Kummer quartics and associated symplectic 6-folds

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Abstract: Kondo(1998) described the automorphism group of the generic Jacobian Kummer surface Km(C) using a Conway-Borcherds chamber in the nef cone. As a sample of higher dimensional analogue, we describe the binational automorphism group of a certain holomorphic symplectic 6-fold associated with Km(C) using the Borcherds(2000) reflection group. Bir-Aut is generated by 864 involutions and an extended extraspecial group 2^{1+8} .2 whose center is Rapagnetta(2007)'s involution.

§1 Motivation/Introduction

General Study Bir-Aut(IHS) similarly to Aut(K3), replacing the nef cone Nef(K3) with the movable cone Mov(IHS)

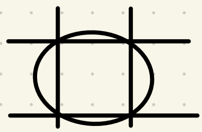
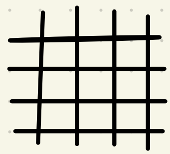
Peculiar (today) **1**. Higher dimensional analogue of Kondo(1998)'s description of Aut(generic Jacobian Kummer)
2. 5 reflection groups in **R**^{1,17}

L := II_{1,17}(2^{+2a}), a = 0, 1, 2, 3, 4
even integral lattice of signature (1, 17), 2-elementary, and

$q_L : L^\vee/L = \mathbf{Z}/2\mathbf{Z}^{2a} \to \mathbf{Q}/2\mathbf{Z}$ is even

a = 0, 1, 2 ⇄ L is realized as the Picard lattice of a K3

Table

2a	0	2	4	*today* ↓ 6	8
L	unimodular U+E8+E8	U+E8+D8	U+D8+D8	U+**Kum**	U+**BW**
realization	$S \longrightarrow \mathbf{P}1$ with $2\widetilde{E8}$	double **P**1×**P**1	Km(E1×E2)	today	not (yet)
# of (-2)'s and (-4)'s	19 0	20 2	24 24	64 896	0 ∞
branch	/	 Barth-Peters surface		/	/

Fact: Reflections generate a subgroup of finite index in $O_{\mathbb{Z}}(L)$ in each case.

Remark II_{1, 17}(2^{+2}), 2a=2, belongs to the Conway-Vinberg chain, on which I gave a talk in the last JES(2022).

Notation $U = (\mathbf{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$,

An, Dn, E6, E7, E8: negative definite root lattices, $\widetilde{An}$ etc.: affine

negative definite rank 16 $\begin{cases} \mathbf{Kum} \supset 16A1 & \text{Kummer lattice} \\ \mathbf{BW} & \text{Barnes-Wall lattice, (Leech lattice)} \end{cases}$ 2A- inv.

Borcherds(2000, 2a=6)

$$(*) \qquad O^{+}_{\mathbb{Z}}(II\_\{1,17\}(2^{+6})) = \begin{pmatrix} \text{group generated by} \\ \text{960 reflections} \end{pmatrix} \rtimes G.L_4(2)$$

$G.L_4(2)$ = (symmetry of fundamental domain), #G = 2^10, $L_4(2) \cong \mathfrak{A}_8$

Today Give a geometric interpretation of (*), that is, describe Bir-Aut(X) for holomorphic symplectic 6-fold X=X(S, h) associated with (Picard general) Kummer quartic surface (S, h).

$$S = \mathrm{Km}(C) \qquad \rho(S)=17,\ h\text{: pull-back of } O_{\mathbf{P}}(1)$$
$$\downarrow$$
$$\mathrm{Jac}\ C/\pm 1 \xhookrightarrow[|2\Theta|]{} \mathbf{P}^3$$

Kum is an overlattice of 16A1 = $\bigoplus_{i \in T_{(2)}} \mathbf{Z}e_i$, $(e_i^2) = -2$

$T = \mathbf{C}^2/\Gamma$, $\Gamma \cong \mathbf{Z}^4$, $T_{(2)}$: group of 2-torsions

Kum = $\bigoplus_{i \in T_{(2)}} \mathbf{Z}e_i + \mathbf{Z}\left\{ \frac{1}{2}\sum_H e_i \,\middle|\, H \subset T_{(2)} \text{ subgroup of order 8, 16} \right\} \subset \bigoplus_{i \in T_{(2)}} \mathbf{Q}e_i$

$\bigcup$

BW*) sublattice of index 2 and of Leech type, i.e., $\nexists$ (-2)-element

§2 Another background

$X_a = M_S(0, h, a)$, $a = 0, 1$, holomorphic symplectic 6-fold

$\begin{cases} \text{birationally equivalent to } S^{[3]} & \text{when } a = 1 \\ \text{not equivalent} & \text{when } a = 0 \end{cases}$

Birational double cover of $\widetilde{K}_{Jac}(0, 2\Theta, 2a)$, which is OG6
(symplectic resolution of Albanese fibration

$M_{Jac}(2v) \longrightarrow \mathrm{Jac} \times \widehat{\mathrm{Jac}}$, $v = (0, \Theta, a)$

of moduli of sheaves on Jac = Jac C. (Studied by Rapagnetta(2004, thesis), Mongardi-Rapagnetta-Sacca(2018), etc.)

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*) **BW** is also constructed from the Reed-Muller code of length 16 (cf. [Conway-Sloane]).

$\xi \in M_S(0, h, a)$ torsion sheaf on S

$$\begin{array}{ccc} \xi & \in & M_S(0,h,a) \\ \downarrow & & \downarrow \\ \mathrm{Supp}\,\xi & \in & |h| \cong \mathbf{P}^3 \ni D, \quad p_a(D) = 3 \end{array}$$

This morphism is a Lagrangian fibration. Generic fiber is $\mathrm{Pic}^a D$ (Abelian 3-fold).

(Pic X, Beauville form) $\cong (0, h, a)^\perp$ in $\mathbf{Z} \oplus \mathrm{Pic}\, S \oplus \mathbf{Z}$

$$= \begin{cases} \mathrm{Pic}\, S + \langle -4 \rangle & \text{when } a = 1 \\ U + (h^\perp \text{ in } \mathrm{Pic}\, S) = U + \mathbf{Kum} & \text{when } a = 0 \end{cases}$$

$X := X_0 = M_S(0, h, 0)$ from now on

Advantage of X: Fourier-M. transform with Kernel $\mathcal{P}$ Poincaré line bdl

descends to $S \times S = S \times M_S(2, -\frac{1}{2}\sum_1^{16} e_i, -2)$

$\downarrow$

Jac $\times$ Jac

$\Phi_1 = \Phi_{|H_1|} : X \longrightarrow \mathbf{P}^{3,*} \quad S, \qquad \xi \mapsto \mathrm{Supp}\, \xi$

interchanged by FM transf.

$\Phi_2 = \Phi_{|H_2|} : X \longrightarrow \mathbf{P}^3 \quad \hat{S} = S, \quad \xi \mapsto \mathrm{Supp}\, \hat{\xi}$

Pic $X \supset U(2) + \mathbf{BW}$

$\langle H_1, H_2 \rangle$

$\uparrow \qquad \uparrow$

$+1 \qquad -1$

Rapagnetta's symplectic birational involution $\longrightarrow$

covering involution of $\begin{array}{c} K3^{[3]} \\ \downarrow \\ OG6 \end{array}$

$\otimes\, \mathcal{O}_S(-\frac{1}{2}\sum_1^{16} e_i)$

§3 Main Theorem

960 reflections = 64 (-2) refl's and 896 (-4) refl's

◇ (-2)'s : ∃ effective irreducible divisors $D \subset X$ with Beauville norm $(D^2)=-2$

Kummer $(16_6 - 16_6)$ configuration

16 (-2)**P**1's $\subset$ S nodes, $\Phi_1 : X \longrightarrow$ **P**^3,* has 16$\tilde{A}1$

16 (-2)**P**1's $\subset$ S tropes, $\Phi_2 : X \longrightarrow$ **P**^3 has 16$\tilde{A}1$

64 (-2) effective divisors = 32 $\tilde{A}1$ 

◇ (-4)'s: ∃ (-4)-divisor classes with divisibility 2, #=896=32×28

864 corresponding to birational involutions of X
32 represented by irreducible effective divisors on X

Main Theorem (S, h): Picard general Kummer, $X = M_S(0, h, 0)$

(1) Bir-Aut(X) = (group generated by 864 involutions) $\rtimes$ G, where G is Borcherds' group of order 2^10. (Corrected on Aug. 18, 2024.)

(2) (Geom. interpretation of G) G is the semi-direct product $H.C_2$ of an extraspecial group, i.e., finite analogue of Heisenberg group,

$1 \longrightarrow$ Rapagnetta's inv. $\rightarrow H \rightarrow (Jac \times \widehat{Jac})_{(2)} \longrightarrow 0$

by C_2 generated by FM transf. (cf. Mongardi-Wandel(2017)).

~~(3) ∃ 28 curves of genus 2 $C=C_1, C_2, \dots, C_{28}$ such that $X \cong M_{S_i}(0, h_2, 0)$ for i=1, ..., 28 (by Torelli type theorem). Each C_i, i≠1, gives 32 birational involutions of X. Hence we have 864 involutions. 16 (co-)tropes of Km(C) yields 32 irred. divisors.~~

References

§1 Motivation/Introduction

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§2 Another background

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§3 Main Theorem

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