The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

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§1. Introduction

§2. The Physical Approach in the p-adic Case

§3. The Modular Approach in the p-adic Case

§1. Introduction

(A.) The Fuchsian Uniformization

Hyperbolic Curve: smooth,
proper connected genus g alg.
curve $-r$ points, s.t. $2g - 2 + r > 0$

Over $\mathbf{C}$: unif. by upper half-plane $\mathcal{H}$

$$X(\mathbf{C}) = \mathcal{X} \cong \mathcal{H}/\Gamma$$

$$\implies \pi_1(\mathcal{X}) \to PSL_2(\mathbf{R}) = \mathrm{Aut}(\mathcal{H})$$

$\exists$ a p-adic analogue of this
Fuchsian uniformization?

Note: Fuchsian $\neq$ Schottky
(cf. D. Mumford's theory), e.g.,

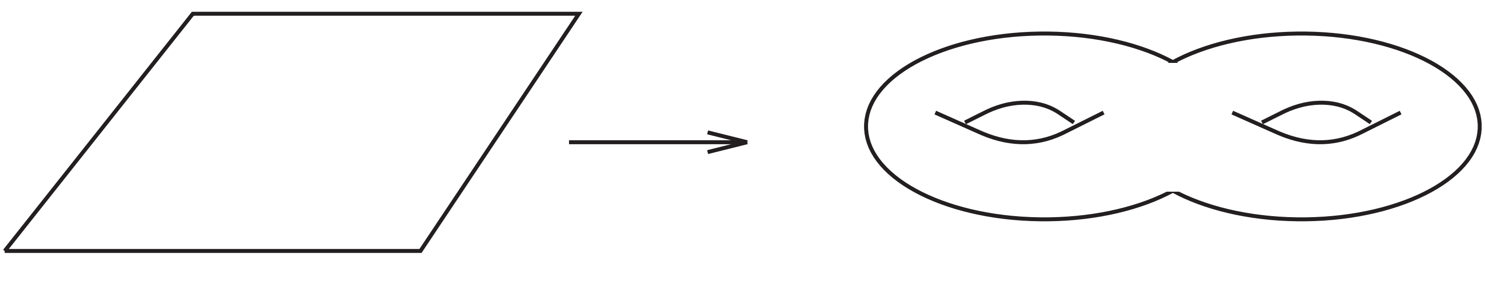
Fuchsian unif. involves arithmetic, i.e.,

real analytic structures $\Longleftrightarrow$
Frobenius at the infinite prime

(B.) The Physical Interpretation

alg. curve $X \Longleftrightarrow SO(2)\backslash PSL_2(\mathbf{R})/\Gamma$
(physical/analytic obj.)
$\Longleftrightarrow \pi_1(\mathcal{X}) + \rho_{\mathcal{X}}$
$\Longleftrightarrow$ top. + arith. str.

Modular forms define first "$\Longleftrightarrow$."



(C.) The Modular Interpretation

$\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(\mathbf{R}) \subseteq PGL_2(\mathbf{C})$
$\Longrightarrow \pi_1(\mathcal{X}) \curvearrowright \mathbf{P}^1_{\mathbf{C}}$

Algebraize quotient $(\mathcal{H} \times \mathbf{P}^1_{\mathbf{C}})/\pi_1(\mathcal{X})$
$\Longrightarrow (P \to X, \nabla_P)$
($\mathbf{P}^1$-bundle + connection)

. . . <u>a</u> (canonical) <u>indigenous bundle</u>

Moduli of I.B.'s: $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$

. . . algebraic "<u>Schwarz torsor</u>" (w.r.t. Ω of $\mathcal{M}_{g,r}$), defined over $\mathbf{Z}[\frac{1}{2}]$.

can. I.B. $\Longrightarrow$ <u>canonical real analytic</u>

$$s : \mathcal{M}_{g,r}(\mathbf{C}) \to \mathcal{S}_{g,r}(\mathbf{C})$$

Teichmüller theory (Bers unif.)

$\iff$ study of can. real an. sect. s

$\iff$ study of quasi-fuchsian deformations of $\rho_{\mathcal{X}}$

(D.) "Intrinsic Hodge Theory"

alg. geom. $\iff$ topology + arith.

<u>Ex.</u>: classical/p-adic Hodge theory:

de Rham coh. $\iff$ sing./ét. coh.+Gal.

<u>Here</u>: alg. geom. = curve itself, moduli

top. + arith. = theory of $\rho_{\mathcal{X}}$

$\Longrightarrow$ "intrinsic"

Not just philosophy; classical/p-adic Hodge theory <u>techniques</u> important.

§2. The Physical Approach in the p-adic case

(A.) The Arithmetic Fundamental Group

$K \stackrel{\text{def}}{=}$ char. 0 field, $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$

X: hyp. curve/K, $\overline{X} \stackrel{\text{def}}{=} X \times_K \overline{K}$.

$\Rightarrow \quad 1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \Gamma_K \to 1$

$\pi_1(\overline{X})$ (geom. π_1): indep. of moduli (but in char. p, may determine moduli! – A. Tamagawa)

Grothendieck's <u>anabelian philosophy</u>: "Extension should determine moduli."

(B.) The Main Theorem

Theorem 1: $K \subseteq$ fin. gen. extn./$\mathbf{Q}_p$,
X: hyperbolic curve/K,
S : smooth variety/K.

$$\Rightarrow X(S)^{\mathrm{dom}} \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(S), \pi_1(X))$$

i.e., alg. curve $X \iff$
phys./an. obj. $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(-, \pi_1(X))$

Builds on work of: H. Nakamura, A. Tamagawa + G. Faltings, Bloch/Kato.

Proof: Consider p-adic analytic diff. forms on $(\mathbf{Z}_p[\ T\ ]^{\mathrm{tame}}_{(p)}\ )^{\wedge}$ (maps to X) – cf. mod. forms on $\mathcal{H}$.

Remark: Also pro-p, function field versions (cf. F. Pop).

(C.) Comparison with the Case of Abelian Varieties

Th.1 resembles Tate Conjecture, i.e.,

Hom(abelian varieties) $\Longleftrightarrow$ Hom(Tate modules)

But T. C. false over *local fields*!

New point of view:

Theorem 1 = p-adic version of physical aspect of of Fuchsian unif.

§3. The Modular Approach in the p-adic case

(A.) The Example of Shimura Curves

∃ a can. p-adic section of Sch. torsor:

$\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ (cf. can. real. an. s)?

Guide: theory of Shimura curves (cf. Y. Ihara's theory)

Ex.: Over $\mathcal{M}_{1,0}$, de Rham coh. of univ. ell. curve $\Longrightarrow$ can. ind. bun.

Note: mod p, p-curv. square nilpotent!

N.B.: p-curvature $\stackrel{\text{def}}{=}$ "[Frob., ∇]"

(B.) The Stack of Nilcurves

$(\mathcal{S}_{g,r})_{\mathbf{F}_p} \supseteq \mathcal{N}_{g,r}$: the stack of nilcurves
(curves + I.B. with sq. nilp. p-curv.)

Theorem 2: $\mathcal{N}_{g,r} \to (\mathcal{M}_{g,r})_{\mathbf{F}_p}$: finite, flat, local complete intersection, degree $= p^{3g-3+r}$, i.e.,

$\mathcal{N}_{g,r}$ "almost" a section of Sch. torsor!

Remarks: (1) $\mathcal{N}_{g,r}$ = central object of study of "p-adic Teichmüller theory."

(2) $\exists$ natural, smooth substacks $\mathcal{N}_{g,r}[d] \subseteq \mathcal{N}_{g,r}$, where d = degree of zero divisor (spikes) of p-curv.

(2) (cont'd) ($d = \infty \implies$ dormant);

$\mathcal{N}_{g,r}[d] \neq \emptyset \Rightarrow \dim = 3g - 3 + r$.

(3) $\mathcal{N}_{g,r}[0]$ affine; this $\implies$

$\mathcal{M}_{g,r}$ connected!

(cf. Teich. th./$\mathbf{C}$; ab. vars. (Oort)!)

(4) molecule $\stackrel{\text{def}}{=}$ nilcurve s.t. curve is

is tot. degen. ($\bigcup$ $\mathbf{P}^1$'s)

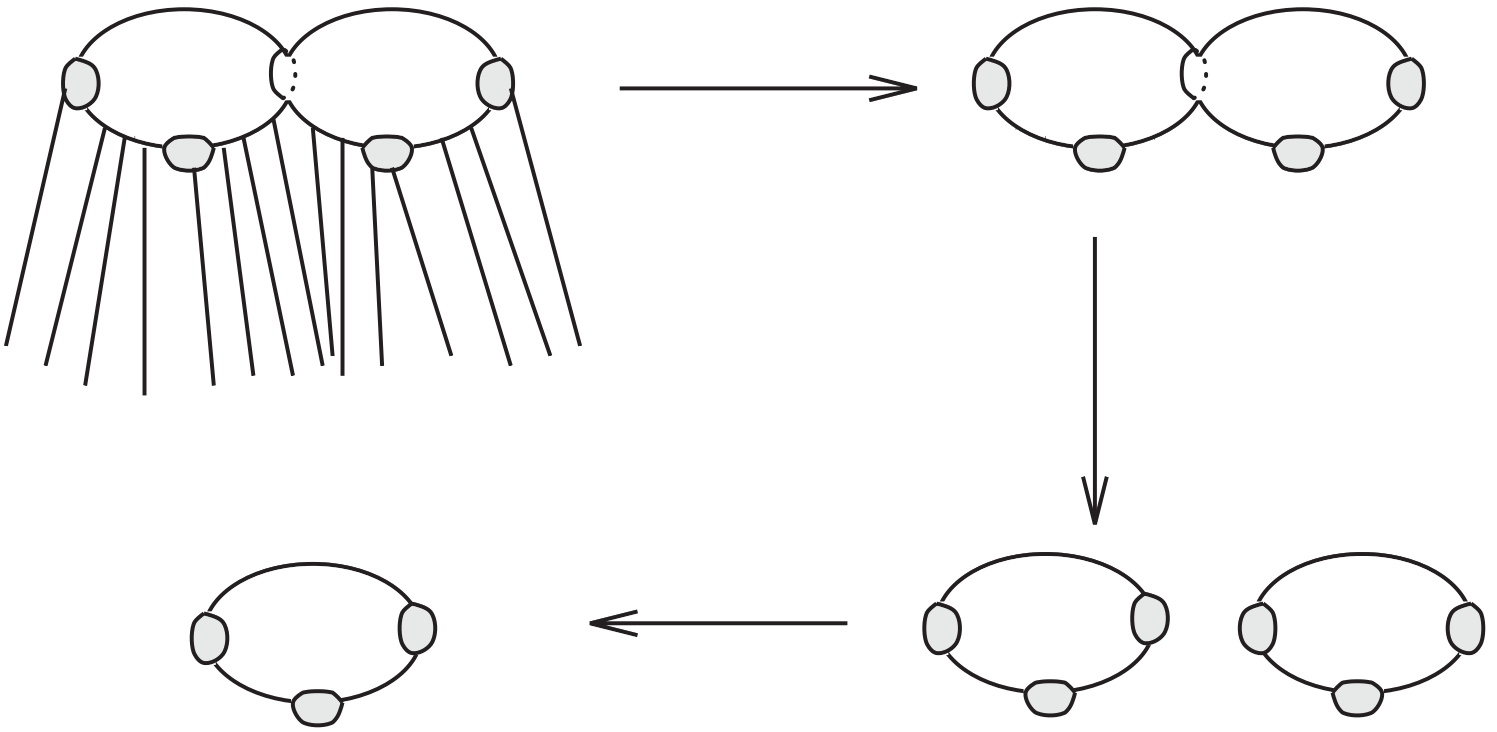
analyze mol.'s $\Rightarrow$ str. of $\mathcal{N}_{g,r}$ at ∞

(5) atom $\stackrel{\text{def}}{=}$ "toral" nilcurve s.t. curve

is $\mathbf{P}^1 - \{0, 1, \infty\}$.

$\{\text{atoms}\} \leftrightarrow$ three radii $\in \mathbf{F}_p/\{\pm 1\}$

(5) (cont'd) — reminiscent of:
"<u>pants</u>" (top. $\cong \mathbf{P}^1 - \{0, 1, \infty\}$)
decomp. of hyp. Riemann surfaces



(C.) Canonical Liftings

$$\mathcal{N}_{g,r} \supseteq (\mathcal{N}_{g,r}^{\mathrm{ord}})_{\mathbf{F}_p} \stackrel{\mathrm{def}}{=} \text{ét. locus}/(\mathcal{M}_{g,r})_{\mathbf{F}_p}$$

$\Rightarrow\ \mathcal{N}^{\text{ord}}_{g,r} \to (\mathcal{M}_{g,r})_{\mathbf{Z}_p}$ ... étale

morph. of p-adic formal stacks

Theorem 3: $\exists\,!\ (s_{\mathcal{N}} : \mathcal{N}^{\text{ord}}_{g,r} \to \mathcal{S}_{g,r};$

Frob. lift. $\Phi_{\mathcal{N}} \curvearrowright \mathcal{N}^{\text{ord}}_{g,r})$

s.t. I.B. def'd by $s_{\mathcal{N}}$ is invariant

w.r.t. Frob. act. def'd by $\Phi_{\mathcal{N}}$, i.e.,

$s_{\mathcal{N}}$ = desired can. sect. of Sch. torsor!

Remarks: (1) $(1/p) \cdot \mathrm{d}\Phi_{\mathcal{N}}$ is isom., i.e.,

$\Phi_{\mathcal{N}}$ is ordinary Frobenius lifting

(2) $\exists$ general theory of ord. F.L.'s $\Rightarrow$

(a.) can. loc. iso. to $\widehat{\mathbf{G}}_{\mathrm{m}} \times \ldots \times \widehat{\mathbf{G}}_{\mathrm{m}}$

(b.) can. Witt vector liftings of

points/char. p perfect fields

(cf. real analytic Kähler metrics)

(3) $\Phi_{\mathcal{N}} \leftrightarrow$ Weil-Petersson metric
can. mult. pars. $\leftrightarrow$ Bers unif.

(4) Serre-Tate Th. for ord. AV's arises from $\exists$ ord. F.L. $\Phi_{\mathcal{A}}$ (e.g., can. mult. coords. $\leftrightarrow$ "S.-T. pars.," etc.)

$\Rightarrow$ Th.3 = Serre-Tate
theory for hyp. curves!

(5) But $\Phi_{\mathcal{N}}$, $\Phi_{\mathcal{A}}$, respective "ord's"
are not compatible!

$\leftrightarrow \quad \mathcal{M}_g \to \mathcal{A}_g$ not isometric/$\mathbf{C}$

(for WP metric, Siegel upper half-plane metric)

(6) $(\mathcal{N}^{\text{ord}}_{g,r})_{\mathbf{F}_p} \subseteq \mathcal{N}_{g,r}[0] \ldots$ for other d,

$\exists$ can. lift. theory with "Lubin-Tate (instead of $\widehat{\mathbf{G}}_{\text{m}}$) uniformizations"!

In fact, <u>the larger</u> d
$\Rightarrow$ <u>the more "Lubin-Tate"</u> the uniformization!

(7) $\exists$ corresponding can. Gal. reps.

ρ : arithmetic π_1(curve)
$\rightarrow$ PGL_2(large ring w/Gal. act.)

...the p-adic analogues of the can. rep. $\rho_{\mathcal{X}}$ arising from the Fuchs. unif.!