

Cubic 4-folds of Mathieu type

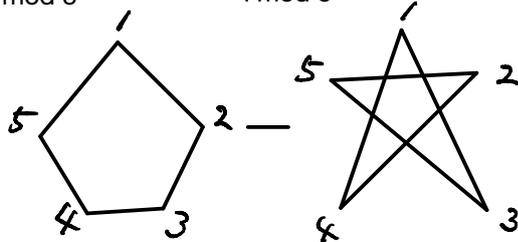
Quiz What is * in the sequence
0, 1, 1, 3, 5, *, 21, 43,

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char. $p=3$

Main Thm \exists cubic 4-fold PP_3 over \mathbf{F}_3 with $\text{Sing}(PP_3) = \{11 \text{ cusps}\}$ and an action of the Mathieu group M_{11} over \mathbf{F}_3 .
Equation in \mathbf{P}^5 is given by

$$z^3 = \sum_{i \bmod 5} x_{i-1} x_i x_{i+1} - \sum_{i \bmod 5} x_{i-2} x_i x_{i+2} \quad (\star)$$



(Pentagon-Pentagram).

char. $p=2$

Remark 1 (Jónsson-McKay 1976) The Mathieu group M_{22} acts on the Fermat (or Hermitian) cubic 4-fold over \mathbf{F}_4 .

Notation cusp = simple singularity of type A_2

$\sim z^3 = q(x)$ for a non-deg. quadratic form q

§1 Mathieu groups and Goley codes

§2 Proof of Main Thm

§3 [Jónsson-McKay] revisited

§4 Toward a geometric construction of McL and Co_2

References (10 items)

§1 Mathieu groups and Golay codes

- ① $M_{24} \curvearrowright \Omega = \{1, 2, \dots, 24\}$ 5-ply transitive
 $M_{24} \supset M_{23} \supset M_{22} \supset M_{21} = L_3(4)$ (stabilizer sequence)
 M_{24} preserves the Steiner system $St(5, 8, 24) \subset \mathbf{P}_8(\Omega)$
 $\#St(5, 8, 24) = (24 \text{ choose } 5)/(8 \text{ choose } 5) = 759$

- ② Binary Golay code $[24, 12, 8]$, 12-dim'l with min. weight 8
 $\mathbf{F}_2^{24} \supset Gol_2 :=$ subspace generated by 759 octads

$$\begin{array}{cccccc} 2^{12} = & 1 & + & 759 & + & 2576 & + & 759 & + & 1 \\ \text{weight} & 0 & & 8 & & 12 & & 16 & & 24 \end{array}$$

- ③ Fix one Ξ among the 2576 dodecads. Then the Mathieu group M_{12} acts on Ξ preserving a new Steiner system $St(5, 6, 12)$.
 $\#St(5, 6, 12) = (12 \text{ choose } 5)/(6 \text{ choose } 5) = 132$.
 $M_{12} \supset M_{11} \supset M_{10} = \mathcal{M}_{12} \supset M_9 = 3^2.Q_8$

- ④ Ternary Golay code $[12, 6, 6]_3$, 6-dim'l with min. weight 6

$M_{12} \curvearrowright \mathbf{F}_3^{12} \supset Gol_3$ M_{12} -invariant subspace

$$\begin{array}{cccccc} 3^6 = & 1 & + & 264 & + & 440 & + & 24 \\ \text{weight} & 0 & & 6 & & 9 & & 12 \end{array}$$

signed hexads s.t. $\text{Supp}(\text{signed hexads}) = St(5, 6, 12)$

Let \mathbf{P}^5 be the projectivization of the space of cocodes, i.e., $(\mathbf{F}_3^{12})/Gol_3$. This \mathbf{P}^5 contains the images e_0, e_1, \dots, e_{11} of standard basis, which are permuted by M_{12} .

Main Thm (cont'd) Our cubic 4-fold PP_3 lives in this \mathbf{P}^5 and has cusps at e_1, \dots, e_{11} . The remaining point e_0 is the unique *inseparable center* (outside PP_3), that is, the projection $PP_3 \rightarrow \mathbf{P}^4$ from e_0 is inseparable, and defined by (\star).

Coordinates: 11 cusps $(x_0 : \dots : x_4; z) =$
 $1(10000; 0), 2(01000; 0), 3(00100; 0), 4(00010; 0), 5(00001; 0), 6(-1-1-1-1-1; 0),$
 $a(01-1-11; 1), b(101-1-1; 1), c(-1101-1; 1), d(-1-1101; 1), e(1-1-110; 1)$
 and the inseparable center $f(00000; 1)$ in the notation of Todd(1959).

§2 Proof of Main Thm

Step 1. Klein's cubic 3-fold $Kl_3 : x^2y + \dots + v^2x = 0 \subset \mathbf{P}^4$ has an action of $L_2(11)$ over $\mathbf{F}_{\{3^5\}}$. Hence the inseparable cubic covering $3\mathbf{P}^4_{KL}$ of \mathbf{P}^4 with branch Kl_3 does so.
 Step 2. Get PP_3 with an action of $L_2(11)$ from $3\mathbf{P}^4_{KL}$ by change of variables with the Vandermonde intertwiner of Adler(1997) and show the action is defined over \mathbf{F}_3 .
 Step 3. This $L_2(11)$ -action extends to an M_{11} -action.*)

§3 [Jónsson-McKay] revisited ($p = 2$)

Fermat (or Hermitian) cubic 4-fold $\sum x_i^3 = 0$, or more conveniently,

$$V : (x^2u + y^2v + z^2w) + (xu^2 + y^2v + z^2w) = 0 \subset \mathbf{P}^5$$

contains 891 planes. Fix a disjoint pair, say, $P: x=y=z=0$ and $Q: u=v=w=0$, blow up along them and we have the diagram:

$$\begin{array}{ccc}
 & \text{Bl}_{P \perp Q} V & \\
 \swarrow & & \searrow \text{b.u.} \\
 \mathbf{P}^2 \times \mathbf{P}^{\{2,*\}} & & V \\
 \cup & &
 \end{array}$$

$$S : x^2u + y^2v + z^2w = xu^2 + y^2v + z^2w = 0$$

The left arrow is also a blow-up. Its center S is $K3^{ss1}(p=2)$, i.e., the supersingular $K3$ surface with Artin invariant 1 in characteristic 2. S is characterized by these two properties: Picard number is 22 (maximal) and discriminant of $\text{Pic } S$ is 2^2 (minimal).

*) $L_2(11) := \text{PSL}(2, \mathbf{F}_{11})$ is a subgroup of M_{11} of index 12.

Remark 2 $K3^{ss1}(p=3)$ is also unique. It is isomorphic to the Fermat (or Hermitian) quartic surface

$$\text{Fer}_4 : x^4 + y^4 + z^4 + t^4 = 0 \subset \mathbf{P}^3.$$

Returning to $p=2$, there exists a set of 22 planes P_1, \dots, P_{22} preserved by M_{22} -action (cf. [ATLAS, p.39]).

Description

$$22 = 6 + 6 + 10 \quad \text{orbit decomposition by } \mathcal{U}_6 \subset M_{22}$$

① First 6 planes in V correspond to linearly general 6 points p_1, \dots, p_6 of $\mathbf{P}^2(\mathbf{F}_4)$.

$$P_i = \langle p_i, (p_i)^\perp \rangle$$

② Second 6 planes correspond to the 6 lines l_1, \dots, l_6 of $\mathbf{P}^2(\mathbf{F}_4)$ which does not pass through any of p_1, \dots, p_6 .

$$P_{\{i+6\}} = \langle l_i, [l_i] \rangle$$

③ $6=3+3$ A pair of triangles $\Delta_{i,j,k}$ and $\Delta_{l,m,n}$ generates an elliptic pencil like $\langle x^3+y^3+z^3, xyz \rangle$ and yields a plane in V . Hence we have 10 more such planes P_{13}, \dots, P_{22} .

Hyperoval p_1, \dots, p_6 in ①, its dual l_1, \dots, l_6 in ② and Hesse pencils in ③ are key ingredients to study the $K3$ surface $S = K3^{ss1}(p=2)$ (see Dolgachev-Kondō(2003)).

Answer to quiz $*$ = 11. The sequence defined by the recurrence relation $2J_{\{n-2\}} + J_{\{n-1\}} = J_n$ is called *Jacobsthal* sequence (OEIS A001045). $\{J_n\}$ is the half of primitive middle Betti number of cubic hypersurfaces, which is 11 for cubic 4-folds.

§4 Toward a geometric construction of McL and Co_2

26 sporadic simple groups

Happy family	$\left\{ \begin{array}{l} 5 \\ 7 \\ 8 \end{array} \right.$	5 Mathieu groups	M_{24}, M_{23}, M_{22}	1st gen.
			M_{12}, M_{11}	
		McLaughlin , HS		2nd gen.
		$\text{Co}_{1,2,3}, \dots$		
		Monster ,		3rd gen.
		$\text{Fi}_{24,23,22}, \dots$		

6 pariah

Conj. 1 ($p=3$) Two symplectic 8-folds of $K3^{[4]}$ -type $\text{LLSvS}(\text{PP}_3 \subset \mathbf{P}^5)$ and the moduli space of vector bundles $M_{\text{Fer}}(3, \alpha, -3)$ are birational, where α is a certain divisor class of the Fermat quartic surface Fer_4 (cf. Remark 2) with self-intersection number $(\alpha^2) = -12$.

Conj. 2 ($p=3$) The symplectic 8-folds in Conj. 1 have a birational action of the sporadic simple group McL , since $\text{LLSvS}(\text{PP}_3)$ and $M_{\text{Fer}}(3, \alpha, -3)$ have that of M_{11} and $U_4(3)$, respectively. (Both are maximal subgroups of McL [ATLAS, p.100].)

Situation becomes more vague in characteristic 2. Even the dimension x is unknown.

Problem ($p=2$) Find a symplectic x -fold with birational action of the 2nd Conway group Co_2 or formulate a conjecture on it replacing M_{11} and $U_4(3)$ in Conj. 2 with M_{22} (or more precisely $2^{10}.M_{22}$) and $U_6(2)$, respectively (cf. [Nagoya22]).

Postscript (Feb. 27, 2026) See [JE-symplectic, §6] for a similar conjectural relation between a symplectic 6-fold in characteristic 2 and the Higman-Sims group HS.

References

(Introduction)

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I. Dolgachev and S. Kondō, A supersingular K3 surface in characteristic 2 and the Leech lattice, Int'l Math. Res. Notices **2003** (2003), 1–23. [$K3^{ss}(p=2)$ as c.i. in $\mathbf{P}^2 \times \mathbf{P}^2$.]

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[JE-symplectic] S. Mukai, Vinberg-Conway chain terminates just after a supersingular symplectic 6-fold with 100 (-2) divisors, talk at Waseda Univ., March, 2022. https://www.kurims.kyoto-u.ac.jp/~mukai/note/JE-symplectic_2022.pdf

[Nagoya22] ---, Fano 3-folds, Lagrangian fibration, and a supersingular OG10 with Co_2 configuration, talk at Nagoya Univ., Nov., 2022, the same homepage.

Abstract: It is known that the Fermat cubic 4-fold has an action of the Mathieu group M_{22} in characteristic 2. As a characteristic 3 analogue I construct a cubic 4-fold PP_3 over the ternary field \mathbf{F}_3 with the following property: 1) $Sing(PP_3)$ consists of 11 cusps. 2) The smallest Mathieu group M_{11} acts on PP_3 over \mathbf{F}_3 permuting the 11 cusps. If time permits I will discuss what is permuted under the action of M_{22} via the supersingular K3 surface with Artin invariant 1, and also an application of our M_{11} -action.