The Hodge-Arakelov Theory of Elliptic Curves

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§1. Main Results

(A.) Simple Version of the Main Comparison Theorem

K: a field of characteristic 0

E: an elliptic curve$/K$

$E^\dagger$: its universal extension

$$= \{ \text{ moduli of } (\mathcal{L}, \nabla_{\mathcal{L}}) : (\mathcal{L}, \nabla_{\mathcal{L}}) : \deg(\mathcal{L}) = 0\}$$

$$\overset{\text{char } 0}{=} H^1_{\mathrm{DR}}(E, \mathcal{O}_E^\times)$$

Over $\mathbf{C}$: $E^\dagger = H^1_{\mathrm{DR}}(E, \mathcal{O}_E)/\Lambda$

where $\Lambda = H^1_{\mathrm{sing}}(E, 2\pi i\mathbf{Z}) \cong \mathbf{Z}^2$

In general:

Tang. sp. to $E^\dagger = H^1_{\mathrm{DR}}(E, \mathcal{O}_E)$

Char. 0: ${}_dE^\dagger \cong {}_dE \stackrel{\text{def}}{=} \ker [d] : E \to E$
(d: a positive integer)

(in mixed char., denominators arise)

$\eta \in E(K)$: torsion point of order m,
s. t. d does not divide m

$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$

Theorem: The restriction morphism

$$\Gamma(E^\dagger, \mathcal{L})^{<d} \xrightarrow{\sim} \mathcal{L}|_{{}_dE^\dagger}$$

is an isomorphism.

Note:

(1.) "$< d$" denotes torsorial degree (relative degree: $E^\dagger/E) < d$.

(2.) Both sides are K-vector spaces of dimension d^2.

(3.) Theorem false if d divides m. (e.g., if $d = m = 1$, then $\Gamma(E, \mathcal{O}_E([0_E]) = \mathcal{L}) \to \mathcal{L}|_{0_E}$ is 0)

(4.) Proof:
Mumford's algebraic theta functions
+ Zhang's theory of admiss. metrics
+ complicated degree computations

(B.) Integral Structures at "Arithmetic" Primes

<u>In general</u>:

$$0 \to \omega_E \to E^\dagger \to E \to 0$$

(ω_E = invariant diffs. on E)

<u>Near "point at infinity" ∞</u>:

$$E = \mathbf{G}_\mathrm{m}/q^{\mathbf{Z}}$$

("Tate curve")

$\Longrightarrow$ Over power series in q ('hat'):

$$\widehat{E} = \widehat{\mathbf{G}}_\mathrm{m}$$

$$\widehat{E}^\dagger = \widehat{\mathbf{G}}_\mathrm{m} \times \widehat{\omega}_E = \widehat{\mathbf{G}}_\mathrm{m} \times \left\langle \frac{dq}{q} \right\rangle$$

Integral structure

at finite primes (mixed char.):

$$\mathcal{O}_{\widehat{E}}[T] = \bigoplus \mathcal{O}_{\widehat{E}} \cdot T^j \Longrightarrow \bigoplus \ \mathcal{O}_{\widehat{E}} \cdot \tfrac{1}{j!} \cdot T^j$$

where T: coord. on ω_E, def'd by $\frac{dq}{q}$

...(p-adic analytically) extends

over all $\overline{\mathcal{M}}_{1,0}$, not just near ∞

Integral Structure Near ∞:

$$\bigoplus \ \mathcal{O}_{\widehat{E}} \cdot \tfrac{1}{j!} \cdot T^j \Longrightarrow$$

$$\bigoplus \ \mathcal{O}_{\widehat{E}} \cdot \tfrac{1}{j!} \cdot q^{\approx -j^2/8d} \cdot T^j$$

"Gaussian poles" (cf. e^{-x^2})

<u>Important Theme</u>:

Gaussian and its derivatives

(cf. Hermite polynomials)

...also, main obstruct. to Dio. applics.

<u>Integral Structure at Arch. Primes</u>:

To relate '<u>DR metric</u>' to '<u>étale metric</u>'

$\Longrightarrow$ approximate by comparison

to special functions — <u>models</u>:

<u>Hermite</u> polys. (slope $= \frac{1}{2}$)

<u>Legendre</u> polys. (slope $= 1$)

$=$ lim (disc. Tchebycheff polys.)

<u>Binomial</u> polys. $= \binom{T}{r}$ (slope $= 0$)

<u>slope</u> $=$ scaling factor as $d \to \infty$

(cf. <u>Frobenius</u> on cryst. coh.)

(C.) Arithmetic Kodaira-Spencer Morphism

Main Theorem is a sort of
function-theoretic comp. isom.:

linear fns. + completion of tors. pts.
$\Longrightarrow$ get classical comp. isoms.:

Over $\mathbf{C}$:

$$\begin{array}{ccc} H^1_{\mathrm{DR}}(E, \mathcal{O}_E) & \supseteq & H^1_{\mathrm{sing}}(E, 2\pi i \cdot \mathbf{R}) \\ \downarrow & & \downarrow \\ E^\dagger & \supseteq & E_{\mathbf{R}} \end{array}$$

Over p-adics:

Hodge-Tate, DR comp. isoms:

$$\text{`}H^1_{\mathrm{DR}} \cong H^1_{\text{ét}}\text{'}$$

also def'able by rest. to p^∞ tors. pts.

In general (global, $\mathbf{C}$, p-adics):

$$\Big\{\text{DR coh.}\Big\} \xrightarrow{\sim} \Big\{\text{ét. coh.}\Big\} \curvearrowleft \underline{\text{Galois}}$$

$\Longrightarrow$ Galois acts on DR coh.!!
$\Longrightarrow$ Look at effect on Hodge filtr.
$\Longrightarrow$ Kodaira-Spencer morphism

motion in base

$\mapsto$ induced motion of Hodge filtr.

Over **C**:

"Galois" = $SL_2(\mathbf{R})$ on upp. half-plane

$\Longrightarrow$ above 'arith. KS' = classical KS

Over p-adics:

$$\mathrm{Gal}(\mathbf{Z}_p[[T]]_{\mathbf{Q}_p}) \approx \text{Tang. bun.}(\mathbf{Z}_p[[T]]_{\mathbf{Q}_p})$$

(Faltings' theory of alm. et. extns.)

$\Longrightarrow$ above 'arith. KS' = classical KS

(cf. Serre-Tate theory)

Hodge-Arakelov (global) Case:

Gal(Base of Fam. of Ell. Curves $\otimes\, \mathbf{Q}$)

$\xrightarrow{\text{arith. KS}}$ {Arak.-theoretic flag bun.} !!

§2. Philosophy: In Search of an Absolute Derivative

(A.) From Differentiation to Comparison Isomorphisms

S: a scheme; $E \to S$ fam. of ell. curves

$\Longrightarrow$ <u>classifying morphism</u> $S \to \mathcal{M}_{1,0}$

$\Longrightarrow$ <u>derivative</u> (KS) $\Omega_{\mathcal{M}_{1,0}}|_S \to \Omega_S$

$$\Downarrow$$

Does $\exists$ <u>arithmetic/absolute analogue</u>

$$`\Omega_{\mathcal{M}_{1,0}}|_S \to \Omega_{\mathbf{Z}/\mathbf{F}_1}'$$

(when $S = \mathrm{Spec}(\mathbf{Z})$,
or $\mathrm{Spec}(\mathcal{O}_F)$, $[F : \mathbf{Q}] < \infty$)?

Observe: $\Omega_{\mathcal{M}_{1,0}}|_S = \omega_E^{\otimes 2}$, and

$$\omega_E \hookrightarrow H^1_{\mathrm{DR}}(E) \xrightarrow{\nabla_{\mathrm{GM}}} H^1_{\mathrm{DR}}(E) \otimes \Omega_S \longrightarrow \tau_E \otimes \Omega_S$$

$\Longrightarrow \Omega_{\mathcal{M}_{1,0}}|_S = \omega_E^{\otimes 2} \to \Omega_S$ (KS)

(∇_{GM}: Gauss-Manin conn. on H^1_{DR})

$$\Downarrow$$

Since $\exists$ H^1_{DR}, Hodge filtr. ($\omega_E \hookrightarrow H^1_{\mathrm{DR}}$) in arith. case, need analogue of ∇_{GM}

$\Longrightarrow$ Recall de Rham isomorphism (= comparison isomorphism/$\mathbf{C}$):

S: Riemann surface $\Longrightarrow$

$$H^1_{\mathrm{DR}}(E/S) \cong H^1_{\mathrm{sing}}(E/S, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{O}_S$$

$\Longrightarrow$ sections of $H^1_{\mathrm{sing}}(E/S, \mathbf{Z})$ are horizontal for ∇_{GM}

$\Longrightarrow$ ∇_{GM} is the unique conn. for which sects. of $H^1_{\mathrm{sing}}(E/S, \mathbf{Z})$ are horiz.

$$\Downarrow$$

Knowledge of comp. isom. $\Longrightarrow$
Knowledge of ∇_{GM}

Conclusion: To construct arith. KS, suffices to construct arith. comp. isom.

(B.) Function-Theoretic Comparison Isomorphisms

So what form should a (global) arith. comp. isom. (ACI) take?

(e.g., over $\mathbf{C}$: $\otimes\ \mathbf{C}$;
over p-adics: $\otimes\ B_{\mathrm{DR}}$, B_{crys}, etc.)

In geometric case/$\mathbf{C}$: one implicit sign of exist. of ∇_{GM} is a sort of 'stability':

$$0 \to \omega_E \to H^1_{\mathrm{DR}}(E/S) \to \tau_E \to 0$$

If this sequ. split — i.e., H^1_{DR} is 'unstable' — then $\exists\ \nabla$ on ω_E (= ample l.b.): ABSURD!

$\Longrightarrow$ Even if can't translate '∇' into arith. case, <u>can translate stability</u> — i.e., of <u>Arakelov</u> bundles

= usual v.b. + metric

$\Longrightarrow$ 'Stability' (e.g., over $\mathbf{Z}$) = '<u>equidistrib. of matter</u> in lattice'

<u>Note</u>: Arakelov degree large (small) $\Longleftrightarrow$ matter dense (sparse)

$$\Downarrow$$

<u>Expected Form I of ACI</u>:

$$\Big\{\text{Matter Distrib. in DR coh.}\Big\} \cong \Big\{\text{Matter Distrib. in étale coh.}\Big\}$$

Note: RHS is ‘equidist.’ by ‘Galois’
$\Longrightarrow$ By ACI, LHS also ‘equidist.’

In no. theory, ‘matter distributions’
typically measured by ‘test fns.’ $\Longrightarrow$

Expected Form II of ACI:

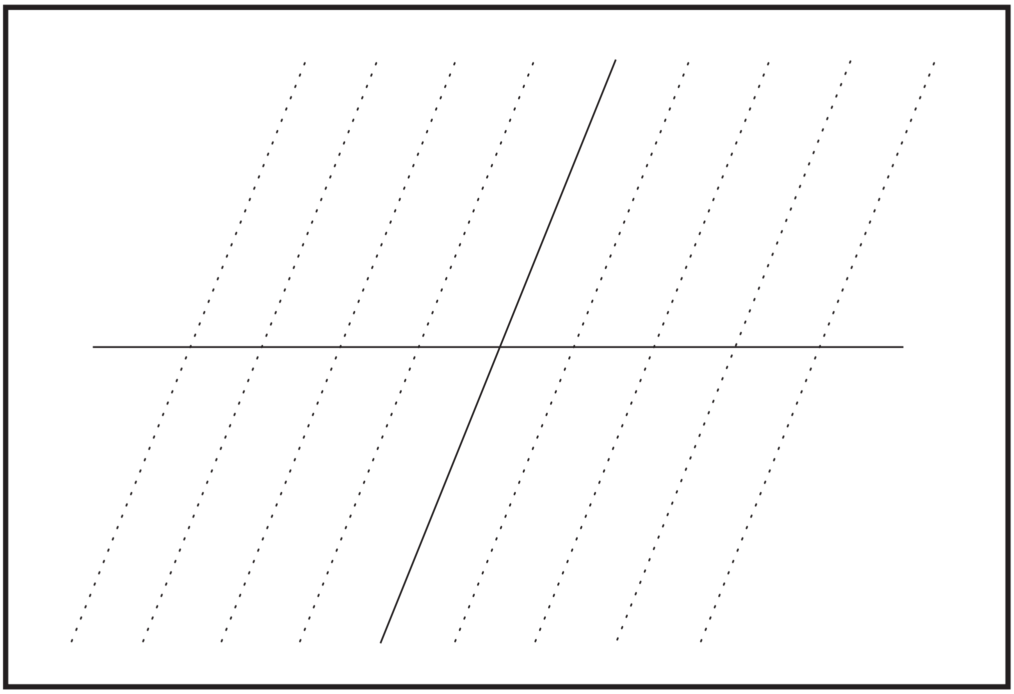
$$\left\{\text{test fns. on DR coh.}\right\}$$
$$\cong \left\{\text{test fns. on étale coh.}\right\}$$

where ‘$\cong$’ is isometry at all primes
of a number field (cf. Arak. theory)

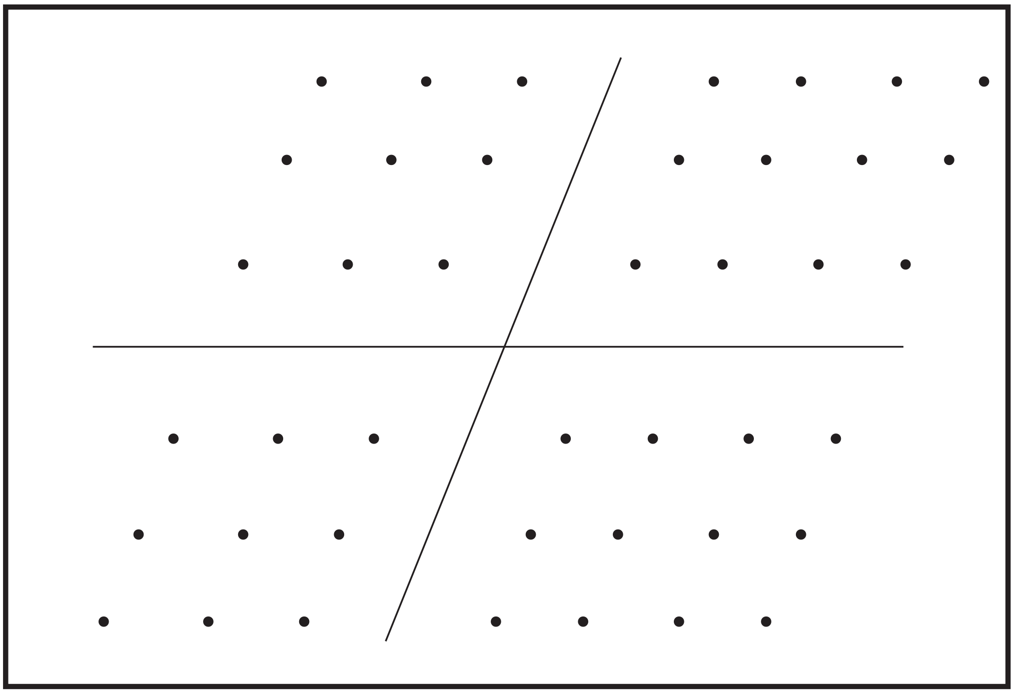
... = the content of the main theorem!!

‘Hodge-Arakelov Comp. Isom.’

‘Split’ distrib. of matter:



‘Equidist.’ distrib. of matter:



(C.) Discretization and the Meaning of Nonlinearity

Note: To measure distribs. in this case, need nonlinear test fns. — cf. linearity of Hodge theory/$\mathbf{C}$, p-adics, additive approach to motive theory.

Reasons for Nonlinearity:

(1.) In Arakelov theory, things tend to become nonlinear (e.g., $H^0(\mathcal{L})$).

(2.) Nonlinear symmetries of noncomm. torus $\approx$ theta gp. $\approx$ Heisenberg alg. (cf. Gaussians!)

Also, related to <u>discreteness</u>:

Hodge-Arakelov Comp. Isom. =
'<u>discretization</u> of loc. Hodge theories'

— e.g.,

Hodge theory/$\mathbf{C}$ ≈ 'calculus on $E_{\mathbf{R}}$'

HACI ≈ 'discrete calc. on tors. pts.'

⟹ <u>periods</u> analogous to

$$2\pi i = \lim_{d\to\infty} d \cdot (e^{2\pi i/d} - 1)$$
$$= \lim_{d\to\infty} (\text{'theta fns.' on } \mathbf{G}_{\mathrm{m}} \text{ evaluated on tors. pts. of } \mathbf{G}_{\mathrm{m}})$$