Lecture Notes on:
The Intrinsic Hodge Theory of
p-adic Hyperbolic Curves

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I. General Introduction

A. The Exterior Galois Representation of a Hyperbolic Curve:

Let p be a prime number. Let K be a finite extension of $\mathbf{Q}_p$. Let X_K be a *hyperbolic* curve over K (i.e., $X_{\overline{K}}$ is obtained by removing r points from a smooth, proper, geometrically connected curve of genus g, and $2g-2+r>0$). Let Γ_K be the absolute Galois group of K. Let $\Pi_{X_K} \stackrel{\text{def}}{=} \pi_1(X_K)$ be the fundamental group of X_K (for some choice of base-point). Then if $\Delta_X \stackrel{\text{def}}{=} \pi_1(X_{\overline{K}})$ is the geometric fundamental group of K, we have an exact sequence

$$1 \to \Delta_X \to \Pi_{X_K} \to \Gamma_K \to 1$$

which induces an exterior Galois representation

$$\rho_X : \Gamma_K \to Out(\Delta_X)$$

(In fact, conversely, the above exact sequence can be recovered from the pair (Δ_X, ρ_X)). Similarly, if $\Delta_X^{(p)}$ is the pro-p completion of Δ_X, we have a "pro-p" version of the above exact sequence $(1 \to \Delta_X^{(p)} \to \Pi_{X_K}^{(p)} \to \Gamma_K \to 1)$, and a pro-$p$ exterior Galois representation $\rho_X^{(p)} : \Gamma_K \to Out(\Delta_X^{(p)})$.

Of course, one can form ρ_X, $\rho_X^{(p)}$ even when $2g-2+r \leq 0$. If $2g-2+r<0$, then ρ_X, $\rho_X^{(p)}$ are uninteresting. If $2g-2+r$ (i.e., essentially the case of elliptic curves), ρ_X, $\rho_X^{(p)}$ have already been extensively studied. Thus, here we would like to study the *hyperbolic case* (i.e., when $2g-2+r>0$).

B. Intrinsic Hodge Theory ("IHT"):

Once one decides that one wants to study the exterior Galois representation ρ_X, one important question is what should one try to prove about it. The theory of the Galois representations defined by Tate modules of abelian varieties (and more generally, p-adic étale cohomology of smooth varieties) provides at least an approximate answer: one wants to study the *Hodge Theory* of such Galois representations. Here by "Hodge theory," we mean a theory giving some sort of equivalence (or at least establishing an intimate relationship between) *étale topological data* (i.e., π_1's, étale cohomology, etc.) and *algebro-geometric data* (i.e., data like differentials, cohomology groups of coherent sheaves, morphisms between varieties, etc., that exists in the purely algebro-geometric category). For instance, p-adic Hodge theory, which relates p-adic étale cohomology groups to de Rham cohomology is clearly a prime example of such a theory.

In the case of ρ_X, however, because one is dealing with a highly nonabelian object such as Δ_X, it is not immediately clear what the appropriate Hodge theory should be. One approach is to consider *nonabelian Hodge theory*, which typically means looking at (étale topological) *spaces of representations of* Δ_X *into an algebraic group* G and relating them to (algebro-geometric) spaces of *G-bundles with connections on* X_K. Although this sort of theory may of interest in its own right, however, there is something fundamentally different about considering this sort of theory relative to understanding X_K and considering, for instance, the Hodge theory of the Tate module of an elliptic curve. It is difficult to summarize this difference in a single sentence, but the rest of this General Introduction will be devoted to trying to explain what we feel is the true hyperbolic analogue of the Hodge theory of the Tate module of an elliptic curve – namely, "intrinsic Hodge theory."

Roughly speaking, *by "intrinsic Hodge theory," we mean a Hodge theory (as defined above) such that the sort of data that appears on the algebro-geometric side is data that is just enough (not too much or too little) to capture the curve* X_K *and/or its moduli.* In the case of the nonabelian Hodge theory referred to in the preceding paragraph, what comes out on the algebro-geometric side, namely, spaces of G-bundles with connection cannot be described as capturing precisely the curve X_K or its moduli.

C. The IHT of Hyperbolic Curves over C: Physical and Modular Aspects:

Perhaps the best way to get a feel for what we mean by "IHT" is to consider the case of hyperbolic curves over $\mathbf{C}$, which are better understood classically than p-adic hyperbolic curves. Thus, let $X_{\mathbf{C}}$ be a hyperbolic curve over $\mathbf{C}$. Let Δ_X be its topological fundamental group. Then, forgetting the modern formalism of "Hodge theory" for a moment, the essence of the intrinsic Hodge theory of $X_{\mathbf{C}}$ is the classical *Köbe uniformization*

$$\widetilde{X}_{\mathbf{C}} \cong H$$

of the universal covering space $\widetilde{X}_{\mathbf{C}}$ of $X_{\mathbf{C}}$ by the upper half-plane H. There are many useful alternative ways to rephrase this uniformization, as follows:

(1) the *canonical representation* $\rho_X : \Delta_X \to PSL_2(\mathbf{R}) = Aut(H)$ defined by the uniformization;

(2) the *hyperbolic metric* μ_X on $X_{\mathbf{C}}$ obtained by pulling back the standard hyperbolic metric $\frac{1}{y^2}(dx^2 + dy^2)$ on H;

(3) the *indigenous bundle* (terminology of Gunning) $(P \to X_{\mathbf{C}}, \nabla_P)$ constructed as follows: We let Δ_X act on $\widetilde{X}_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ by means of the natural action on the first factor and ρ_X on the second factor. Taking the quotient of this product by Δ_X then gives rise to a natural *algebraic* $\mathbf{P}^1$-bundle $P \to X_{\mathbf{C}}$, equipped with a connection ∇_P, and a section $\sigma : X_{\mathbf{C}} \to P$ (induced by the section $\widetilde{X}_{\mathbf{C}} \to \widetilde{X}_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ given by the uniformization mapping $\widetilde{X}_{\mathbf{C}} \cong H \subseteq \mathbf{P}^1_{\mathbf{C}}$). Moreover, if one differentiates the section σ by means of ∇_P, one obtains a Kodaira-Spencer mapping $\tau_{X_{\mathbf{C}}} \to \sigma^* \tau_{P/X_{\mathbf{C}}}$ which is an *isomorphism*. This sort of data $(P \to X_{\mathbf{C}}, \nabla_P)$ (i.e., such that there exists a σ whose Kodaira-Spencer morphism is an isomorphism) is called an *indigenous bundle*. Lots of indigenous bundles exist on $X_{\mathbf{C}}$, but the one just constructed from the upper half-plane uniformization – called *canonical* – is a particularly special one.

If one wants to formalize things according to the definition of "Hodge theory" given above, two natural approaches are the following:

The Physical Picture: One can physically recover the *algebraic curve* $X_{\mathbf{C}}$ (algebro-geometric data) from the π_1-theoretic (i.e., topological) datum ρ_X via the following well-known double-coset recipe:

$$Im(\rho_X) \backslash PSL_2(\mathbf{R}) / PSO_2$$

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The Modular Picture: Let $\mathcal{M}_{g,r}$ be the moduli stack of hyperbolic curves of type (g, r) over $\mathbf{C}$. Then over $\mathcal{M}_{g,r}$, there is a natural stack $\mathcal{S} \to \mathcal{M}_{g,r}$ of hyperbolic curves equipped with an indigenous bundle. Moreover, $\mathcal{S}$ has the natural structure of $\Omega_{\mathcal{M}_{g,r}}$-torsor over $\mathcal{M}_{g,r}$. (In fact, this torsor is the Hodge-theoretic first Chern class of a certain line bundle that can be written down explicitly.) The canonical indigenous bundle constructed above defines a *real analytic section* $s_H : \mathcal{M}_{g,r} \to \mathcal{S}$. Moreover, if $m \in \mathcal{M} \stackrel{\text{def}}{=} \mathcal{M}_{g,r}$, then by localizing at m, we see that s_H defines a morphism of holomorphic germs $\mathcal{M}_m \times \mathcal{M}^c_m \to \mathcal{S}_m$ (where $\mathcal{M}^c$ is the complex conjugate stack to $\mathcal{M}$). Restricting this morphism to $m \in \mathcal{M}_m$, and letting Q_m be the $(3g - 3 + r)$-dimensional (over $\mathbf{C}$) affine space which is the fiber of $\mathcal{S} \to \mathcal{M}$ at m, we see that we get an anti-holomorphic morphism

$$\beta : \mathcal{M}_m \to Q_m$$

which turns out to be an embedding. In fact, this morphism extends (uniquely) to all of $\widetilde{\mathcal{M}}$ (the universal covering space of $\mathcal{M}$) to define an anti-holomorphic uniformization $\widetilde{\mathcal{M}} \hookrightarrow Q_m$ which is usually referred to as the *Bers embedding*. Thus, in summary, the theory just discussed relates the π_1-theoretic ρ_X to the algebro-geometric $\mathcal{M}_m$ by showing how ρ_X defines *canonical coordinates* on $\mathcal{M}_m$.

The intrinsic Hodge theory of p-adic hyperbolic curves to be discussed in the rest of this lecture can be regarded as the generalization to the p-adic case of the physical and modular aspects of the intrinsic Hodge theory of hyperbolic curves over $\mathbf{C}$ discussed above.

D. The IHT of Abelian Varieties:

Since the intrinsic Hodge theory of abelian varieties has already been p-adicized, it is useful to recall what happens for abelian varieties. Namely, one has the following:

The Physical Picture: Over $\mathbf{C}$, one has the uniformization of an (algebraic) abelian variety $A_{\mathbf{C}}$ by $T_{A,e}/\pi_1(A_{\mathbf{C}})$ (where $T_{A,e}$ is the tangent space to $A_{\mathbf{C}}$ at the origin). In the p-adic case, there are several candidates for an analogue. One is the *Tate conjecture* (Faltings' theorem – see [Falt]), which states that if A and B are abelian varieties over a number field F with p-adic Tate modules $T_p(A)$ and $T_p(B)$, respectively, then the natural morphism

$$Hom_F(A, B) \otimes_{\mathbf{Z}} \mathbf{Z}_p \to Hom_{\Gamma_F}(T_p(A), T_p(B))$$

is bijective. Another candidate is the result that states that for an abelian variety A over a local p-adic field K (i.e., a finite extension of $\mathbf{Q}_p$), the K-valued points $A(K)$ (algebro-geometric data) may be recovered from the full (i.e., $\widehat{\mathbf{Z}}$-flat) Tate module $T(A)$ as the kernel of

$$H^1(\Gamma_K, T(A)) \to H^1(\Gamma_K, T(A) \otimes_{\widehat{\mathbf{Z}}} \mathbf{C}_p)$$

(see [BK], §3). Both of these analogues are related to the result that we obtain for p-adic hyperbolic curves.

The Modular Picture: Over $\mathbf{C}$, this is given by the uniformization of $(\mathcal{A}_g)_{\mathbf{C}}$ (the moduli stack of principally polarized abelian varieties over $\mathbf{C}$) by the Siegel upper half-plane. In the p-adic case, over $(\mathcal{A}_g^{ord})_{\mathbf{Z}_p}$ (the moduli stack of principally polarized abelian varieties with ordinary mod p reduction over $\mathbf{Z}_p$), one has the Serre-Tate theory (see, e.g., [Mess]) of *canonical coordinates* and *canonical liftings*.

II. The IHT of p-adic Hyperbolic Curves: the Modular Picture

We begin with the *modular picture*, since this was discovered first (see [Mzk1], [Mzk2], [Mzk3]). First observe that the notion of an *indigenous bundle*, being entirely algebraic, exists over $\mathbf{Z}[\frac{1}{2}]$. Thus, for instance, in characteristic p (where p is odd), we can consider *nilpotent indigenous bundles*, i.e., indigenous bundles whose p-curvature forms a nilpotent matrix. This gives rise to a stack $\overline{\mathcal{N}}_{g,r}$ of stable curves (of type (g,r)) equipped with nilpotent indigenous bundles. Moreover, the natural morphism

$$\overline{\mathcal{N}}_{g,r} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$$

is finite and flat of degree p^{3g-3+r}. We denote by $\overline{\mathcal{N}}_{g,r}^{ord} \subseteq \overline{\mathcal{N}}_{g,r}$ the *ordinary locus*, i.e., the locus of points of $\overline{\mathcal{N}}_{g,r}$ that are *étale* over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$. Since $\overline{\mathcal{N}}_{g,r}^{ord} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is étale, it lifts uniquely to an étale morphism $(\overline{\mathcal{N}}_{g,r}^{ord})_{\mathbf{Z}_p} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{Z}_p}$, where $(\overline{\mathcal{N}}_{g,r}^{ord})_{\mathbf{Z}_p}$ is a *formal* p-adic stack.

Over $\mathcal{N} \stackrel{\text{def}}{=} (\overline{\mathcal{N}}_{g,r})^{ord}_{\mathbf{Z}_p}$, one has the following *ordinary theory* ([Mzk1]): First, there is a natural Frobenius lifting

$$\Phi_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$$

(analogous to the Frobenius lifting

$$\Phi_{\mathcal{A}} : (\mathcal{A}_g^{ord})_{\mathbf{Z}_p} \to (\mathcal{A}_g^{ord})_{\mathbf{Z}_p}$$

of Serre-Tate theory given by mapping an abelian variety A (with ordinary reduction modulo p) to the quotient of A by the multiplicative part of the kernel of $p \cdot : A \to A$). Just as in Serre-Tate theory, there are unique *multiplicative canonical coordinates* fixed by $\Phi_{\mathcal{N}}$, as well as a notion of *canonical liftings*. Finally, if $\mathcal{S}_{\mathcal{N}} \to \mathcal{N}$ is the pull-back to $\mathcal{N}$ of the torsor of indigenous bundles over $\overline{\mathcal{M}}_{g,r}$, then there is a canonical section $\mathcal{N} \to \mathcal{S}_{\mathcal{N}}$ (cf. "s_H" in the complex case) that corresponds to a unique *Frobenius-invariant indigenous bundle*.

In fact, $\overline{\mathcal{N}}_{g,r}$ consists (in general) of irreducible components which are not generically ordinary. The degree of the ordinary (and nonordinary) locus of $\overline{\mathcal{N}}_{g,r}$ over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ can be computed (at least, in principle) by means of a complicated *combinatorial algorithm* ([Mzk3]). Moreover, in general there exist irreducible components whose generic points correspond to *dormant indigenous bundles* (indigenous bundles whose p-curvatures are identically zero) and *spiked indigenous bundles* (indigenous bundles whose p-curvatures have zeroes, but are not identically zero). Roughly speaking, these loci also admit theories with Frobenius liftings, etc. analogous to the ordinary theory of the preceding paragraph, but instead of multiplicative canonical coordinates (i.e., local uniformizations by products of $\mathbf{G}_m$), in the *generalized ordinary theory* ([Mzk2]), one gets local uniformizations by more general sorts of Lubin-Tate groups, or twisted (fiber) products of Lubin-Tate groups in the dormant and spiked cases. This sort of generalized ordinary theory is a phenomenon essentially without analogue in the case of hyperbolic curves over $\mathbf{C}$ or p-adic abelian varieties.

III. The IHT of p-adic Hyperbolic Curves: the Physical Picture

The physical side of the intrinsic Hodge theory of a p-adic hyperbolic curve consists of the following Theorem (roughly conjectured by Grothendieck in a letter to Faltings): First, some terminology: If K is a field, we shall refer to a K-scheme S_K as a *smooth pro-variety* (respectively, *hyperbolic pro-curve*) over K if S_K can be written as the projective limit of a projective system of smooth varieties (respectively, hyperbolic curves) over K in which the transition morphisms are all birational. Note that the notion of a "smooth pro-variety" (respectively, "hyperbolic pro-curve") has as special cases: (i) a smooth variety (respectively, hyperbolic curve) over K; (ii) the spectrum of a function field of arbitrary dimension (respectively, function field of dimension one) over K. Then we have the following π_1*-theoretic recipe for recovering the nonconstant* S_K*-valued points* (where S_K is a smooth pro-variety over K) *of a hyperbolic pro-curve* X_K ([Mzk5]):

Theorem: *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbf{Q}_p$. Let X_K be a hyperbolic pro-curve over K. Then for any smooth pro-variety S_K over K, the natural map*

$$X_K(S_K)^{nonconst} \to Hom^{open}_{\Gamma_K}(\Pi^{(p)}_{S_K}, \Pi^{(p)}_{X_K})$$

is bijective. (Here $Hom^{open}_{\Gamma_K}(\Pi^{(p)}_{S_K}, \Pi^{(p)}_{X_K})$ denotes the set of open, continuous group homomorphisms $\Pi^{(p)}_{S_K} \to \Pi^{(p)}_{X_K}$ over Γ_K, considered up to composition with an inner homomorphism arising from $\Delta^{(p)}_X$.)

Note that although this result is formally analogous to the Tate conjecture for abelian varieties over a number field (especially if one takes S_K also to be a hyperbolic pro-curve), one major difference is that the above Theorem is valid over *local fields*, whereas the Tate conjecture fails over local fields.

One also has the following application (unrelated to curves) of the above Theorem:

Corollary: *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbf{Q}_p$. Let L and M be function fields of arbitrary dimension over K. Then the natural map*

$$Hom_K(Spec(L), Spec(M)) \to Hom^{open}_{\Gamma_K}(\Gamma_L, \Gamma_M)$$

is bijective. (Here, $Hom^{open}_{\Gamma_K}(\Gamma_L, \Gamma_M)$ is the set of open, continuous group homomorphisms $\Gamma_L \to \Gamma_M$ over Γ_K, considered up to composition with an inner homomorphism arising from $Ker(\Gamma_M \to \Gamma_K)$.)

Note that in characteristic zero, this generalizes the result of [Pop], where a similar result to this Corollary is obtained, except that the morphisms $\Gamma_L \to \Gamma_M$, $Spec(L) \to Spec(M)$ are required to be *isomorphisms*, and K is required to be finitely generated over $\mathbf{Q}$.

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