-morphisms $\phi_1, \ldots, \phi_n$ such that the domain of ϕ_1 is equal to A, it holds that the *cardinality* of the set of j [where $j = 1, \ldots, n$] such that ϕ_j *fails* to project to an *isomorphism* of $\mathcal{D}$ is $\leq N$. If ϕ_j projects to an isomorphism of $\mathcal{D}$, then by assertion (iv), it follows that ϕ_j projects to an *FSMI-morphism* of $\mathcal{F}_0$. Thus, by our discussion of *composites of [not necessarily FSMI-!] morphisms* in $\mathcal{F}_0$, it follows that there exists an N' [depending only on A!] such that the *cardinality* of the set of $j \in \{1, \ldots, n\}$ for which ϕ_j projects to an isomorphism of $\mathcal{D}$ is $\leq N'$. But this implies that $n \leq N + N'$.

Next, let ϕ be an *arbitrary FSM-morphism* of $\mathcal{F}$. Observe that [by [Mzk5], Definition 1.3, (iv), (a), applied to the Frobenioid $\mathcal{A}_0$] the projection ϕ_0 of ϕ to $\mathcal{F}_0$ admits a *factorization* $\phi_0 = \alpha_0 \circ \beta_0$, where α_0 is a *pull-back morphism*, and β_0 is a *base-isomorphism*. Since the projection of β_0 to $\mathcal{D}_0$ is an isomorphism, it thus follows that there exists a factorization $\phi = \alpha \circ \beta$ in $\mathcal{F}$, where α, β *lift* α_0, β_0 respectively, and $\beta_{\mathcal{D}} \stackrel{\text{def}}{=} \mathrm{Base}(\beta) \in \mathrm{Arr}(\mathcal{D})$ is an *isomorphism*. Since $\phi_{\mathcal{D}} \stackrel{\text{def}}{=} \mathrm{Base}(\phi) \in \mathrm{Arr}(\mathcal{D})$ is an *FSM-morphism* [cf. assertion (iii)], it thus follows that $\alpha_{\mathcal{D}} \stackrel{\text{def}}{=} \mathrm{Base}(\alpha) = \phi_{\mathcal{D}} \circ \beta_{\mathcal{D}}^{-1} \in \mathrm{Arr}(\mathcal{D})$ is an *FSM-morphism*, hence [by assertion (vii)] that α is an *FSM-morphism*. Moreover, since α is a *monomorphism*, it follows *formally* from the fact that ϕ is an FSM-morphism that β is an *FSM-morphism*, hence [by assertion (iv), since $\beta_{\mathcal{D}}$ is an *isomorphism*] that β_0 is an *FSM-morphism* of $\mathcal{F}_0$. Thus, by factoring β_0 as a composite of FSMI-morphisms of $\mathcal{F}_0$ [since we have already shown that $\mathcal{F}_0$ is *of FSMFF-type*], we conclude that we may write $\beta = \beta_b \circ \ldots \circ \beta_1$, where the β_j [for $j = 1, \ldots, b$] are morphisms of $\mathcal{F}$ that project to FSMI-morphisms of $\mathcal{F}_0$ and to isomorphisms of $\mathcal{D}$. Now it follows *formally* that the β_j are *monomorphisms*, hence [again by a *formal* argument, since β is an *FSM-morphism*] that the β_j are *FSM-morphisms*, hence [cf. assertion (v)] *FSMI-morphisms*. Moreover, by factoring $\alpha_{\mathcal{D}}$ as a composite of FSMI-morphisms of $\mathcal{D}$ [since $\mathcal{D}$ is, by assumption, *of FSMFF-type*] and applying assertion (vii), we conclude that α, β, hence also ϕ admits a factorization as a composite of FSMI-morphisms of $\mathcal{F}$. This completes the proof of assertion (viii). ◯

Remark 3.4.1. Note that there exist monomorphisms of $\mathcal{G}$ that do *not* satisfy either of the conditions of Proposition 3.4, (ii), and which *fail* to project to monomorphisms of $\mathcal{D}$. Indeed, by using *angular regions* whose projections to $\mathbb{S}^1$ *fail to intersect* $\{1, -1\}$ [cf. Lemma 3.2, (ii)], one may construct examples of *linear monomorphisms* $\phi : A \to B$ of $\mathcal{G}_0$, where A is *complex*, and B is *real* [so the projection of ϕ to $\mathcal{D}_0$ is *not* a monomorphism].

Proposition 3.5. **(Iso-subanchors)** *In the notation and terminology of Example 3.3, let $\mathcal{F}$ be one of the categories $\mathcal{C}$, $\mathcal{A}$; $\mathcal{G}$ one of the categories $\mathcal{N}$, $\mathcal{R}$; $\mathcal{H}$ one of the categories $\mathcal{F}$, $\mathcal{G}$. If $\mathcal{F} = \mathcal{C}$ (respectively, $\mathcal{F} = \mathcal{A}$; $\mathcal{G} = \mathcal{N}$; $\mathcal{G} = \mathcal{R}$; $\mathcal{H} = \mathcal{F}$; $\mathcal{H} = \mathcal{G}$), then set $\mathcal{F}_0 = \mathcal{C}_0$ (respectively, $\mathcal{F}_0 = \mathcal{A}_0$; $\mathcal{G}_0 = \mathcal{N}_0$; $\mathcal{G}_0 = \mathcal{R}_0$; $\mathcal{H}_0 = \mathcal{F}_0$; $\mathcal{H}_0 = \mathcal{G}_0$). Suppose further that $\mathcal{D}$ is of* **RC-iso-subanchor** *type. Then:*

(i) Let $A \in \mathrm{Ob}(\mathcal{H})$; suppose that $B_{\mathcal{D}} \to A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A)$ is a mono-minimal categorical quotient of $B_{\mathcal{D}}$ by a group $G_{\mathcal{D}} \subseteq \mathrm{Aut}_{\mathcal{D}}(B_{\mathcal{D}})$ in $\mathcal{D}$. Then there exists a **pull-back morphism** *$B \to A$ that lifts $B_{\mathcal{D}} \to A_{\mathcal{D}}$ and a group $G \subseteq \mathrm{Aut}_{\mathcal{H}}(B)$ that maps isomorphically to $G_{\mathcal{D}}$ such that $B \to A$ is a* **mono-minimal categorical quotient** *of B by G in $\mathcal{H}$.*

(ii) The Frobenioid $\mathcal{F}$ is **quasi-isotropic***, i.e., an object of $\mathcal{F}$ is* **non-isotropic** *if and only if it is an* **iso-subanchor** *of $\mathcal{F}$.*

(iii) $\mathcal{G}$ is of **RC-iso-subanchor** *type.*

(iv) $\mathcal{F}$ is **not** *of* **RC-iso-subanchor** *type.*

Proof. First, we consider assertion (i). By [Mzk5], Definition 1.3, (i), (c), there exists a *pull-back morphism* $B \to A$ of $\mathcal{H}$ that lifts $B_{\mathcal{D}} \to A_{\mathcal{D}}$. Write $B_0 \to A_0$, $B_{\mathcal{D}_0} \to A_{\mathcal{D}_0}$ for the respective projections of $B \to A$ to $\mathcal{H}_0$, $\mathcal{D}_0$. Now observe that it follows immediately from the simple, explicit structure of $\mathcal{H}_0$ [cf. Example 3.3, (i), (ii), (iii), (iv)] that the natural *surjection*

$$\mathrm{Aut}_{(\mathcal{H}_0)_{A_0}}(B_0) \to \mathrm{Aut}_{(\mathcal{D}_0)_{A_{\mathcal{D}_0}}}(B_{\mathcal{D}_0})$$

splits, hence that there exists a group $G \subseteq \mathrm{Aut}_{\mathcal{H}_A}(B)$ that maps isomorphically to $G_{\mathcal{D}}$. Since $B_{\mathcal{D}} \to A_{\mathcal{D}}$ is a *categorical quotient* of $B_{\mathcal{D}}$ by $G_{\mathcal{D}}$ in $\mathcal{D}$, it follows again from the simple, explicit structure of $\mathcal{H}_0$ [cf. Example 3.3, (i), (ii), (iii), (iv)] that $B \to A$ is a *categorical quotient* of B by G in $\mathcal{H}$. If $B \rightarrowtail B'$ is a *monomorphism* that violates the mono-minimality of $B \to A$, then it follows from the fact that $B \to A$ is a *pull-back morphism*, hence, in particular, an *isometry*, that $B \rightarrowtail B'$ is an *isometry* [which implies that even if $\mathcal{H} = \mathcal{C}$, the arrow $B \rightarrowtail B'$ lies in $\mathcal{A}$], hence, by Proposition 3.4, (ii), that the projection $B_{\mathcal{D}} \to B'_{\mathcal{D}}$ of $B \rightarrowtail B'$ to $\mathcal{D}$ is a *monomorphism* that violates the mono-minimality of $B_{\mathcal{D}} \to A_{\mathcal{D}}$. [Here, we observe that if $B \rightarrowtail B'$ fails to project to an isomorphism of $\mathcal{D}_0$ [i.e., fails to satisfy condition (a) of Proposition 3.4, (ii)], then the pull-back morphism $B \to A$ also fails to project to an isomorphism of $\mathcal{D}_0$; but this implies that B is *isotropic*, so the morphism $B \rightarrowtail B'$ necessarily satisfies condition (b) of Proposition 3.4, (ii).] Thus, we conclude that $B_{\mathcal{D}} \rightarrowtail B'_{\mathcal{D}}$ is an *isomorphism*. In particular, the *pull-back morphism* $B \to A$ admits a factorization $B \rightarrowtail B' \to A$, where $B \rightarrowtail B'$ is a *base-isomorphism* which [cf. [Mzk5], Proposition 1.7, (v)] is also a *pull-back morphism*. But this implies [cf. [Mzk5], Remark 1.2.1] that $B \rightarrowtail B'$ is an *isomorphism*, hence that $B \to A$ is *mono-minimal*, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). By [Mzk5], Remark 3.1.1, it suffices to show that every *non-isotropic* [hence necessarily complex!] $A \in \mathrm{Ob}(\mathcal{F})$ is an *iso-subanchor* of $\mathcal{F}$. Since $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D}[\mathbb{C}])$ is an *RC-iso-subanchor* of $\mathcal{D}$, it follows that there exists a *mono-minimal categorical quotient* [in $\mathcal{D}$] $B_{\mathcal{D}} \to A_{\mathcal{D}}$ of $B_{\mathcal{D}}$ by a group $G_{\mathcal{D}} \subseteq \mathrm{Aut}_{\mathcal{D}}(B_{\mathcal{D}})$, where $B_{\mathcal{D}}$ is an RC-subanchor of $\mathcal{D}$. Thus, by assertion (i), we obtain a *pull-back morphism* $B \to A$ of $\mathcal{F}$ that lifts $B_{\mathcal{D}} \to A_{\mathcal{D}}$ and which is a *mono-minimal categorical quotient* of B by some group $G \subseteq \mathrm{Aut}_{\mathcal{F}}(B)$. Here, we note that since A is *non-isotropic*, it follows [cf. [Mzk5], Definition 1.3, (vii), (b)] that B is *non-isotropic*. Since $B_{\mathcal{D}}$ is an RC-subanchor of $\mathcal{D}$, it follows that there exists a morphism $B_{\mathcal{D}} \to C_{\mathcal{D}}$ in $\mathcal{D}[\mathbb{C}]$, where $C_{\mathcal{D}}$ is an *anchor* of $\mathcal{D}[\mathbb{C}]$; note, moreover, that since every morphism of $\mathcal{D}_0[\mathbb{C}]$ is an *isomorphism*, we may assume that this morphism $B_{\mathcal{D}} \to C_{\mathcal{D}}$ is the projection to $\mathcal{D}$ of a *pull-back morphism* $B \to C$ of $\mathcal{F}$ [so C is *non-isotropic*]. Thus, to show that A is an *iso-subanchor* of $\mathcal{F}$, it suffices to show that C is an *anchor* of $\mathcal{F}$.

Thus, let $\phi : C \to C'$ be an *irreducible* morphism of $\mathcal{F}$. Since ϕ is irreducible, it follows [cf. the factorizations of [Mzk5], Definition 1.3, (iv), (v)] that ϕ is either a *pull-back morphism*, an *isometric pre-step*, a *co-angular pre-step* with *irreducible zero divisor*, or a *prime-Frobenius morphism*. Since $\mathbb{R}_{\geq 0}$ has *no irreducible elements* [cf. [Mzk5], §0], it follows that ϕ is *not* a co-angular pre-step. If ϕ is a pull-back morphism or a prime-Frobenius morphism [hence a *non-pre-step* — cf. [Mzk5], Remark 1.2.1!], then since C is *non-isotropic*, it follows [by considering the factorization of ϕ through an *isotropic hull* of C, together with the *irreducibility of* ϕ] that C' is *non-isotropic*, hence, in particular, *complex*. Thus, the finiteness of the collection of isomorphism classes of ${}_{C}\mathcal{F}$ arising from [irreducible] ϕ which are pull-back morphisms (respectively, prime-Frobenius morphisms) then follows immediately from the fact that $C_{\mathcal{D}}$ is an *anchor* of $\mathcal{D}[\mathbb{C}]$ (respectively, from Lemma 3.2, (v)). Finally, [again since C is *non-isotropic*!] the finiteness of the collection of isomorphism classes of ${}_{C}\mathcal{F}$ arising from [irreducible] ϕ which are isometric pre-steps follows formally from Lemma 3.2, (iii), (vii). This completes the proof of assertion (ii).

Next, we consider assertion (iii). Just as in the case of assertion (ii), we may apply assertion (i) to conclude that to complete the proof of assertion (iii), it suffices to show that any $C \in \mathrm{Ob}(\mathcal{G}[\mathbb{C}])$ such that $C_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(C) \in \mathrm{Ob}(\mathcal{D}[\mathbb{C}])$ is an anchor of $\mathcal{D}[\mathbb{C}]$ is itself an *anchor of* $\mathcal{G}[\mathbb{C}]$. Thus, let $\phi : C \to C'$ be an *irreducible* morphism of $\mathcal{G}[\mathbb{C}]$. Just as in the argument used to prove assertion (ii), since ϕ is irreducible, it follows [cf. the factorizations of [Mzk5], Definition 1.3, (iv), (v)] that ϕ is either a *pull-back morphism* or an *isometric pre-step* [since all morphisms of $\mathcal{G}$ are *linear isometries*]. Now [cf. the argument used to prove assertion (ii)] the finiteness of the collection of isomorphism classes of ${}_{C}\mathcal{F}$ arising from [irreducible] ϕ which are pull-back morphisms (respectively, isometric pre-steps) follows formally from the fact that $C_{\mathcal{D}}$ is an anchor of $\mathcal{D}[\mathbb{C}]$ (respectively, from Lemma 3.2, (iii), (vii)). This completes the proof of assertion (iii).

Finally, we consider assertion (iv). Suppose that $\mathcal{F}$ is of *RC-iso-subanchor* type. Then it follows formally that $\mathcal{F}$ admits a *complex object*, hence, in particular, a *complex isotropic object* $A \in \mathrm{Ob}(\mathcal{F})$. Since A is an *RC-iso-subanchor*, it follows immediately, by taking *isotropic hulls* [cf. the argument of [Mzk5], Remark 3.1.1], that $\mathcal{F}$ admits a *complex isotropic RC-subanchor* B, hence [cf. [Mzk5], Definition 1.3, (vii), (b)] that $\mathcal{F}$ admits a *complex isotropic RC-anchor* C, i.e., that

$\mathcal{F}[\mathbb{C}]$ admits a *complex isotropic anchor* C. But this contradicts the existence of *prime-Frobenius morphisms*, of arbitrary prime degree, with domain equal to C [cf. [Mzk5], Proposition 1.10, (iv)]. This completes the proof of assertion (iv). $\bigcirc$

Remark 3.5.1. In the context of Proposition 3.5, we observe that it is important to consider *not just anchors, but also subanchors* since, for instance in the case of the base categories associated to *connected temperoids* [cf. Example 1.3], it is not difficult to construct examples of base categories that appear naturally in arithmetic geometry and which have objects that are subanchors, but not anchors. [Indeed, this phenomenon occurs in the case of the temperoid associated to a tempered group [such as a nonabelian discrete finitely generated free group] which admits a topological subquotient isomorphic to an *infinite direct sum*

$$\bigoplus \mathbb{Z}/p\mathbb{Z}$$

of copies of $\mathbb{Z}/p\mathbb{Z}$, equipped with the discrete topology.] Moreover, it is important to consider not just anchors and subanchors, but also *RC-anchors and RC-subanchors* — i.e., in short, *complex anchors and subanchors* — since the arguments applied in the proof of Proposition 3.5 require one to work with *non-isotropic* [hence *complex*] objects that map to anchors of the base category. Thus, one is obliged to work with *RC-iso-subanchors* in order to accommodate *real* objects.

Our *main result* concerning the *Frobenioids* of Example 3.3 is the following:

Theorem 3.6. (Basic Properties of Archimedean, Angular Frobenioids) *In the notation and terminology of Example 3.3, let $\mathcal{F}$ be one of the following Frobenioids: $\mathcal{C}^{\Lambda}$, $\mathcal{A}$ [where, when $\mathcal{F} = \mathcal{A}$, we take $\Lambda = \mathbb{Z}$]. Also, denote by $\Phi^{\measuredangle}$ the restriction $\Phi_0^{\measuredangle}|_{\mathcal{D}}$ to $\mathcal{D}$ of the functor $\Phi_0^{\measuredangle} : \mathcal{D}_0 \to \mathfrak{Mon}$ determined by the assignment*

$$(\mathrm{Ob}(\mathcal{D}_0) \ni) \ \mathrm{Spec}(K) \mapsto \mathcal{O}_K^{\times}$$

and by Φ^{fld} the functor $\mathcal{D} \to \mathfrak{Mon}$ given by $(\mathrm{Ob}(\mathcal{D}) \ni D) \mapsto \Phi^{\mathrm{gp}}(D) \times \Phi^{\measuredangle}(D)$ [so projecting to the two factors of the product monoid yields natural transformations $\Phi^{\mathrm{fld}} \to \Phi^{\measuredangle}$, $\Phi^{\mathrm{fld}} \to \Phi^{\mathrm{gp}}$]. If $\Lambda = \mathbb{Z}$ (respectively, $\Lambda = \mathbb{Q}$; $\Lambda = \mathbb{R}$), then write $(\Phi^{\mathrm{fld}})^{\Lambda} \stackrel{\mathrm{def}}{=} \Phi^{\mathrm{fld}}$ (respectively, $(\Phi^{\mathrm{fld}})^{\Lambda} \stackrel{\mathrm{def}}{=} (\Phi^{\mathrm{fld}})^{\mathrm{pf}}$; $(\Phi^{\mathrm{fld}})^{\Lambda} \stackrel{\mathrm{def}}{=} \Phi^{\mathrm{gp}}$). Then:

(i) The Frobenioid $(\mathcal{C}^{\Lambda})^{\mathrm{istr}}$ is of **isotropic**, **base-trivial**, *and* **model** *type, with* **rational function monoid** *naturally isomorphic to $(\Phi^{\mathrm{fld}})^{\Lambda}$; the* **canonical decomposition** *of Definition 3.1, (ii), determines a* **characteristic splitting** *[cf. [Mzk5], Definition 2.3] on $\mathcal{C}^{\Lambda}$. If $\Lambda \geq \mathbb{Q}$, then $(\mathcal{C}^{\Lambda})^{\mathrm{istr}} = \mathcal{C}^{\Lambda}$. For arbitrary Λ, there is a natural equivalence of categories $(\mathcal{C}^{\Lambda})^{\text{un-tr}} \xrightarrow{\sim} \mathcal{C}^{\mathbb{R}}$, compatible with the Frobenioid structures; moreover, the Frobenioid $\mathcal{C}^{\Lambda}$ is of* Aut**-ample**, Aut$^{\mathrm{sub}}$**-ample**, End**-ample**, *and* **metrically trivial** *type, but* **not** *of group-like type. If, moreover, $\mathcal{D}$ is of* **FSMFF-** *and* **RC-iso-subanchor** *type, then $\mathcal{C}^{\Lambda}$ is of* **rationally standard** *type.*

(ii) The Frobenioid $\mathcal{A}^{\mathrm{istr}}$ *is of* **base-trivial** *and* **model** *type, with* **rational function monoid** *naturally isomorphic to* $\Phi^{\measuredangle}$. *The Frobenioid* $\mathcal{A}$ *is of* Aut**-ample**, $\mathrm{Aut}^{\mathrm{sub}}$**-ample**, End**-ample**, **group-like**, *and* **metrically trivial** *type. If, moreover,* $\mathcal{D}$ *is of* **FSMFF-** *and* **RC-iso-subanchor** *type, then* $\mathcal{A}$ *is of* **standard** *[but* **not** *of rationally standard] type.*

(iii) Every morphism $\phi : B \to A$ *of* $\mathcal{C}$ **factors uniquely** *as a composite* $\phi = \beta \circ \alpha$, *where* α *is an* **isometry** *[hence belongs to* $\mathcal{A}$*], and* $\beta \in \mathcal{O}^{\rhd}(A) \subseteq \Phi^{\mathrm{fld}}(A)$ *belongs to the submonoid* $\Phi^{\mathrm{gp}}(A) \times \{1\} \subseteq \Phi^{\mathrm{fld}}(A)$ *[cf. the characteristic splitting of (i)]. Moreover, the assignment* $\phi \mapsto \alpha$ *determines an* **"isometrization functor"**

$$\mathcal{C} \to \mathcal{A}$$

which, together with the **isotropification functors** $\mathcal{C} \to \mathcal{C}^{\mathrm{istr}}$, $\mathcal{A} \to \mathcal{A}^{\mathrm{istr}}$ *[cf. [Mzk5], Proposition 1.9, (v)], determines an* **equivalence of categories**

$$\mathcal{C} \xrightarrow{\sim} \mathcal{A} \times_{\mathcal{A}^{\mathrm{istr}}} \mathcal{C}^{\mathrm{istr}}$$

that is 1**-compatible** *with the natural functors to* $\mathbb{F}_\Phi$ *on both sides.*

(iv) Let $A \in \mathrm{Ob}(\mathcal{F})$; $A_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(A) \in \mathrm{Ob}(\mathcal{D})$. *Write* $A_0 \in \mathrm{Ob}(\mathcal{D}_0)$ *for the image of* $A_{\mathcal{D}}$ *in* $\mathcal{D}_0$. *Then the natural action of* $\mathrm{Aut}_{\mathcal{F}}(A)$ *on* $\mathcal{O}^{\rhd}(A)$, $\mathcal{O}^{\times}(A)$ **factors** *through* $\mathrm{Aut}_{\mathcal{D}_0}(A_0)$. *If, moreover,* $\Lambda \in \{\mathbb{Z}, \mathbb{Q}\}$, *then this factorization determines a* **faithful** *action of the image of* $\mathrm{Aut}_{\mathcal{F}}(A)$ *in* $\mathrm{Aut}_{\mathcal{D}_0}(A_0)$ *on* $\mathcal{O}^{\rhd}(A)$, $\mathcal{O}^{\times}(A)$.

(v) Let $A \in \mathrm{Ob}(\mathcal{F})$. *Then the group* $\mathcal{O}^{\times}(A)$ *is* **trivial** *if and only if one of the following holds: (a)* $\Lambda = \mathbb{Z}$ *and* A *is* **complex non-isotropic***; (b)* $\Lambda = \mathbb{Q}$ *and* A *is* **real***; (c)* $\Lambda = \mathbb{R}$. *The group* $\mathcal{O}^{\times}(A)$ *is* **nontrivial** *and* **torsion free** *[and in fact isomorphic to* $\mathbb{S}^1 \otimes_{\mathbb{Z}} \mathbb{Q}$*] if and only if* $\Lambda = \mathbb{Q}$ *and* A *is* **complex**. *The group* $\mathcal{O}^{\times}(A)$ *is* **of order two** *if and only if* $\Lambda = \mathbb{Z}$ *and* A *is* **real**. *The group* $\mathcal{O}^{\times}(A)$ **has infinitely many torsion elements** *[and is in fact isomorphic to* $\mathbb{S}^1$*] if and only if* $\Lambda = \mathbb{Z}$ *and* A *is* **complex isotropic**.

(vi) If $\mathcal{D}$ *admits a* **pseudo-terminal object**, *then* $\mathcal{F}$ *admits a* **pseudo-terminal** *object.*

(vii) Suppose that $\Lambda = \mathbb{Z}$. *Let* $A \in \mathrm{Ob}(\mathcal{F})$ *be* **complex**. *Then the assignment that maps an* **isometric pre-step**

$$B \to A$$

of $\mathcal{F}$ *to the image of the boundary* $\partial \mathbb{A}_B$ *of the angular region* $\mathbb{A}_B$ *of* B *in the boundary* $\partial \mathbb{A}_A$ *of the angular region* $\mathbb{A}_A$ *of* A *determines an* **equivalence of categories** *— which is* **functorial** *[cf. [Mzk5], Proposition 1.9, (ii), (iii)] in* A *—*

$$\mathcal{F}_A^{\mathrm{imtr\text{-}pre}} \xrightarrow{\sim} \mathrm{Open}^0(\partial \mathbb{A}_A)$$

— where $\mathcal{F}^{\mathrm{imtr\text{-}pre}} \subseteq \mathcal{F}$ *denotes the full subcategory determined by the arrows which are isometric pre-steps [cf. [Mzk5], Proposition 1.9];* $\mathcal{F}_A^{\mathrm{imtr\text{-}pre}} \stackrel{\mathrm{def}}{=} (\mathcal{F}^{\mathrm{imtr\text{-}pre}})_A$;

$\mathrm{Open}^0(\partial\mathbb{A}_V)$ *is the category of connected open subsets of* $\partial\mathbb{A}_V$ *[cf. the Appendix]. In particular, [cf. Theorem A.2, (vi)] the* **topological space** $\partial\mathbb{A}_A$ *may be* **recovered functorially** *from the category* $\mathcal{F}_A^{\text{imtr-pre}}$. *Finally, if* A *is* **isotropic**, *then the action of* $\mathcal{O}^\times(A)$ *on* $\partial\mathbb{A}_A$ *determines on* $\partial\mathbb{A}_A$ *a structure of* **torsor** *over this group.*

(viii) If $\Lambda = \mathbb{Z}$, *then* $\mathcal{F}[\mathbb{C}]$ *is of* **weakly dissectible** *type [cf. §0].*

(ix) Suppose that $\mathcal{D}$ *is of strongly indissectible type. If* $\mathcal{D}$ *is* **not complexifiable**, *then we assume further that* $\Lambda \neq \mathbb{Z}$. *Then* $\mathcal{F}^{\text{istr}}$ *is of* **strongly indissectible** *type.*

(x) If $\mathcal{D}$ *is* **slim**, *and* $\Lambda \in \{\mathbb{Z}, \mathbb{R}\}$, *then* $\mathcal{F}$ *is also* **slim**.

Proof. First, we consider assertion (i). By [the earlier portion of] Lemma 3.2, (ii), it is immediate from the construction of $\mathcal{C}^\Lambda$ that $\mathcal{C}^\Lambda$ is of Aut-*ample*, $\mathrm{Aut}^{\mathrm{sub}}$-*ample*, and End-*ample* type. Also, it is immediate from the construction of $\mathcal{C}^\Lambda$ that $\mathcal{C}^\Lambda$ is of *metrically trivial*, *[strictly] rational*, and *birationally Frobenius-normalized* type, but *not* of group-like type, and from the construction of $(\mathcal{C}^\Lambda)^{\text{istr}}$ that $(\mathcal{C}^\Lambda)^{\text{istr}}$ is of *base-trivial* and *isotropic* type. Since $\mathcal{D}_0$ admits a *terminal object*, it thus follows from [Mzk5], Proposition 2.9, (i), that $(\mathcal{C}_0^\Lambda)^{\text{istr}}$, hence also $(\mathcal{C}^\Lambda)^{\text{istr}}$ [which may be constructed as a categorical fiber product $(\mathcal{C}_0^\Lambda)^{\text{istr}} \times_{\mathcal{D}_0} \mathcal{D}$], is of *model* type. Moreover, it is immediate from the construction of $(\mathcal{C}^\Lambda)^{\text{istr}}$ that the *rational function monoid* of $(\mathcal{C}^\Lambda)^{\text{istr}}$ is naturally isomorphic to $(\Phi^{\text{fld}})^\Lambda$ [cf. also the *canonical decomposition* of Definition 3.1, (ii)]; in a similar vein, it is immediate from the definitions that the *canonical decomposition* of Definition 3.1, (ii), determines a *characteristic splitting* on $\mathcal{C}^\Lambda$. The fact that if $\Lambda \geq \mathbb{Q}$, then $(\mathcal{C}^\Lambda)^{\text{istr}} = \mathcal{C}^\Lambda$ is immediate from the definitions. The existence of a natural equivalence of categories $(\mathcal{C}^\Lambda)^{\text{un-tr}} \xrightarrow{\sim} \mathcal{C}^{\mathbb{R}}$, compatible with the Frobenioid structures, follows immediately from the construction of $\mathcal{C}^\Lambda$. Also, it is immediate from the construction of $\mathcal{C}^{\mathbb{R}}$ that every object of $((\mathcal{C}^\Lambda)^{\text{un-tr}})^{\text{birat}} = (\mathcal{C}^{\mathbb{R}})^{\text{birat}}$ is *Frobenius-compact*. Thus, if, moreover, $\mathcal{D}$ is of *FSMFF*- and *RC-iso-subanchor* type, then, [since Φ is manifestly *non-dilating*] to complete the proof of assertion (i), it suffices to observe that by Proposition 3.5, (ii), $\mathcal{C}$ is of *quasi-isotropic* type. This completes the proof of assertion (i).

Next, we consider assertion (ii). The fact that the Frobenioid $\mathcal{A}$ is of Aut-*ample*, $\mathrm{Aut}^{\mathrm{sub}}$-*ample*, End-*ample*, *group-like*, and *metrically trivial* type, as well as the fact that the Frobenioid $\mathcal{A}^{\text{istr}}$ is of *base-trivial* and *model* type, together with the description of the *rational function monoid* of the Frobenioid $\mathcal{A}^{\text{istr}}$, follow immediately from the corresponding facts for $\mathcal{C}$, $\mathcal{C}^{\text{istr}}$ proven in assertion (i) [together with the construction of $\mathcal{A}$, $\mathcal{A}^{\text{istr}}$]. If, moreover, $\mathcal{D}$ is of *FSMFF*- and *RC-iso-subanchor* type [which implies, in particular, that $\mathcal{D}$ admits *complex* objects], then one verifies immediately that the complex isotropic objects of $\mathcal{A}$ are *Frobenius-compact*; moreover, the *quasi-isotropicity* of $\mathcal{A}$ follows from Proposition 3.5, (ii), while the *Frobenius-isotropicity*, and *Frobenius-normalizedness* of $\mathcal{A}$ follows immediately from the corresponding properties for $\mathcal{C}$ [together with the construction of $\mathcal{A}$, $\mathcal{C}$]; thus, we conclude that $\mathcal{A}$ is of *standard* type. On the other hand, since, as is easily verified, $(\mathcal{A}^{\text{un-tr}})^{\text{birat}}$ is of *unit-trivial* type, it follows that $\mathcal{A}$ is *not* of rationally standard type. This completes the proof of assertion (ii).

Next, we consider assertion (iii). The existence and uniqueness of the *factorization* asserted in the statement of assertion (iii) follows immediately from the construction of $\mathcal{C}$; also, it is immediate that this factorization yields an *"isometrization functor"* $\mathcal{C} \to \mathcal{A}$. The resulting functor

$$\mathcal{C} \xrightarrow{\sim} \mathcal{A} \times_{\mathcal{A}^{\mathrm{istr}}} \mathcal{C}^{\mathrm{istr}}$$

is then manifestly *essentially surjective* and 1-*compatible* with the natural functors to $\mathbb{F}_\Phi$ on both sides. Now the *fully faithfulness* of this functor is immediate from the factorization just discussed, together with the 1-compatibility with the natural functors to $\mathbb{F}_\Phi$ on both sides. This completes the proof of assertion (iii).

Assertions (iv), (v), (vi) follow immediately from the definitions. Assertion (vii) follows immediately from the definitions and Theorem A.2, (vi), of the Appendix [since $\mathbb{S}^1$ is clearly *sober* and *locally connected*]. Assertion (viii) follows by considering *disjoint angular regions*. Assertion (ix) follows, in light of the strong indissectibility assumption on $\mathcal{D}$, by reducing [cf. our assumption concerning the case when $\mathcal{D}$ is *not complexifiable*] to the easily verified fact that the categories $(\mathcal{C}_0^\Lambda)^{\mathrm{istr}}[\mathbb{R}]$ [when $\Lambda \neq \mathbb{Z}$] , $(\mathcal{C}_0^\Lambda)^{\mathrm{istr}}[\mathbb{C}]$ [for arbitrary Λ] are of *strongly indissectible* type. Assertion (x) follows formally from [Mzk5], Proposition 1.13, (iii) [since, by assertions (v), (vii) of the present Theorem 3.6, either "condition (a)" or "condition (b)" of *loc. cit.* is always satisfied by objects of $\mathcal{F}$]. ◯

Remark 3.6.1. Note that the *topology* of $\mathcal{O}^\times(A)$ ($\cong \mathbb{S}^1$), for *complex isotropic* $A \in \mathrm{Ob}(\mathcal{C})$, may be recovered from the category-theoretic structure of $\mathcal{C}$ [cf. Theorem 3.6, (vii)] precisely because of the existence of the *non-isotropic* objects. This is the *principal reason* for the inclusion of non-isotropic objects in the theory of Frobenioids.

Section 4: Angloids as Base Categories

In the present §4, we show [cf. Corollary 4.2, (v); Remark 4.2.1] that the categories called *"angloids"*, which were constructed in Example 3.3, (iii), (iv), satisfy those properties required of a *base category of an archimedean or angular Frobenioid* [cf. Theorem 3.6, (i), (ii)], in order for the *main results* of the general theory of Frobenioids developed in [Mzk5] to apply. In addition, we study various other *basic properties of angloids*, certain of which may be regarded [cf. Remark 4.2.1] as a sort of *archimedean analogue* of the category-theoreticity of the *[nonarchimedean] local reciprocity map*, as discussed in Theorem 2.4, (i), (ii). Finally, we discuss the *main motivating examples* [cf. Examples 4.3, 4.4] from arithmetic geometry of the theory developed thus far for *archimedean* primes.

Proposition 4.1. **(Basic Properties of Angloids)** *In the notation and terminology of Example 3.3, let $\mathcal{G}$ be one of the following categories $\mathcal{N}$, $\mathcal{R}$; if $\mathcal{G} = \mathcal{N}$ (respectively, $\mathcal{G} = \mathcal{R}$), then write $\mathcal{G}_0 \stackrel{\mathrm{def}}{=} \mathcal{N}_0$ (respectively, $\mathcal{G}_0 \stackrel{\mathrm{def}}{=} \mathcal{R}_0$). Then:*

(i) Let $A \in \mathrm{Ob}(\mathcal{G})$ be **complex**. *Then the assignment that maps a* **base-isomorphism** *[or, equivalently, an* **isometric pre-step** *— cf. the definition of $\mathcal{G}$ in Example 3.3, (iii), (iv)]*

$$B \to A$$

of $\mathcal{G}$ to the image of the boundary $\partial\mathbb{A}_B$ of the angular region $\mathbb{A}_B$ of B in the boundary $\partial\mathbb{A}_A$ of the angular region $\mathbb{A}_A$ of A determines an **equivalence of categories** *— which is* **functorial** *[cf. [Mzk5], Proposition 1.9, (ii), (iii)] in A —*

$$\mathcal{G}_A^{\text{imtr-pre}} \xrightarrow{\sim} \mathrm{Open}^0(\partial\mathbb{A}_A)$$

— where $\mathcal{G}^{\text{imtr-pre}} \subseteq \mathcal{G}$ denotes the full subcategory determined by the arrows which are isometric pre-steps [cf. [Mzk5], Proposition 1.9]; $\mathcal{G}_A^{\text{imtr-pre}} \stackrel{\mathrm{def}}{=} (\mathcal{G}^{\text{imtr-pre}})_A$; $\mathrm{Open}^0(\partial\mathbb{A}_V)$ is the category of connected open subsets of $\partial\mathbb{A}_V$ [cf. the Appendix]. In particular, [cf. Theorem A.2, (vi)] the **topological space** *$\partial\mathbb{A}_A$ may be* **recovered functorially** *from the category $\mathcal{G}_A^{\text{imtr-pre}}$. Finally, if A is* **isotropic**, *and $\mathcal{G} = \mathcal{N}$, then the action of $\mathcal{O}^\times(A)$ on $\partial\mathbb{A}_A$ determines on $\partial\mathbb{A}_A$ a structure of* **torsor** *over this group.*

(ii) A morphism of $\mathcal{G}$ with **real** *domain or codomain is an* **isomorphism** *if and only if it is a* **base-isomorphism**.

(iii) Suppose that $\mathcal{D}$ is of **discontinuously ordered** *type. Then every* **continuously ordered** *[cf. §0] morphism of $\mathcal{G}$ is a* **base-isomorphism** *[i.e., an* **isometric pre-step** *— cf. the definition of $\mathcal{G}$ in Example 3.3, (iii), (iv)].*

(iv) Suppose that $\mathcal{D}$ is of **discontinuously ordered** *type. Then a morphism $\phi : A \to B$ of $\mathcal{G}$ is a* **base-isomorphism** *[or, equivalently, an* **isometric pre-step** *— cf. the definition of $\mathcal{G}$ in Example 3.3, (iii), (iv)] if and only if there exists a morphism $\psi : C \to A$ of $\mathcal{G}$ and* **quasi-continuously ordered** *[cf. §0] morphisms*

$C \to C_1$, $C \to C_2$ of $\mathcal{G}$ such that the object of ${}_C\mathcal{G}$ constituted by the composite arrow $\phi \circ \psi : C \to B$ is an **inductive limit** *of the diagram*

$$C_2 \longleftarrow C \longrightarrow C_1$$

in ${}_C\mathcal{G}$.

(v) If $\mathcal{D}$ is of **strongly dissectible** *type, then so is $\mathcal{G}$. If $\mathcal{D}$ is of* **weakly indissectible** *type, then so is $\mathcal{G}$.*

Proof. Assertion (i) follows immediately from the definitions and Theorem A.2, (vi), of the Appendix [cf. the proof of Theorem 3.6, (vii)]. Assertion (ii) follows immediately from the definition of $\mathcal{G}$ in Example 3.3, (iii), (iv).

Next, we consider assertion (iii). Let $\phi : A \to B$ be a *continuously ordered* [hence, in particular, *totally ordered*] morphism of $\mathcal{G}$; $\phi_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\phi) \in \mathrm{Arr}(\mathcal{D})$. Let $\phi = \beta \circ \alpha$ be a *factorization* of ϕ, where α, β are *monomorphisms* which are *not* isomorphisms. Thus, α, β are *continuously ordered* [hence, in particular, *totally ordered*]. Then I *claim* that neither of the projections α_0, β_0 of α, β to $\mathcal{G}_0$ is an isomorphism. Indeed, suppose that the projection γ_0 of $\gamma \in \{\alpha, \beta\}$ to $\mathcal{G}_0$ is an isomorphism. Then one verifies immediately that the natural projection functor determines an *equivalence of categories* $\mathcal{G}_{\gamma}^{\rightarrowtail} \xrightarrow{\sim} \mathcal{D}_{\gamma_{\mathcal{D}}}^{\rightarrowtail}$ [where we write $\gamma_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\gamma)$]. Since γ is continuously ordered, we thus conclude that $\gamma_{\mathcal{D}}$ is *continuously ordered.* On the other hand, since $\mathcal{D}$ is of *discontinuously ordered* type, this implies that $\gamma_{\mathcal{D}}$ is an isomorphism, hence that γ is an isomorphism, a contradiction. This completes the proof of the *claim.* Since every morphism between *real* objects of $\mathcal{G}_0$ is an isomorphism [cf. assertion (ii)], we thus conclude that the codomain of α is *complex.*

Next, I *claim* that α is a *base-isomorphism.* Indeed, suppose that $\alpha_{\mathcal{D}} \stackrel{\mathrm{def}}{=} \mathrm{Base}(\alpha)$ is *not* an isomorphism. Since α_0 is *not* an isomorphism, it thus follows that by lifting $\alpha_{\mathcal{D}}$ to a morphism of $\mathcal{G}$ that projects to an isomorphism of $\mathcal{G}_0$ and lifting α_0 to a morphism of $\mathcal{G}$ that projects to an isomorphism of $\mathcal{D}$, we thus obtain *two factorizations* of α, i.e., two objects of $\mathcal{G}_{\alpha}^{\rightarrowtail}$, neither of which "dominates" the other [cf. the fact that $\mathcal{D}$, $\mathcal{G}_0$ are *totally epimorphic*!] — a contradiction [since α is *totally ordered*]. This completes the proof of the *claim.* Thus, in summary, α is an *isometric pre-step.* More concretely, the morphism α may be thought of as an *"enlargement of the angular region that defines A".* In particular, by taking the *union* of the angular regions that arise from such enlargements, one obtains a factorization $\phi = \beta^{\infty} \circ \alpha^{\infty}$, where $\alpha^{\infty} : A \to A^{\infty}$ is an *isometric pre-step* [hence, in particular, a *base-isomorphism*], such that every α appearing in a factorization $\phi = \beta \circ \alpha$ as above determines a factorization $\alpha^{\infty} = \alpha' \circ \alpha$.

Next, I *claim* that β^{∞} is an *isomorphism.* Indeed, suppose that β^{∞} is *not* an isomorphism. Now if β^{∞} is a *monomorphism*, then α^{∞}, β^{∞} determine an object of $\mathcal{G}_{\phi}^{\rightarrowtail}$ which [by the definition of α^{∞}; our assumption that β^{∞} is *not* an isomorphism] *contradicts* the fact that ϕ is *continuously ordered.* Thus, we conclude that $\beta^{\infty} : A^{\infty} \to B$ is *not* a monomorphism, i.e., that there exist *distinct* morphisms

$\gamma', \gamma'' : C \to A^\infty$ in $\mathcal{G}$ such that $\beta^\infty \circ \gamma' = \beta^\infty \circ \gamma''$. On the other hand, by the definition of α^∞ in terms of angular regions, it follows immediately that there exists a factorization $\phi = \beta \circ \alpha$, where we write $\alpha : A \to E$, $\beta : E \to B$, that forms an object of $\mathcal{G}_\phi^{\hookrightarrow}$, together with morphisms $\delta : D \to C$ and $\epsilon', \epsilon'' : D \to E$ of $\mathcal{G}$ such that $\gamma' \circ \delta = \zeta \circ \epsilon'$, $\gamma'' \circ \delta = \zeta \circ \epsilon''$, where $\zeta : E \to A^\infty$ is the morphism arising from the definition of α^∞ [so $\beta^\infty \circ \zeta = \beta$], and δ is an isometric pre-step. Thus, we conclude that $\beta^\infty \circ \zeta \circ \epsilon' = \beta^\infty \circ \gamma' \circ \delta = \beta^\infty \circ \gamma'' \circ \delta = \beta^\infty \circ \zeta \circ \epsilon''$, i.e., that $\beta \circ \epsilon' = \beta \circ \epsilon''$. But since β is a *monomorphism*, this implies that $\epsilon' = \epsilon''$, hence that $\gamma' \circ \delta = \gamma'' \circ \delta$, i.e., [since δ is an *epimorphism*!] that $\gamma' = \gamma''$ — a contradiction. This completes the proof of the *claim*. Since $\phi = \beta^\infty \circ \alpha^\infty$, and α^∞ is a *base-isomorphism*, we thus conclude that ϕ is also a *base-isomorphism*, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). To verify the *sufficiency* of the condition in the statement of assertion (iv), observe that by assertion (iii), the morphisms $C \to C_1$, $C \to C_2$ are *base-isomorphisms* [i.e., *isometric pre-steps*]; thus, any isotropic hull $C \to C'$ of C factors through $C \to B$ [cf. [Mzk5], Proposition 1.9, (vii)], which, by the *total epimorphicity* of $\mathcal{D}$, implies that $C \to B$, hence also $A \to B$, is a *base-isomorphism*, as desired. Next, we consider *necessity*. In the *real* case, this necessity follows immediately from assertion (ii). In the *complex case*, the necessity in question follows from Lemma 3.2, (viii) [which we apply to construct the morphisms $C \to C_1$, $C \to C_2$ for an appropriate isometric pre-step $C \to A$] and Lemma 3.2, (vii) [which we apply to show that these morphisms $C \to C_1$, $C \to C_2$ are *quasi-continuously ordered* isometric pre-steps]. [Here, we note that *"continuously ordered pairs"* in the context of Lemma 3.2 give rise to *"continuously ordered isometric pre-steps"*.] This completes the proof of assertion (iv).

Finally, we observe that assertion (v) follows by considering appropriate *isometric pre-steps* and *pull-back morphisms* to relate "dissections in $\mathcal{G}$" to "dissections in $\mathcal{D}$" [cf. the discussion of §0]. ◯

Corollary 4.2. **(Category-theoreticity of Factors of Angloids)** *For $i = 1, 2$, let $\mathcal{D}_i$ be a* **connected, totally epimorphic** *category of* **discontinuously ordered** *type, equipped with a functor $\mathcal{D}_i \to \mathcal{D}_0$, with respect to which $\mathcal{D}_i$ is* **complexifiable***; $\mathcal{G}_i$ one of the categories $\mathcal{N}_i \stackrel{\text{def}}{=} \mathcal{N}_0 \times_{\mathcal{D}_0} \mathcal{D}_i$, $\mathcal{R}_i \stackrel{\text{def}}{=} \mathcal{R}_0 \times_{\mathcal{D}_0} \mathcal{D}_i$ [cf. Example 3.3, (iii), (iv)];*

$$\Psi : \mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$$

an **equivalence of categories***. If $\mathcal{G}_i = \mathcal{N}_i$ (respectively, $\mathcal{G}_i = \mathcal{R}_i$), then set $\mathcal{G}_{i,0} \stackrel{\text{def}}{=} \mathcal{N}_0$ (respectively, $\mathcal{G}_{i,0} \stackrel{\text{def}}{=} \mathcal{R}_0$). Then:*

(i) If $\mathcal{G}_{1,0} = \mathcal{N}_0$ (respectively, $\mathcal{G}_{1,0} = \mathcal{R}_0$), then $\mathcal{G}_{2,0} = \mathcal{N}_0$ (respectively, $\mathcal{G}_{2,0} = \mathcal{R}_0$). In particular, it makes sense to write $\mathcal{G}_0 \stackrel{\text{def}}{=} \mathcal{G}_{1,0} = \mathcal{G}_{2,0}$.

(ii) The category of $\mathcal{G}_0$ is **slim***. If $\mathcal{G}_0 = \mathcal{N}_0$ (respectively, $\mathcal{G}_0 = \mathcal{R}_0$), then there is a natural outer isomorphism between the group of isomorphism classes of* **self-equivalences** *of the category $\mathcal{G}_0$ and the group of* **translations** *and* **reflections**

$\mathrm{Refl}(\mathbb{S}^1)$ *of Lemma 3.2, (xii) (respectively, the group of* **self-homeomorphisms** *of* $\mathbb{S}^1$ *that* **commute** *with the complex conjugation automorphism* $\iota : \mathbb{S}^1 \xrightarrow{\sim} \mathbb{S}^1$*).*

(iii) Suppose that $\mathcal{D}_i$ *is* **RC-connected**. *Then there exists a* **1-unique** *1-commutative diagram*

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\Psi} & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ \mathcal{G}_0 & \xrightarrow{\Psi^0} & \mathcal{G}_0 \end{array}$$

— where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories.

(iv) If $\mathcal{D}_1$, $\mathcal{D}_2$ *are* **Frobenius-slim**, *then so are* $\mathcal{G}_1$, $\mathcal{G}_2$. *Suppose that* $\mathcal{D}_1$, $\mathcal{D}_2$ *are* **slim**. *Then* $\mathcal{G}_1$, $\mathcal{G}_2$ *are* **slim**, *and there exists a* **1-unique** *1-commutative diagram*

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\Psi} & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\mathrm{Base}}} & \mathcal{D}_2 \end{array}$$

— where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories; both composite functors are **rigid**.

(v) If [for $i = 1, 2$*]* $\mathcal{D}_i$ *is of* **RC-standard** *type, then so is* $\mathcal{G}_i$. *In particular, in this case,* $\mathcal{G}_i$ *satisfies* **all** *of the hypotheses on "*$\mathcal{D}$*" in Theorem 3.6, (i), (ii).*

Proof. First, we observe that Ψ preserves *base-isomorphisms* [cf. Proposition 4.1, (iv)]. Since the *complex* objects of $\mathcal{G}_i$ are precisely the objects that appear as domains or codomains of base-isomorphisms which are *not* isomorphisms [cf. Proposition 4.1, (i), (ii)], it thus follows that Ψ preserves *real* and *complex* objects. For $i = 1, 2$, suppose that $A_i \in \mathrm{Ob}(\mathcal{G}_i)$ is *complex*, and that $A_2 = \Psi(A_1)$. [Here, we note that since $\mathcal{D}_i$ is *complexifiable*, it follows that such A_1, A_2 always *exist*.] Then Ψ induces a *homeomorphism*

$$\partial\mathbb{A}_{A_1} \xrightarrow{\sim} \partial\mathbb{A}_{A_2}$$

[cf. Proposition 4.1, (i); Theorem A.2, (vi)], which is *functorial* in A_1, A_2. In particular, it follows [cf. Lemma 3.2, (x)] that Ψ preserves *isotropic objects* and *isotropic hulls*.

Next, we consider assertion (i). Observe [cf. Theorem 3.6, (v)] that $\mathcal{O}^\times(A_i) = \{1\}$ if and only if either of the following conditions holds: (a) $\mathcal{G}_i = \mathcal{R}_i$; (b) $\mathcal{G}_i = \mathcal{N}_i$, and A_i is *non-isotropic*. Moreover, one verifies immediately that there exist *non-isotropic* A_i such that the natural homomorphism $\mathrm{Aut}_{\mathcal{G}_i}(A_i) \to \mathrm{Aut}_{\mathcal{D}_i}(\mathrm{Base}(A_i))$

[whose kernel is equal to $\mathcal{O}^\times(A_i)$] is *surjective* [cf. the Aut-*ampleness* of Theorem 3.6, (ii)]. In particular, we conclude that $\mathcal{G}_i = \mathcal{R}_i$ if and only if there exist non-isotropic A_i such that if $A_i \to A_i^{\mathrm{istr}}$ is an *isotropic hull*, then the natural homomorphism $\mathrm{Aut}_{\mathcal{G}_i}(A_i) \to \mathrm{Aut}_{\mathcal{G}_i}(A_i^{\mathrm{istr}})$ [induced by the defining property of the isotropic hull] is an *isomorphism*. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us observe that the existence of the *functorial homeomorphism* $\partial \mathbb{A}_{A_1} \xrightarrow{\sim} \partial \mathbb{A}_{A_2}$ discussed above [when $\mathcal{D}_i \stackrel{\mathrm{def}}{=} \mathcal{D}_0$] implies that $\mathcal{G}_0$ is *slim*. Now let us consider this functorial homeomorphism for *complex isotropic* A_i. Note that since all complex isotropic objects of $\mathcal{G}_0$ are isomorphic, we may assume [for the remainder of the proof of the present assertion (ii)], without loss of generality, that $A_1 = A_2$. Write $I \subseteq \mathrm{Homeo}(\mathbb{S}^1)$ for the image in $\mathrm{Homeo}(\mathbb{S}^1)$ [cf. Lemma 3.2, (xii)] of $\mathrm{Aut}_{\mathcal{G}_0}(A_i)$. Observe that this image is given [for a suitable identification of $\partial \mathbb{A}_{A_i}$ with $\mathbb{S}^1$] by the subgroup $\mathrm{Refl}(\mathbb{S}^1)$ [cf. Lemma 3.2, (xii)], in the case $\mathcal{G}_0 = \mathcal{N}_0$, and by the subgroup [of order 2] generated by ι in the case $\mathcal{G}_0 = \mathcal{R}_0$. In particular, the subset of I consisting of elements that *reverse* the orientation of $\mathbb{S}^1$ — i.e., of elements that determine the structure of *real* objects of $\mathcal{G}_0$ as *categorical quotients* of *complex isotropic objects* — is *preserved* by conjugation by arbitrary elements of the *normalizer* of I in $\mathrm{Homeo}(\mathbb{S}^1)$ [cf. Lemma 3.2, (xii)]. Thus, by thinking of the *functorial homeomorphism* $\partial \mathbb{A}_{A_1} \xrightarrow{\sim} \partial \mathbb{A}_{A_2}$ as an element of $\mathrm{Homeo}(\mathbb{S}^1)$, we conclude that the isomorphism classes of *self-equivalences* of $\mathcal{G}_0$ may be identified with the *normalizer* of I in $\mathrm{Homeo}(\mathbb{S}^1)$ [which may be computed when $\mathcal{G}_0 = \mathcal{N}_0$ by applying Lemma 3.2, (xii)]. This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us observe that since $\mathcal{D}_i$ is *complexifiable*, it follows that we may choose A_i to be *[complex] isotropic* and such that the natural homomorphism $\mathrm{Aut}_{\mathcal{G}_i}(A_i) \to \mathrm{Aut}_{\mathcal{D}_i}(\mathrm{Base}(A_i))$ is *surjective* [a condition that may be phrased in "category-theoretic" terms as the condition that A_i admit an automorphism [in $\mathcal{G}_i$] that *reverses* the orientation of $\partial \mathbb{A}_{A_i}$]. Let B_i be a *complex isotropic* object of $\mathcal{G}_i$; write $\mathrm{Aut}_\partial(B_i)$ for the group of automorphisms of $\partial \mathbb{A}_{B_i}$ induced by automorphisms of B_i [in $\mathcal{G}_i$]. Since $\mathcal{D}_i$ is *RC-connected*, we obtain a *homeomorphism* $\partial(A_i, B_i) : \partial \mathbb{A}_{A_i} \xrightarrow{\sim} \partial \mathbb{A}_{B_i}$ by applying [Mzk5], Proposition 1.9, (ii), (iii), to various morphisms of $\mathcal{G}_i[\mathbb{C}]$ to *"connect"* A_i to B_i. Moreover, by projecting to $\mathcal{G}_0$, one verifies immediately that the $\mathrm{Aut}_\partial(A_i)$-*orbit* — which we denote by $\partial[A_i, B_i]$ — of this homeomorphism $\partial(A_i, B_i)$ *depends only on* B_i [i.e., not on the *choice* of morphisms used to "connect" A_i to B_i]. In a similar vein, by projecting to $\mathcal{G}_0$, one verifies immediately that conjugation by any homeomorphism $\in \partial[A_i, B_i]$ maps $\mathrm{Aut}_\partial(B_i)$ into $\mathrm{Aut}_\partial(A_i)$. Next, let us observe that the *functoriality* of the *functorial homeomorphism* $\partial \mathbb{A}_{A_1} \xrightarrow{\sim} \partial \mathbb{A}_{A_2}$ implies that this functorial homeomorphism is *compatible* with the $\partial[A_i, B_i]$'s and induces an isomorphism $\mathrm{Aut}_\partial(A_1) \xrightarrow{\sim} \mathrm{Aut}_\partial(A_2)$. In particular, by applying the *complexifiability* of $\mathcal{D}_i$ in the case of *real* objects of $\mathcal{G}_i$, we conclude as in the proof of assertion (ii) that this *functorial homeomorphism* $\partial \mathbb{A}_{A_1} \xrightarrow{\sim} \partial \mathbb{A}_{A_2}$ determines a 1-*unique* 1-commutative diagram as in the statement of assertion (iii).

Next, we consider assertion (iv). First, let us observe that the portion of assertion (iv) concerning the *Frobenius-slimness* and *slimness* of $\mathcal{G}_1$, $\mathcal{G}_2$ follows immediately from Proposition 4.1, (i) [cf. the use of condition (a) in the proof of [Mzk5], Proposition 1.13, (iii)], together with the assumption that $\mathcal{D}_1$, $\mathcal{D}_2$ are Frobenius-slim or slim; similarly, the *rigidity* of the composite functors in the diagram appearing in the statement of assertion (iv) [if such a diagram exists] follows immediately from the slimness of $\mathcal{D}_1$, $\mathcal{D}_2$ [cf. [Mzk5], Proposition 1.13, (i)]. Now assume that $\mathcal{D}_1$, $\mathcal{D}_2$ are *slim*. Next, let us observe that the *functor*

$$\mathcal{G}_i \to \mathcal{G}_i^{\text{istr}}$$

[where $\mathcal{G}_i^{\text{istr}} \subseteq \mathcal{G}_i$ is the full subcategory determined by the *isotropic* objects] induced by the *isotropification functor* of [Mzk5], Proposition 1.9, (v), may be reconstructed, up to isomorphism, as the *left adjoint* to the natural inclusion functor $\mathcal{G}_i^{\text{istr}} \hookrightarrow \mathcal{G}_i$ [cf. the analogous fact for Frobenioids, discussed in [Mzk5], Proposition 1.9, (v)]. Moreover, one verifies immediately that the natural projection functor $\mathcal{G}_i^{\text{istr}} \to \mathcal{D}_i$ is *full* and *essentially surjective*, and that if $A, B \in \text{Ob}(\mathcal{G}_i^{\text{istr}})$, then

$$\text{Hom}_{\mathcal{D}_i}(\text{Base}(A), \text{Base}(B))$$

may be reconstructed as the *quotient* of $\text{Hom}_{\mathcal{G}_i^{\text{istr}}}(A, B) = \text{Hom}_{\mathcal{G}_i}(A, B)$ by the natural action of $\mathcal{O}^\times(A)$ [cf. the analogous fact for Frobenioids, i.e., the *faithfulness* discussed in [Mzk5], Proposition 3.3, (iv)]. Since we have already seen that Ψ preserves *isotropic* objects [hence, in particular, the subcategory $\mathcal{G}_i^{\text{istr}} \subseteq \mathcal{G}_i$], to complete the proof of assertion (iv), it suffices to show that Ψ preserves "$\mathcal{O}^\times(-)$".

Let $A \in \text{Ob}(\mathcal{G}_i^{\text{istr}})$. Then the projection functor $\mathcal{G}_i \to \mathcal{D}_i$ induces an *equivalence of categories*

$$(\mathcal{G}_i)_A^{\text{istr}} \xrightarrow{\sim} (\mathcal{D}_i)_{\text{Base}(A)}$$

— cf. [Mzk5], Definition 1.3, (i), (c); the fact [cf. [Mzk5], Proposition 1.4, (ii)] that a morphism of a Frobenioid is *LB-invertible* and *linear* if and only if it is a *pull-back morphism*. Thus, since $\mathcal{D}_i$ is *slim*, the group

$$\text{Aut}((\mathcal{G}_i)_A^{\text{istr}} \to \mathcal{D}_i) \cong \text{Aut}((\mathcal{D}_i)_{\text{Base}(A)} \to \mathcal{D}_i)$$

is *trivial*. In particular, [composition with the projection functor $\mathcal{G}_i \to \mathcal{D}_i$ reveals that] any

$$\alpha \in \text{Aut}((\mathcal{G}_i)_A^{\text{istr}} \to \mathcal{G}_i)$$

consists of *base-identity* automorphisms, i.e., that by considering the automorphism of A determined by α, we obtain a *natural isomorphism*

$$\mathcal{O}^\times(A) \xrightarrow{\sim} \text{Aut}((\mathcal{G}_i)_A^{\text{istr}} \to \mathcal{G}_i)$$

[cf. the analogue of this argument for Frobenioids given in the proof of [Mzk5], Corollary 4.11, (i)]. In particular, we conclude that Ψ preserves "$\mathcal{O}^\times(-)$", as desired. This completes the proof of assertion (iv).

Finally, we observe that assertion (v) follows formally from Proposition 3.4, (viii); 3.5, (iii). ◯

Remark 4.2.1.

(i) Thus, by Corollary 4.2, (v), if $\mathcal{D}$ is any category of *discontinuously ordered* type, equipped with a functor $\mathcal{D} \to \mathcal{D}_0$ such that $\mathcal{D}$ is *RC-standard* type, then the resulting categories $\mathcal{N}$, $\mathcal{R}$ form *"suitable base categories"*, from the point of view of Theorem 3.6, (i), (ii). That is to say:

> *The main results of the theory of [Mzk5] may be applied to archimedean or angloid Frobenioids over the base categories $\mathcal{N}$, $\mathcal{R}$.*

Moreover, in the case of $\mathcal{N}$, if A is a complex isotropic object of such an *archimedean or angloid Frobenioid* that projects to a complex isotropic object $A_{\mathcal{N}}$ of the *base category* $\mathcal{N}$, then one obtains natural *category-theoretic isomorphisms* of the *topological group structure* of the "$\mathbb{S}^1$" [i.e., the "$\mathcal{O}^\times(-)$"] of A with the "$\mathbb{S}^1$" of $A_{\mathcal{N}}$ via Theorem 3.6, (vii); Proposition 4.1, (i); Corollary 4.2, (ii), (iii). Furthermore:

> Such category-theoretic isomorphisms between the *"Frobenioid-theoretic"* and *"base category-theoretic"* copies of "$\mathbb{S}^1$" may be regarded as an *archimedean analogue* of the category-theoreticity of the *[nonarchimedean] local reciprocity map*, as discussed in Theorem 2.4, (i), (ii).

Indeed, this sort of "archimedean analogue of the local reciprocity map" constituted one of the *main goals* of the author in the development of the [technically somewhat cumbersome — cf., especially, Propositions 3.4, 3.5] theory of §3.

(ii) It is perhaps of interest to note that in the "archimedean analogue of the category-theoreticity of the local reciprocity map" discussed in (i), if one forgets the vast collection of terminology and "general nonsense" developed so far in the theory of §3 and the present §4, then the *main "substantive ingredients"* of the theory appear to be the following:

(a) the *reconstruction* of $\mathbb{S}^1$ from the associated category of open connected subsets $\mathrm{Open}^0(\mathbb{S}^1)$ [cf. the theory of the Appendix];

(b) the fact that $\mathrm{Refl}(\mathbb{S}^1)$ is equal to its own *normalizer* in $\mathrm{Homeo}(\mathbb{S}^1)$ [cf. Lemma 3.2, (xii)].

Here, we recall that (a) was applied in the proof of Theorem 3.6, (vii), and Proposition 4.1, (i), while (b) was applied in the proof of Corollary 4.2, (ii).

(iii) Write G_K for the *absolute Galois group of a nonarchimedean [mixed-characteristic] local field K*. Then relative to the point of view discussed in (i), in which $\mathcal{G}_0$ [especially $\mathcal{N}_0$] plays the role of a sort of *archimedean version of G_K*, the *slimness* of $\mathcal{G}_0$ [cf. Corollary 4.2, (ii)] may be thought of as a sort of *archimedean analogue* of the slimness of G_K [cf., e.g. [Mzk1], Theorem 1.1.1, (ii)]. In a similar vein, the *"category-theoreticity of the projection to $\mathcal{G}_0$"* given in Corollary 4.2, (iii), may be thought of as a sort of *archimedean version of the group-theoreticity of the projection to G_K* given in [Mzk1], Lemma 1.3.8.

Finally, we discuss some examples from *arithmetic geometry at archimedean primes* of categories "$\mathcal{D}$" which satisfy the hypotheses of Corollary 4.2, (v).

Example 4.3. Categories Associated to Hyperbolic Riemann Surfaces.

(i) In the notation and terminology of [Mzk3], §2: Let (X^*, ϕ^*) be an *RC-Teichmüller pair* [roughly speaking: a hyperbolic Riemann surface considered up to complex conjugation and equipped with a nonzero *square differential*]; $\pi_1(X^*) \to \Pi$ [where $\pi_1(X^*)$ is the topological fundamental group of X^*] a *dense morphism of tempered topological groups* such that Π is *"totally ramified at infinity"* and *"stack-resolving"*; $\Box \in \{\mathfrak{P}, \mathfrak{R}, \mathfrak{S}\}$. Then associated to this data, one may define [cf. [Mzk3], the discussion preceding Proposition 2.2] a *category*

$$\mathcal{D} \stackrel{\text{def}}{=} \mathfrak{Loc}_{\Pi}^{\Box}(X^*, \phi^*)$$

of parallelogram, rectangle, or square localizations [depending on the choice of $\Box$] of (X^*, ϕ^*). Roughly speaking, this category consists of two [mutually exclusive] types of objects: (a) *complete objects*, which are coverings of the hyperbolic Riemann surface in question [as prescribed by Π]; (b) *parallelogram objects*, which are [pre-compact] parallelograms, rectangles, or squares [depending on the choice of $\Box$] on the [universal covering of the noncritical locus of the] Riemann surface in question, relative to the natural parameters [in the sense of Teichmüller theory] determined by the given square differential. The main result proven concerning this category $\mathcal{D}$ in [Mzk3] is that if $\Box = \mathfrak{R}, \mathfrak{S}$ (respectively, $\Box = \mathfrak{P}$), then one may *reconstruct* the given Riemann surface, together with its *conformal structure* (respectively, *quasiconformal structure*) — up to possible confusion with the corresponding complex conjugate structures — in an entirely *category-theoretic* fashion from the category $\mathcal{D}$ [cf. [Mzk3], Theorem 2.3]. It is immediate from the definition of $\mathcal{D}$ that $\mathcal{D}$ is equipped with a *natural functor* $\mathcal{D} \to \mathcal{D}_0$ [that maps "objects of complex type" of $\mathcal{D}$ [cf. [Mzk3], Proposition 1.5, (i)] to complex objects of $\mathcal{D}_0$ and "objects of real type" of $\mathcal{D}$ [cf. [Mzk3], Proposition 1.5, (ii)] to real objects of $\mathcal{D}_0$].

(ii) Here, we wish to observe that:

> $\mathcal{D}$ [equipped with the natural functor $\mathcal{D} \to \mathcal{D}_0$] is a *slim* category of *RC-standard*, *FSM-*, and *strictly partially ordered* [hence, a fortiori, of *discontinuously ordered*] type [cf. Remark 4.2.1].

Indeed, $\mathcal{D}$ is *slim* [cf. [Mzk3], Theorem 2.3, (i)], *totally epimorphic* [cf. [Mzk3], Proposition 2.2, (v)], *RC-connected* [cf. the definition of $\mathcal{D}$ in *loc. cit.*], and *complexifiable* [cf. our assumption that Π is "stack-resolving"]. Since the complete objects of $\mathcal{D}$ that arise from *finite Galois coverings* of X^* of *complex* type [where we note that such objects always *exist*, by our assumption that Π is "stack-resolving"] form *RC-anchors* of $\mathcal{D}$, it follows immediately from the definition of $\mathcal{D}$ in *loc. cit.* that $\mathcal{D}$ is of *RC-iso-subanchor* type. Next, let us observe that it follows immediately from the description of *monomorphisms* of $\mathcal{D}$ given in [Mzk3], Proposition 2.2, (ix), that $\mathcal{D}$ is of *FSM-* [hence, a fortiori, of *FSMFF-*] type. Since, moreover, any parallelogram P_1 [of type "$\Box$"] contained in [a compact subset of] a parallelogram P_2 [of type "$\Box$"] of the Euclidean plane admits *"intermediary parallelograms"* [of type "$\Box$"] P_3 [i.e., $P_1 \subseteq P_3 \subseteq P_2$, $P_3 \neq P_1, P_2$] in *two dimensions* [hence, in particular, intermediary P_3, P_4 [of type "$\Box$"] neither of which is contained in the other], it follows from the description of monomorphisms of $\mathcal{D}$ given in [Mzk3], Proposition 2.2, (ix), that $\mathcal{D}$ is of *strictly partially ordered* type.

(iii) Now, suppose that, in the notation of Example 1.4, we take "v" to be an *archimedean* valuation. We shall refer to the Galois extension $\widetilde{F}/F$ as *amply quadratic* if there exist *totally complex* extensions $F_1 \subseteq \widetilde{F}$ of F such that $[F_1 : F] \leq 2$. Relative to the notation $\mathcal{D} \to \mathcal{D}_0$ of (i), observe that we have a natural functor $\mathcal{P}_0 \to \mathcal{D}_0$; set $\mathcal{P} \stackrel{\mathrm{def}}{=} \mathcal{D} \times_{\mathcal{D}_0} \mathcal{P}_0$, $\mathcal{E} \stackrel{\mathrm{def}}{=} \mathcal{P} \times_{\mathcal{P}_0} \mathcal{E}_0$. Thus, $\mathcal{E}$ may be identified with $\mathcal{D} \times_{\mathcal{D}_0} \mathcal{E}_0$, and we note that, so long as $\widetilde{F}/F$ is *amply quadratic*, even if $\mathcal{P}_0 \to \mathcal{D}_0$ *fails* to be an equivalence of categories, the difference between $\mathcal{D}$ and $\mathcal{P}$ amounts, in effect, to forming "$\mathcal{D}$" as above, but with the original "X^*" replaced by the result of "tensoring X^* over $\mathbb{R}$ with $\mathbb{C}$". [Put another way, the assumption that $\widetilde{F}/F$ is *amply quadratic* allows one to rule the case where *every* object in the image of $\mathcal{P}_0$ in $\mathcal{D}_0$ is *real*.] Then:

> Suppose that $\widetilde{F}/F$ is *amply quadratic*. Then $\mathcal{E}$ [equipped with the natural functor $\mathcal{E} \to \mathcal{D}_0$] is a *slim* category of *RC-standard* and *discontinuously ordered* type [cf. Remark 4.2.1].

Indeed, the *slimness* of $\mathcal{E}$ follows immediately from the slimness of $\mathcal{D}$ [cf. (ii) above] and Proposition 1.5, (iv); the *total epimorphicity* of $\mathcal{E}$ follows immediately from the total epimorphicity of $\mathcal{D}$ [cf. (ii) above] and Proposition 1.5, (iii); the *RC-connectedness* follows immediately from the RC-connectedness of $\mathcal{D}$ [cf. (ii) above] and Proposition 1.5, (ii); the *complexifiability* of $\mathcal{E}$ follows immediately from the complexifiability of $\mathcal{D}$ [cf. (ii) above] and our assumption that $\widetilde{F}/F$ is *amply quadratic*; the fact that $\mathcal{E}$ is of *FSMFF-type* follows from the fact that $\mathcal{D}$ is *totally epimorphic* and of *FSM-type* [cf. (ii) above], and Proposition 1.5, (viii); the fact that $\mathcal{E}$ is of *RC-iso-subanchor* type follows from the corresponding fact for $\mathcal{D}$ [cf. (ii) above] and our assumption that $\widetilde{F}/F$ is *amply quadratic*. Finally, by Proposition 1.5, (ix), it follows that *totally ordered* morphisms of $\mathcal{E}$ project to totally ordered morphisms of $\mathcal{D}$, which are *isomorphisms* [since $\mathcal{D}$ is of *strictly partially ordered* type — cf. (ii) above]; thus, it follows immediately from the "discrete structure" of $\mathcal{E}_0$ that $\mathcal{E}$ is of *discontinuously ordered* type, as desired.

(iv) Finally, we observe that by considering parallelogram objects whose images in $X^*[\mathbb{C}]$ are *sufficiently small* and *disjoint*, it follows immediately that the categories $\mathcal{D}$, $\mathcal{E}$ are of *strongly dissectible* type [cf. Proposition 1.5, (vii)].

Remark 4.3.1. Note that, by contrast to the categories of Riemann surfaces discussed in Example 4.3, the categories of Riemann surfaces "$\mathfrak{Loc}_{\Pi}(X^*)$" of [Mzk3],

§1 — which are intended to embody the *upper half-plane uniformization* of a *hyperbolic Riemann surface* — *fail* to be *of FSMFF-type*. Indeed, if D is the *open unit disc* in the complex plane, and $V \subseteq D$, $U \subseteq V$ are *connected open subsets*, then the inclusion $U \hookrightarrow V$ determines a monomomorphism of this category "$\mathfrak{Loc}_\Pi(X^*)$". Thus, if, for instance, U is the *complement of a(n) [infinite] compact line segment* in D, then $U \hookrightarrow D$ determines an *FSM-morphism* [which is not an isomorphism] such that monomorphism factorizations of this morphism correspond to connected open subsets $W \subseteq D$ containing U. In particular, it follows immediately that this morphism $U \hookrightarrow D$ *fails* to factor as a composite of finitely many FSMI-morphisms. Also, we observe that if U is the *complement of an infinite discrete subset* of D, then "filling in successive points of the complement $D \backslash U$" [cf. the *slit morphisms* of Example 3.3, (v)] yields an *infinite sequence* of FSMI-morphisms $U \to U_1 \to U_2 \to \dots$. Thus, in summary, the category "$\mathfrak{Loc}_\Pi(X^*)$" does not satisfy *either* of the two conditions in the definition of a "category of FSMFF-type" [cf. [Mzk5], §0]. Moreover, we observe that if, for instance, X^* is *non-compact*, then its upper half-plane uniformization constitutes a morphism from an "object which is not of finite type" [in the terminology of [Mzk3], §1] to an "object of finite type" which does *not* have *pre-compact* image. Thus, it does not appear that this situation may be remedied by imposing a simple "pre-compact image condition" of the sort that occurs in the definition of the categories of Example 4.3 [cf. [Mzk3], §2].

Example 4.4. Archimedean Quasi-temperoids.

(i) Let

$$\mathcal{D} \stackrel{\text{def}}{=} \mathcal{B}^{\text{temp}}(\Pi, \Pi^\circ)^0$$

be any category as in Example 1.3, (iii), where $\Pi \to Q$, $G_{\mathbb{R}} \twoheadrightarrow Q$ [i.e., we take "F" of *loc. cit.* to be $\mathbb{R}$], $\Pi^\circ \subseteq \Pi$ are as in *loc. cit.*, and we assume further that the surjection $G_{\mathbb{R}} \twoheadrightarrow Q$ is an *isomorphism*. Thus, we have a *natural functor* $\mathcal{D} = \mathcal{B}^{\text{temp}}(\Pi, \Pi^\circ)^0 \to \mathcal{D}_0$ [cf. Example 1.3, (ii), (iii)].

(ii) Now we observe that:

> $\mathcal{D}$ [equipped with the natural functor $\mathcal{D} \to \mathcal{D}_0$] is a category of *RC-standard*, *FSM-*, and *strictly partially ordered* [hence, a fortiori, of *discontinuously ordered*] type [cf. Remark 4.2.1]. If, moreover, Π is *temp-slim*, then $\mathcal{D}$ is *slim*.

Indeed, this follows via the a similar [but somewhat easier] argument to the argument applied in Example 4.3, (ii) [cf. also Example 1.3, (i)].

(iii) Now, suppose that, in the notation of Example 1.4, we take "v" to be an *archimedean* valuation. Observe that we have a natural functor $\mathcal{P}_0 \to \mathcal{D}_0$. Set $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{D} \times_{\mathcal{D}_0} \mathcal{P}_0$; $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{P} \times_{\mathcal{P}_0} \mathcal{E}_0 \stackrel{\sim}{\to} \mathcal{D} \times_{\mathcal{D}_0} \mathcal{E}_0$ [cf. Example 4.3, (iii)], and we note that, so long as $\widetilde{F}/F$ is *amply quadratic*, even if $\mathcal{P}_0 \to \mathcal{D}_0$ *fails* to be an equivalence of categories, the difference between $\mathcal{D}$ and $\mathcal{P}$ amounts, in effect, to forming "$\mathcal{D}$" as above, but with the original "Π", "Π°" replaced by the kernel of the given morphism $\Pi \to Q \cong G_{\mathbb{R}}$ and its intersection with Π°, respectively. Then:

> If $\widetilde{F}/F$ is *amply quadratic*, then $\mathcal{E}$ [equipped with the natural functor $\mathcal{E} \to \mathcal{D}_0$] is a category of *RC-standard* and *discontinuously ordered* type [cf. Remark 4.2.1]. If, moreover, Π is *temp-slim*, then $\mathcal{E}$ is *slim*.

Indeed, this follows via the a similar [but somewhat easier] argument to the argument applied in Example 4.3, (iii).

(iv) Finally, we observe that the categories $\mathcal{D}$, $\mathcal{E}$ are of *strongly indissectible* type [cf. Example 1.3, (i); Proposition 1.5, (vii)].

Section 5: Poly-Frobenioids

In the present §5, we develop the "general nonsense" necessary to construct categories obtained by *"gluing together"* various Frobenioids in a certain fashion. We shall refer to the categories obtained in this way as *poly-Frobenioids*. This "gluing problem" is motivated by the goal of "gluing together" the Frobenioids of [Mzk5], Example 6.3 [which embody the *global arithmetic of number fields*] to the p-adic and archimedean Frobenioids of Examples 1.1, 3.3 [which embody the *arithmetic of local fields*] in a fashion that reflects, at the level of *category theory*, the way in which localizations of number fields are typically treated in arithmetic geometry.

Definition 5.1.

(i) We shall refer to as a *collection of grafting data* any collection of data

$$\mathfrak{A} \stackrel{\mathrm{def}}{=} \left(\mathcal{A}_{\circledcirc}, \{\mathcal{A}_\iota\}_{\iota\in I}, \{\underline{\mathcal{A}}_\iota\}_{\iota\in I}, \{\zeta_\iota : \mathcal{A}_{\circledcirc} \to \underline{\mathcal{A}}_\iota^{\top}\}_{\iota\in I}, \{\eta_\iota : \mathcal{A}_\iota \to \underline{\mathcal{A}}_\iota\}_{\iota\in I}\right)$$

where I is a set; the $\mathcal{A}_{\circledcirc}$, $\mathcal{A}_\iota$, $\underline{\mathcal{A}}_\iota$ are *connected, totally epimorphic categories*; "$\top$" is as in [Mzk5], §0; the arrows ζ_ι are functors, which we shall refer to as *global contact functors*; the functors η_ι, which we shall refer to as *local contact functors*, are *relatively initial* [cf. §0]. We shall refer to $\mathcal{A}_{\circledcirc}$ as the *global component* of $\mathfrak{A}$ and to the $\mathcal{A}_\iota$ as the *local components* of $\mathfrak{A}$. The *grafted category* associated to $\mathfrak{A}$ is defined to be the category

$$\mathcal{G} \stackrel{\mathrm{def}}{=} \mathcal{A}_{\circledcirc} \dashv_{\left(\prod_{\iota\in I} \underline{\mathcal{A}}_\iota^{\top}\right)} \left(\coprod_{\iota\in I} \mathcal{A}_\iota\right)$$

where "$\dashv$" is defined relative to the functors

$$\mathcal{A}_{\circledcirc} \to \prod_{\iota\in I} \underline{\mathcal{A}}_\iota^{\top}$$
$$\coprod_{\iota\in I} \mathcal{A}_\iota \to \prod_{\iota\in I} \mathcal{A}_\iota^{\top} \to \prod_{\iota\in I} \underline{\mathcal{A}}_\iota^{\top}$$

[cf. §0]; the arrow in the first line is the functor determined by the ζ_ι's; the first arrow in the second line is the natural functor [cf. §0]; the second arrow in the second line is the functor determined by the η_ι's. Thus, [cf. §0] we may think of the *global component*, as well as the *local components*, of $\mathfrak{A}$ as *full subcategories* of $\mathcal{G}$. An object or arrow that belongs to the global component (respectively, a local component $\iota \in I$) of $\mathcal{G}$ will be referred to as *global* (respectively, *[ι-]local*). Denote by $\mathcal{G}_{\circledcirc} \subseteq \mathcal{G}$ (respectively, $\mathcal{G}_\iota \subseteq \mathcal{G}$ [for $\iota \in I$]; $\mathcal{G}_{\dots} \subseteq \mathcal{G}$) the full subcategory determined by the global (respectively, ι-local; local) objects. A morphism of $\mathcal{G}$ from an ι-local object to a global object will be referred to as *ι-heterogeneous*.

(ii) We shall say that $\mathfrak{A}$ is *of uniformly dissectible type* if the following conditions are satisfied: (a) there exist at least 2 elements $\iota \in I$ for which ζ_ι is *totally non-initial* [cf. §0]; (b) for each $\iota \in I$, $\mathcal{A}_\iota$ is *either* of weakly indissectible type *or* of strongly dissectible type; (c) there exists an $\iota \in I$ such that $\mathcal{A}_\iota$ is of weakly indissectible type, and ζ_ι is totally non-initial. If, for $\iota \in I$, η_ι is an *equivalence of categories*, then we shall say that $\mathfrak{A}$ is *of ι-tactile type*; if $\mathfrak{A}$ is of ι-tactile type for all $\iota \in I$, then we shall say that $\mathfrak{A}$ is *of totally tactile type.* A morphism of $\mathcal{G}$ whose domain is [ι-]local [for some $\iota \in I$], whose codomain is global, and which is *minimal-adjoint* [cf. [Mzk5], §0] to the morphisms with local codomain, will be referred to as *[ι-]pre-tactile.* A morphism $A \to B$ of $\mathcal{G}$ whose domain is [ι-]local [for some $\iota \in I$], whose codomain is global, and which corresponds [cf. the definition of $\mathcal{G}$] to an isomorphism of $\eta_\iota(A)$ onto a connected component of $\zeta_\iota(B)$, will be referred to as *[ι-]tactile.* If a(n) [ι-]heterogeneous morphism $\phi : A \to B$ of $\mathcal{G}$ factors as a composite $A \to A' \to B$, where $A' \to B$ is [ι-]tactile, then we shall refer to this factorization as a(n) *[ι-]tactile factorization* of ϕ. If, for some $\iota \in I$, $\{A_j \to B\}_{j \in J}$ forms a collection of ι-tactile morphisms of $\mathcal{G}$ that corresponds [cf. the definition of $\mathcal{G}$] to an isomorphism $\coprod_{j \in J} \eta_\iota(A_j) \xrightarrow{\sim} \zeta_\iota(B)$, then we shall refer to the collection of morphisms $\{A_j \to B\}_{j \in J}$ as a *complete [ι-]tactile collection [for B].*

(iii) If

$$\mathfrak{A}' \stackrel{\text{def}}{=} \left(\mathcal{A}'_{\circledcirc}, \{\mathcal{A}'_{\iota'}\}_{\iota' \in I'}, \{\underline{\mathcal{A}}'_{\iota'}\}_{\iota' \in I'}, \{\zeta'_{\iota'} : \mathcal{A}'_{\circledcirc} \to \underline{\mathcal{A}}'_{\iota'}\}_{\iota' \in I'}, \{\eta'_{\iota'} : \mathcal{A}'_{\iota'} \to \underline{\mathcal{A}}'_{\iota'}\}_{\iota' \in I'}\right)$$

is another collection of grafting data, then we shall refer to as an *equivalence of collections of grafting data*

$$\Psi : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$$

a bijection $\Psi_I : I \xrightarrow{\sim} I'$, together with equivalences of categories

$$\Psi_{\circledcirc} : \mathcal{A}_{\circledcirc} \xrightarrow{\sim} \mathcal{A}'_{\circledcirc}$$

and

$$\Psi_\iota : \mathcal{A}_\iota \xrightarrow{\sim} \mathcal{A}'_{\iota'}; \quad \underline{\Psi}_\iota : \underline{\mathcal{A}}_\iota \xrightarrow{\sim} \underline{\mathcal{A}}'_{\iota'}$$

for $\iota \in I$, $\iota' \stackrel{\text{def}}{=} \Psi_I(\iota)$, which are 1-compatible with the various functors ζ_ι, $\zeta'_{\iota'}$, η_ι, $\eta'_{\iota'}$. [Thus, the equivalences of collections of grafting data $\mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$, together with isomorphisms [in the evident sense] between such equivalences, form a *category.*] If $\mathcal{G}'$ is the grafted category associated to $\mathfrak{A}'$, then it is immediate that Ψ determines an *equivalence of categories* $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$.

Remark 5.1.1. Since the *local components* of $\mathcal{G}$ are assumed to be *connected* [cf. Definition 5.1, (i)], it follows immediately that the $\mathcal{G}_\iota$ are precisely the *connected components* [cf. [Mzk5], §0] of the category $\mathcal{G}_{\dots}$, and that the category $\mathcal{G}_{\dots}$ is naturally equivalent to the *coproduct* [cf. §0] of the categories $\mathcal{G}_\iota$.

Remark 5.1.2. Let $\iota \in I$. Then the following assertions follow immediately from the definitions and the *total epimorphicity* assumption in Definition 5.1, (i): Every *ι-tactile* morphism of $\mathcal{G}$ is *ι-pre-tactile.* Suppose further that $\mathfrak{A}$ is of *ι-tactile* type. Then any two *ι-tactile factorizations* of an *ι-heterogeneous* morphism $\phi : A \to B$ of $\mathcal{G}$ determine *isomorphic* objects of the *category of factorizations* $\mathcal{G}_\phi$ [cf. §0]; every global object B of $\mathcal{G}$ admits a *complete* ι*-tactile collection*; if $\{A_j \to B\}_{j \in J}$ is any collection of ι-tactile morphisms of $\mathcal{G}$, then this collection is a *complete* ι*-tactile collection* for B if and only if every ι-heterogeneous morphism $C \to B$ factors through precisely one of the $A_j \to B$; finally, a morphism of $\mathcal{G}$ is ι*-tactile* if and only if it is ι*-pre-tactile*.

Proposition 5.2. **(Dissection of Grafted Categories)** *Let* $\mathfrak{A}$, $\mathfrak{A}'$ *be* **collections of grafting data** *of* **uniformly dissectible** *type, whose associated categories we denote by* $\mathcal{G}$, $\mathcal{G}'$*, respectively. Then:*

(i) The **global** *objects of* $\mathcal{G}$ *are* **strongly dissectible***.*

(ii) Let $\iota \in I$*. Suppose that* $\mathcal{A}_\iota$ *is of* **weakly indissectible** *(respectively,* **strongly dissectible***) type. Then the* ι**-local** *objects of* $\mathcal{G}$ *are* **weakly indissectible** *(respectively,* **strongly dissectible***).*

(iii) An object of $\mathcal{G}$ *is* **global** *if and only if it is a* **strongly dissectible** *object that appears as the codomain of a morphism whose domain is* **weakly indissectible***.*

(iv) Every **equivalence of categories** $\Xi : \mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ *induces an* **equivalence of categories** $\mathcal{G}_\circledcirc \xrightarrow{\sim} \mathcal{G}'_\circledcirc$*, a* **bijection** $\Xi_I : I \xrightarrow{\sim} I'$*, and, for each* $I \ni \iota \mapsto \iota' \in I'$*, an* **equivalence of categories** $\mathcal{G}_\iota \xrightarrow{\sim} \mathcal{G}'_{\iota'}$*. In particular,* Ξ *preserves* **global** *and* **local** *objects.*

(v) Suppose that $\mathfrak{A}$, $\mathfrak{A}'$ *are of* **totally tactile** *type. Then there is a* **natural equivalence of categories** *between the category of equivalences* $\mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ *and the category of equivalences* $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$*.*

Proof. To verify assertion (i), let $A \in \mathrm{Ob}(\mathcal{A}_\circledcirc)$. Since [cf. Definition 5.1, (ii), (a); Definition 5.1, (i)] there exist *distinct* $\iota_1, \iota_2 \in I$ such that [for $j = 1, 2$] ζ_{ι_j} is *totally non-initial*, and η_{ι_j} is *relatively initial*, it follows that there exists [for $j = 1, 2$] a morphism $A_j \to A$, where $A_j \in \mathrm{Ob}(\mathcal{G})$ is ι_j*-local*. On the other hand, since no object of $\mathcal{G}$ admits a morphism to both A_1 and A_2, it follows that $A_1 \to A$, $A_2 \to A$ form a *strongly dissecting* pair of arrows, as desired. Assertion (ii) follows immediately from the definitions [together with the fact that the domain of any morphism of $\mathcal{G}$ whose codomain is ι-local is itself ι-local]. Assertion (iii) follows immediately from assertions (i), (ii) [cf. also Definition 5.1, (ii), (b)], together with the existence [cf. Definition 5.1, (ii), (c); Definition 5.1, (i)] of an $\iota \in I$ such that $\mathcal{A}_\iota$ is of *weakly indissectible* type, ζ_ι is *totally non-initial*, and η_ι is *relatively initial*. Assertion (iv) follows immediately from assertion (iii), Remark 5.1.1. Finally, assertion (v) follows immediately from assertion (iv), by considering *complete tactile collections* of morphisms [cf. Remark 5.1.2]. ◯

Observe that our discussion so far has nothing to do with Frobenioids [at least in an explicit sense]. We now return to discussing *Frobenioids*.

Definition 5.3. For $i = 1, 2$, let $\mathcal{C}_i$ be a *Frobenioid* over a *base category* $\mathcal{D}_i$, with *perf-factorial divisor monoid* Φ_i. Note that Φ_i extends naturally to a functor $\Phi_i^{\top}$ on $\mathcal{D}_i^{\top}$ by assigning to a coproduct of objects $\{A_j\}_{j \in J}$ of $\mathcal{D}_i$ the direct sum of the monoids $\Phi_i(A_j)$.

(i) Suppose that $\mathcal{C}_1$ is of *isotropic* type. Then we shall refer to a functor

$$\Psi : \mathcal{C}_1 \to \mathcal{C}_2^{\top}$$

as *GC-admissible* [i.e., "global contact-admissible"] if the following conditions are satisfied: (a) there exists a functor $\Psi^{\mathrm{Base}} : \mathcal{D}_1 \to \mathcal{D}_2^{\top}$ which is 1-compatible with Ψ, [relative to the *natural projection functors* $\mathcal{C}_i \to \mathcal{D}_i$, for $i = 1, 2$]; (b) Ψ maps an arrow of *Frobenius degree* $d \in \mathbb{N}_{\geq 1}$ of $\mathcal{C}_1$ to an arrow of $\mathcal{C}_2^{\top}$ each of whose component arrows [$\in \mathrm{Arr}(\mathcal{C}_2)$] is of Frobenius degree d; (c) there exists a natural transformation $\Psi^{\Phi} : \Phi_1 \to \Phi_2^{\top}|_{\mathcal{C}_1}$ that is compatible with Ψ relative to the functors $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ [for $i = 1, 2$] that define the Frobenioid structures of the $\mathcal{C}_i$ [in other words: Ψ^{Φ} is compatible with Ψ relative to the operation of taking the *zero divisor*]; (d) every component object $\in \mathrm{Ob}(\mathcal{C}_2)$ of an object in the image of Ψ is *isotropic*.

(ii) We shall refer to a functor

$$\Psi : \mathcal{C}_1 \to \mathcal{C}_2$$

as *LC-admissible* [i.e., "local contact-admissible"] if there exists a subfunctor of group-like monoids

$$\Phi^{\mathrm{cnst}} \subseteq (\Phi_1^{\mathrm{rlf}})^{\mathrm{birat}}$$

[where "rlf" is as in [Mzk5], Definition 2.4, (i); cf. also [Mzk5], Proposition 5.3] on $\mathcal{D}_1$ of *"constant realified rational functions"* such that the following conditions are satisfied: (a) there exists an *equivalence of categories* $\Psi^{\mathrm{Base}} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$ which is 1-compatible with Ψ [relative to the *natural projection functors* $\mathcal{C}_i \to \mathcal{D}_i$, for $i = 1, 2$]; (b) Ψ is compatible with *Frobenius degrees*; (c) the natural inclusion $\Phi_1 \hookrightarrow \Phi_1^{\mathrm{rlf}}$ factors as a composite of inclusions

$$\Phi_1 \hookrightarrow \Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\mathrm{rlf}}$$

of functors of monoids on $\mathcal{D}_1$ that are compatible with Ψ relative to the functors $\mathcal{C}_i \to \mathbb{F}_{\Phi_i}$ [for $i = 1, 2$] that define the Frobenioid structures of the $\mathcal{C}_i$ [in other words: relative to the operation of taking the *zero divisor*]; (d) applying the operation of "groupification" to the monoids of (c) induces inclusions

$$\Phi_1^{\mathrm{birat}} \hookrightarrow \Phi_2^{\mathrm{birat}}|_{\mathcal{D}_1} \hookrightarrow (\Phi_1^{\mathrm{rlf}})^{\mathrm{birat}} = \mathbb{R} \cdot \Phi_1^{\mathrm{birat}}$$

[where we recall that $\Phi_i^{\mathrm{birat}} \subseteq \Phi_i^{\mathrm{gp}}$, for $i = 1, 2$ — cf. [Mzk5], Proposition 4.4, (iii)] of functors of monoids on $\mathcal{D}_1$; (e) the image of the inclusion $\Phi_2^{\mathrm{birat}}|_{\mathcal{D}_1} \hookrightarrow (\Phi_1^{\mathrm{rlf}})^{\mathrm{birat}}$ of (d) is equal to the subfunctor of [group-like] monoids $\Phi_1^{\mathrm{birat}} + \Phi^{\mathrm{cnst}} \subseteq (\Phi_1^{\mathrm{rlf}})^{\mathrm{birat}}$; (f) the functor Φ^{cnst} is *"constant"* on $\mathcal{D}_1$ in the sense that Φ^{cnst} maps every arrow of $\mathcal{D}_1$ to an isomorphism; (g) Ψ preserves *isotropic objects* and *co-angular morphisms*. We shall say that Ψ is *LC-unit-admissible* if Ψ is LC-admissible, and, moreover, the following condition is satisfied: (h) if $A_2 = \Psi(A_1) \in \mathrm{Ob}(\mathcal{C}_2)$, where $A_1 \in \mathrm{Ob}(\mathcal{C}_1)$, then the induced homomorphism of abelian groups $\mathcal{O}^\times(A_1) \to \mathcal{O}^\times(A_2)$ induces a *surjection on perfections* $\mathcal{O}^\times(A_1)^{\mathrm{pf}} \twoheadrightarrow \mathcal{O}^\times(A_2)^{\mathrm{pf}}$.

(iii) We shall refer to as a *collection of Frobenioid-theoretic grafting data* any collection of grafting data

$$\mathfrak{C} \stackrel{\mathrm{def}}{=} \left(\mathcal{C}_{\circledcirc}, \{\mathcal{C}_\iota\}_{\iota \in I}, \{\underline{\mathcal{C}}_\iota\}_{\iota \in I}, \{\zeta_\iota : \mathcal{C}_{\circledcirc} \to \underline{\mathcal{C}}_\iota^{\top}\}_{\iota \in I}, \{\eta_\iota : \mathcal{C}_\iota \to \underline{\mathcal{C}}_\iota\}_{\iota \in I}\right)$$

such that the following conditions are satisfied: (a) the categories $\mathcal{C}_{\circledcirc}$, $\mathcal{C}_\iota$, $\underline{\mathcal{C}}_\iota$ [for $\iota \in I$] are equipped with *Frobenioid* structures, the base categories of which we denote by $\mathcal{D}_{\circledcirc}$, $\mathcal{D}_\iota$, $\underline{\mathcal{D}}_\iota$, respectively, and the divisor monoids of which we denote by $\Phi_{\circledcirc}$, Φ_ι, $\underline{\Phi}_\iota$, respectively; (b) the divisor monoids $\Phi_{\circledcirc}$, Φ_ι, $\underline{\Phi}_\iota$ [for $\iota \in I$] are all *perf-factorial*; (c) the *global contact functors* ζ_ι [for $\iota \in I$] are *GC-admissible* [which implies, in particular, that $\mathcal{C}_{\circledcirc}$ is of *isotropic* type]; (d) the *local contact functors* η_ι [for $\iota \in I$] are *LC-admissible.* A category obtained as the grafted category associated to such data $\mathfrak{C}$ will be referred to as a *poly-Frobenioid* $\mathcal{C}$. If all of the η_ι [for $\iota \in I$] are LC-unit-admissible, then we shall say that $\mathfrak{C}$ or $\mathcal{C}$ is *of LC-unit-admissible type.* Observe that $\mathfrak{C}$ determines a *collection of base-category grafting data* $\mathfrak{D}$, which is of *totally tactile* type [cf. (ii), (a)]; the associated grafted category $\mathcal{D}$ will be referred to as the *poly-base category* of the poly-Frobenioid $\mathcal{C}$. [Here, we observe that, by a [harmless!] abuse of notation, "$\mathcal{C}_{\circledcirc}$" denotes both the category "$\mathcal{A}_{\circledcirc}$" and the category "$\mathcal{G}_{\circledcirc}$" of Definition 5.1, (i); a similar remark holds for "$\mathcal{D}_{\circledcirc}$", "$\mathcal{C}_\iota$", "$\mathcal{D}_\iota$", where $\iota \in I$.] Thus, we have a *natural projection functor* $\mathcal{C} \to \mathcal{D}$. If $\mathbb{P}$ is a *property of Frobenioids* (respectively, *base categories of Frobenioids*), then we shall say that $\mathfrak{C}$ or $\mathcal{C}$ (respectively, $\mathfrak{D}$ or $\mathcal{D}$) satisfies this property $\mathbb{P}$ if the Frobenioids $\mathcal{C}_{\circledcirc}$, $\mathcal{C}_\iota$, $\underline{\mathcal{C}}_\iota$ [for $\iota \in I$] (respectively, the base categories $\mathcal{D}_{\circledcirc}$, $\mathcal{D}_\iota$, $\underline{\mathcal{D}}_\iota$ [for $\iota \in I$]) all satisfy this property $\mathbb{P}$. [In particular, one must be careful to distinguish between the assertion that "$\mathcal{D}$ satisfies $\mathbb{P}$ as an *abstract category*" and the assertion that "$\mathcal{D}$ satisfies $\mathbb{P}$ as a *poly-base category* of a poly-Frobenioid".]

(iv) Let $\phi : A \to B$ be a morphism of the poly-Frobenioid $\mathcal{C}$. We shall denote the projection of ϕ to $\mathcal{D}$ by $\mathrm{Base}(\phi)$. Observe that by conditions (i), (b), and (ii), (b), above, it makes sense to speak of the *Frobenius degree* $\deg_{\mathrm{Fr}}(\phi) \in \mathbb{N}_{\geq 1}$ of ϕ; if $\deg_{\mathrm{Fr}}(\phi) = 1$, then we shall say that ϕ is *linear.* If, for $\iota \in I$, ϕ is ι-local (respectively, ι-heterogeneous), then it makes sense to speak of the *zero divisor* $\mathrm{Div}(\phi) \in \Phi_\iota(A)$ (respectively, $\mathrm{Div}(\phi) \in \underline{\Phi}_\iota(A)$); if ϕ is global, then it makes sense to speak of the *zero divisor* $\mathrm{Div}(\phi) \in \Phi_{\circledcirc}(A)$. We shall say that ϕ satisfies a *property* $\mathbb{P}$ *of arrows of Frobenioids* if ϕ is *homogeneous*, and, moreover, satisfies the property $\mathbb{P}$ when regarded as an arrow of one of the local or global component Frobenioids of $\mathcal{C}$; a similar convention will be applied to objects of $\mathcal{C}$. We shall say that ϕ is *base-[ι-]tactile* if it projects to a(n) [ι-]tactile morphism of $\mathcal{D}$ [for $\iota \in I$]. We shall say that ϕ is *birationally [ι-]tactile* [i.e., "base-linear-tactile"] if it is linear and base-[ι-]tactile [for $\iota \in I$], with isotropic domain. We shall say that ϕ is *base-[ι-]pre-tactile* if ϕ is [ι-]heterogeneous, and, moreover, for every factorization $A \to A' \to B$ of ϕ, where A' is [ι-]local, it holds that $A \to A'$ is a base-isomorphism [for $\iota \in I$]. We shall say that ϕ is *birationally [ι-]pre-tactile* if ϕ is [ι-]heterogeneous, and, moreover, for every factorization $A \to A' \to B$ of ϕ, where A' is [ι-]local, it holds that $A \to A'$ is a co-angular pre-step [for $\iota \in I$]. If ϕ factors as a composite $A \to A' \to B$, where $A' \to B$ is base-[ι-]tactile (respectively, birationally [ι-]tactile) [for some $\iota \in I$], then we shall refer to this factorization as a *base-[ι-]tactile* (respectively, *birationally [ι-]tactile*) *factorization* of ϕ. If, for some $\iota \in I$, $\{A_j \to B\}_{j \in J}$ forms a collection of birationally ι-tactile morphisms of $\mathcal{C}$ that projects to a complete ι-tactile collection of Base(B), then we shall refer to the collection of morphisms $\{A_j \to B\}_{j \in J}$ as a *complete birationally [ι-]tactile collection [for B]*.

(v) A routine check reveals [cf. Remarks 5.3.1, 5.3.3, below; [Mzk5], Definition 2.4, (i); [Mzk5], Proposition 3.2, (ii), (iii); [Mzk5], Proposition 4.4, (iv)] that applying the operations

"pf", "un-tr", "rlf", "birat", or "istr"

[where "pf" is defined whenever $\mathfrak{C}$ is of *Frobenius-isotropic* type] to each of the Frobenioids $\mathcal{C}_{\circledcirc}$, $\mathcal{C}_\iota$, $\underline{\mathcal{C}}_\iota$ [for $\iota \in I$] yields a new collection of Frobenioid-theoretic grafting data "$\mathfrak{C}^{\mathrm{pf}}$", "$\mathfrak{C}^{\mathrm{un\text{-}tr}}$", "$\mathfrak{C}^{\mathrm{rlf}}$", "$\mathfrak{C}^{\mathrm{birat}}$", or "$\mathfrak{C}^{\mathrm{istr}}$", whose associated *poly-Frobenioids* "$\mathcal{C}^{\mathrm{pf}}$", "$\mathcal{C}^{\mathrm{un\text{-}tr}}$", "$\mathcal{C}^{\mathrm{rlf}}$", "$\mathcal{C}^{\mathrm{birat}}$", or "$\mathcal{C}^{\mathrm{istr}}$" we shall refer to as the *perfection*, *unit-trivialization*, *realification*, *birationalization*, or *isotropification* of the poly-Frobenioid $\mathcal{C}$. Observe, moreover, that each of the operations "un-tr", "rlf", "istr" preserves the property of being of *LC-unit-admissible* type; if $\mathfrak{C}$ is of *Frobenius-normalized* type, then the operation "pf" preserves the property of being of *LC-unit-admissible* type [cf. [Mzk5], Proposition 5.5, (i)].

(vi) If $\mathfrak{C}$ is of *uniformly dissectible*, *LC-unit-admissible*, and *standard* [cf. the convention concerning "$\mathbb{P}$" in (iii) above] type, then we shall say that $\mathfrak{C}$ or $\mathcal{C}$ is *of poly-standard type*. If $\mathfrak{C}$ is of *uniformly dissectible*, *LC-unit-admissible*, and *rationally standard* [cf. the convention concerning "$\mathbb{P}$" in (iii) above] type, then we shall say that $\mathfrak{C}$ or $\mathcal{C}$ is *of poly-rationally standard type*. If each of the Frobenioids $\mathcal{C}_{\circledcirc}$, $\mathcal{C}_\iota$, $\underline{\mathcal{C}}_\iota$ [for $\iota \in I$] contains a non-group-like object, then we shall say that $\mathfrak{C}$ or $\mathcal{C}$ is *of poly-non-group-like type*.

Remark 5.3.1. Observe that if a morphism ϕ of the domain Frobenioid of a *GC-admissible* functor Ψ is a *morphism of Frobenius type* (respectively, *pre-step*; *pull-back morphism*; *linear morphism*; *isometry*; *base-isomorphism*; *base-identity endomorphism*), then so is each of the component arrows of $\Psi(\phi)$ [cf. Definition 5.3, (i); [Mzk5], Definition 1.3, (vii), (b); [Mzk5], Proposition 1.4, (ii)].

Remark 5.3.2. We observe in passing that, in the notation of Definition 5.3, (i), the functor $\Psi : \mathcal{C}_1 \to \mathcal{C}_2^{\top}$ that maps every object of $\mathcal{C}_1$ to the *initial* object of $\mathcal{C}_2^{\top}$ is a *GC-admissible* functor which is *not totally non-initial*. In particular, Definition 5.3, (iii), does not rule out the possibility that *some* [but, at least in the case of data of *uniformly dissectible* type, *not all* — cf. Definition 5.1, (ii), (a), (c)] of the global contact functors that appear in a collection of Frobenioid-theoretic grafting data are functors of this form [i.e., functors that map every object to the initial object].

Remark 5.3.3. Observe that [in the notation of Definition 5.3, (ii)] any *LC-admissible* functor $\Psi : \mathcal{C}_1 \to \mathcal{C}_2$ induces an *equivalence of categories* $\mathcal{C}_1^{\mathrm{rlf}} \xrightarrow{\sim} \mathcal{C}_2^{\mathrm{rlf}}$ between the associated *realifications*. Indeed, this follows immediately from Definition 5.3, (ii), (a), (c), (d) [cf. also the definition of the "realification" in [Mzk5], Proposition 5.3]. In particular, [cf. also Definition 5.3, (ii), (g); [Mzk5], Proposition 1.4, (ii)] Ψ preserves *morphisms of Frobenius type*, *pre-steps*, *pull-back morphisms*, *linear morphisms*, *isometries*, *isotropic hulls*, *co-angular morphisms*, *base-isomorphisms*, and *base-identity endomorphisms*.

Remark 5.3.4. It is immediate from the definitions that *"birationally pre-tactile"* implies *"base-pre-tactile"*, that *"birationally tactile"* implies *"base-tactile"*, that *"birationally tactile"* implies *"birationally pre-tactile"*, and that *"base-tactile"* implies *"base-pre-tactile"* [cf. Remark 5.1.2].

Proposition 5.4. **(Birationally Tactile Factorizations)** *In the notation of Definition 5.3, (iii), suppose that the* **poly-Frobenioid** $\mathcal{C}$ *is of* **LC-unit-admissible**, **Frobenius-normalized**, *and* **perfect** *type. Let* $\phi : A \to B$ *be an* ι**-heterogeneous** *morphism, where* $\iota \in I$, *of* $\mathcal{C}$, *whose image in the* **birationalization** *[cf. Definition 5.3, (v)]* $\mathcal{C}^{\mathrm{birat}}$ *of* $\mathcal{C}$ *we denote by* $\phi^{\mathrm{birat}} : A^{\mathrm{birat}} \to B^{\mathrm{birat}}$. *Then:*

(i) ϕ^{birat} *admits a* **tactile factorization**.

(ii) ϕ^{birat} *is* **tactile** *if and only if it is* **pre-tactile**.

(iii) There exists a co-angular pre-step $\psi : C \to A$ *such that* $\phi \circ \psi$ *admits a* **birationally tactile factorization**.

(iv) There exists a co-angular pre-step $\psi : C \to A$ *such that* $\phi \circ \psi$ *is* **base-tactile** *(respectively,* **birationally tactile**) *if and only if* $\phi \circ \psi$ *is* **base-pre-tactile** *(respectively,* **birationally pre-tactile**).

Proof. First, we consider assertion (i). Since every connected component of an object in the essential image of the global contact functor labeled ι is *isotropic* [cf. Definition 5.3, (i), (d)], it follows that ϕ^{birat} factors through any *isotropic hull* of A^{birat}; thus, we may assume without loss of generality that A, A^{birat} [cf. [Mzk5], Proposition 4.4, (iv)] are *isotropic*. Also, by factoring ϕ through a morphism of Frobenius type with domain A [cf. [Mzk5], Definition 1.3, (ii), (iv)], we may assume without loss of generality that ϕ is *linear*. Since the isomorphism class of an isotropic [hence *Frobenius-trivial* — cf. [Mzk5], Proposition 1.10, (vi)] object of $\underline{\mathcal{C}}_{\iota}^{\mathrm{birat}}$ is determined by the isomorphism class of its projection to $\mathcal{D}_{\iota}$ [cf. [Mzk5], Theorem 5.1, (iii)], and $\mathfrak{D}$ is of *totally tactile* type, it follows that there exists an object $D \in \mathrm{Ob}(\mathcal{C}_{\iota}^{\mathrm{istr}})$ such that the morphism $\eta_{\iota}(A)^{\mathrm{birat}} \to \zeta_{\iota}(B)^{\mathrm{birat}}$ determined by ϕ admits a *factorization*

$$\eta_{\iota}(A)^{\mathrm{birat}} \xrightarrow{\alpha} \eta_{\iota}(D)^{\mathrm{birat}} \xrightarrow{\delta} \zeta_{\iota}(B)^{\mathrm{birat}}$$

[where δ determines a *tactile* morphism of $\mathcal{C}^{\mathrm{birat}}$; we use the superscript "birat" to denote the images of objects or arrows of Frobenioids or poly-Frobenioids in their respective *birationalizations*]. Since A, A^{birat}, hence also $\eta_\iota(A)^{\text{birat}}$ [cf. Definition 5.3, (ii), (g)], are *isotropic*, it follows that the first arrow $\alpha : \eta_\iota(A)^{\text{birat}} \to \eta_\iota(D)^{\text{birat}}$ of this factorization is *linear*, *co-angular*, and [cf. the definition of "birat"!] *isometric*. It thus follows that α is a *pull-back morphism* [cf. [Mzk5], Proposition 1.4, (ii)], hence [cf. [Mzk5], Definition 1.3, (i), (c); the definition of a "pull-back morphism" in [Mzk5], Definition 1.2, (ii)] that this arrow differs from the image in $\underline{\mathcal{C}}_\iota^{\text{birat}}$ of a pull-back morphism $\beta : A^{\text{birat}} \to D^{\text{birat}}$ in $\mathcal{C}_\iota^{\text{birat}}$ by composition with a *unit* of $\underline{\mathcal{C}}_\iota^{\text{birat}}$, i.e., an element $u \in \mathcal{O}^\times(\eta_\iota(A)^{\text{birat}})$ [so $\alpha = \eta_\iota(\beta) \circ u$]. Moreover, if we apply the isomorphism "$\Phi_2^{\text{birat}}|_{\mathcal{D}_1} \xrightarrow{\sim} \Phi_1^{\text{birat}} + \Phi^{\text{cnst}}$ ($\subseteq (\Phi_1^{\text{rlf}})^{\text{birat}}$)" of Definition 5.3, (ii), (e), it follows that we may write $u = u_1 \cdot u_{\text{cnst}}$, where $u_1 = \eta_\iota(v_1)$ for some $v_1 \in \mathcal{O}^\times(A^{\text{birat}})$, and the image of $u_{\text{cnst}} \in \mathcal{O}^\times(\eta_\iota(A)^{\text{birat}})$ in $(\Phi_1^{\text{rlf}})^{\text{birat}}$ lies in Φ^{cnst}; by applying Definition 5.3, (ii), (f), (h) [cf. also our assumption that the poly-Frobenioid $\mathcal{C}$ is of *LC-unit-admissible*, *Frobenius-normalized*, and *perfect* type; [Mzk5], Proposition 5.5, (i)], it follows that we may write

$$u_{\text{cnst}} = v_{\text{cnst}}|_{\eta_\iota(A)^{\text{birat}}} \cdot \eta_\iota^{\text{birat}}(w_{\text{cnst}})$$

where $v_{\text{cnst}} \in \mathcal{O}^\times(\eta_\iota(D)^{\text{birat}})$, $w_{\text{cnst}} \in \mathcal{O}^\times(A^{\text{birat}})$. Thus, by replacing α by $v_{\text{cnst}}^{-1} \circ \alpha$ and β by $\beta \circ w_{\text{cnst}} \circ v_1$, we conclude that we may assume that $\alpha = \eta_\iota^{\text{birat}}(\beta)$. Then β, δ determine a *tactile factorization* of ϕ^{birat}, as desired. This completes the proof of assertion (i).

Now assertion (ii) follows immediately from the *existence of tactile factorizations* [cf. assertion (i)] and the *total epimorphicity* of $\underline{\mathcal{C}}_\iota^{\text{birat}}$ [cf. Remark 5.1.2]. Assertion (iii) follows immediately from assertion (i); the definition of the *"birationalization"* [cf. [Mzk5], Proposition 4.4, (i)]; [Mzk5], Proposition 1.11, (vii); and the inclusions "$\Phi_1 \hookrightarrow \Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\text{rlf}}$" of Definition 5.3, (ii), (c) [which imply that pre-steps of the *underlined* local components may be dominated by pre-steps of the *non-underlined* local components]. Finally, assertion (iv) follows immediately from assertion (iii) and the *total epimorphicity* of $\mathcal{D}_\iota$ [cf. also Remark 5.3.4; [Mzk5], Proposition 1.9, (iv); the derivation of assertion (ii) from assertion (i)]. ◯

Theorem 5.5. **(Category-theoreticity of Poly-Frobenioids)** *For $i = 1, 2$, let $\mathcal{C}_i$ be a* **poly-Frobenioid** *of* **poly-standard** *type;*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

an **equivalence of categories**. *Then:*

(i) Ψ preserves **global** *and* **local** *objects and induces a* **compatible bijection** *$I_1 \xrightarrow{\sim} I_2$ between the sets I_1, I_2 of local components of $\mathcal{C}_1$, $\mathcal{C}_2$. Moreover, $\mathcal{C}_1$ is of* **poly-non-group-like** *type if and only if $\mathcal{C}_2$ is.*

(ii) Ψ preserves the **full subcategories** *$\mathcal{C}_i^{\text{istr}} \subseteq \mathcal{C}_i$.*

(iii) Let $\square \in \{\text{pf}, \text{un-tr}, \text{rlf}, \text{birat}\}$. If $\square = \text{un-tr}$, then we assume further that the poly-base category $\mathcal{D}_i$ is **Frobenius-slim**. *If $\square = \text{rlf}$, then we assume further that $\mathcal{C}_i$ is of* **poly-rationally standard** *type, and that the poly-base category $\mathcal{D}_i$*

is **slim**. *If* $\square \neq$ birat, *then we assume further that* $\mathcal{C}_i$ *is of* **poly-non-group-like** *type. Then there exists a* 1**-unique** 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{C}_1^{\square} & \xrightarrow{\Psi^{\square}} & \mathcal{C}_2^{\square} \end{array}$$

[where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories].

(iv) Suppose that the poly-base category $\mathcal{D}_i$ *is* **slim**, *and that* $\mathcal{C}_i$ *is of* **poly-non-group-like** *type. Then there exists a* 1**-unique** 1*-commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{D}_1 & \xrightarrow{\Psi^{\mathrm{Base}}} & \mathcal{D}_2 \end{array}$$

[where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories].

(v) Suppose that $\mathcal{C}_i$ *is of* **poly-rationally standard** *and* **poly-non-group-like** *type, and that the poly-base category* $\mathcal{D}_i$ *is* **slim**. *Then* Ψ *induces* **isomorphisms** *between the* **divisor monoids** *of corresponding [cf. (i)] local or global components of* $\mathcal{C}_1$, $\mathcal{C}_2$. *Moreover, these isomorphisms are* **compatible** *with the* **local** *and* **global contact functors** *that determine the poly-Frobenioid structures of* $\mathcal{C}_1$, $\mathcal{C}_2$ *[a statement that makes sense in light of the inclusions "*$\Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\mathrm{rlf}}$*" of Definition 5.3, (ii), (c)].*

Proof. Assertion (i) follows formally from Proposition 5.2, (iv), and [Mzk5], Theorem 3.4, (ii). In light of assertion (i), assertion (ii) follows from [Mzk5], Theorem 3.4, (i). Next, let us observe that $\mathcal{C}_i^{\mathrm{pf}}$ may be constructed from $\mathcal{C}_i$ by considering appropriate *inductive limits* involving *pairs of morphisms of Frobenius type*, just as in the case of Frobenioids [cf. [Mzk5], Definition 3.1, (ii), (iii)]. Since, whenever $\square \neq$ birat [so $\mathcal{C}_i$ is of *poly-non-group-like* type], Ψ preserves *pairs of morphisms of Frobenius type of the same Frobenius degree* [cf. assertion (i); [Mzk5], Theorem 3.4, (iii)], we thus conclude that assertion (iii) holds when $\square =$ pf. In a similar vein, it follows from the inclusions "$\Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\mathrm{rlf}}$" of Definition 5.3, (ii), (c) [which imply that pre-steps of the *underlined* local components may be dominated by pre-steps of the *non-underlined* local components], that $\mathcal{C}_i^{\mathrm{birat}}$ may be constructed from $\mathcal{C}_i$ by *"inverting the co-angular pre-steps"*, just as in the case of Frobenioids [cf. [Mzk5], Proposition 4.4, (i)]. Since Ψ preserves *co-angular pre-steps* [cf. assertion (i); [Mzk5], Theorem 3.4, (ii); [Mzk5], Corollary 4.10], we thus conclude that assertion (iii) holds when $\square =$ birat. Thus, in the following, we assume that $\square \neq$ birat [so $\mathcal{C}_i$ is of *poly-non-group-like* type].

Next, observe that by [Mzk5], Proposition 5.5, (iii), it follows that $\mathcal{C}_i^{\mathrm{pf}}$ is of *standard* type [i.e., each of its local and global component Frobenioids is of standard type]. Thus, by *"applying the last two arguments in succession"*, it follows that we obtain a 1-*unique* 1-commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ (\mathcal{C}_1^{\mathrm{pf}})^{\mathrm{birat}} & \xrightarrow{\Psi^{\Box}} & (\mathcal{C}_2^{\mathrm{pf}})^{\mathrm{birat}} \end{array}$$

[where the vertical arrows are the natural functors; the horizontal arrows are equivalences of categories]. [Note that here, it is not clear that the *uniform dissectibility* — hence, a fortiori, the property of being of *poly-standard* type — of $\mathcal{C}_i$ implies that of $\mathcal{C}_i^{\mathrm{pf}}$! Thus, one cannot apply to "$\mathcal{C}_i^{\mathrm{pf}}$" the portion of Theorem 5.5 already proven for "$\mathcal{C}_i$". Nevertheless, this does not cause any problems since uniform dissectibility is only used to *"dissect"* the poly-Frobenioid in question; thus, the necessary "dissection" of $\mathcal{C}_i^{\mathrm{pf}}$ follows immediately from the dissection already discussed of $\mathcal{C}_i$.]

Next, let us observe that since, for $A \in \mathrm{Ob}(\mathcal{C}_i)$ with image $A^{\mathrm{pf}} \in \mathrm{Ob}(\mathcal{C}_i^{\mathrm{pf}})$, we have a natural isomorphism $\mathcal{O}^{\times}(A)^{\mathrm{pf}} \xrightarrow{\sim} \mathcal{O}^{\times}(A^{\mathrm{pf}})$ [cf. [Mzk5], Proposition 5.5, (i)], it follows that the *LC-unit-admissibility* of $\mathcal{C}_i$ implies that of $\mathcal{C}_i^{\mathrm{pf}}$ [cf. Definition 5.3, (v)]. Thus, $\mathcal{C}_i^{\mathrm{pf}}$ is of *standard*, *perfect* [cf. [Mzk5], Proposition 3.2, (iii)], and *LC-unit-admissible* type. In particular, we may apply Proposition 5.4 to $\mathcal{C}_i^{\mathrm{pf}}$. Thus, it follows that global objects of $(\mathcal{C}_i^{\mathrm{pf}})^{\mathrm{birat}}$ admit *complete tactile collections* [cf. Proposition 5.4, (i)], which are, moreover, *preserved by* Ψ [cf. Proposition 5.4, (ii); the fact that, in light of assertion (i), the definition of "pre-tactile" is *manifestly category-theoretic*]. [Here, we note in passing that if one wishes to restrict oneself to operating in $\mathcal{C}_i^{\mathrm{pf}}$, then instead of working with complete tactile collections in $(\mathcal{C}_i^{\mathrm{pf}})^{\mathrm{birat}}$, one can instead work with *complete birationally tactile collections* in $\mathcal{C}_i^{\mathrm{pf}}$ — cf. Proposition 5.4, (iii), (iv).]

Now let us consider assertion (iv). Since the poly-base category $\mathcal{D}_i$ is assumed to be *slim*, it follows from [Mzk5], Theorem 3.4, (v) [cf. also assertion (i); our assumption that $\mathcal{C}_i$ is of *poly-non-group-like* type], that Ψ induces equivalences of categories between the various local and global components of $\mathcal{D}_i$ which are *compatible* with the equivalences of categories induced by Ψ between the various local and global components of $\mathcal{C}_i$. Moreover, by considering *complete tactile collections* in $(\mathcal{C}_i^{\mathrm{pf}})^{\mathrm{birat}}$, it follows immediately that these equivalences of categories between the various local and global components of $\mathcal{D}_i$ are *compatible* with the *local and global contact functors* of $\mathcal{D}_i$. Thus, we obtain a 1-unique 1-commutative diagram as in assertion (iv).

Next, we consider *unit-trivializations* [i.e., the case "$\Box$ = un-tr"]. By the *Frobenius-slimness* assumption, it follows from [Mzk5], Theorem 3.4, (iv), that Ψ induces equivalences of categories between the various local and global components of $\mathcal{C}_i^{\text{un-tr}}$ which are *compatible* with the equivalences of categories induced by Ψ between the various local and global components of $\mathcal{C}_i$. Thus, to obtain a diagram as in assertion (iii), it suffices to show that Ψ preserves *unit-equivalent pairs of co-objective heterogeneous morphisms* of $\mathcal{C}_i^{\mathrm{istr}}$ [i.e., pairs that map to the same arrow in $\mathcal{C}_i^{\text{un-tr}}$]. By the *faithfulness* portion of [Mzk5], Proposition 3.3, (iv) [applied to the *underlined local components* of the $\mathcal{C}_i$!], it follows that it suffices to show that Ψ maps a unit-equivalent pair of heterogeneous morphisms $\phi_1, \psi_1 : A \to B$ of $\mathcal{C}_1$ to a pair of morphisms $\phi_2 \stackrel{\mathrm{def}}{=} \Psi(\phi_1)$, $\psi_2 \stackrel{\mathrm{def}}{=} \Psi(\psi_1)$ of $\mathcal{C}_2$ such that $\deg_{\mathrm{Fr}}(\phi_2) = \deg_{\mathrm{Fr}}(\psi_2)$, $\mathrm{Div}(\phi_2) = \mathrm{Div}(\psi_2)$, $\mathrm{Base}(\phi_2) = \mathrm{Base}(\psi_2)$.

To do this, it suffices to show that "$\deg_{\mathrm{Fr}}(-)$", "$\mathrm{Div}(-)$", and "$\mathrm{Base}(-)$" may be *"computed"* entirely in terms of *homogeneous morphisms* [whose behavior with respect to Ψ and the operation of taking the unit-trivialization is already well-understood]. Thus, it suffices to observe that the following fact: The *Frobenius degree* may be computed by considering the existence of *factorizations* $A \to A' \to B$, where $A \to A'$ is a [necessarily homogeneous!] morphism of Frobenius type [cf. [Mzk5], Definition 1.3, (iv), (a)]. The property of *metric equivalence* [i.e., "$\mathrm{Div}(\phi_2) = \mathrm{Div}(\psi_2)$"] of a pair of linear co-objective morphisms $A \to B$ may be described [cf. the inclusions "$\Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\mathrm{rlf}}$" of Definition 5.3, (ii), (c)] by projecting these morphisms to morphisms $A^{\mathrm{pf}} \to B^{\mathrm{pf}}$ of $\mathcal{C}_i^{\mathrm{pf}}$ and considering the condition that one of these morphisms $A^{\mathrm{pf}} \to B^{\mathrm{pf}}$ *factors* through an arbitrary given co-angular pre-step $A^{\mathrm{pf}} \to (A')^{\mathrm{pf}}$ [cf. [Mzk5], Definition 1.3, (iii), (d); [Mzk5], Definition 1.3, (iv), (a)] if and only if the other does. [Here, we use the fact that Ψ^{pf} preserves *co-angular pre-steps* of $\mathcal{C}_i^{\mathrm{pf}}$ — cf. [Mzk5], Theorem 3.4, (ii).] Finally, the property of *base-equivalence* [i.e., "$\mathrm{Base}(\phi_2) = \mathrm{Base}(\psi_2)$"] of a pair of linear, metrically equivalent, co-objective morphisms $A \to B$ may be described by projecting these morphisms to morphisms $A^{\mathrm{pf}} \to B^{\mathrm{pf}}$ of $\mathcal{C}_i^{\mathrm{pf}}$ and considering the condition [cf. [Mzk5], Definition 1.3, (iii), (c); [Mzk5], Definition 1.3, (iv), (a); [Mzk5], Definition 1.3, (vi)] that these morphisms may be obtained from one another by composition with a [manifestly homogeneous!] element $\in \mathcal{O}^\times(A^{\mathrm{pf}}) \cong \mathcal{O}^\times(A)^{\mathrm{pf}}$. [Note that here, we must apply the *LC-unit-admissibility* of $\mathcal{C}_i$.] This completes the proof of assertion (iii) in the case "$\Box = \text{un-tr}$".

Next, we consider *divisor monoids* [i.e., assertion (v)]. By [Mzk5], Corollary 4.11, (iii), it follows that Ψ induces isomorphisms between the *divisor monoids* of corresponding local or global components of $\mathcal{C}_1, \mathcal{C}_2$. The asserted *compatibility* then follows by considering global co-angular pre-steps $B^{\mathrm{pf}} \to C^{\mathrm{pf}}$ of $\mathcal{C}_i^{\mathrm{pf}}$ [cf. [Mzk5], Definition 1.3, (iii), (d)], together with birationally tactile morphisms $A^{\mathrm{pf}} \to B^{\mathrm{pf}}$ [whose *existence* follows from the fact that $\mathcal{D}_i$ is of *totally tactile* type; [Mzk5], Definition 1.3, (i), (b); the inclusions "$\Phi_2|_{\mathcal{D}_1} \hookrightarrow \Phi_1^{\mathrm{rlf}}$" of Definition 5.3, (ii), (c)], which allow one to compute the image of $\mathrm{Div}(B^{\mathrm{pf}} \to C^{\mathrm{pf}})$ in [the value of the appropriate divisor monoid at] A^{pf} as the *difference*

$$\mathrm{Div}(A^{\mathrm{pf}} \to B^{\mathrm{pf}} \to C^{\mathrm{pf}}) - \mathrm{Div}(A^{\mathrm{pf}} \to B^{\mathrm{pf}})$$

— where we note that each of the terms in this difference may be computed as the *supremum* of the $\mathrm{Div}(A^{\mathrm{pf}} \to (A')^{\mathrm{pf}})$, where $A^{\mathrm{pf}} \to (A')^{\mathrm{pf}})$ is a [necessarily homogeneous!] co-angular pre-step through which the heterogeneous morphism in question [i.e., $A^{\mathrm{pf}} \to B^{\mathrm{pf}} \to C^{\mathrm{pf}}$ or $A^{\mathrm{pf}} \to B^{\mathrm{pf}}$] *factors.* This completes the proof of assertion (v). Finally, assertion (iii) in the case $\Box = \mathrm{rlf}$ follows by applying

[Mzk5], Corollary 5.4, to the various local and global components of $\mathcal{C}_i$, in light of the *compatibility* of assertion (v). $\bigcirc$

We are now ready to discuss the *main motivating example* of the theory of [Mzk5] and the present paper, an example to which most of the *main results* of this theory may be applied. The example arises, in effect, by *"grafting"* the global example of [Mzk5], Example 6.3, onto the local examples of Examples 1.1, 3.3, of the present paper.

Example 5.6. Poly-Frobenioids Associated to Number Fields.

(i) Let $\widetilde{F}/F$ be a *Galois extension* of a *number field* F; write $G_{\circledcirc} \stackrel{\text{def}}{=} \text{Gal}(\widetilde{F}/F)$, $\mathbb{V}(F)$ for the set of *valuations* on F [where we identify complex archimedean valuations with their complex conjugates —cf. [Mzk5], Example 6.3]. For $v \in \mathbb{V}(F)$, write F_v for the *completion of* F *at* v, $\widetilde{F}_v$ for the Galois extension of F_v determined by $\widetilde{F}$, $\mathfrak{D}_v \stackrel{\text{def}}{=} \text{Gal}(\widetilde{F}_v/F_v) \subseteq G_{\circledcirc}$ [so $\mathfrak{D}_v$ is well-defined up to conjugation in $G_{\circledcirc}$]. Also, we assume that we have been given *monoid types*

$$\Lambda_{\circledcirc}; \quad \{\Lambda_v\}_{v\in\mathbb{V}(F)}; \quad \{\underline{\Lambda}_v\}_{v\in\mathbb{V}(F)}$$

satisfying $\underline{\Lambda}_v \geq \Lambda_{\circledcirc}$, $\underline{\Lambda}_v \geq \Lambda_v$, for all $v \in \mathbb{V}(F)$. Note that [if we take the "$\widetilde{F}/F$", "v" of Example 1.4 to be the "$\widetilde{F}/F$", "v" of the present discussion, then] the category "$\mathcal{P}_0$" of Example 1.4 may be identified with the category $\mathcal{P}_v \stackrel{\text{def}}{=} \mathcal{B}(\mathfrak{D}_v)^0$. Thus, the functor "$\mathcal{E}_0 \to \mathcal{P}_0$" of Example 1.4 determines a functor which we denote by $\mathcal{E}_v \to \mathcal{P}_v$.

(ii) Let

$$\Pi_{\circledcirc} \twoheadrightarrow G_{\circledcirc}$$

be a *surjection of residually finite tempered groups*. Then the category

$$\mathcal{D}_{\circledcirc} \stackrel{\text{def}}{=} \mathcal{B}(\Pi_{\circledcirc})^0$$

is *connected*, *totally epimorphic*, *Frobenius-slim*, and of *FSM*- and *strongly indissectible* [hence, in particular, weakly indissectible] type — cf. Example 1.3, (i). Thus, if "$\mathcal{C}_{\widetilde{F}/F}$" is as in [Mzk5], Example 6.3, then

$$\mathcal{C}^{\mathbb{Z}}_{\circledcirc} \stackrel{\text{def}}{=} \mathcal{C}_{\circledcirc} \stackrel{\text{def}}{=} \mathcal{C}_{\widetilde{F}/F} \times_{\mathcal{B}(G_{\circledcirc})} \mathcal{D}_{\circledcirc}; \quad \mathcal{C}^{\mathbb{Q}}_{\circledcirc} \stackrel{\text{def}}{=} \mathcal{C}^{\text{pf}}_{\circledcirc}; \quad \mathcal{C}^{\mathbb{R}}_{\circledcirc} \stackrel{\text{def}}{=} \mathcal{C}^{\text{rlf}}_{\circledcirc}$$

determines a *Frobenioid* $\mathcal{C}^{\Lambda_{\circledcirc}}_{\circledcirc}$ which is of *rationally standard* type over a *Frobenius-slim* base category [cf. [Mzk5], Theorem 6.4, (i), for the case $\Pi_{\circledcirc} = G_{\circledcirc}$; the case of arbitrary $\Pi_{\circledcirc}$ is entirely similar]. If, moreover, $\Pi_{\circledcirc}$ is *temp-slim*, then $\mathcal{D}_{\circledcirc}$ is *slim* [cf. Example 1.3, (i)].

(iii) Let $v \in \mathbb{V}(F)$ be *nonarchimedean*; $\widetilde{K}_v/F_v$ a *Galois extension* containing $\widetilde{F}_v$; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_v/F_v)$. Thus, we have a natural surjection $G_v \twoheadrightarrow \mathfrak{D}_v \subseteq G_{\circledcirc}$. Suppose further that we have been given a *commutative diagram*

$$\begin{array}{ccc} \Pi_v & \twoheadrightarrow & G_v \\ \big\downarrow & & \big\downarrow \\ \Pi_{\circledcirc} & \twoheadrightarrow & G_{\circledcirc} \end{array}$$

of *residually finite tempered groups* [where the lower horizontal surjection and the vertical homomorphism on the right are the morphisms that were given previously]. Let

$$\mathcal{D}_v$$

be one of the categories $\mathcal{B}(\Pi_v)^0$, $\mathcal{B}(\Pi_v)^0 \times_{\mathcal{P}_v} \mathcal{E}_v$ [where the functor $\mathcal{B}(\Pi_v)^0 \to \mathcal{P}_v$ is the functor determined by the surjection $\Pi_v \twoheadrightarrow G_v \twoheadrightarrow \mathfrak{D}_v$ — cf. Example 1.3, (ii)]. Then the category $\mathcal{D}_v$ is *connected*, *totally epimorphic*, *Frobenius-slim*, and of *FSMFF*- and *strongly indissectible* [hence, in particular, weakly indissectible] type [cf. Example 1.3, (i); Example 1.4, (iii)]. If, moreover, Π_v is *temp-slim*, then $\mathcal{D}_v$ is *slim* [cf. Example 1.3, (i); Example 1.4, (iii)]. Write

$$\mathcal{C}_v^{\Lambda_v}; \quad \underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$$

for the respective categories "$\mathcal{C}$" of Example 1.1, (ii), obtained by taking the "$\mathcal{D}$" of *loc. cit.* to be $\mathcal{D}_v$, the "Λ" of *loc. cit.* to be Λ_v, $\underline{\Lambda}_v$, and the "Φ" of *loc. cit.* to be functors $\Phi_v, \underline{\Phi}_v : \mathcal{D}_v \to \mathfrak{Mon}$ as in *loc. cit.* such that the following conditions are satsfied:

(a) $\Phi_v \subseteq \underline{\Phi}_v$;

(b) the resulting functors Φ_v^{birat}, $\underline{\Phi}_v^{\mathrm{birat}}$ satisfy $\underline{\Phi}_v^{\mathrm{birat}} = \Phi_v^{\mathrm{birat}} + \Phi_v^{\mathrm{cnst}}$ for some group-like functor $\mathcal{D}_v \to \mathfrak{Mon}$ which is *constant* [i.e., maps all arrows of $\mathcal{D}_v$ to isomorphisms of $\mathfrak{Mon}$];

(c) $\underline{\Phi}_v$ is *fieldwise saturated* [cf. Example 1.1, (ii)] and *supported* by $\underline{\Lambda}_v$ [cf. [Mzk5], Definition 2.4, (ii)].

Thus, $\mathcal{C}_v^{\Lambda_v}$, $\underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$ are *Frobenioids* which are of *rationally standard* type over a *Frobenius-slim* base category [cf. Theorem 1.2, (i)]; the underlying category of $\mathcal{C}_v^{\Lambda_v}$, $\underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$ is of *weakly indissectible* type [cf. the discussion of §0]. Moreover, [since $\underline{\Lambda}_v \geq \Lambda_{\circledcirc}$, $\underline{\Lambda}_v \geq \Lambda_v$, for all $v \in \mathbb{V}(F)$] it follows immediately from the definitions [by restricting *arithmetic divisors* on number fields to divisors on their localizations and nonzero elements of number fields to elements of their completions] that we obtain *functors*

$$\mathcal{C}_{\circledcirc}^{\Lambda_{\circledcirc}} \to (\underline{\mathcal{C}}_v^{\underline{\Lambda}_v})^{\top}; \quad \mathcal{C}_v^{\Lambda_v} \to \underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$$

the first of which is *GC-admissible* and 1-*compatible* with the natural localization functor $\mathcal{D}_{\circledcirc} \to \mathcal{D}_v^{\top}$ [arising from the homomorphism "$\Pi_v \to \Pi_{\circledcirc}$" — cf. the "pull-back functor" of Example 1.3, (ii)], and the second of which is *LC-unit-admissible* [cf. [Mzk5], Proposition 5.5, (i)]; also, we note that the "restriction of arithmetic divisors" portion of the first of these functors is possible precisely because of condition (c).

(iv) Let $v \in \mathbb{V}(F)$ be *archimedean*; $\widetilde{K}_v/F_v$ a *Galois extension* containing $\widetilde{F}_v$ such that $\widetilde{K}_v$ is *complex*; $G_v \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_v/F_v)$. Thus, we have a natural surjection $G_v \twoheadrightarrow \mathfrak{D}_v \subseteq G_{\circledcirc}$. Suppose further that we have been given a *commutative diagram*

$$\begin{array}{ccc} \Pi_v & \twoheadrightarrow & G_v \\ \downarrow & & \downarrow \\ \Pi_{\circledcirc} & \twoheadrightarrow & G_{\circledcirc} \end{array}$$

of *residually finite tempered groups* [where the lower horizontal surjection and the vertical homomorphism on the right are the morphisms that were given previously]. Let

$$\mathcal{D}_v$$

be one of the following categories:

(a) $\mathcal{B}(\Pi_v)^0$, $\mathcal{B}(\Pi_v)^0 \times_{\mathcal{P}_v} \mathcal{E}_v$ [where the functor $\mathcal{B}(\Pi_v)^0 \to \mathcal{P}_v$ is the functor determined by the surjection $\Pi_v \twoheadrightarrow G_v \twoheadrightarrow \mathfrak{D}_v$ — cf. Example 1.3, (ii)];

(b) one of the *categories of hyperbolic Riemann surfaces* "$\mathcal{D}$" or "$\mathcal{E}$" of Example 4.3, (i), (ii), (iii), where the tempered group "Π" of *loc. cit.* is equal to Π_v [in a fashion compatible with the surjection $\Pi_v \twoheadrightarrow G_v$];

(c) a *nonrigidified* or *rigidifed angloid* [i.e., "$\mathcal{N}$", "$\mathcal{R}$"] over a base category as in (a) above [cf. Remark 4.2.1];

(d) a *nonrigidified* or *rigidifed angloid* [i.e., "$\mathcal{N}$", "$\mathcal{R}$"] over a base category as in (b) above [cf. Remark 4.2.1].

If $\mathcal{D}_v$ is taken to be one of the categories that involves $\mathcal{E}_v$, then let us assume further that the extension $\widetilde{F}/F$ is *amply quadratic* [cf. Example 4.3, (iii)]. Then [cf. Example 1.3, (i); Proposition 1.5, (iv); Proposition 4.1, (v); Corollary 4.2, (iv), (v); Example 4.3, (ii), (iii), (iv); Example 4.4, (ii), (iii), (iv)] the category $\mathcal{D}_v$ is *Frobenius-slim* and of *RC-standard* [hence, in particular, *connected*, *totally epimorphic*, and of *FSMFF-type*] and either *strongly dissectible* or *weakly indissectible* type; moreover, in cases (b), (d), $\mathcal{D}_v$ is always *slim*, while in cases (a), (c), $\mathcal{D}_v$ is *slim* whenever Π_v is *temp-slim*. Write

$$\mathcal{C}_v^{\Lambda_v}; \quad \underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$$

for the respective categories "$\mathcal{C}^{\Lambda}$" of Example 3.3, (ii), obtained by taking the "$\mathcal{D}$" of *loc. cit.* to be $\mathcal{D}_v$ and the "Λ" of *loc. cit.* to be Λ_v, $\underline{\Lambda}_v$. Thus, $\mathcal{C}_v^{\Lambda_v}$, $\underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$ are *Frobenioids* which are of *rationally standard* type over a *Frobenius-slim* base category [cf. Theorem 3.6, (i)]; the underlying category of $\mathcal{C}_v^{\Lambda_v}$, $\underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$ is of either *strongly dissectible* or *weakly indissectible* type [cf. the discussion of §0]. Moreover, [since $\underline{\Lambda}_v \geq \Lambda_{\circledcirc}$, $\underline{\Lambda}_v \geq \Lambda_v$, for all $v \in \mathbb{V}(F)$] it follows immediately from the definitions [by restricting *arithmetic divisors* on number fields to divisors on their localizations and nonzero elements of number fields to elements of their completions] that we obtain *functors*

$$\mathcal{C}_{\circledcirc}^{\Lambda_{\circledcirc}} \to (\underline{\mathcal{C}}_v^{\underline{\Lambda}_v})^{\top}; \quad \mathcal{C}_v^{\Lambda_v} \to \underline{\mathcal{C}}_v^{\underline{\Lambda}_v}$$

the first of which is *GC-admissible* and 1-*compatible* with the natural localization functor $\mathcal{D}_{\circledcirc} \to \mathcal{D}_v^{\top}$ [arising from the homomorphism "$\Pi_v \to \Pi_{\circledcirc}$" — cf. the "pull-back functor" of Example 1.3, (ii)], and the second of which is *LC-unit-admissible* [cf. [Mzk5], Proposition 5.5, (i)].

(v) Thus, the data

$$\mathfrak{C} \stackrel{\mathrm{def}}{=} \left(\mathcal{C}_{\circledcirc}^{\Lambda_{\circledcirc}}, \{\mathcal{C}_v^{\Lambda_v}\}_{v\in\mathbb{V}(F)}, \{\underline{\mathcal{C}}_v^{\underline{\Lambda}_v}\}_{v\in\mathbb{V}(F)}, \{\mathcal{C}_{\circledcirc}^{\Lambda_{\circledcirc}} \to (\underline{\mathcal{C}}_v^{\underline{\Lambda}_v})^{\top}\}_{v\in\mathbb{V}(F)}, \{\mathcal{C}_v^{\Lambda_v} \to \underline{\mathcal{C}}_v^{\underline{\Lambda}_v}\}_{v\in\mathbb{V}(F)}\right)$$

determines a *poly-Frobenioid* $\mathcal{C}$ of *poly-rationally standard* and *poly-non-group-like* type over a *Frobenius-slim poly-base category* $\mathcal{D}$. In particular, all but the portion requiring "slimness" of Theorem 5.5 applies to $\mathcal{C}$; the conditions on the data of the present Example 5.6 necessary for the poly-base category to be *slim* are as discussed above.

Remark 5.6.1. Observe that unlike the case with Frobenioids, *poly-Frobenioids* are *not* necessarily *totally epimorphic*! In a similar vein [cf. the use of the *injectivity* condition of [Mzk5], Definition 1.1, (ii), (a), in the proof of the *total epimorphicity* portion of [Mzk5], Proposition 1.5], we observe that [again, unlike the case with Frobenioids] the map induced by a heterogeneous morphism between the values of the divisor monoids at the domain and codomain is *not* necessarily *injective*! Indeed, these phenomena occur, for instance, if one considers the poly-Frobenioids of Example 5.6, when $\Lambda_{\circledcirc} = \mathbb{R}$.

Remark 5.6.2. Just as in the case of *archimedean* v, it is natural to consider $\Pi_{\circledcirc}$, Π_v [for *nonarchimedean* v] arising from the *arithmetic fundamental group* of a *hyperbolic curve*. One may then apply to such $\Pi_{\circledcirc}$, Π_v various results from the *absolute anabelian geometry of hyperbolic curves* [cf., e.g., [Mzk4]].

Remark 5.6.3. One verifies immediately that, when considering *equivalences of categories* between poly-Frobenioids of the sort discussed in Example 5.6, (v), the induced equivalences of categories between the respective *global components* determine a *"degree"* $\in \mathbb{R}_{>0}$, as in [Mzk5], Theorem 6.4, (ii). Moreover, just as in [Mzk5], Theorem 6.4, (iii), this degree is $\in \mathbb{Q}_{>0}$ [even if $\Lambda_{\circledcirc} = \mathbb{R}$!] whenever it holds that the $\Lambda_v \leq \mathbb{Q}$ for all $v \in \mathbb{V}(F)$. Similarly, just as in [Mzk5], Theorem 6.4, (iv), this degree $= 1$ [even if $\Lambda_{\circledcirc} = \mathbb{R}$!] whenever it holds that the $\Lambda_v = \mathbb{Z}$ for all $v \in \mathbb{V}(F)$.

Appendix: Categorical Representation of Topological Spaces

In this Appendix, we discuss certain classical results concerning how a *topological space* may be represented by means of a *category.*

Let X be a *topological space*. Then we shall write

$$\text{Subset}(X)$$

for the category whose *objects* are the subsets of X [including X, the empty set], and whose *morphisms* are the inclusions of subsets of X. Write

$$\text{Open}(X) \subseteq \text{Subset}(X); \quad \text{Closed}(X) \subseteq \text{Subset}(X)$$

for the full subcategories determined by the *open* and *closed* subsets, respectively. Also, we shall denote by

$$\text{Open}^0(X) \subseteq \text{Open}(X)$$

the full subcategory determined by the [nonempty] *connected open subsets* and by

$$\text{Shv}(X)$$

the *category of sheaves* [valued in sets, relative to some universe fixed throughout the discussion] on X.

Definition A.1.

(i) We shall say that X is *locally connected* if, for every open subset $U \subseteq X$ and every point $x \in U$, there exists a connected open subset $V \subseteq U$ such that $x \in V$. We shall say that X is *sober* if, for every irreducible closed subset $F \subseteq X$, there exists a unique point $x \in F$ such that F is equal to the closure of the set $\{x\}$ in X [cf. [John], p. 230].

(ii) We shall refer to a collection $\{A_i\}_{i \in I}$ of distinct objects of $\text{Open}^0(X)$ as a *collection of disjoint objects* if, for any pair of distinct elements i, j of I, there does not exist an object $C \in \text{Ob}(\text{Open}^0(X))$ that admits a morphism in $\text{Open}^0(X)$ to both A_i and A_j.

(iii) Denote by

$$\text{Disjt}(\text{Open}^0(X))$$

the category defined as follows: An *object* of this category is a collection of disjoint objects $\{U_i\}_{i \in I}$ [where I is a [possibly empty] set]. A *morphism* of this category

$$\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$$

consists of a function $f : I \to J$ and a collection of morphisms [in $\text{Open}^0(X)$] $U_i \to V_{f(i)}$ [where i ranges over the elements of I]. Thus, by assigning to an object of $\text{Open}^0(X)$ the collection of objects of $\text{Open}^0(X)$ consisting of this single object, and to a collection $\{U_i\}_{i \in I}$ the object of $\mathrm{Open}(X)$ constituted by the *union* of the connected open sets determined by the U_i, we obtain *natural functors*

$$\mathrm{Open}^0(X) \to \mathrm{Disjt}(\mathrm{Open}^0(X)); \quad \mathrm{Disjt}(\mathrm{Open}^0(X)) \to \mathrm{Open}(X)$$

the first of which is easily verified to be *fully faithful*.

Remark A.1.1. Thus, if X is *locally connected*, then one verifies immediately that, relative to the notation of Definition A.1, (ii), the A_i are disjoint if and only if the open subsets of X to which the A_i correspond are pair-wise mutually "disjoint" in the usual sense.

Theorem A.2. **(Categorical Representation of Topological Spaces)** *Let* X, Y *be* **topological spaces**. *Then:*

(i) The category $\mathrm{Open}(X)$ *is equivalent to the* **opposite category** *to* $\mathrm{Closed}(X)$.

(ii) The categories $\mathrm{Subset}(X)$, $\mathrm{Open}(X)$, $\mathrm{Open}^0(X)$, $\mathrm{Closed}(X)$ *are* **slim**.

(iii) By assigning to an object of $\mathrm{Open}(X)$ *the sheaf on* $\mathrm{Open}(X)$ *represented by the given object, we obtain a* **natural functor**

$$\mathrm{Open}(X) \to \mathrm{Shv}(X)$$

which is **fully faithful**.

(iv) Suppose that X, Y *are* **sober**. *Then passing to the induced equivalence on the categories "*$\mathrm{Open}(-)$*" determines a* **bijection** *between the equivalences of categories*

$$\mathrm{Open}(X) \xrightarrow{\sim} \mathrm{Open}(Y)$$

[considered up to isomorphism] and the **homeomorphisms** $X \xrightarrow{\sim} Y$.

(v) Suppose that X *is* **locally connected**. *Then the natural functor*

$$\mathrm{Disjt}(\mathrm{Open}^0(X)) \to \mathrm{Open}(X)$$

is an **equivalence of categories**.

(vi) Suppose that X, Y *are* **sober** *and* **locally connected**. *Then passing to the induced equivalence on the categories "*$\mathrm{Open}^0(-)$*" determines a* **bijection** *between the equivalences of categories*

$$\mathrm{Open}^0(X) \xrightarrow{\sim} \mathrm{Open}^0(Y)$$

[considered up to isomorphism] and the **homeomorphisms** $X \xrightarrow{\sim} Y$.

Proof. The equivalence of assertion (i) is obtained by associating to an open set of X the closed set of X given by its *complement*. Assertion (ii) follows immediately from the fact that, by definition, the categories in question have *no nontrivial automorphisms.* Assertion (iii) follows immediately from the definitions and *"Yoneda's Lemma"*. To verify assertion (iv), we recall [cf. [John], Theorem 7.24] that passing to the induced equivalence on the categories "Shv(−)" determines a *bijection* between the equivalences of categories

$$\mathrm{Shv}(X) \xrightarrow{\sim} \mathrm{Shv}(Y)$$

[considered up to isomorphism] and the homeomorphisms $X \xrightarrow{\sim} Y$. Note, moreover, that an open V of X is a *union* of opens $\{U_\alpha\}_{\alpha \in I}$ of X if and only if V forms [i.e., in $\mathrm{Open}(X)$] an *inductive limit* [a purely category-theoretic notion!] of the system constituted by the U_α. Thus, to complete the proof of assertion (iv), it suffices, in light of the natural embedding of assertion (iii), to observe that $\mathrm{Shv}(X)$ may be *reconstructed* [by the definition of a *"sheaf"*!] directly from the category $\mathrm{Open}(X)$, in a fashion that is *compatible* with the natural embedding of assertion (iii). This completes the proof of assertion (iv).

Next, we consider assertion (v). First, I *claim* that every open V of X is a disjoint union of *connected opens* of X. Indeed, consider the equivalence relation on objects of $\mathrm{Open}^0(V)$ generated by the pre-equivalence relation that two distinct objects A, B of $\mathrm{Open}^0(V)$ are "pre-equivalent" if the pair of objects A, B *fails* to form a collection of disjoint objects of $\mathrm{Open}^0(V)$ [i.e., the connected opens corresponding to A, B *intersect*]. Denote by I the set of equivalence classes of objects of $\mathrm{Open}^0(V)$, relative to this equivalence relation. For $i \in I$, write $U_i \in \mathrm{Ob}(\mathrm{Open}(X))$ for the union of the connected opens that lie in the class i. Then it follows immediately from the definitions that U_i is a *connected open*, hence forms an object of $\mathrm{Open}^0(X)$. Since, moreover, X is *locally connected*, it follows that the union of U_i [as i ranges over the elements of I] is equal to V. This completes the proof of the *claim.* Now assertion (v) follows formally.

Finally, we observe that assertion (vi) follows immediately from assertions (iv), (v), together with the easily verified observation that the category $\mathrm{Disjt}(\mathrm{Open}^0(-))$ may be reconstructed directly from the category $\mathrm{Open}^0(-)$, in a fashion that is *compatible* with the natural embedding $\mathrm{Open}^0(X) \hookrightarrow \mathrm{Disjt}(\mathrm{Open}^0(X))$. ◯

Remark A.2.1. Note that *neither of the two conditions of "soberness" and "local connectedness" implies the other.* Indeed, suppose that the underlying set of X is *countably infinite*, and that the proper [i.e., $\neq X$] closed subsets of X are precisely the finite subsets of X. [Consider, for instance, the *Zariski topology on the set of closed points of the affine line* over a countable algebraically closed field.] Then observe that X is *irreducible*, and that any nonempty open subset $Y \subseteq X$ is *homeomorphic to* X, hence, in particular, *irreducible.* Since irreducible topological spaces are *connected*, it thus follows that X is *locally connected.* On the other hand, since X is irreducible, but clearly *fails* to admit a generic point, it follows that X is not *sober.* In the "opposite direction", any *infinite "profinite set"* [i.e., a projective limit of finite sets — e.g., the underlying topological space of a *profinite group*] is *Hausdorff*, hence *sober*, but satisfies the property that every open subset is *totally disconnected*, hence fails to be *locally connected.*

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