Holomorphic symplectic 6-folds associated with Kummer quartics

10/21/24(M), Beijing
Sino-French Conf. on Alg. Geom.

Abstract: The orthogonal group of the lattice II_{1,17}(2^{+6}) has a fundamental domain with 896+64 facets by Borcherds(2000). This is the Picard lattice (w.r.t. Beauville form) of the holomorphic symplectic 6-fold Jac lhl of a very general Kummer quartic surface (S, h). I will explain the following

Theorem: The birational automorphism group Bir(Jac lhl) is generated by 864 modified reflections among the 896 above and a 2-group whose center is Rapagnetta(2007)'s involution.

Corrected on 2/22/25(S).

Motivation

General	Aut(K3) nef cone	$\xrightarrow{\text{h.d.}}$	Bir(IHS) movable cone
Peculiar	Kondo[K]	$\rightsquigarrow$	Main Thm (§3)
(today)	Aut(Km C)		

$$= \left\langle \begin{matrix} \text{32 projections, 32 collineations} \\ \text{60 Gopels, 192 Webers} \end{matrix} \right\rangle \rtimes 2^{4+1}$$

2^{4+1} ↑ translations and switches

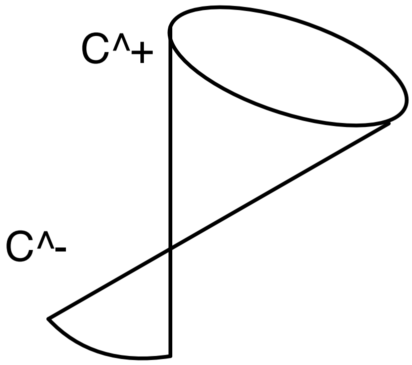
§1 Reflective lattices

(L, < >) lattice, i.e., free Z-module L with < > : L × L → Z
sgn (1,*), {<x^2> > 0} = C^+ ⊔ C^- ⊂ L⊗R positive cone C^+

Reflection by m with <m^2> < 0 ε O^+(L)

r_m : x ⟶ x - 2<x,m>m/<m^2>

is a *reflection* of L if r_m(L)⊂ L.



L is *reflective* $\Leftrightarrow$ Subgroup $\langle$all reflections r_m of $L\rangle$ is of finite index in $O^+(L)$.

Exists only up to rk $L = 22$ (Esselmann[E]).
Conway-Vinberg[CS] chain $U+D_{20}$ maximal
$\cancel{U+D_{19}}$
$U+D_{18}$
$U+D_{17}$
are reflective.

Notation A_n, D_n, $E_{\{6,7,8\}}$ negative root lattices, $U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, $\mathrm{Disc}(L) = \mathrm{Coke}[L \to \mathrm{Hom}(L, \mathbb{Z})]$ (with $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form), $II_{1,*}(**)$: even lattice of signature $(1, *)$ with discriminant type $**$.

§2 Known geometric realization

Picard #
$\rho=22$ $U+D_{20} = II_{1,21}(2^{+2})$
$\rho=20$ $U+D_{18} = II_{1,19}(2^{+2})$
$U+D_{17}$ not 2-elementary
$\rho=18$ $U+E_8+E_8$ $U+D_{16}$ $U+D_8+D_7$ $U+\dots$
$\| \quad \| \quad \| \quad \|$
$II_{1,17}$ $II_{1,17}(2^{+2})$ $II_{1,17}(2^{+4})$ $II_{1,19}(2^{+6})$

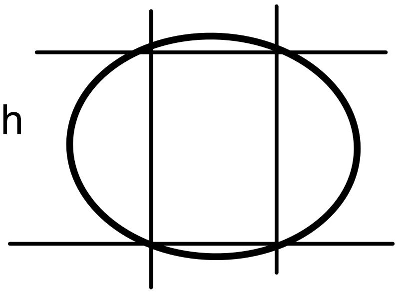
today

C-V chain

$II_{1,17}(2^{+2a})$ is realized as $\mathrm{Pic}(S)$, S: K3, and gives explicit description of $\mathrm{Aut}(S)$ for $a = 0,1,2$.

0) $II_{1,17}$ $S \to \mathbb{P}^1$ elliptic K3 with 2 singular fibers of type E_8, $\mathrm{Aut}(S)$ is finite

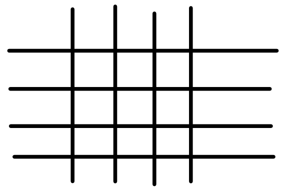
1) $II_{1,17}(2^{+2})$ $S \to \mathbb{P}^1 \times \mathbb{P}^1$ with branch



$\mathrm{Aut}(S) \sim \mathbb{Z}$ (Horikawa, Dolgachev, Brth-Peters, Namikawa-M. etc.). Here "$\sim$" means "the same up to finite groups".

2) $II_{1,17}(2^{+4})$ $S = \mathrm{Km}(E_1 \times E_2)$ Kummer of product type

$$\begin{array}{ccc} & S = \mathrm{Km}(E_1 \times E_2) & \\ \text{double cover} & \downarrow & \\ & P^1 \times P^1 & \text{branch} = \end{array}$$



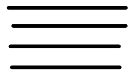
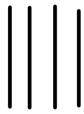
$$O^+(II_{1,17}(2^{+4})) \approx \left\langle \begin{array}{l} 24 \ \ (-2)\text{-reflections} \\ 24 \ \ (-4) \text{ reflections} \end{array} \right\rangle \rtimes 2^4.\mathfrak{S}_{4.4}$$

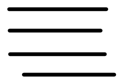
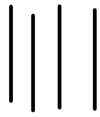
nef cone $\supset$ (Conway-Borcherds domain[B1])

bounded by (24+24) walls

(-2)-walls $\Leftrightarrow$ basic (-2)P^1's on S

$$24 \begin{cases} 8 & \text{strict transforms of} \\ 16 & \text{exceptional divisors of 16 nodes at the intersection of} \end{cases}$$

 & 

 & 

(-4)-walls $\Leftrightarrow$ Each (-4)-reflection is realized by an involution of S coming from a Cremona transformation of $P^1 \times P^1$, after multiplying a suitable $g \in \mathfrak{S}_{4,4}$.

Thm(Keum-Kondo[KK]) $\mathrm{Aut}(\mathrm{Km}(E_1 \times E_2))$ for general E_1 and E_2 is generated by the above 24 involutions.

3) The next lattice $II_{1,17}(2^{+6})$ is no more Pic (K3), but …

§3 Main theorem

$II_{1,17}(2^{+6}) = \mathrm{Pic}(\mathrm{Jac}^2 |h|)$ with Beauville form

(S, h): min. res. of Kummer quartic $\mathrm{Km}(C) \subset P^3$ embedded by $|2\Theta|$

$$\begin{array}{ccccc} \mathrm{Jac}^d |h| & = & \coprod_{D \in |h|} & \mathrm{Jac}^d D & \\ \downarrow & & & \mapsto & \\ |2\Theta| & & \ni & [D] & g(D)=3 \end{array}$$

$\mathrm{Jac}^d |h| = M_S(0, h, d-2)$ hol. symplectic 6-fold of $K3^{[2]}$-deformation type

$\begin{cases} d\text{: odd} & \text{birationally equivalent to } (\mathrm{Km}\, C)^{[2]} \\ d\text{: even} & \text{not} \end{cases}$

$X := \mathrm{Jac}^2 |h| = M_S(0, h, 0)$

$\mathrm{Pic}\, X = (h^{\perp} \text{ in } U + \mathrm{Pic}\, S) = U + \mathrm{Kum}$, where Kum denotes the (negative definite) Kummer lattice

Thm(Borcherds[B2]):

$O^+(II_{1,17}(2^{+6})) = \left\langle \begin{matrix} 896\ (-4)\text{-reflections} \\ 64\ (-2)\text{-reflections} \end{matrix} \right\rangle \rtimes (2\text{-group}).\mathfrak{S}_8$

Main Thm: Birational automorphism group Bir(X) is the semi-direct product

$\langle 864\ (\mathfrak{S}_8\text{-modified})\ (-4)\text{-reflections} \rangle \rtimes (2\text{-group})$,

where the 2-group is an extended extraspecial group $2^{1+8}.2$ with center Mongardi-Rapagnetta-Sacca[MRS] involution.

§4 Sketch of proof

Similar to §2 (case of $\mathrm{Km}(E_1 \times E_2)$), that is, find a geometric meaning for each reflection r_m: either

$\begin{cases} \text{the mirror } m \text{ is represented by an effective divisor, or} \\ r_m \text{ is represented by an invoultion.} \end{cases}$

One difference is that not only (-2) classes but some of (-4)-classes are also represented by effective divisors.

In our case,

64 (-2)-classes come from Kummer's $(16_6\text{-}16_6)$ configuration and their *complements*. These divisors appear in (16+16) reducible fibers of two Lagrangian fibrations in pairs:

$X = Jac^2\ |h|$ ↓ $P^{4,*}$ and its Fourier-M. transform $\hat{X} = Jac^2\ \widehat{|h|}$ ↓ P^4.

896 (-4)-reflection (This part is subtle.)

Claim: 32 (-4)-mirrors are represented by effective divisors others are not

By general theory of MMP for M_S(v) for v=(r, *, s) (Bayer-Macri[BM], Hassett-Tschinkel[HT]), there are 3 types of divisorial contractions:

- **Brill-Noether** induced from rigid objects like (-2)P^1's on S
- **Hilbert-Chow** S^[n] → S^(n) is typical.
- **Li-Gieseker-Uhlenbeck** Original one is the contraction (moduli of Gieseker-semi-stable rank 2 sheaves) → (Uhlenbeck-Yau compactification of μ-stable moduli).

(HC)-type does not happen in our case since M. vector v=(0,h,0) has divisibility 2.

(LGU)-type happens for 32 (-4)-mirrors in our case in the following manner:

$$D - D \subset Jac^0\ D \subset Jac^0\ |h| \supset \coprod_{D \in |h|} D - D =: \mathfrak{D} \searrow S \times S$$

$$\downarrow \qquad \downarrow$$

$$[D] \quad \in \quad |h|$$

from divisor (p)-(q) on a curve D, forget D

The divisor $\mathfrak{D}$ is contracted to S x S [redacted] by the extremal contraction $X = Jac^0\ |h| \to \bar{X}$ of (LGU)-type.

Let t be one of 16 tropes of $S \subset P^3$.
Tensoring the line bundle $O_S(-t)$ we obtain an isomorphism

$$Jac^2 |h| \xrightarrow{\sim} Jac^0 |h|.$$

Hence we have 16 divisorial contractions and another 16's by replacing $Jac^0 |h|$ by $Jac^0 |\hat{h}|$.

The remaining 864 (-4)-reflections are realized by ~~involutions~~ (automorphisms after multiplying suitable $g \in \mathfrak{A}_8$) by virtue of Torelli-type theorem. (Explicit constructions are desirable.)

References

[K] Kondo, S.: The automorphism group of a generic Jacobian Kummer surface, J. Alg. Geom. **7**(1998), 589-609.

§1 [E] Esselmann, F.: Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe, J. Number Th., 61(1996), 103-144.

[N2] Nikulin, V.: Integral symmetric bilinear forms and some of their applications, Math. USSR Izv., **14**(1980), 105-167.

[CS] Conway, J.H. and Sloane, N.J.A,: Leech roots and Vinberg groups, Proc. Roy. Soc. London, A384(1983), 233-258.

§2 [B1] Borcherds, R.: Automorphism groups of Lorentzian lattices, J. Algebra 111(1987), 133-153.

[KK] Keum, J. and Kondo, S.: The automorphism groups of Kummer surfaces associated with the product of two elliptic curves, Trans. AMS., **353**(2000), 1469-1487.

§3 [N1] Nikulin, V.: On Kummer surfaces, Math. USSR Izv., **9**(1975), 261-275.

[B2] Borcherds, R.E.: Reflection groups of Lorentzian lattices, Duke Math. J., **104**(2000), 319-366.

[MRS] Mongardi, G., Rapagnetta, A. and Sacca, G.: The Hodge diamond of O'Grady's six-dimensional example, Comp. Math. **154**(2018), 984-1013.

§4 [BM] Bayer, A. and Macri, E.: MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations, Invent. math. **198**(2014), 505-590.

[HT] Hassett, B. and Tshinkel, Y.: Extremal rays and automorphisms of holomorphic symplectic varieties, in "*K3 surfaces and their moduli*", Progr. in Math., 2016, pp. 73-95.