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Unirationality of $\mathcal{M}_{11}$ and prime Fano 3-folds V_{22} of genus 12 — genus 11 and two poristic neighbors —

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in occasion of Prof. Mori's 70th Birthday

§1 mini-history

• G. Fano (1871–1952)

Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche, 1937

$$C = C_{2g-2} \xrightarrow[\Phi_{|K|}]{} \mathbb{P}^{g-1},\ z \longmapsto \underbrace{(\omega_1 : \cdots : \omega_g)}_{\text{basis of } H^0(C, K_C)}$$

"K3 surface" in this lecture

= surface $[S \subset \mathbb{P}^g]$ w. $[S \cap \mathbb{P}^{g}] \cap H$ canonical

• Iskovskikh (1977, 78) Modernization / Deciphering of classification of prime Fano 3-folds

Answer $g = 2, \ldots, 10, 12$ — overlooked by Fano. # of moduli = 6

Method Fano's double projection from a line

• Mori Theory of extremal ray, PNAS (1980)

Mori–M. Classification of Fano 3-folds w. $B_2 \geq 2$

1981–82 IHS @ Princeton AG Year

§2. Main subject 1.

$$\mathcal{M}_g := \{\text{curves of genus } g\} / \text{isom.} \qquad \dim = \begin{cases} g & g=0,1 \\ 3g-3 & g \geq 2 \end{cases}$$

Alg. variety X is uniruled $\Longleftrightarrow$ $\exists\, Y$ $\exists$ dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow\!\!\!\to X$ with $\dim Y = \dim X - 1$

Mori - M. (1983) $\mathcal{M}_{11}$ is uniruled. More precisely
(Work @ Princeton)

$\exists$ dominant rat'l map $\mathbb{P}^{11} \times \mathcal{F} \dashrightarrow\!\!\!\to \mathcal{M}_{11}$. Furthermore,

19-dimensional

$\mathcal{F}$ is moduli of polarized K3 surface (S, h) of genus 11.

<u>Remark</u> 1. (a) Chang - Ran (1984) $\mathcal{M}_{11}$ is unirational, i.e., $\exists\ \mathbb{P}^{30} \dashrightarrow\!\!\!\to \mathcal{M}_{11}$.

(b) Harris - Mumford (1983) $\mathcal{M}_g$ is of general type, i.e. $\kappa(\mathcal{M}_g) = \dim \mathcal{M}_g$ when $g >> 0$.

<u>Problem discussed in MM'83</u>

$$\mathcal{F}_g := \left\{ \begin{array}{c} \text{polarized K3 surface } (S, h) \\ \text{with } (h^2) = 2g-2 \end{array} \right\} / \text{isom.}$$

$$\Downarrow$$

$$[S_{2g-2} \subset \mathbb{P}^g_S] \cap H = [C_{2g-2} \subset_{\Phi_{|K|}} \mathbb{P}^{g-1}]$$

$$\begin{array}{ccc} H & \longmapsto & [C \subset S \ni H] \\ \cap & & \cap \\ \mathcal{P}_g = \coprod_{(S,h)} \mathbb{P}_S^{g,*} & \xrightarrow[\text{forgetful map}]{\varphi_g} & \mathcal{M}_g \\ \downarrow & & \\ \mathcal{F}_g & & \end{array}$$

$\mathbb{P}^g$-bundle

$K_g := \operatorname{Im} \varphi_g$ Locus of "K3 curves"

Obviously, we have

$$\dim K_g \leq \min\{\underset{g+19}{\dim \mathcal{P}_g}, \underset{3g-3}{\dim \mathcal{M}_g}\} =: \text{exp-dim.}$$

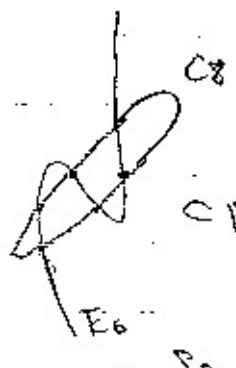
<u>Theorem</u> 2. ([MM'83] + α) "=" holds except for $g = 10, 12$.

"pf ($g=11$)" C <u>hexagonal</u> curve of $g=11$, i.e., $\exists\, C \longrightarrow \mathbb{P}^1$ of degree 6

$C_{14} \subset \mathbb{P}^5$ $\quad \alpha = \mathcal{O}_C(1)$, $\beta = K_C \otimes \mathcal{O}_C(-1)$ (BN # $= 11 - 2\times 6 = -1$)

$h^0 = 2$, degree 6 $\quad h^0 = 6$, 14

$$\Phi_{|\beta|} : C_{14} \hookrightarrow \mathbb{P}^5$$



$2\cdot 14 - 10 = 18$ $\quad H^0(\mathcal{O}_C(2)) \longleftarrow H^0(\mathcal{O}_{\mathbb{P}}(2)) = S^2\mathbb{C}^6$ 21 dim.

$\exists\, Q_1, Q_2, Q_3 \supset C \qquad S = Q_1 \cap Q_2 \cap Q_3 \supset C$ K3 surface

"$\varphi_{11}^{-1}[C] = \{S\}$"

inside S_8

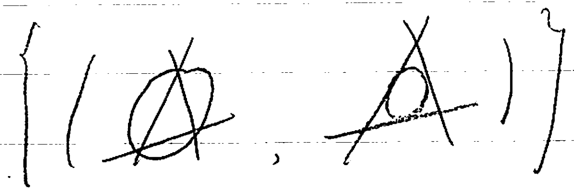
Hence φ_{11} is generically finite by u.s.c.! upper $\frac{1}{2}$-cont. of fiber dimension.

<u>Remark 3</u> Proof in [MM'83] uses DM-stable curve $\underset{g=5}{C_8} + \underset{g=1}{E_6} \subset \mathbb{P}^5$, a degeneration of the above $C_{14} \subset \mathbb{P}^5$, instead. Not easy to make the above proof a real one.

§3. Porism

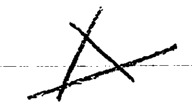
Elementary geometry on $\mathbb{P}^2$

pair of conics

$6+2+2=10$  $\xrightarrow[\varphi]{\text{tangential}}$

$5+5=10$

$\downarrow$

$2+2+2=6$ triangle 

Q. Is φ of maximal rk?

A. No, if a pair contains a biscribed triangle, then it contains ∞^1 such pairs (Poncelet's porism, or closure thm).

Poristic two neighbors: $g=10, 12 \Rightarrow \dim \mathcal{K}_g = (\text{exp. dim}) - 1$

$\boxed{\text{Genus } 12}$

$\varphi_{12}: \mathcal{P}_{12} \dashrightarrow \mathcal{M}_{12}$ 31-dim. $\dashrightarrow$ 33-dim.

Existence of a K3-extension (of $C_{22} \subset \mathbb{P}^{10}_{|K|}$) involves that of ∞^1 such extension.

Reason = Existence of Fano 3-folds $V_{22} \subset \mathbb{P}^{13}$ of genus 12 (due to Iskovskikh).

$$[C_{22} \subset \mathbb{P}^{11}] = [V_{22} \subset \mathbb{P}^{13}] \cap H_1 \cap H_2$$

$$\subset [S_{22} \subset \mathbb{P}^{12}] \cap (aH_1 + bH_2), \quad (a:b) \in \mathbb{P}^1$$

∞^1 K3-extension

$C \xrightarrow[\Phi_{|\alpha|}]{\text{birat}} \overline{C}_q \subset \mathbb{P}^2$

<u>Example 4.</u> $G^3_q = \left\{ \begin{matrix} C \text{ with} \\ a\ g^2_q \end{matrix} \right\} \subset \mathcal{M}_{12}$ codim 3

$BN\# = 12 - 3 \times 5 = -3$

$\alpha = \mathcal{O}_C(1)$, $\beta = K_C \otimes \mathcal{O}_C(-1)$

$h^0 = 3$ $h^0 = 5$

Consider $\Phi_{|\beta|} : C_{|\beta|} \hookrightarrow \mathbb{P}^4$ (as in the case $g=11$).

and $26 - 11 = 15$ $H^0(\mathcal{O}_C(2)) \xleftarrow{r_2} H^0(\mathcal{O}_{\mathbb{P}}(2)) = S^2\mathbb{C}^5$ 15

~~28 $H^0(\mathcal{O}_C(3)) \xleftarrow{r_3} H^0(\mathcal{O}_{\mathbb{P}}(3)) = S^3\mathbb{C}^5$ 35~~

$G^{3,sp}_2 := \left\{ \begin{matrix} C \text{ with} \\ \det r_2 = 0 \end{matrix} \right\} \subset G^3_q$ divisor

$G^{3,sp}_2 \subset \mathcal{K}_{12}$ divisor

Because $C \subset \exists Q \subset \mathbb{P}^4$ and $\exists D_1, D_2 \subset \mathbb{P}^4$

~~cubic s.t. $C \subset Q \cap D_2$ K3 surface of degree 6.~~

~~$(35 - 28 - 5 - 1 = 1)$. $\exists$ 1-dim'l family of such K3's, (possible)~~

$C \subset Q \cap (aD_2 + bD_1)$,

$(a:b) \in \mathbb{P}^1$

<u>§4. Main subject 2.</u>

$\mathcal{K}_{12} \subset \mathcal{M}_{12}$ contains 4 divisors $\mathcal{D}^{(i)}$, $i = 0, 2, 3, 4$.

corresponding to 1-nodal degeneration $V_{22}^{(i)}$ of smooth

Fano 3-folds. $i = \#$ of lines l, $(l.-K_V) = 1$, passing

through the unique node.

The node is algebraically non-factorial

and $V_{22}^{(i)}$ has two small resolutions.

node = O.D.P.
$\sim (x^2 + y^2 + z^2 + t^2 = 0)$
analytic.

V_{22}^a and V_{22}^b. R_a, R_b ext. ray

$i \neq 0$
$\Rightarrow i = -(BN\#$ of gen. member of $\mathcal{D}^{(i)})$

i	0	2	3	4
$\mathcal{D}^{(i)}$	G^1_5	$G^{1,sp}_6$	$G^{2,sp}_9$	G^3_{11}
V_{22}^a and cont. R_4	$\mathbb{P}(\mathcal{E})/\mathbb{P}^2$ 2-bundle $c_1=0$, $c_2=4$ (Barth, 1977)	$Bl_{R_4} V_5$	$Bl_{R_5} Q^3$	$Bl_{R_5} \mathbb{P}^3$
V_{22}^b and cont. R_6	$V_{22}^1 \longrightarrow \mathbb{P}^1$ dP_5	dP_6-fibration	conic bundle $/\mathbb{P}^2$ w. cubic discriminant	$Bl_{R_5'} \mathbb{P}^3$

Table 5. $\mathcal{D}^{(i)}$ and $V_{22}^{(i)}$

(Linear section) Thm 6 For $[C] \in \mathcal{K}_{12}$, $C_{22} \underset{\Phi_{|K|}}{\hookrightarrow} \mathbb{P}^{11}$

(1) $C \notin \bigcup_{i=0,2,3,4} \mathcal{D}^{(i)} \Rightarrow [C_{22} \subset \mathbb{P}^{11}] \overset{\exists !}{\cong} [\underset{\text{smooth}}{V_{22}} \subset \mathbb{P}^{13}] \cap H_1 \cap H_2$

(2) $[C] \in \mathcal{D}^{(i)}$ general member

$\Longrightarrow C \cong V_{22}^{(i)} \cap H_1 \cap H_2$

Example 4 (cont'd). $[C] \in \mathcal{D}^{(3)} = G^{2,sp}_9$

$\Phi_{|\beta|} : C_{13} \hookrightarrow Q^3 \subset \mathbb{P}^4$ $\qquad C_{13} \subset Q \cap D_1 \cap D_2$

$Q \cap D_1 \cap D_2 = C_{13} \cup R_5$ rat'l quintic

$R_5 \subset Q^3 \subset \mathbb{P}^4$ has a unique 3-secant line

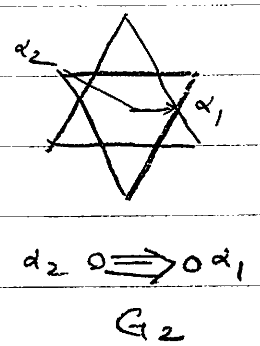
$l \subset Q^3$, $\#(R_5 \cap l) = 3$

$$\begin{array}{ccccccc}
l' \subset & Bl_R Q^3 = V_{22}^a & \overset{flop}{\longleftarrow\!\!-\!\!-\!\!\longrightarrow} & V_{22}^b & & \\
\downarrow ? & \downarrow & \searrow & \swarrow & \searrow \text{conic bundle} & \\
l \subset & Q^3 \supset R_5 & & V_{22}^{(3)} & & \Delta_3 \subset \mathbb{P}^2 \\
 & & & \Downarrow & & \\
 & & & \text{ODP} & &
\end{array}$$

Remark 7. Similar results hold for $g = 7, 8, 9, 10$ replacing $V_{22} \subset \mathbb{P}^{13}$ by rat'l homogeneous spaces

g	7	8	9	10
G/P	$OG(5,10)^+ \subset \mathbb{P}^{15}$	$G(2,6) \subset \mathbb{P}^{14}$	$SpG(3,6) \subset \mathbb{P}^{13}$	$G_2/P \subset \mathbb{P}(\mathfrak{g}_2)$

and show nodal degenerations.



Title：Uniruledness of M_{11}, and prime Fano 3-folds V_{22} of genus 12

Abstract：Let M_g be the moduli space of curves of genus g >1 and K_g the locus of those which are embeddable in K3 surfaces. K_g is of expected dimension min{3g-3, g+19} except for g = 10, 12. In the case of genus 11, this implies the uniruledness of M_{11} (Mori-M. 1983). In the case of genus 12, 30-dimensional K_{12} contains four subloci [Clebsch-Luroth], [2.7], [3.5] and [4.4], corresponding to four 1-nodal degenerations of V_{22}. If a curve C in K_{12} belong to none of these loci, then C is the intersection of two anticanonical members of a Fano 3-fold V_{22} in a unique way (the last linear section theorem).

講演題目：M_{11}の単線織性と種数12の3次元素Fano多様体

講演概要：K3曲面に埋込可な種数 g>1 の曲線の軌跡 K_g の次元は、二つの例外 g=10, 12 を除いて、期待される値 min{3g-3, g+19} に一致する。種数11の場合、これより曲線のモジュライM_{11}の単線織性が従う (Mori-M. 1983)。例外の一つである種数 12 の場合、30次元の軌跡 K_{12} は、V_{22}の4つの1-nodal な退化に対応して、4個の因子[Clebsch-Luroth], [2.7], [3.5] and [4.4]を含む。K_{12}内の曲線 C は、これらの因子に属さなければ、非特異な V_{22} の二つの反標準因子の交わりである（最後の線型切断定理）。