Non-abelian Brill-Noether loci of curves and application to the unirationality of moduli of K3 surfaces

(Berlin Humboldt 大学にて, 50分)

S. Mukai

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$\mathcal{F}_g$ = moduli of primitively polarized K3's (S,h) of genus g, i.e. degree $(h^2) = 2g-2$, with only RDP's. = Orthogonal modular variety D^{19}/Γ_{g-1} quasi-proj. 19-dim. (19-dim's)

<u>Thm</u> $\mathcal{F}_g$ is unirational for $g \leq 13$ (sub $g=11$), $g=17, 18$ and 20 (new).

GHS'07 $\kappa(\mathcal{F}_g) \geq 0$ for $g = 41, 43, 44, 47$ and $g \geq 49$.

Easy for $g \leq 5$.

<u>Example</u> ($g=5$) $\mathcal{F}_5 \approx \{\text{C.I. } Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5\}$

$\amalg \{\text{trigonal}\} \amalg \{\text{h.e.}\} \amalg \{\text{monogonal}\}$

$\underset{\text{birat.}}{\sim} G(3, S^2\mathbb{C}^6)/\!/PGL(6)$

Vector bundle method for $g \geq 6$.

① Grassmannian embedding gives explicit description of generic member $(S,h) \in \mathcal{F}_g$ for $g \neq 11, 17$

<u>Key Fact</u> $g = rs$ $\exists!$ stable r-bundle E s.t. $c_1(E) = h$, $h^0(E) = r+s$ and

$$\Phi_{|E|} : S \longrightarrow G(H^0(E), r) \quad g\text{-dim Grassmannian}$$

is an embedding. (rigid bundle, or exceptional)

<u>Example</u> $g = 20 = 4 \times 5$ $\quad S \hookrightarrow G(9,4)$ 20-dim'l Grassmannian

(M'92) S is C.I. w.r.t. $(\bigwedge^2 \mathcal{E})^{\oplus 3}$, where $\mathcal{E}$ is the universal quotient bundle. Namely, $\exists\, s_1, s_2, s_3 \in H^0(\bigwedge^2 \mathcal{E})$

$S = (\Delta_1)_0 \cap (\Delta_2)_0 \cap (\Delta_3)_0 \quad \dim = 20 - 3(\mathrm{rk}\, \overset{2}{\wedge} E) = 2$

$$\mathcal{F}_{20} \underset{\text{birat}}{\sim} G(3, \overset{2}{\wedge} \underset{\underset{H^0(\wedge E)}{\parallel}}{\mathbb{C}^9}) / PGL(9)$$

② Moduli method for $g = 11, 17$

Description of S in a suitable moduli space.

In fact, S itself is a moduli!

$g=11$ "non-abelian BN locus" + moduli K3

$g=17$ " + "isogeny of K3s"

<u>Fact</u> $(d=) g-1 = rs$

$T = M_S(r, h, s)$ = moduli of stable r-bundles with $c_1(E) = h$ and $\chi(E) = r+s$.

$M_S(r, h, s)$ is again a K3 surface and carries a natural polarization $\hat{h}$ of degree $(-h^2)/(r,s)^2$.

$g=11$: $g-1 = 10 = 2 \times 5$ $\quad M_S(2, h, 5)$ is again K3 of genus 11

$g=17$: $g-1 = 16 = 2 \times 8$ $\quad M_S(2, h, 8)$ is a K3 of genus 5

$\rightsquigarrow$ non-trivial involution σ_{11} of $\mathcal{F}_{11}$, and rational map $\mathcal{F}_{17} \longrightarrow \mathcal{F}_5$

σ_{17}:

$\boxed{\text{Example}}$ $g=5$: $S = S_5 = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ $\quad g-1 = 4 = 2 \times 2$

$M_S(2, h, 2) \longrightarrow \mathbb{P}^2 = \langle Q_1, Q_2, Q_3 \rangle$ net of quadrics

$\tau^2 = \det(x_1 Q_1 + x_2 Q_2 + x_3 Q_3)$

$M_S(2, h, 2)$ is K3 of deg 2.

§1 Genus 11

$(S, h) \in \mathcal{F}_g$, $C \in |h| \cong \mathbb{P}^g$ $g(C) = g$

$$\begin{tikzcd} \mathcal{P}_g = \{(S, C)\} \arrow[r, dashed, "\varphi_g"] \arrow[d, "\pi"] & \mathcal{M}_g \\ \mathcal{F}_g \end{tikzcd}$$

<u>CK3-Problem</u> When is φ_g dominant? generically finite?

$\Downarrow$ $\dim \mathcal{P}_g \geq \dim \mathcal{M}_g$ $\Leftrightarrow$ $g+19 \geq 3g-3$, $g \leq 11$

$\Downarrow$ $\dim \mathcal{P}_g \leq \dim \mathcal{M}_g$ $\Leftrightarrow$ $g \geq 11$

<u>Answer</u> φ_g is dominant $\Leftrightarrow$ $g \leq 9$ or $g = 11$ ($g=10$ is missing)

φ_g is generically finite $\Leftrightarrow$ $g = 11$ or $g \geq 13$ ($g=12$ is missing)

<u>Theorem</u> (M'06) $\varphi_{11} : \mathcal{P}_{11} \dashrightarrow \mathcal{M}_{11}$ is birational.
Moreover $\pi \circ \varphi_{11}^{-1}$ is the composite of $\underbrace{\ \ }_{\text{the involution}}$ and $SL(2)$-BN map.

$$\mathcal{M}_{11} \overset{\beta}{\dashrightarrow} \mathcal{F}_{11}, \quad C \longmapsto SU_C(2, K, 5).$$

By Chang-Ran, $\mathcal{M}_{11}$ is uniruled.

<u>Cor</u> $\mathcal{F}_{11}$ is uniruled.

$$\begin{tikzcd} \mathcal{P}_{11} \arrow[r, dashed, "\varphi_{11}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{11} \arrow[d, "\beta"] \\ \mathcal{F}_{11} & \mathcal{F}_{11} \arrow[l, dashed, "\alpha_{11}"] \end{tikzcd}$$

$S \supset C$ is recovered as double moduli

$C \longmapsto T = M_C(2, K, 5) \longmapsto S = M_T(2, h, 5)$

(Generalization of $\mathrm{Alb}(X) = \mathrm{Pic}^0 \mathrm{Pic}^0 X$).

§2 SL(2)-BN-locus

C curve of genus g, $\xi \in \mathrm{Pic}\, C$

$$SU_C(2, \xi) = \{\text{stable 2-bundles with } \det E \cong \xi\} / \text{isom.}$$

quasi-proj. of dim $3g-3$

BN locus of type III

$$SU_C(2, K, s) = \{h^0(E) \geq s+2\} \subset SU_C(2, K)$$

Petri map $H^0(E) \times H^0(E) \longrightarrow H^0(S^2 E)$ symmetric

Expected dim $\rho = 3g-3 - \dfrac{(s+2)(s+3)}{2}$

or BN number

$$g=11,\ s=5 \Rightarrow \rho = 30 - \frac{7 \cdot 8}{2} = 2 \qquad \text{surface? Yes.}$$

Thm (cont'd) $C \in \mathcal{M}_{11}$ general $\Rightarrow (SU_C(2, K, 5), \text{polarization}) \in \mathcal{F}_{11}$

BN locus of type II $\qquad F$ 2-bundle on C

$$SU_C(2, K; k\, F) = \{E \mid \dim \mathrm{Hom}(F, E) \geq k\}$$
$$\subset SU_C(2, K \otimes \det F)$$

省略する

~~$\mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(\wedge^2 F, \wedge^2 E) \cong K_C$~~

~~$f \longmapsto \wedge^2 f$~~

~~is non-deg. quadratic form with value in K_C.~~

~~By Atiyah-Mumford, $\dim \mathrm{Hom}(F, E) \equiv 2$ is locally constant.~~

~~More explicitly, $\dim \mathrm{Hom}(F, E) \equiv \deg E$ mod 2 holds.~~

Petri map $\mathrm{Hom}(F, E) \times \mathrm{Hom}(F, E) \longrightarrow \mathrm{Hom}(\wedge^2 F, S^2 E)$

symmetric

Expected dimension $\rho = 3g-3 - \dfrac{k(k+1)}{2}$

or BN number

$g=5,\ k=5 \Rightarrow \rho = 12-10$ surface? Yes.

Thm $C \in \mathcal{M}_5$, F general stable bundle $\Rightarrow SU_C(2,K;5F)$ is K3. Moreover $(SU_C(2,K;5F), \text{tautological}) \in \mathcal{F}_{17}$

and $\{(C,F)\} \dashrightarrow \mathcal{F}_{17}$ is dominant.

$\dim = 12+12 = 24$

$$\begin{tikzcd} \{(C,F)\} \arrow[d] \\ \mathcal{M}_5 \end{tikzcd}$$

12-dim

More precisely, we have the following diagram

$$\begin{tikzcd} SU(2) \ni \{(C,F)\} \arrow[d] & \arrow[l, "\cong"'] & \mathcal{P}_5 \times_{\mathcal{F}_5} \mathcal{F}_{11} \arrow[l, dashed, "\text{birat}"'] \arrow[r] \arrow[d] & \mathcal{P}_5 \arrow[d, "\mathbb{P}^1\text{-bundle}"] \\ \mathcal{M}_5 & & \mathcal{F}_{17} \arrow[r, dashed, "\varphi_{11}"'] & \mathcal{F}_5 \end{tikzcd}$$

$\mathcal{P}^{(1)}_{17} :=$

Voisin $g=16$

$(S,h) \in \mathcal{F}_{17}$, $(T = M_S(2,h,2), \hat{h}) \in \mathcal{F}_5$

$\exists!$ $\mathbb{P}^1$-bundle P on T. by Tyurin

(not coming from V.b.) $\quad P|_C = \mathbb{P}(F)$

$$(S, C) \overset{\varphi^{(1)}}{\dashrightarrow} (C, P|_C) \in SU(2)$$

$g=17 \qquad g=5$

BN: $C \dashrightarrow SU_C(2,K;5F)$ are each other dual.

Future project

$(h^2)=36$

$g=19 \quad (S,h) \in \mathcal{F}_{19} \quad (T = M_S(3,h,6), \hat{h}) \in \mathcal{F}_3$

$\exists$ $\mathbb{P}^2$-bundle Q on T.

$$\mathcal{P}^{(3)}_{19} := \mathcal{P}_3 \times_{\mathcal{F}_3} \mathcal{F}_{19} \overset{\varphi^{(3)}}{\dashrightarrow} SU(3) \quad \dim = 6+16 = 22$$

$(S,C) \longmapsto (C, Q|_C)$ $\downarrow$ $\mathcal{M}_3$

$\dim = 19+3 = 22$. $g=19 \quad g=2$

Is $\varphi^{(3)}$ birational? $\varphi^{(1)}$ by $SL(3)$?

BN

61